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Existence of Spanning $\mathcal{F}$-Free Subgraphs with Large Minimum Degree

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Let $\mathcal{F}$ be a family of graphs and let d be large enough. For every d-regular graph G, we study the existence of a spanning $\mathcal{F}$-free subgraph of G with large minimum degree. This problem is well understood if $\mathcal{F}$ does not contain bipartite graphs. Here we provide asymptotically tight results for many families of bipartite graphs such as cycles or complete bipartite graphs. To prove these results, we study a locally injective analogue of the question.

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1. Introduction

Let $G = (V, E)$ be a d-regular graph on n vertices. It is well known that every d-regular graph has a spanning triangle-free subgraph H with minimum degree at least $d/2$. Let $\mathcal{F}$ denote a family of graphs. We say that G is $\mathcal{F}$-free if, for every $F \in \mathcal{F}$, G does not contain any subgraph isomorphic to F. In this paper we study which is the largest minimum degree of a spanning $\mathcal{F}$-free subgraph H of a d-regular graph G.

Let $\mathrm{ex}(n, \mathcal{F})$ be the maximum number of edges in an $\mathcal{F}$-free graph on n vertices. Since the complete graph K_n on $n = d + 1$ vertices is d-regular, the minimum degree of a spanning $\mathcal{F}$-free subgraph H of K_{d+1} is at most $2\,\mathrm{ex}(d+1, \mathcal{F})/(d+1)$. In [10], Foucaud, Krivelevich and Perarnau conjectured the following.

Conjecture 1.1 ([10]). *For every family $\mathcal{F}$ there exists a constant $c_{\mathcal{F}} > 0$ such that, for every $d \geqslant 1$ and every d-regular graph G, there exists a spanning $\mathcal{F}$-free subgraph H of G with minimum degree*

$$\delta(H) \geqslant c_{\mathcal{F}} \cdot \frac{\mathrm{ex}(d, \mathcal{F})}{d}.$$

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Here and throughout the paper, for any two functions f, g that depend on several variables, including d, we will use the standard asymptotic notation $g = \Omega(f)$ to denote the following: given that all variables apart from d are fixed, we have $\liminf_{d\to\infty}(g/f) > 0$. We also use $g = O(f)$ to denote $f = \Omega(g)$, and $g = \Theta(f)$ if both $g = \Omega(f)$ and $g = O(f)$ hold. For instance, Conjecture 1.1 can be written as follows: given a family $\mathcal{F}$, for every d and every d-regular graph G, there exists a spanning $\mathcal{F}$-free subgraph H of G with minimum degree $\delta(H) = \Omega(\mathrm{ex}(d, \mathcal{F})/d)$.

If the chromatic number of $\mathcal{F}$[1] is at least 3 (*i.e.* there are no bipartite graphs in $\mathcal{F}$) it is easy to verify the conjecture (see Proposition 1 in [10] for a precise result).

The family $\mathcal{F}_g = \{C_3, \dots, C_{g-1}\}$ is of special interest since a graph H is $\mathcal{F}_g$-free if and only if H has girth at least g. Even if the order of $\mathrm{ex}(d, \mathcal{F}_g)$ is not known in general, there are some partial results towards Conjecture 1.1 for $\mathcal{F}_g$. Kun [15] showed that for every $g \geqslant 4$, every d-regular graph G admits a spanning $\mathcal{F}_g$-free subgraph with minimum degree $\Omega(d^{1/g})$. Foucaud, Krivelevich and Perarnau [10] improved the lower bound on the minimum degree to $\Omega(\mathrm{ex}(d, \mathcal{F}_g)/(d \log d))$. This shows that Conjecture 1.1 holds up to a logarithmic factor for $\mathcal{F}_g$.

In this paper we prove Conjecture 1.1 for a large number of families $\mathcal{F}$ with chromatic number 2, which in particular includes $\mathcal{F}_g$.

Before stating our main result we need to introduce some definitions. For any two graphs F and J, we say that $\varphi : V(F) \to V(J)$ is a *homomorphism* if, for every $uv \in E(F)$, we have $\varphi(u)\varphi(v) \in E(J)$. We say that φ is *locally injective* if, for every $v \in F$, the restriction of φ onto the neighbours of v in F is injective. In other words, if u_1 and u_2 are neighbours of v in F, then $\varphi(u_1) \neq \varphi(u_2)$. Let $\hom^*(F, J)$ be the number of locally injective homomorphisms from F to J. Observe that the condition $\hom^*(F, J) = 0$, for every $F \in \mathcal{F}$, is stronger than $\mathcal{F}$-freeness. Any copy of F in J induces an injective homomorphism from F to J which, in particular, is also locally injective.

For every graph J, we denote by $\delta(J)$ its minimum degree and by $\Delta(J)$ its maximum degree. For every $\beta \geqslant 1$, we say that J is *β-almost regular* if $\Delta(J) \leqslant \beta\delta(J)$.

Our first result is a locally injective version of Conjecture 1.1.

Theorem 1.2. *Let $\mathcal{F}$ be a family of graphs, $\epsilon > 0$, $\beta \geqslant 1$ and $\alpha \geqslant 25\beta^2$, and let d be large enough. If J is a graph on αd vertices such that:*

– J is β-almost regular,
– J has minimum degree $\delta(J) > d^{\epsilon}$, and
– $\hom^*(F, J) = 0$, for every $F \in \mathcal{F}$,

then, for every d-regular graph G there exists a spanning subgraph H of G with $\delta(H) = \Omega(\delta(J))$ and $\hom^*(F, H) = 0$, for every $F \in \mathcal{F}$.

The proof goes as follows. Initially, we select a bipartite subgraph G' of G with stable sets A and B and large minimum degree. The main part of the proof consists in finding a vertex colouring χ using as a colour palette $V(J)$. We colour A in a two-step procedure, the

[1] The chromatic number of $\mathcal{F}$ is the smallest chromatic number of $F \in \mathcal{F}$.

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first random and the second deterministic. At the same time we delete some edges of G' to obtain a subgraph H' that has large minimum degree and where vertices in B have rainbow coloured neighbourhoods. We colour B afterwards via an iterative colouring procedure which also finishes with a deterministic step. Again, during the colouring procedure we delete some edges to obtain a subgraph H that has minimum degree $\Omega(\delta(J))$ and where all the neighbourhoods are rainbow coloured. Moreover, the colouring satisfies that $\chi(a)\chi(b)$ is an edge of J for every edge ab of H. Thus, χ naturally provides a locally injective homomorphism from H to J. Since J has no locally injective copies of F, neither does H.

Theorem 1.2 has some direct implications for Conjecture 1.1. A family of graphs $\mathcal{F}$ is *closed* if for every graph J we have the following: $\hom^*(F, J) = 0$ for every $F \in \mathcal{F}$ if and only if J is $\mathcal{F}$-free. That is, if J is $\mathcal{F}$-free, then there are no locally injective copies of $F \in \mathcal{F}$ in J. As a consequence of Theorem 1.2, we obtain that Conjecture 1.1 holds for every closed family.

Theorem 1.3. *Conjecture 1.1 holds for every closed family $\mathcal{F}$ of graphs: every d-regular graph G has a spanning $\mathcal{F}$-free subgraph H with minimum degree $\delta(H) = \Omega(\mathrm{ex}(d, \mathcal{F})/d)$.*

Theorem 1.3 gives a lower bound on the minimum degree in terms of $\mathrm{ex}(d, \mathcal{F})$. We now provide some explicit results for some important families $\mathcal{F}$ with chromatic number 2 that are closed. The problem of determining $\mathrm{ex}(d, \mathcal{F})$ when $\mathcal{F}$ contains bipartite graphs is one of the most important in extremal graph theory (see [11] for a complete survey on the topic). We use some well-known constructions of extremal graphs to provide explicit corollaries of Theorem 1.3.

- Let $\mathcal{F} = \mathcal{F}_g = \{C_3, \ldots, C_{g-1}\}$. Observe that $\mathcal{F}$ is a closed family. However, the asymptotic order of $\mathrm{ex}(d, \mathcal{F})$ is not known in general. Using the Erdős–Rényi random graph $G(d, p)$ for some suitably chosen probability $p = p(d, g)$, one can show the existence of an $\mathcal{F}$-free graph of order d and $\Theta(d^{1+1/(g-3)})$ edges. This provides a lower bound for $\mathrm{ex}(d, \mathcal{F})$. By Theorem 1.3, we have that every d-regular graph G has a spanning subgraph with girth at least g and minimum degree $\Omega(d^{1/(g-3)})$. This improves the result of Kun [15].
- Let $\mathcal{F} = \mathcal{F}_6 = \{C_3, C_4, C_5\}$. From the constructions of extremal C_4-free graphs provided by Erdős, Rényi and Sós [8] and Brown [2], one can obtain an $\mathcal{F}$-free graph of order d and $\Theta(d^{3/2})$ edges. Again, by Theorem 1.3, for every d-regular graph G we can show the existence of a spanning subgraph with girth at least 6 and minimum degree $\Omega(\sqrt{d})$. Extremal constructions for graphs with girth at least 8 and 12 are also known [16, 17]. From these we can obtain tight explicit lower bounds for $\mathcal{F}_8 = \{C_3, \ldots, C_7\}$ and $\mathcal{F}_{12} = \{C_3, \ldots, C_{11}\}$.
- Let $\mathcal{F} = \{K_{a,b}\}$ with $a \leqslant b$. Since $K_{a,b}$ has diameter two, each locally injective homomorphism of $K_{a,b}$ onto a graph J is also injective. Thus, if $\hom^*(K_{a,b}, J) = 0$, then J is $K_{a,b}$-free, and $\mathcal{F}$ is closed. In particular, any family composed of complete bipartite graphs is closed. It is conjectured that $\mathrm{ex}(d, K_{a,b}) = \Theta(n^{2-1/a})$. While the upper bound has been proved for all values of a and b (Kovári, Sós and Turán [13]), the lower bound is still wide open. However, it is known to be true in the following cases: $a = 2$ and $b \geqslant 2$, $a = 3$ and $b \geqslant 3$ (Brown [2]), and $b > (a-1)!$ (Kollár, Rónyai and Szabó

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[12] and Alon, Rónyai and Szabó [1]). Using these results we can get tight explicit lower bounds on the minimum degree of the largest subgraph of a d-regular graph that induces no complete bipartite subgraph of a given size.

– Let $\mathcal{F} = \{Q_3\}$, where Q_s is the s-dimensional hypercube. The family $\mathcal{F}$ is not closed since Q_3 admits a locally injective homomorphism to K_4, that is not injective. As before, the family $\mathcal{F}' = \{K_4, Q_3\}$ is closed. Since $\chi(K_4) = 4$, we have $\mathrm{ex}(d, \mathcal{F}) = \Theta(\mathrm{ex}(d, \mathcal{F}'))$ and Conjecture 1.1 is true for $\mathcal{F}$. It is conjectured in [9] that $\mathrm{ex}(d, \mathcal{F}) = \Theta(d^{8/5})$, but no better lower bound that $\mathrm{ex}(d, \mathcal{F}) \geqslant \mathrm{ex}(d, C_4) = \Omega(d^{3/2})$ is known.

Theorem 1.3 solves in the affirmative Conjecture 1.1 for families of graphs satisfying a 'local' condition. However, there are many families that contain bipartite graphs and that are not closed. In order to solve the conjecture for every family of graph $\mathcal{F}$, one needs to extend the idea of local injectivity in Theorem 1.2 to injectivity. In terms of colourings, it would suffice to prove the existence of a spanning subgraph H with relatively large minimum degree and a colouring χ such that *all* copies of F in H are rainbow. This is stronger than the rainbow condition in the neighbourhoods of H that we impose here.

Finally, Theorems 1.2 and 1.3 study the case where G is d-regular. Similar results in terms of the maximum and minimum degree have been given in [10]. We believe that the same techniques used here could be extended to the non-regular case, possibly adding a mild condition between the maximum and the minimum degree.

Related work. Conjecture 1.1 is closely related to the following very general question: given a graph parameter ρ, a value k and a family of graphs $\mathcal{F}$, determine the largest value of ℓ such that for every graph G with $\rho(G) \geqslant k$ there exists an $\mathcal{F}$-free subgraph H of G with $\rho(H) \geqslant \ell$. Here we list some interesting results for other important graph parameters.

– Let ρ be the average degree. Thomassen [19] conjectured that every graph with average degree at least d has an $\mathcal{F}_g$-free subgraph with average degree $f(d)$, for some $f(d) \to \infty$ as $d \to \infty$. Kühn and Osthus [14] showed that Thomassen's conjecture is true for $g = 6$, but the general conjecture is still wide open.

– Let ρ be the number of edges. Bollobás and Erdős asked this problem for the case $\mathcal{F} = \{C_4\}$ in a workshop in 1966 [7]. The problem was rediscovered by Foucaud, Krivelevich and Perarnau [10], and they provided lower bounds for the case $\mathcal{F}_g$ that are tight up to logarithmic factors. Conlon, Fox and Sudakov [3] proved tight lower bounds when $\mathcal{F}$ is composed of complete bipartite graphs. In particular, this answers the question of Bollobás and Erdős.

– Let ρ be the vertex-connectivity. Thomassen [20] also conjectured that every k-connected graph has a bipartite subgraph ($\mathcal{F}$-free for $\mathcal{F} = \{C_3, C_5, C_7, \dots\}$) with connectivity $f(k)$, for some $f(k) \to \infty$ as $k \to \infty$. A first step towards the proof of this conjecture is the result of Delcourt and Ferber [4].

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2. Proof of Theorem 1.3 assuming Theorem 1.2

First, we may assume that the chromatic number of $\mathcal{F}$ is 2. Otherwise, Theorem 1.2 can be easily proved (see Proposition 1 in [10] for a stronger version). We can also assume that $\mathcal{F}$ does not contain any forest. If $\mathcal{F}$ contains a forest T, since $\mathrm{ex}(d,\mathcal{F}) \leqslant \mathrm{ex}(d,T) = O(d)$, then Theorem 1.2 is trivially true.

Recall that for every $\beta \geqslant 1$, we say that J is *β-almost regular* if $\Delta(J) \leqslant \beta\delta(J)$. Erdős and Simonovits [9] showed that for every $\gamma \in (0,1)$, there exists $\beta \geqslant 1$ such that the following holds for every family of graphs $\mathcal{F}$ and every positive integer m: if every $\mathcal{F}$-free β-almost regular graph of order m has at most $O(m^{1+\gamma})$ edges, then $\mathrm{ex}(m,\mathcal{F}) = O(m^{1+\gamma})$, where the asymptotic notation here corresponds to $m \to \infty$. Otherwise stated, for every $\mathcal{F}$ and every m, there exists a β-almost regular graph J on m vertices and $\Omega(\mathrm{ex}(m,\mathcal{F}))$ edges. In particular, J satisfies $\delta(J) = \Omega(\mathrm{ex}(m,\mathcal{F})/m)$.

A classic result of Erdős [5, 6] states that if $\mathcal{F}$ does not contain any forest, then there exists a constant $\varepsilon_0 = \varepsilon_0(\mathcal{F}) > 0$ such that for every sufficiently large m $\mathrm{ex}(m,\mathcal{F}) > m^{1+\varepsilon_0}$.

From these two results, we conclude that for every $\mathcal{F}$ that does not contain a forest there exist $\epsilon > 0$ and $\beta \geqslant 1$ such that for every sufficiently large m, there is an $\mathcal{F}$-free β-almost regular graph J on m vertices with $\delta(J) = \Omega(\mathrm{ex}(m,\mathcal{F})/m) > m^{\varepsilon}$.

Since $\mathcal{F}$ is closed by the hypothesis of Theorem 1.3, by the definition of closed, any $\mathcal{F}$-free graph J also satisfies $\hom^*(F,J) = 0$, for every $F \in \mathcal{F}$.

Given the family $\mathcal{F}$, the choice of $\alpha \geqslant 25\beta^2$ and setting $m = \alpha d$, we note that

$$\delta(J) = \Omega\left(\frac{\mathrm{ex}(\alpha d,\mathcal{F})}{\alpha d}\right) = \Omega\left(\frac{\mathrm{ex}(d,\mathcal{F})}{d}\right).$$

Thus, we can use J in Theorem 1.2 to find a spanning $\mathcal{F}$-free subgraph H of G with minimum degree $\delta(H) = \Omega(\mathrm{ex}(d,\mathcal{F})/d)$, concluding the proof of Theorem 1.3.

3. Notation and probabilistic tools

For every $v \in V(G)$ we let $N_G(v)$ denote the set of vertices adjacent to v in G, we let $d_G(v) = |N_G(v)|$ be the degree of v in G and let $N^2_G(v)$ be the set of vertices at distance two from v in G. If the graph G is clear from the context, we use $N(v)$, $d(v)$ and $N^2(v)$. We let $\Delta(G)$ and $\delta(G)$ denote the maximum and the minimum degree of G respectively. We let χ be a vertex (partial) colouring of a graph G. Throughout the paper, we identify the colour palette with the vertices of a graph J of order αd and talk indistinctly of an αd colouring or a $V(J)$ colouring. We use $\chi(G)$ to denote the chromatic number of G, and for every family of graphs $\mathcal{F}$, we also use $\chi(\mathcal{F}) = \min_{F\in\mathcal{F}} \chi(F)$. For every vertex $v \in V$, we let $\chi(v)$ denote its colour and for every set $S \subseteq V(G)$, we let $\chi(S)$ be the set of colours appearing in S. We call $S \subseteq V$ *rainbow* if $\chi(u) \neq \chi(v)$ for every $u,v \in S$, $u \neq v$, that were assigned a colour by χ.

Below, we introduce some standard tools from the probabilistic method that can be found in [18] and that will be of use in the proof of Theorem 1.2.

Lemma 3.1 (Chernoff's inequality). *For any $0 \leqslant t \leqslant np$,*

$$\mathbb{P}(|\mathrm{Bin}(n,p) - np| > t) < 2e^{-t^2/3np}.$$

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Moreover, for every $t \geqslant 7np$,

$$\mathbb{P}(\mathrm{Bin}(n,p) > t) < e^{-t}.$$

Lemma 3.2 (Talagrand's inequality). *Let X be a non-negative random variable not identically 0, which is determined by n independent trials $T_1, \ldots, T_n$ and satisfying the following for some $c_1, c_2 > 0$:*

- *changing the outcome of any one trial can affect X by at most c_1, and*
- *for any s, if $X \geqslant s$ then there is a set of at most $c_2 s$ trials whose outcomes certify that $X \geqslant s$.*

Then for any $0 \leqslant t \leqslant \mathbb{E}(X)$,

$$\mathbb{P}(|X - \mathbb{E}(X)| > t + 60c_1\sqrt{c_2\mathbb{E}(X)}) \leqslant 4e^{-t^2/(8c_1^2c_2\mathbb{E}(X))}.$$

Lemma 3.3 (McDiarmid's inequality). *Let X be a non-negative random variable not identically 0, which is determined by n independent trials $T_1, \ldots, T_n$ and m independent uniform random permutations $\pi_1, \ldots, \pi_m$, and satisfying the following for some $c_1, c_2 > 0$:*

- *changing the outcome of any one trial can affect X by at most c_1,*
- *interchanging two elements in any one permutation can affect X by at most c_1, and*
- *for any s, if $X \geqslant s$ then there is a set of at most $c_2 s$ trials whose outcomes certify that $X \geqslant s$.*

Then for any $0 \leqslant t \leqslant \mathbb{E}(X)$,

$$\mathbb{P}(|X - \mathbb{E}(X)| > t + 60c_1\sqrt{c_2\mathbb{E}(X)}) \leqslant 4e^{-t^2/(8c_1^2c_2\mathbb{E}(X))}.$$

Lemma 3.4 (Lovász Local Lemma). *Consider a set $\mathcal{E}$ of events such that, for each $E \in \mathcal{E}$,*

- *$\mathbb{P}(E) \leqslant p < 1$, and*
- *E is mutually independent of the set of all but at most D of other events.*

If $4pD \leqslant 1$, then with positive probability, none of the events in $\mathcal{E}$ occur.

4. Proof of Theorem 1.2

The rest of the paper is devoted to the proof of Theorem 1.2. Throughout this section we will fix a family of graphs $\mathcal{F}$, $\epsilon > 0$, $\beta \geqslant 1$ and $\alpha \geqslant 25\beta^2$. In order for some inequalities to hold, we will require d to be large enough with respect to the previous parameters. Recall that, by assumption, there exists a graph J on αd vertices that satisfies the conditions of Theorem 1.2. We first describe the main ideas of the proof.

4.1. Outline of the proof

First of all, we choose a large bipartite subgraph G' of G, by selecting a maximum edge-cut and keeping the edges in it. Observe that every vertex in G' has degree at least $d/2$. Let A and B be the two stable sets of G'.

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The goal of the proof is to select a spanning subgraph H of G' and a $V(J)$ colouring χ of H that satisfies the following properties:

– for every $v \in V(H)$, $d_H(v) = \Omega(\delta(J))$,
– for every $v \in V(H)$, $N_H(v)$ is rainbow, and
– for every $uv \in E(H)$, $\chi(u)\chi(v) \in E(J)$.

From these properties, it will be easy to deduce that the subgraph H satisfies the statement of Theorem 1.2. This is done at the end of the paper.

The subgraph H is selected by deleting some edges of G' depending on the colour assigned to its vertices. Here and throughout the proof of the theorem, we always identify the set of αd colours with $V(J)$.

We design a colouring procedure that is divided into two phases.

In Phase I we colour the vertices of A and delete some of the edges of G'. This is done in two steps, the first random and the second deterministic. We obtain a spanning subgraph H' of G and a colouring χ' of A with some desirable properties (see Lemma 4.2 for a precise description).

In Phase II we colour the vertices of B and delete some of the edges of H'. In contrast to Phase I, here we perform a randomized iterative colouring procedure to colour most of the vertices of B. We also complete the colouring of B with a deterministic step. After Phase II, we obtain the spanning subgraph H of G and the colouring χ of H with the properties described above.

4.2. Phase I: colouring A

For every graph G, partial colouring χ of G and $v \in V$, we define

$$\mathbf{Bad}(v, \chi, G) = |\{u \in N_G(v) : \exists v' \in N_G(u) \setminus \{v\} \text{ and } \chi(v') = \chi(v)\}|,$$

that is, the number of vertices $u \in N_G(v)$ such that $\chi(v)$ appears more than once in $N_G(u)$. This quantity will be crucial throughout the paper.

We will colour the set A as follows.

1 For every $a \in A$, let $\chi'_0(a) = c$, where $c \in V(J)$ is chosen independently and uniformly at random.

2 Uncolour $a \in A$ if

$$\mathbf{Bad}(a, \chi'_0, G') \geqslant \frac{d}{\sqrt{\alpha}}.$$

Let A_1 be the set of uncoloured vertices in A.

3 Delete all the edges between $a \in A \setminus A_1$ and $N_{G'}(a)$ that may cause conflicts, that is, delete ab if $\exists a' \in N_{G'}(b) \setminus \{a\}$ such that $\chi'_0(a') = \chi'_0(a)$. Let H'_0 be the subgraph obtained by removing these edges from G'.

4 Consider an arbitrary order of the vertices in $A_1 = (a_1, \ldots, a_s)$, where $s = |A_1|$.

5 For every i from 1 to s,

(a) Assign to a_i the colour in $V(J)$ that minimizes $\mathbf{Bad}(a_i, \chi'_{i-1}, H'_{i-1})$. Let χ'_i be the partial colouring of A obtained from χ'_{i-1} and the coloured vertex a_i.

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(b) Delete all the edges between a_i and $N_{H'_{i-1}}(a_i)$ that may cause conflicts, that is, delete a_ib if $\exists a' \in N_{H'_{i-1}}(b) \setminus \{a_i\}$ such that and $\chi'_i(a') = \chi'_i(a_i)$. Let H'_i be the subgraph obtained by removing these edges from H'_{i-1}.

6 Let $\chi' = \chi'_s$ and $H' = H'_s$.

For a random map $\chi : V(G) \to V(J)$, let p be the probability that an edge $uv \in E(G)$ gets mapped to an edge $\chi(u)\chi(v) \in E(J)$. Then we have

$$p^- := \frac{\delta(J)}{\alpha d} \leqslant p = \frac{2|E(J)|}{(\alpha d)^2} \leqslant \frac{\Delta(J)}{\alpha d} =: p^+.$$

Observe that since J is β-almost regular, $p^+ \leqslant \beta p^-$.

The next lemma is going to be useful to prove that χ' has some good properties with respect to each $b \in B$.

Lemma 4.1. *Let $\alpha \geqslant 25$ and let d be large enough. Let J be a graph on αd vertices. Let G' be a bipartite graph with stable sets A and B, maximum degree at most d and minimum degree at least $d/2$. Let χ be a random colouring of A where each vertex is assigned a colour from $V(J)$ independently and uniformly at random. For every $b \in B$ and $c \in V(J)$, let $W_{b,c}$ be the number of vertices $a \in N_{G'}(b)$ such that:*

(1) *a is the only vertex with colour $\chi(a)$ in $N_{G'}(b)$,*
(2) ***Bad**$(a, \chi, G') \leqslant d/\sqrt{\alpha}$, and*
(3) *$\chi(a)c \in E(J)$.*

Then,

$$\mathbb{P}\left(W_{b,c} \leqslant \frac{\delta(J)}{4\alpha}\right) = e^{-\Omega(\delta(J))}.$$

Proof. Let X be the number of neighbours $a \in N(b)$ such that at least one of these two conditions holds:

(F1) $\exists a' \in N(b) \setminus \{a\}$ with $\chi(a') = \chi(a)$,
(F2) $|\{b' \in N(a) \setminus \{b\} : \exists a' \in N(b') \setminus N(b) \text{ and } \chi(a') = \chi(a)\}| \geqslant d/\sqrt{\alpha} - 1$.

Observe that if a does not satisfy conditions (F1) and (F2), then there are at most $d/\sqrt{\alpha}$ vertices $b' \in N(a)$ (this might include $b' = b$) such that there exists $a' \in N(b') \setminus \{a\}$ with $\chi(a') = \chi(a)$. In other words, **Bad**$(a, \chi, G') \leqslant d/\sqrt{\alpha}$ (condition 2 of the lemma). Moreover, condition (F1) also implies that a is the only vertex with colour $\chi(a)$ in $N(b)$ (condition 1 of the lemma).

We first show that, with high probability, X is not too large. We will prove a stronger statement: X is concentrated around its expected value. Indeed, we will fix a colouring of $A \setminus N(b)$ and prove that, conditional on any such colouring, X is concentrated.

For the event (F1), since $d(a) \leqslant d$ and there are $|V(J)| = \alpha d$ colours, the probability that a vertex a receives the same colour as one of the other neighbours of b is at most $1/\alpha$.

We call a colour c' *dangerous* for $a \in N(b)$ if (F2) is satisfied when $\chi(a) = c'$. The fact that c' is dangerous for a depends only on the colouring of $A \setminus N(b)$, which is already

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fixed. In particular, since there are at most d^2 edges $a'b'$ with $b' \in N(a)$, the number of dangerous colour classes is at most

$$\frac{d^2}{(d/\sqrt{\alpha})-1} \leqslant 2\sqrt{\alpha}d.$$

Thus, the probability that a vertex $a \in N(b)$ satisfies the event (F2) is at most $2/\sqrt{\alpha}$.

Using a union bound for the events (F1) and (F2), we obtain

$$\mathbb{E}(X \mid \chi(A \setminus N(b))) \leqslant \left(\frac{1}{\alpha} + \frac{2}{\sqrt{\alpha}}\right) d(b) \leqslant \frac{3d(b)}{\sqrt{\alpha}}.$$

Since we have conditioned on the colouring given to $A \setminus N(b)$, X only depends on the colours assigned to $N(b)$. Changing the colour of a vertex $a \in N(b)$ from c' to c'' can change the value of X by at most 2. This change can create or destroy at most two vertices with the property (F1). It can also be the case that c'' is a dangerous colour for a and that c' is not (or *vice versa*); in this case the change is by at most 1.

Furthermore, if we have $X \geqslant s$ and given that the colouring in $A \setminus N(b)$ has been fixed previously, there exists a set of s choices of colours that certify $X \geqslant s$; if a vertex $a \in N(b)$ satisfies (F1), it suffices to reveal all the vertices in $N(b)$ with colour $\chi(a)$ (and all of them count for X), and if it satisfies (F2), it suffices to reveal $\chi(a)$, since the fact that it is a dangerous colour is certified by the colouring of $A \setminus N(b)$, which is fixed.

Since $\alpha \geqslant 25$, by Talagrand's inequality with $c_1 = 2$ and $c_2 = 1$, we have

$$\mathbb{P}\left(X \geqslant \frac{4d(b)}{\sqrt{\alpha}} \mid \chi(A \setminus N(b))\right) \leqslant \mathbb{P}\left(|X - \mathbb{E}(X)| \geqslant \frac{d(b)}{\sqrt{\alpha}} \mid \chi(A \setminus N(b))\right) = e^{-\Omega(d)}. \tag{4.1}$$

Since the previous inequality holds conditional on any colouring of $A \setminus N(b)$, it also holds unconditionally.

We now count the number of vertices $a \in N(b)$ for which neither (F1) nor (F2) hold and that satisfy $\chi(a)c \in E(J)$. Such vertices are in $W_{b,c}$ since they fulfil conditions 1, 2 and 3. Observe that colouring each vertex in A with a colour chosen independently and uniformly at random in $V(J)$ is equivalent to colouring A in the same way and then permuting the colour classes according to a permutation π of length $\alpha d = |V(J)|$ chosen uniformly at random. These two steps can be done independently and thus we can analyse them separately. In the first step the colour classes are set, while in the second one each colour class is assigned a particular colour according to π. Observe that the condition $\chi(a)c \in E(J)$ only depends on the second step, while (F1) and (F2) only depend on the partition induced by the colour classes.

Let M be the set of vertices $a \in N(b)$ such that, after the first step, conditions (F1) and (F2) are not satisfied. Notice that every vertex in M receives a different colour. Since π is a uniformly chosen permutation, the set of $|M|$ colours assigned to M is chosen uniformly from all the sets of size $|M| \subseteq [\alpha d]$. Then, for every $I \subseteq N(b)$ with $|I| = i$,

$$\mathbb{E}(W_{b,c} \mid M = I) = \sum_{a \in I} \mathbb{P}(\chi(a)c \in E(J) \mid M = I) = \sum_{a \in I} \mathbb{P}(\chi(a)c \in E(J)) \geqslant p^{-}i.$$

Recall that, by the independence of the two-step colouring, to show that $W_{b,c}$ is large with high probability, we only need to focus on the second step. Once the colour classes have

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been set, a swap between two positions of π can change $W_{b,c}$ by at most 2 and if $W_{b,c} \geqslant s$ we can certify it by only revealing the permutation π. By McDiarmid's inequality, for all $0 < \gamma < 1$,

$$\begin{aligned}\mathbb{P}\big(W_{b,c} \leqslant (1-\gamma)p^- i \mid M = I\big) &\leqslant \mathbb{P}\big(|W_{b,c} - \mathbb{E}(W_{b,c})| \geqslant \gamma p^- i \mid M = I\big) \\ &= e^{-\Omega(p^- i)}. \end{aligned} \tag{4.2}$$

Since $d(b) \geqslant d/2$, we have

$$\frac{p^- d(b)}{2} \geqslant \frac{\delta(J)}{4\alpha}.$$

Now, using (4.1) and (4.2), we obtain the desired result:

$$\begin{aligned}\mathbb{P}\left(W_{b,c} \leqslant \frac{\delta(J)}{4\alpha}\right) &\leqslant \mathbb{P}\left(W_{b,c} \leqslant \frac{p^- d(b)}{2}\right) \\ &= \sum_{I \subseteq N(b)} \mathbb{P}\left(W_{b,c} \leqslant \frac{p^- d(b)}{2} \mid M = I\right)\mathbb{P}(M = I) \\ &\leqslant \sum_{\substack{I \subseteq N(b) \\ i \geqslant (1-4/\sqrt{\alpha})d(b)}} \mathbb{P}\left(W_{b,c} \leqslant \frac{p^- d(b)}{2} \mid M = I\right)\mathbb{P}(M = I) + e^{-\Omega(d)} \\ &\leqslant \sum_{\substack{I \subseteq N(b) \\ i \geqslant (1-4/\sqrt{\alpha})d(b)}} \mathbb{P}\left(W_{b,c} \leqslant \frac{p^- i}{2\big(1-4/\sqrt{\alpha}\big)} \mid M = I\right)\mathbb{P}(M = I) + e^{-\Omega(d)} \\ &\leqslant e^{-\Omega(\delta(J))} \sum_{\substack{I \subseteq N(b) \\ i \geqslant (1-4/\sqrt{\alpha})d(b)}} \mathbb{P}(M = I) + e^{-\Omega(d)} \\ &= e^{-\Omega(\delta(J))}. \end{aligned}$$

□

Lemma 4.2. *Let $\epsilon > 0$ and $\alpha \geqslant 25$ and let d be large enough. Let J be a graph on αd vertices such that $\delta(J) > d^\epsilon$.*

With positive probability, Phase I provides a spanning subgraph H' of G and a colouring χ' of A with the following properties:

(P1) *for every $a \in A$, $d_{H'}(a) \geqslant d/4$,*
(P2) *for every $b \in B$, $N_{H'}(b)$ is rainbow, and*
(P3) *for every $b \in B$ and every $c \in V(J)$, there are at least $\delta(J)/4\alpha$ vertices $a \in N_{H'}(b)$ such that $\chi'(a)c \in E(J)$.*

Proof. Let us first show that (P1) is satisfied. If $a \in A \setminus A_1$, we have

$$d_{H'}(a) = d_{H'_0}(a) \geqslant (1 - 1/\sqrt{\alpha})d_{G'}(a),$$

since by the choice of $\chi'_0(a)$ we deleted at most $d/\sqrt{\alpha}$ edges incident to it. For every $a \in A_1$, there is an $1 \leqslant i \leqslant s$ such that $a = a_i$. Since there are at most d^2 edges incident to

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$N_{H'_{i-1}}(a_i)$, there exists a colour $c \in V(J)$ (recall that $|V(J)| = \alpha d$) such that if $\chi'_i(a_i) = c$,

$$\mathbf{Bad}(a_i, \chi'_i, H'_{i-1}) = |\{b \in N_{H'_{i-1}}(a_i) : \exists a' \in N_{H'_{i-1}}(b) \setminus \{a_i\} \text{ and } \chi'_i(a') = \chi'_i(a_i)\}| \leqslant \frac{d}{\alpha}.$$

Since in Phase I we only delete edges incident to a_i when its colour is assigned, $d_{H'}(a_i) = d_{H'_i}(a_i) \geqslant (1 - 1/\alpha)d_{G'}(a)$ for all $a_i \in A_1$. Since $\alpha \geqslant 8$ and since $d_{G'}(a) \geqslant d/2$, for all $a \in A$ we have $d_{H'}(a) \geqslant d/4$.

Property (P2) also holds deterministically. Consider a vertex $b \in B$. Note that an edge ab can only be deleted in the iteration when a retains its colour. Suppose that a retains its colour at the ith iteration and that the edge ab is not deleted. Then, the colour $\chi'_i(a)$ only appears once in the partial colouring χ'_i restricted to $N_{H'_{i-1}}(b)$. Moreover, if a vertex $a' \in N_{H'_i}(b) \setminus \{a\}$ retains the colour $\chi'_i(a)$ in a further iteration, then the edge $a'b$ is deleted. Thus, a is the only neighbour of b in H' with colour $\chi'_i(a)$. Since the choice of a is arbitrary, $N_{H'}(b)$ is rainbow.

Let us show that (P3) holds with positive probability. In order to show this, it suffices to look at the first partial colouring χ'_0. For every $b \in B$ and $c \in V(J)$, let $E_{b,c}$ be the event that there are at most $\delta(J)/4\alpha$ vertices $a \in N_{H'_0}(b)$ such that a is the only vertex with colour $\chi'_0(a)$ in $N_{H'_0}(b)$, a retains its colour (*i.e.* $\mathbf{Bad}(a, \chi'_0, G') \geqslant d/\sqrt{\alpha}$) and $\chi'_0(a)c \in E(J)$.

By applying Lemma 4.1 to G' with $\chi = \chi'_0$ and noting that by assumption $\delta(J) > d^{\varepsilon}$, we obtain $\mathbb{P}(E_{b,c}) = e^{-\Omega(d^{\varepsilon})}$. The event $E_{b,c}$ is mutually independent of the other events $E_{b',c'}$ where both b' and c' are at distance larger than 6 from b and c in the respective graphs G and J. Then, each event is mutually independent of all but at most $d^6\Delta(J)^6 \leqslant d^{12}$ other events. Since $e^{-\Omega(d^{\varepsilon})} = O(d^{-13})$, provided that d is large enough with respect to ϵ and α, we can use the Local Lemma to show that none of the events in $E_{b,c}$ holds with positive probability.

By colouring the vertices $a_i \in A_1$ and deleting edges incident to a_i, one can neither decrease the degree of b in $A \setminus A_1$ nor change the fact that a given $a \in A \setminus A_1$ is the only neighbour of b with colour $\chi'_0(a)$. Thus, with positive probability (P3) is satisfied. □

4.3. Phase II: colouring *B*

In the second colouring phase it will be convenient to redefine $\mathbf{Bad}(v, \chi, G)$ in a way that it takes into account the compatibilities between colours given by the edges of J. For every graph G, every set $F \subseteq E(G)$ (we say that the edges in F are *activated*), every partial colouring χ of G, every $v \in V$ and every graph J, we define

$$\mathbf{Bad}_J(v, \chi, G) = \left|\left\{u \in N_G(v) : \begin{array}{c}\chi(u)\chi(v) \in E(J), \exists v' \in N_G(u) \setminus \{v\}, \\ uv' \in F \text{ and } \chi(v') = \chi(v)\end{array}\right\}\right|,$$

We will colour B as follows.

1 Let $B_0 = B$, $H_0 = H'$, $\chi_0 = \chi'$ and $i = 1$. For every $a \in A$, let $B_0(a) = B_0 \cap N_{H_0}(a)$.

2 While there exists $a \in A$ with $|B_{i-1}(a)| > d^{\epsilon/2}$:

(a) For every edge in H_{i-1}, activate it independently with probability $1/\alpha$.

(b) Construct χ_i as follows. For every $v \in A \cup (B \setminus B_{i-1})$, let $\chi_i(v) = \chi_{i-1}(v)$ and for every $b \in B_{i-1}$, let $\chi_i(b) = c$, where c is chosen independently and uniformly at random in $V(J)$.

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(c) Uncolour $b \in B_{i-1}$ if

$$\mathbf{Bad}_J(b, \chi_i, H_{i-1}) \geqslant \frac{\delta(J)}{8\alpha}.$$

Let B_i be the set of uncoloured vertices.

(d) Construct H_i from H_{i-1} by deleting the following edges: for every $b \in B_{i-1} \setminus B_i$, delete all the edges ab in H_{i-1} that either can cause conflicts (there exists a vertex $b' \in N_{H_{i-1}}(a) \setminus \{b\}$ such that ab' is activated, $\chi_i(a)\chi_i(b) \in E(J)$ and $\chi_i(b') = \chi_i(b)$), $\chi_i(a)\chi_i(b) \notin E(J)$ or ab has not been activated.

Let $B_i(a) = B_i \cap N_{H_i}(a)$ and increase i by one.

3 Consider an arbitrary order of the vertices in $B_\tau = (b_{\tau+1}, \ldots, b_t)$, where $t = |B_\tau| + \tau$.

4 For every i from $\tau + 1$ to t:

(a) Activate all the edges of H_{i-1}.

(b) Assign to b_i the colour in $\in V(J)$ that minimizes $\mathbf{Bad}_J(b_i, \chi_{i-1}, H_{i-1})$. Let χ_i be the partial colouring of $V(G)$ obtained from χ_{i-1} and the coloured vertex b_i.

(c) Construct H_i from H_{i-1} by deleting all the edges ab_i that either can cause conflicts (there exists a vertex $b' \in N_{H_{i-1}}(a) \setminus \{b_i\}$ such that $\chi_i(a)\chi_i(b_i) \in E(J)$ and $\chi_i(b') = \chi_i(b_i)$) or $\chi_i(a)\chi_i(b_i) \notin E(J)$.

5 Let $\chi = \chi_t$ and $H = H_t$.

We define τ to be the smallest i such that $|B_i(a)| \leqslant d^{\varepsilon/2}$ for every $a \in A$, that is, the number of iterations in step 2 of Phase II. Observe that τ is a random variable.

Lemma 4.3. *With positive probability, Phase II of the colouring procedure ends and provides a spanning subgraph H of G and a colouring χ of H with the following properties:*

(Q1) *for every $v \in V(H)$, $d_H(v) \geqslant \delta(J)/(16\alpha^2)$,*
(Q2) *for every $v \in V(H)$, $N_H(v)$ is rainbow, and*
(Q3) *for every $uv \in E(H)$, $\chi(u)\chi(v) \in E(J)$.*

Let $\ell_i(a)$ be the number of vertices in $b \in B_{i-1}(a) \setminus B_i(a)$ such that ab is active in H_{i-1} and $\chi_i(a)\chi_i(b) \in E(J)$. Recall that $p^+ = \Delta(J)/\alpha d$.

In order to prove Lemma 4.3 we will make sure that, for every $1 \leqslant i < \tau$, the following three conditions are satisfied:

(C1) for every $a \in A$, $\ell_i(a) \leqslant \max\{(2/\alpha) \cdot p^+ |B_{i-1}(a)|, d^{\varepsilon/2}\}$,
(C2) for every $a \in A$, $|B_i(a)| \leqslant d/\alpha^{i/2}$, and
(C3) for every $b \in B_{i-1} \setminus B_i$, $d_{H_i}(b) \geqslant \delta(J)/(16\alpha^2)$.

In words, (C1) ensures that the degree of a in $B \setminus B_\tau$ is not too large, (C2) implies that the number of uncoloured vertices in each neighbourhood of A decreases exponentially with the number of iterations, and (C3) ensures that $b \in B \setminus B_\tau$ has a large minimum degree in H. We will use (C1) and (C2) to prove that $b \in B_\tau$ also has a large minimum degree in H.

In particular, we will require that the following condition is satisfied after the first iteration of Phase II:

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(C4) for every $a \in A$, the degree of a to $B \setminus B_1$ in H_1 is at least $\delta(J)/(16\alpha^2)$.

Condition (C4) ensures that the degree of A in H is large.

Let us show that the probability that any of these four conditions is violated is exponentially small in terms of d. In order to control the condition (C1), for every $a \in A$, we define $D_1^i(a)$ to be the event that $\ell_i(a) \geqslant \max\{(2/\alpha) \cdot p^+|B_{i-1}(a)|, d^{\varepsilon/2}\}$.

Lemma 4.4. *For every $1 \leqslant i < \tau$ and for every $a \in A$, we have*

$$\mathbb{P}(D_1^i(a)) = e^{-\Omega(d^{\varepsilon/4})}.$$

Proof. Each edge ab is activated independently with probability $1/\alpha$. Moreover, by definition of p^+, and independently from the fact that ab has been activated, when b is assigned a random colour c, it satisfies $\chi_i(a)c \in E(J)$ with probability at most p^+. Since the choice of colour for each $b \in B_{i-1}(a)$ is done independently, $\ell_i(a)$ follows a binomial distribution with $|B_{i-1}(a)|$ trials and probability at most p^+/α. In particular, $\mathbb{E}(\ell_i(a)) \leqslant (1/\alpha) \cdot p^+|B_{i-1}(a)|$.

Suppose first that

$$|B_{i-1}(a)| \geqslant \frac{\alpha d^{\varepsilon/4}}{p^+}.$$

By Chernoff's inequality,

$$\mathbb{P}\left(\ell_i(a) \geqslant \frac{2}{\alpha} \cdot p^+|B_{i-1}(a)|\right) \leqslant \mathbb{P}\left(|\ell_i(a) - \mathbb{E}(\ell_i(a))| \geqslant \frac{1}{\alpha} \cdot p^+|B_{i-1}(a)|\right)$$
$$= e^{-\Omega(p^+|B_{i-1}(a)|)} = e^{-\Omega(d^{\varepsilon/4})}.$$

Suppose now that

$$|B_{i-1}(a)| < \frac{\alpha d^{\varepsilon/4}}{p^+}.$$

Then, $\mathbb{E}(\ell_i(a)) < d^{\varepsilon/4}$. Provided that d is large enough, $d^{\varepsilon/2} \geqslant 7\mathbb{E}(\ell_i(a))$ and by Chernoff's inequality,

$$\mathbb{P}(\ell_i(a) \geqslant d^{\varepsilon/2}) = e^{-\Omega(d^{\varepsilon/2})}. \qquad \square$$

In order to control the condition (C2), for every $a \in A$, we define $D_2^i(a)$ to be the event that $|B_i(a)| \geqslant d/\alpha^{i/2}$.

Lemma 4.5. *For every $1 \leqslant i < \tau$ and for every $a \in A$, if* (C1) *and* (C2) *hold for every $1 \leqslant j < i$, then*

$$\mathbb{P}(D_2^i(a)) = e^{-\Omega(d^{\varepsilon/4})}.$$

Proof. We call a colour c *dangerous* for $b \in B_{i-1}(a)$ if the number of vertices $a' \in N_{H_{i-1}}(b) \setminus \{a\}$ such that $\chi_i(a')\chi_i(b) \in E(J)$ and there exists $b' \in N_{H_{i-1}}(a') \setminus B_{i-1}(a)$ with $a'b'$

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activated and $\chi_i(b') = c$, is at least $\delta(J)/8\alpha - 1$. The fact that c is a dangerous colour is fully determined by the colouring of $B \setminus B_{i-1}(a)$ and the choice of activated edges between A and $B \setminus B_{i-1}(a)$.

We now show that conditional on a certain event that holds with very high probability, for every $b \in B_{i-1}(a)$ there are few dangerous colours.

Let $\mathbf{Free}_i(a')$ be the number of colours $c \in V(J)$ with $\chi_i(a')c \in E(J)$ such that at least one of the following two conditions is satisfied:

– c does not appear in $N_{H_{i-1}}(a') \setminus B_{i-1}(a)$,
– for every $b' \in N_{H_{i-1}}(a') \setminus B_{i-1}(a)$ with $\chi_i(b') = c$, $a'b'$ is not activated.

In other words, the number of colours such that if we assign one of them to b, then a' will not count for $\mathbf{Bad}_J(b, \chi_i, H_{i-1})$.

Let E be the event that $\mathbf{Free}_i(a') \geqslant (1 - 1/\alpha)\delta(J)$, for every $a' \in N^2_{H_{i-1}}(a)$. Observe that the event E only depends on the colouring of A, that of $B \setminus B_{i-1}(a)$ and the activated edges between A and $B \setminus B_{i-1}(a)$.

Next claim shows that the probability of E is very large.

Claim 4.6. *If* (C1) *and* (C2) *are satisfied for every* $1 \leqslant j < i$, *then* $\mathbb{P}(E) = 1 - e^{-\Omega(d^{\epsilon/4})}$.

Proof. Observe that $\mathbf{Free}_i(a')$ can be controlled with $\ell_j(a')$, for $j \leqslant i$. Since (C1) is satisfied for every $j < i$, we have that

$$\ell_j(a') < \max\left\{\frac{2}{\alpha} \cdot p^+ |B_j(a')|, d^{\epsilon/2}\right\}.$$

Moreover, by Lemma 4.6, $D^i_1(a')$ does not hold with probability at least $1 - e^{-\Omega(d^{\epsilon/4})}$. Recall that $D^i_1(a')$ is defined as $\ell_i(a') < \max\{(2/\alpha) \cdot p^+ |B_i(a')|, d^{\epsilon/2}\}$. A union bound shows that this is true for all $a' \in N^2_{H_{i-1}}(a)$ with probability $1 - e^{-\Omega(d^{\epsilon/4})}$.

Since (C2) holds for every $j < i$ and $|B_i(a')| \leqslant |B_{i-1}(a')|$, we have that $\sum_{j=1}^{i} |B_j(a')| \leqslant 2d$. Thus, for every $a' \in N^2_{H_{i-1}}(a)$ we obtain

$$\mathbf{Free}_i(a') \geqslant \delta(J) - \sum_{j=1}^{i} \ell_j(a') \geqslant \delta(J) - \frac{4}{\alpha} \cdot p^+ d \geqslant \left(1 - \frac{1}{\alpha}\right)\delta(J),$$

where we used that $\alpha \geqslant 25\beta^2$ in the last inequality. We conclude that E is satisfied with probability $1 - e^{-\Omega(d^{\epsilon/4})}$. □

Moreover, the event E allows us to control the number of dangerous colours for the neighbours of a.

Claim 4.7. *If E is satisfied, then for each $b \in N_{H_{i-1}}(a)$, the number of dangerous colours for b is at most $9d$.*

Proof. Since E implies $\mathbf{Free}_i(a') \geqslant (1 - (1/\alpha))\delta(J)$ for every $a' \in N_{H_{i-1}}(b) \setminus \{a\}$, we have that there are at most $(\delta(J)/\alpha) \cdot d_{H_{i-1}}(b)$ edges $a'b'$ with $\chi_i(a')\chi_i(b') \in E(J)$. Recall that if c

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is dangerous for b, then there are at least $\delta(J)/8\alpha - 1$ vertices $a' \in N_{H_{i-1}}(b) \setminus \{a\}$ that are incident to an (activated) edge $a'b'$ with $\chi_i(a')\chi_i(b') \in E(J)$ and $\chi_i(b') = c$.

Thus, the number of dangerous colours is at most

$$\frac{(\delta(J)/\alpha) \cdot d_{H_{i-1}}(b)}{(\delta(J)/8\alpha) - 1} \leqslant 9d. \qquad \square$$

Let us prove that the probability $|B_i(a)|$ is large is exponentially small. Let X be the number of vertices $b \in B_{i-1}(a)$ that satisfy at least one of these two conditions:

(F1) there exists $b' \in N_{H_{i-1}}(a) \setminus \{b\}$ with $\chi_i(b') = \chi_i(b)$,
(F2) $\chi_i(b)$ is a dangerous colour for b.

We claim that $|B_i(a)| \leqslant X$. Suppose that b does not satisfy (F1) and (F2). Since (F1) does not hold, all the other vertices in $B_{i-1}(a)$ have a colour different from $\chi_i(b)$. Together with (F2) not holding, we obtain that $\mathbf{Bad}_J(b, \chi_i, H_{i-1}) \leqslant \delta(J)/8\alpha$ and b retains its colour.

Thus, we will show that X is small with very high probability. Indeed, it will suffice to show it in the case that the event E holds. We fix a colouring on $B \setminus B_{i-1}(a)$ and an activation of edges between A and $B \setminus B_{i-1}(a)$ that is compatible with E and show that, conditional on this, X is concentrated. Recall that the choice of the colouring of $B \setminus B_{i-1}(a)$ and of the activation of the edges between A and $B \setminus B_{i-1}(a)$ fully determines whether E is satisfied.

For $b \in B_{i-1}(a)$, the probability that $\chi_i(b)$ is assigned a colour that already appears in $N_{H_{i-1}}(a)$ (*i.e.* of condition (F1)) is at most $1/\alpha$, since $d(a) \leqslant d$ and there are αd colours. Since E is satisfied, by Claim 4.7, for each $b \in B_{i-1}(a)$ there are at most $9d$ dangerous colours. Thus, the probability that b is assigned a dangerous colour is at most $9/\alpha$.

By the hypothesis of the lemma, (C2) holds for $i-1$ and there are $|B_{i-1}(a)| \leqslant d/\alpha^{(i-1)/2}$ candidates for X. Hence, conditional on $\chi_i(B \setminus B_{i-1}(a))$ and on the activation of edges,

$$\mathbb{E}(X) \leqslant \frac{10}{\alpha} \cdot |B_{i-1}(a)| \leqslant 10\alpha^{-1/2} \cdot \frac{d}{\alpha^{i/2}}.$$

As in the proof of Lemma 4.1, by changing the colour of a vertex $b \in B_{i-1}(a)$ one can change X by at most 2 and, if $X \geqslant s$, then there exists a set of s coloured vertices that certifies $X \geqslant s$. By applying Talagrand's inequality to X with $c_1 = 2$ and $c_2 = 1$, and conditional on $\chi_i(B \setminus B_{i-1}(a))$ and on the activation of edges,

$$\begin{aligned}\mathbb{P}(D_2^i(a)) &\leqslant \mathbb{P}\left(X \geqslant \frac{d}{\alpha^{i/2}}\right) \leqslant \mathbb{P}\left(|X - \mathbb{E}(X)| > (1 - 10\alpha^{-1/2})\frac{d}{\alpha^{i/2}}\right) \\ &= e^{-\Omega(d/\alpha^{i/2})} = e^{-\Omega(d^{\varepsilon/2})},\end{aligned}$$

if $\alpha \geqslant 400$. Since the previous statement holds for any choice of $\chi_i(B \setminus B_{i-1}(a))$ and of activation of edges that are compatible with E, it also holds if we only condition on E. Since by Claim 4.6, $\mathbb{P}(E) = 1 - e^{-\Omega(d^{\varepsilon/4})}$, we obtain

$$\mathbb{P}(D_2^i(a)) \leqslant \mathbb{P}(\overline{E}) + \mathbb{P}(D_2^i(a) \mid E) = e^{-\Omega(d^{\varepsilon/4})}. \qquad \square$$

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In order to control the condition (C3), for every $b \in B_{i-1} \setminus B_i$, we define $D_3^i(b)$ to be the event that $d_{H_i}(b) \leqslant \delta(J)/8\alpha^2$.

Lemma 4.8. *For every $b \in B_{i-1} \setminus B_i$,*

$$\mathbb{P}(D_3^i(b)) = e^{-\Omega(d^\epsilon)}.$$

Proof. If $b \in B_{i-1} \setminus B_i$, then $\mathbf{Bad}_J(b, \chi_i, H_{i-1}) \leqslant \delta(J)/8\alpha$; that is, there are at most $\delta(J)/8\alpha$ vertices $a \in N_{H_{i-1}}(b)$ with $\chi_i(a)\chi_i(b) \in E(J)$ such that there exists $b' \in N_{H_{i-1}}(a) \setminus \{b\}$ with ab' activated and $\chi_i(b') = \chi_i(b)$. By (P3) with $c = \chi_i(b)$, there are at least $\delta(J)/4\alpha$ vertices $a \in N_{H_{i-1}}(b)$ with $\chi_i(a)\chi_i(b) \in E(J)$. Hence, there are at least $\delta(J)/8\alpha$ vertices $a \in N_{H_{i-1}}(b)$ with $\chi_i(a)\chi_i(b) \in E(J)$ such that either b is the only neighbour of a with colour $\chi_i(b)$, or if there is another neighbour b' with $\chi_i(b') = \chi_i(b)$, then ab' is not activated and will be deleted in the case that b' retained its colour.

We activate every such edge ab independently with probability $1/\alpha$. Therefore, the probability that $d_{H_i}(b)$ is smaller than k is at most the probability that a binomial random variable with $\delta(J)/8\alpha$ trials and probability $1/\alpha$ is smaller than k. Since $\delta(J) \geqslant d^\epsilon$, Chernoff's inequality implies that

$$\mathbb{P}(D_3^i(b)) = \mathbb{P}\left(d_{H_i}(b) \leqslant \frac{\delta(J)}{16\alpha^2}\right) = e^{-\Omega(d^\epsilon)}. \qquad \square$$

In order to control condition (C4), for every $a \in A$, we define $D_4^1(a)$ to be the event that the degree of a to $B \setminus B_1$ in H_1 is at most $\delta(J)/16\alpha^2$, that is, $|N_{H_1}(a) \cap (B \setminus B_1)| \leqslant \delta(J)/16\alpha^2$.

Lemma 4.9. *For every $a \in A$,*

$$\mathbb{P}(D_4^1(a)) = e^{-\Omega(d^\epsilon)}.$$

Proof. Note that in the first iteration of Phase II we can colour B in the following equivalent way:

(i) colour each vertex in $B \setminus N_{H_0}(a)$ independently,
(ii) colour each vertex in $N_{H_0}(a)$ independently,
(iii) keep the colour classes in $B \setminus N_{H_0}(a)$ but permute the colours assigned to each class.

After step (i), we say that a colour class $C \subseteq B \setminus N_{H_0}(a)$ (with no colour assigned yet) is *dangerous* for $b \in N_{H_0}(a)$ if there are at least $\delta(J)/8\alpha - 1$ activated edges $a'b'$ with $a' \in N_{H_0}(b) \setminus \{a\}$ and $b' \in C$.

For a $b \in N_{H_0}(a)$, we consider the following set of conditions:

(F1) there exists $b' \in N_{H_0}(a) \setminus \{b\}$ with $\chi_1(b') = \chi_1(b)$,
(F2) $\chi_1(b)$ is a dangerous colour for b,
(F3) $\chi_1(a)\chi_1(b) \notin E(J)$.

Let Y be the number of vertices $b \in N_{H_0}(a)$ that do not satisfy (F1), (F2) and (F3).

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Observe that the number of vertices $b \in N_{H_0}(a)$ that do not satisfy (F1) and (F3) is determined by the colouring of $N_{H_0}(a)$ (step (ii)). The number of vertices that do not satisfy (F2) is determined by the final colouring of $B \setminus N_{H_0}(a)$ (steps (i) and (iii)) and the activation of the edges between A and $B \setminus N_{H_0}(a)$. We will use the fact that these steps are independent to bound Y.

Let Z be the number of vertices $b \in N_{H_0}(a)$ that do not satisfy (F1) and (F3). For a given vertex b, the probability that $\chi_1(a)\chi_1(b) \in E(J)$ is at least $p^- = \delta(J)/\alpha d$. Given that a colour has been assigned to b, the probability that this colour is unique in $N_{H_0}(a)$ is at least $(1-(1/\alpha d))^d \geqslant 1 - 1/\alpha$. Since $d_{H_0}(a) \geqslant d/4$, we have

$$\mathbb{E}(Z) \geqslant \left(1 - \frac{1}{\alpha}\right) p^- d_{H_0}(a) \geqslant \frac{(1-1/\alpha)\delta(J)}{4\alpha}.$$

Similarly to the proof of Lemma 4.1, we can apply Talagrand's inequality to obtain that

$$\mathbb{P}\left(Z \leqslant \frac{\delta(J)}{8\alpha}\right) \leqslant e^{-\Omega(\delta(J))}. \tag{4.3}$$

To show that Y is typically large, consider M to be the set of vertices that do not satisfy (F1) and (F3). Let $b \in M$. At step (iii), each colour class will be assigned a random (and different) colour and, if in step (i) we have created a small number of dangerous colour classes, it will be likely that b does not satisfy (F2).

We define E to be the event that, for every $a' \in N^2_{H_0}(a)$, there are at most $2d/\alpha$ activated edges $a'b'$ with $b' \notin N_{H_0}(a)$. A simple use of Chernoff's inequality and a union bound gives $\mathbb{P}(E) = 1 - e^{-\Omega(d)}$. The event E is fully determined by the activation of edges between A and $B \setminus N_{H_0}(a)$. Moreover, it allows us to control the number of dangerous colour classes.

Claim 4.10. *If E is satisfied, then for each $b \in N_{H_0}(a)$, the number of dangerous colour classes for b is at most $\alpha^{1/2}d$.*

Proof. Since E is satisfied, for any $a' \in N_{H_0}(b)$, there are at most $2d/\alpha$ activated edges $a'b'$ with $b' \notin N_{H_0}(a)$. Since $N_{H_0}(b)$ is rainbow, there are at most $\Delta(J) \leqslant \beta\delta(J)$ vertices $a' \in N_{H_0}(b)$ with $\chi_1(a')\chi_1(b) \in E(J)$ (recall that $\chi_1(b)$ has been fixed at step (ii)). Therefore, there are at most $(2\beta/\alpha)\delta(J)d$ activated edges $a'b'$ with $a' \in N_{H_0}(b)$, $\chi_1(a')\chi_1(b) \in E(J)$ and $b' \notin N_{H_0}(a)$.

Recall that a colour class $C \subseteq B \setminus N_{H_0}(a)$ is dangerous for b if there are at least

$$\frac{\delta(J)}{8\alpha} - 1 \geqslant \frac{\delta(J)}{9\alpha}$$

activated edges $a'b'$ with $a' \in N_{H_0}(b)$, $\chi_1(a')\chi_1(b) \in E(J)$ and $b' \in C$.

Thus, there are at most $18\beta d \leqslant \alpha^{1/2}d$ dangerous colour classes, as desired. □

We now proceed to show that, conditional on E, most of the vertices in M do not satisfy (F2). By Claim 4.10 and since there are at least αd colours, the probability that all the dangerous classes for b are assigned in step (iii) a colour different to $\chi_1(b)$ is at least $1 - \alpha^{-1/2}$. Thus,

$$\mathbb{E}(Y \mid E, M) \geqslant (1 - \alpha^{-1/2})|M|.$$

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Observe that swapping the colours in two colour classes of $B \setminus N_{H_0}(a)$ can only change Y by at most 2, since every vertex in M has a unique colour. We can use McDiarmid's inequality to prove that

$$\mathbb{P}\big(Y \leqslant (1-2\alpha^{-1/2})|M| \mid E,\, M\big) \leqslant e^{-\Omega(|M|)}.$$

Using (4.3) and $\mathbb{P}(E) = 1 - e^{-\Omega(d)}$, we obtain a bound on the unconditional probability,

$$\mathbb{P}\left(Y \leqslant \frac{(1-2\alpha^{-1/2})\delta(J)}{8\alpha}\right) \leqslant e^{-\Omega(d^{\epsilon})},$$

where we used that $\delta(J) \geqslant d^{\epsilon}$.

If $b \in N_{H_0}(a)$ does not satisfy (F1), (F2) and (F3), then $b \in B \setminus B_1$. The edge ab is retained in H_1 if it was activated in H_0. Observe that the events (F1), (F2) and (F3) are independent of the activation of the edges between a and $N_{H_0}(a)$: (F1) and (F3) do not depend on edge activations, and (F2) only depends on the activation of edges between A and $B \setminus N_{H_0}(a)$. Such an edge ab is activated independently with probability $1/\alpha$. Using Chernoff's inequality we conclude the proof:

$$\mathbb{P}(D_4^1(a)) = \mathbb{P}\left(|N_{H_1}(a) \cap (B \setminus B_1)| \leqslant \frac{\delta(J)}{16\alpha^2}\right) = e^{-\Omega(d^{\epsilon})}. \qquad \square$$

Lemma 4.11. *Conditions* (C1), (C2), (C3) *and* (C4) *hold after the first iteration with positive probability.*

Proof. By Lemmas 4.4, 4.5, 4.8 and 4.9, we have $\mathbb{P}(D_k^1(a)) = e^{-\Omega(d^{\varepsilon/4})}$, for every $k \in \{1,2,3,4\}$. On the other hand, $D_1^1(a)$, $D_2^1(a)$, $D_3^1(b)$ and $D_4^1(a)$ are mutually independent of every other $D_1^1(a')$, $D_2^1(a')$, $D_3^1(b')$ and $D_4^1(a')$ such that a' (or b') is at distance larger than 6 from a (or b) in H_0. Thus, each event is mutually independent of all but at most $4d^6$ other events. Since $e^{-\Omega(d^{\varepsilon/4})} = O(d^{-7})$, provided that d is large enough with respect to ϵ, β and α, we can use the Local Lemma to show that after the first iteration, (C1), (C2), (C3) and (C4) hold with positive probability. $\square$

The same argument suffices to show the following.

Lemma 4.12. *Let* $1 \leqslant i < \tau$. *If* (C1), (C2) *and* (C3) *hold for every* $1 \leqslant j < i$, *then* (C1), (C2) *and* (C3) *hold after the ith iteration with positive probability.*

Proof of Lemma 4.3. Observe that (Q2) and (Q3) are satisfied deterministically, by Phases I and II of the colouring procedure.

We first show that Phase II stops with positive probability. By Lemma 4.12, condition (C2) is satisfied with positive probability for every $i \geqslant 1$. Since the stopping condition of the iterative part of Phase II is $|B_i(a)| \leqslant d^{\varepsilon/2}$ for every $a \in A$, with positive probability the iterative part of Phase II ends after $O(\log d)$ iterations.

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It remains to show that (Q1) is satisfied with positive probability at the end of Phase II. By Lemma 4.11, condition (C4) is satisfied after the first iteration. Hence,

$$d_H(a) \geqslant |N_H(a) \cap (B \setminus B_1)| = |N_{H_1}(a) \cap (B \setminus B_1)| \geqslant \frac{\delta(J)}{16\alpha^2}.$$

Let us show that the degrees of $b \in B$ in H are also large. First notice that the degree of a vertex $b \in B$ only decreases in the iteration when b retains its colour.

Suppose first that $b \in B \setminus B_\tau$. By Lemma 4.12, condition (C3) is satisfied at every iteration. Therefore,

$$d_H(b) \geqslant \frac{\delta(J)}{16\alpha^2}.$$

Now suppose that $b \in B_\tau$. Then $b = b_i$, for some $i \geqslant \tau + 1$. Recall that Phase II assigns to b_i the colour that minimizes $\mathbf{Bad}_J(b_i, \chi_{i-1}, H_{i-1})$. First, we show that for every $a \in A$, $d_{H_{i-1}}(a)$ is not too large. By Lemma 4.12, (C2) holds at each iteration, which implies that the number of uncoloured neighbours decreases exponentially fast (in particular, $\tau = O(\log d)$). Recall that since J is β-almost regular, $\Delta(J) \leqslant \beta\delta(J)$. Since (C1) and (C2) are satisfied at every iteration, we have

$$\begin{aligned}
|N_{H_{i-1}}(a) \cap (B \setminus B_\tau)| &\leqslant \sum_{i=1}^{\tau} \ell_i(a) \leqslant \sum_{i=1}^{\tau} \frac{2}{\alpha} p^+ |B_i(a)| + O(d^{\varepsilon/2} \log d) \\
&\leqslant \left(\sum_{i=1}^{\infty} \frac{1}{\alpha^{i/2}} \right) \frac{2\Delta(J)}{\alpha^2 d} |B_0(a)| + O(d^{\varepsilon/2} \log d) \\
&\leqslant \frac{4\Delta(J)}{\alpha^2} \leqslant \frac{\delta(J)}{2\alpha},
\end{aligned}$$

provided that d is large enough and since $\alpha \geqslant 25\beta^2$. Here we also used that $|B_0(a)| \leqslant d$. By the definition of τ, we also have $|N_{H_{i-1}}(a) \cap B_\tau| = |B_\tau(a)| \leqslant d^{\varepsilon/2}$. This implies that, for any $i > \tau$,

$$d_{H_{i-1}}(a) \leqslant \frac{\delta(J)}{2\alpha} + d^{\varepsilon/2} \leqslant \frac{\delta(J)}{\alpha}. \tag{4.4}$$

Now we can lower-bound the degree of b_i in H. By (4.4), there are at most $(\delta(J)d)/\alpha$ edges in H_{i-1} incident to a neighbour of b_i. Thus, there is a colour $c \in V(J)$ such that if $\chi_i(b_i) = c$, $\mathbf{Bad}_J(b_i, \chi_i, H_{i-1})$ is small, in fact

$$|\{a \in N_{H_{i-1}}(b_i) : \exists b' \in N_{H_{i-1}}(b_i) \setminus \{b_i\} \text{ and } \chi_i(b') = c\}| \leqslant \frac{\delta(J)}{\alpha^2}.$$

We set $\chi_i(b_i) = c$ and delete all the edges ab_i for some $a \in A$ that either cause conflicts (at most $\delta(J)/\alpha^2$) or $\chi_i(a)\chi_i(b_i) \notin E(J)$. Since property (P3) holds for every $b \in B$ and every colour $c \in V(J)$, the degree of b_i in H is

$$d_H(b_i) \geqslant \frac{\delta(J)}{4\alpha} - \frac{\delta(J)}{\alpha^2} \geqslant \frac{\delta(J)}{16\alpha^2},$$

provided that $\alpha \geqslant 5$. □

We conclude with the proof of the main theorem.

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Proof of Theorem 1.2. Let H be the spanning subgraph and let χ be the partial colouring of H provided by Lemma 4.3. For the sake of contradiction, suppose that H contains a locally injective copy of $F \in \mathcal{F}$. Consider the colouring of F induced by the colours given by χ to the locally injective copy of F in H. Since for every edge ab in H we have $\chi(a)\chi(b) \in E(J)$, the colouring of F induces a homomorphism from F to J. Moreover, since for every vertex $v \in V(H)$, $N_H(v)$ is rainbow, this homomorphism is locally injective. However, $\hom^*(F, J) = 0$ by the hypothesis of the theorem and we obtain a contradiction. Thus, $\hom^*(F, H) = 0$. Finally, the subgraph H also satisfies $\delta(H) = \Omega(\delta(J))$. □

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