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Asymptotic periodicity in outer billiards with contraction

JOSÉ PEDRO GAIVÃO

Departamento de Matemática e CEMAPRE, ISEG, Universidade de Lisboa, Rua do Quelhas 6, 1200-781 Lisboa, Portugal
(e-mail: jpgaivao@iseg.utl.pt)

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Abstract. We show that for almost every (P, λ), where P is a convex polygon and $\lambda \in (0, 1)$, the corresponding outer billiard about P with contraction λ is asymptotically periodic, i.e., has a finite number of periodic orbits and every orbit is attracted to one of them.

1. *Introduction*

Outer billiards about convex compact planar regions were introduced by Day and popularized by Neumann and Moser [**9**, **23**, **24**]. In this paper we are interested in the class of polygonal outer billiards which we now describe. Let $P \subset \mathbb{C}$ be a convex d-gon with cyclically ordered vertices $\boldsymbol{v} = (v_1, \ldots, v_d)$. For $z \in X := \mathbb{C} \setminus P$, let $L_z \subset \mathbb{C}$ be the supporting line of P passing through z such that the interior of P lies on the left-hand side of L_z with respect to the obvious orientation. The set of points $z \in X$ for which the supporting line L_z contains an edge of the polygon is denoted by $\mathcal{S}$. The set $\mathcal{S}$ is a union of d half-lines. Given $z \in X \setminus \mathcal{S}$, the polygon P and the supporting line L_z intersect at a single vertex v_k. Denote by z' the point in L_z obtained by reflecting z with respect to v_k. The map $T_P : X \setminus \mathcal{S} \to X$ defined by $z \mapsto z'$ is called the *outer billiard map* of P. The outer billiard map is not defined on $\mathcal{S}$ and has jump discontinuities across its lines. We call $\mathcal{S}$ the *singular set of* T_P and we shall occasionally write $\mathcal{S}_P$ to express the dependency on P. The complement $X \setminus \mathcal{S}$ is a disjoint union of d open cones $A_1, \ldots, A_d$. The apex of A_k is a vertex $v_k \in P$ and the restriction $T_P|_{A_k}$ is the Euclidean reflection about v_k. Thus, T_P is a planar piecewise isometry on d cones.

Polygonal outer billiards constitute a very interesting and special class of piecewise isometries. See [**31**] for a nice introduction to the subject and [**13**] for a survey. Several important questions remain open regarding the dynamics of polygonal outer billiards. Notably, the Neumann–Moser question about the existence of unbounded orbits. There is a class of polygons, called quasirational, where every orbit stays bounded [**14**, **21**, **32**].

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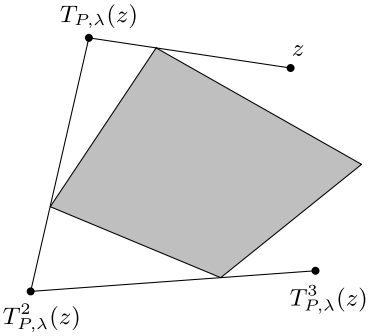


FIGURE 1. Example of a polygonal outer billiard with contraction. Four points of the orbit of z are depicted.

On the other hand, Schwartz gave the first example of a polygon, the Penrose kite, whose outer billiard has unbounded orbits [**27**]. Building on this work, he wrote a rather detailed study of the dynamics of outer billiards about kites [**28**]. In particular, he proved that the outer billiard has unbounded orbits if and only if the kite is irrational. Other classes of piecewise isometries have been studied in detail, mostly in two dimensions [**1**, **2**, **7**, **12**, **15**, **16**, **29**].

We will now embed the outer billiard about P into a one-parameter family of dynamical systems. Let $\lambda \in (0, 1]$. Define

$$T_{P,\lambda}(z) = -\lambda z + (1+\lambda)v_k \quad \text{for } z \in A_k. \tag{1.1}$$

When $\lambda < 1$, the map $T_{P,\lambda}$ is a piecewise affine contraction on the finite collection of disjoint open cones $\mathcal{A}_P := \{A_1, \ldots, A_d\}$. We will call the map $T_{P,\lambda}$ the *outer billiard map of P with contraction* λ. See Figure 1. For $\lambda = 1$, we recover the outer billiard map T_P.

We say that a point $z \in X$ is *non-singular* if it has a forward orbit, i.e., $T^n_{P,\lambda}(z) \notin \mathcal{S}_P$ for every $n \geq 0$. For such points, we denote by $\omega(z)$ the ω-limit set of the $T_{P,\lambda}$-orbit of z. The following notion is central to our study.

Definition 1.1. We say that $T_{P,\lambda}$ is *asymptotically periodic* if $T_{P,\lambda}$ has only a finite number of periodic orbits and the ω-limit set of any non-singular point is a periodic orbit.

Polygonal outer billiards with contraction were introduced to us by Eugene Gutkin. When the polygon is either a triangle or a parallelogram, we obtained a complete description of the dynamics using elementary methods. More precisely, for this class of polygons, we proved that the outer billiard with contraction λ is asymptotically periodic and the number of periodic orbits grows as $\lambda \to 1$ [**10**]. We also described the sequence of bifurcations that accumulate at $\lambda = 1$. A similar description can be obtained for affine-regular hexagons and small perturbations of parallelograms. When $\lambda = 1$, the dynamics of the outer billiard about regular d-gons with $d \in \{3, 4, 6\}$ is very simple, i.e., each point is periodic and the periodic points tile the plane in a regular way. See Figure 2 for the $d = 3$ case. Using the symmetries of the regular polygons, the method in [**10**] consisted in reducing the outer billiard map with contraction to a single cone and studying a first return map to a certain subset of the cone where the dynamics is easier to describe. Because the

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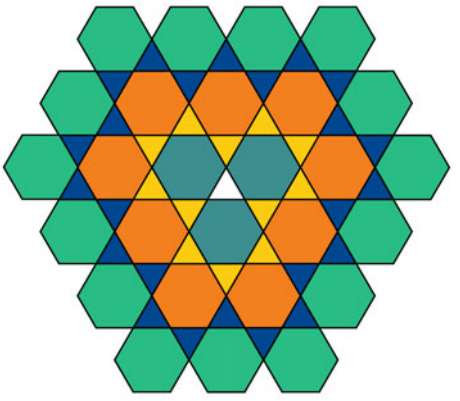


FIGURE 2. Periodic dynamics of the outer billiard about the equilateral triangle.

outer billiard with contraction is affine-equivariant, i.e., $A \circ T_{P,\lambda} = T_{A(P),\lambda} \circ A$ for any affine transformation $A \in \mathrm{Aff}(2, \mathbb{R})$, the outer billiard with contraction about any affine-regular polygon can be studied in a similar way. Independently, Jeong obtained the same results for the equilateral triangle and the square using a different and simpler method [**17**, **19**]. Taking advantage of the periodic dynamics when $\lambda = 1$, Jeong used the index of the periodic tiles to define a Lyapunov function for $\lambda < 1$. However, this method does not seem to apply to other regular polygons. Besides the equilateral triangle, square and regular hexagon, we do not know if the outer billiard with contraction about any other regular polygon is asymptotically periodic. See Figure 3 for the regular heptagon.

Regarding general results, it is known that polygonal outer billiards with contraction are asymptotically periodic when the contraction λ is sufficiently small [**10**, Corollary 3.4]. On the other hand, there are uncountably many pairs (P, λ) such that the outer billiard about P with contraction λ has an attracting Cantor set [**18**]. Nevertheless, numerical experiments suggest that, for generic (P, λ), the corresponding polygonal outer billiard with contraction is asymptotically periodic [**10**, **17**].

The set $\mathcal{P}_d$ of convex polygons with d sides can be identified with a subset of $\mathbb{C}^d$. Naturally, it inherits the Euclidean topology and the Lebesgue measure of $\mathbb{C}^d$. The following theorem is our main result.

THEOREM 1.1. *For almost every pair $(P, \lambda) \in \mathcal{P}_d \times (0, 1)$, the outer billiard map of P with contraction λ is asymptotically periodic.*

Polygonal outer billiards with contraction form a special class of piecewise affine contractions on the plane,

$$G(z) = \lambda_k z + (1 - \lambda_k) w_k, \quad z \in X_k,$$

where $\{X_k\}_{k=1}^K$ is a finite partition of $\mathbb{C}$, $|\lambda_k| < 1$ and $w_k \in \mathbb{C}$. For simplicity we assume that the singular set $\bigcup_k \partial X_k$ consists of finitely many straight lines or segments. In [**6**], Bruin and Deane proved that for almost every choice of the branch fixed points $(w_1, \ldots, w_K) \in \mathbb{C}^K$, the map G is asymptotically periodic. The dynamics of other classes

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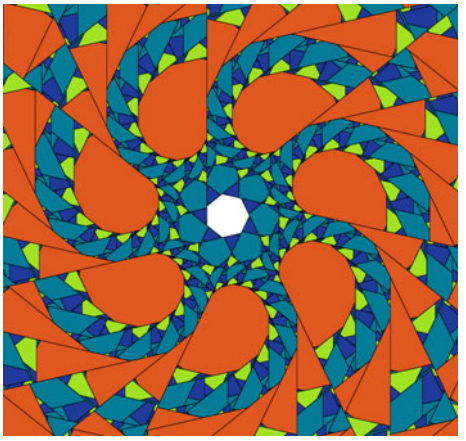


FIGURE 3. Basins of attraction of the periodic orbits of the outer billiard about the regular heptagon with contraction $\lambda = 0.9$.

of piecewise contractions has been recently investigated in [**5**, **8**, **25**, **26**]. The idea of the proof in [**6**] was to show that for generic choices of the branch fixed points, the attractor does not intersect the singular set. This was achieved by proving that the attractor has zero Hausdorff dimension, since a generically parametrized set with zero Hausdorff dimension should be disjoint from the singular set. Theorem 1.1 does not follow from the results of [**6**]. In polygonal outer billiards with contraction the branch fixed points are the vertices of the polygon which also define the partition of the outer billiard map. Thus, moving the fixed points also moves the partition of the map, whereas in [**6**] the partition is assumed to be fixed from the very beginning. So, what is generic for outer billiards with contraction may not be generic for planar piecewise affine contractions, since outer billiards with contraction belong to a positive-co-dimension subspace. To prove Theorem 1.1, we had to include the contraction λ as a perturbation parameter. By varying λ we hope to separate the attractor from the singular set. To that end we have derived a new quantitative Łojasiewicz-type inequality for polynomials of one variable (Theorem 4.6) that may be of independent interest. As far as we are aware, Theorem 1.1 is the first result in the class of piecewise affine contractions where asymptotic periodicity is obtained by varying the contraction coefficients.

The rest of the paper is structured as follows. Sections 2 and 3 contain some preliminary results. In §4 we derive a Łojasiewicz-type inequality for polynomials through a concept of transversality of power series. Finally, in §5 we prove Theorem 1.1.

2. *Itineraries*

Denote by $\|\cdot\|$ the sup-norm in $\mathbb{C}^d$. Throughout this section we fix $(P', \lambda') \in \mathcal{P}_d \times (0, 1)$. Any convex polygon $P \in \mathcal{P}_d$ can be represented by a tuple $\boldsymbol{v} = (v_1, \ldots, v_d)$ of vertices in the complex plane. For $\epsilon > 0$ sufficiently small, any $\boldsymbol{v}$ belonging to the ball

$$\mathcal{B}_\epsilon(P') := \{\boldsymbol{v} \in \mathbb{C}^d : \|\boldsymbol{v} - \boldsymbol{v}'\| < \epsilon\}$$

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defines a convex polygon $P \in \mathcal{P}_d$ which is *ϵ-close to P'* and has the same cyclic order of its vertices. By slightly abusing the notation, we write $P \in \mathcal{B}_\epsilon(P')$ and $\|P\| := \|\boldsymbol{v}\|$.

Define

$$a_\epsilon := \lambda' + \epsilon, \quad b_\epsilon := \sup\{\|P\| \colon P \in \mathcal{B}_\epsilon(P')\} \tag{2.1}$$

and the ϵ-neighbourhood Δ_ϵ of (P', λ') by

$$\Delta_\epsilon = \Delta_\epsilon(P', \lambda') := \mathcal{B}_\epsilon(P') \times (\lambda' - \epsilon, \lambda' + \epsilon).$$

We assume that $\epsilon > 0$ is small, so that $(\lambda' - \epsilon, \lambda' + \epsilon) \subset (0, 1)$.

Denote by $K_\epsilon := \mathbb{D}_{r_\epsilon}(0) \subset \mathbb{C}$ the closed disc in the complex plane centred at the origin and having radius

$$r_\epsilon := b_\epsilon \frac{1 + a_\epsilon}{(1 - a_\epsilon)^2}.$$

Also let $\omega(P, \lambda)$ denote the union of the ω-limit sets of non-singular points of $T_{P,\lambda}$. The following inclusions are easy to check.

LEMMA 2.1. *For every $(P, \lambda) \in \Delta_\epsilon$, we have that*

$$\omega(P, \lambda) \subset T_{P,\lambda}(K_\epsilon) \subset K_\epsilon.$$

Proof. Given any non-singular point $z \in X$, there exists a sequence $(i_j)_{j \geq 0}$ with $i_j \in \{1, \ldots, d\}$ such that for every $n \in \mathbb{N}$,

$$T^n_{P,\lambda}(z) = (-\lambda)^n z + (1 + \lambda) \sum_{j=0}^{n-1} (-\lambda)^{n-j-1} v_{i_j}.$$

Thus,

$$|T^n_{P,\lambda}(z)| \leq a^n_\epsilon |z| + \frac{(1 + a_\epsilon)}{1 - a_\epsilon} b_\epsilon$$

and the claim follows. □

For every $(P, \lambda) \in \Delta_\epsilon$, Lemma 2.1 tells us that the limit dynamics of $T_{P,\lambda}$ occurs inside the compact set K_ϵ. Therefore, we restrict the open cones $A_k \in \mathcal{A}_P$ to K_ϵ where $T_{P,\lambda}$ is defined. To simplify the notation we use the same letters to denote the restricted cones of any polygon P which is ϵ-close to P'.

Let $\mathcal{A}_n$ be the collection of non-empty sets

$$A_{i_0, i_1, \ldots, i_{n-1}} = A_{i_0} \cap T^{-1}_{P,\lambda}(A_{i_1}) \cap \cdots \cap T^{-n+1}_{P,\lambda}(A_{i_{n-1}}),$$

where $A_{i_k} \in \mathcal{A}_P$ and $i_0, i_1, \ldots, i_{n-1} \in \{1, \ldots, d\}$. Each element of $\mathcal{A}_n$ is a domain of continuity of $T^n_{P,\lambda}$. We can represent each element of $\mathcal{A}_n$ using the corresponding itinerary.

Definition 2.1. The set of *admissible itineraries of order n of $T_{P,\lambda}$* is the set of n-tuples $(i_0, \ldots, i_{n-1}) \in \{1, \ldots, d\}^n$ such that

$$A_{i_0, \ldots, i_{n-1}} \neq \emptyset.$$

We denote this set by $\mathcal{I}_n(P, \lambda)$. The union of the sets $\mathcal{I}_n(P, \lambda)$ over $(P, \lambda) \in \Delta_\epsilon$ is denoted by $\mathcal{I}^\epsilon_n = \mathcal{I}^\epsilon_n(P', \lambda')$.

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Let $T_{P,\lambda,k} = T_{P,\lambda}|_{A_k}$ denote the branch map on the kth cone. Naturally, $T_{P,\lambda,k}$ extends to an affine contraction on all of $\mathbb{C}$. Given $\vec{i} = (i_0, \ldots, i_{n-1}) \in \mathcal{I}_n^\epsilon$, we define

$$\begin{aligned} H(P, \lambda, \vec{i}\,) &:= T_{P,\lambda,i_{n-1}} \circ \cdots \circ T_{P,\lambda,i_0}(0) \\ &= (1+\lambda) \sum_{j=0}^{n-1} (-\lambda)^{n-j-1} v_{i_j} \end{aligned}$$

and

$$\Lambda_\epsilon(P, \lambda) := \bigcap_{m \geq 1} \overline{\bigcup_{n \geq m} \{H(P, \lambda, \vec{i}\,) : \vec{i} \in \mathcal{I}_n^\epsilon\}}.$$

LEMMA 2.2. *For every $(P, \lambda) \in \Delta_\epsilon$, we have that*

$$\omega(P, \lambda) \subset \Lambda_\epsilon(P, \lambda) \subset K_\epsilon.$$

Proof. Let $y \in \omega(P, \lambda)$. By Lemma 2.1 and the definition of the omega-limit set $\omega(P, \lambda)$, there exist $z \in K_\epsilon$ and an increasing sequence $(n_k)_{k \geq 1}$ such that $T_{P,\lambda}^{n_k}(z) \to y$ as $k \to \infty$. Let $(i_0, \ldots, i_{n_k-1}) \in \mathcal{I}_{n_k}^\epsilon$ be the itinerary of z. Define $x_k := H(P, \lambda, (i_0, \ldots, i_{n_k-1}))$. Then

$$|T_{P,\lambda}^{n_k}(z) - x_k| \leq r_\epsilon a_\epsilon^{n_k}.$$

This implies that $x_k \to y$ as $k \to \infty$, i.e., $y \in \Lambda_\epsilon(P, \lambda)$. Finally, the inclusion $\Lambda_\epsilon(P, \lambda) \subset K_\epsilon$ follows from the bound

$$|H(P, \lambda, \vec{i}\,)| \leq \frac{(1 + a_\epsilon)}{1 - a_\epsilon} b_\epsilon \leq r_\epsilon. \qquad \square$$

LEMMA 2.3. *For every $(P, \lambda) \in \Delta_\epsilon$ and for every $n \geq 1$, the set $\Lambda_\epsilon(P, \lambda)$ can be covered by a finite union of discs of radius $2r_\epsilon a_\epsilon^n$ centred at the points $\{H(P, \lambda, \vec{i}\,) : \vec{i} \in \mathcal{I}_n^\epsilon\}$.*

Proof. Given $z \in \Lambda_\epsilon(P, \lambda)$, there exist an increasing sequence $(m_k)_{k \geq 1}$ and itineraries $\vec{i}_{m_k} \in \mathcal{I}_{m_k}^\epsilon$ such that $y_k \to z$ as $k \to \infty$, where $y_k := H(P, \lambda, \vec{i}_{m_k})$. Choose $k > 0$ sufficiently large so that $m_k \geq n$ and $|z - y_k| \leq r_\epsilon a_\epsilon^n$. Define $x := H(P, \lambda, (d_0, \ldots, d_{n-1}))$, where $d_0, \ldots, d_{n-1}$ are the last n entries of $\vec{i}_{m_k}$. Hence, $|y_k - x| \leq r_\epsilon a_\epsilon^n$. This implies that

$$|z - x| \leq |z - y_k| + |y_k - x| \leq 2r_\epsilon a_\epsilon^n. \qquad \square$$

LEMMA 2.4. *Every admissible itinerary of order $n \geq 2$ has at least two distinct symbols. Moreover, there exists an integer $N \geq 1$ such that for every $\epsilon > 0$ sufficiently small, every itinerary of $\mathcal{I}_N^\epsilon$ contains at least three distinct symbols (labels of the vertices).*

Proof. The first claim is obvious since consecutive reflections cannot take place at the same vertex of the polygon. We now prove the second claim. Suppose that for every integer $n \geq 1$, there exist $\epsilon_n > 0$ converging to zero as $n \to \infty$ and itineraries $\vec{i}_n \in \mathcal{I}_n^{\epsilon_n}$ containing at most two symbols. In fact, by the first claim of the lemma, we can assume that $\vec{i}_n$ contains exactly two symbols. We also assume that n is even. The odd case follows from similar considerations. Then

$$\vec{i}_n = (k_n, j_n, k_n, j_n, \ldots, k_n, j_n),$$

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where $k_n, j_n \in \{1, \ldots, d\}$ represent the labels of distinct vertices. Since $\vec{i}_n$ is an admissible itinerary, it is associated with a pair $(P^{(n)}, \lambda^{(n)}) \in \Delta_{\epsilon_n}$. Notice that $(P^{(n)}, \lambda^{(n)}) \to (P', \lambda')$. Denote the vertices of $P^{(n)}$ by $\boldsymbol{v}^{(n)} = (v_1^{(n)}, \ldots, v_d^{(n)})$. The affine contraction $T_{P^{(n)},\lambda^{(n)},j_n} \circ T_{P^{(n)},\lambda^{(n)},k_n}$ has a single fixed point $x^{(n)}$ which is on the line spanned by the vertices $v_{k_n}^{(n)}$ and $v_{j_n}^{(n)}$, i.e.,

$$x^{(n)} = \frac{v_{j_n}^{(n)} - \lambda^{(n)} v_{k_n}^{(n)}}{1 - \lambda^{(n)}}.$$

Hence, taking n sufficiently large, we conclude that the itinerary $\vec{i}_n$ cannot be admissible, which is a contradiction. This concludes the proof. □

For any integer $n \geq 1$, let $\mathcal{S}_{P,\lambda}^n$ denote the set of points $z \in X$ for which there exists an integer $0 \leq k < n$ such that $T_{P,\lambda}^k(z)$ belongs to $\mathcal{S}_P$ (so $\mathcal{S}_{P,\lambda}^1 := \mathcal{S}_P$). We call $\mathcal{S}_{P,\lambda}^n$ the *singular set of order n* of $T_{P,\lambda}$. Notice that $\mathcal{S}_{P,\lambda}^n$ is a finite union of half-lines and segments, and $\mathcal{S}_{P,\lambda}^n \subseteq \mathcal{S}_{P,\lambda}^{n+1}$ for every $n \geq 1$.

LEMMA 2.5. *Let* $(P, \lambda) \in \Delta_\epsilon$. *If*

$$\Lambda_\epsilon(P, \lambda) \cap \mathcal{S}_P = \emptyset,$$

then $T_{P,\lambda}$ *is asymptotically periodic.*

Proof. Since $\Lambda_\epsilon(P, \lambda) \cap \mathcal{S}_P = \emptyset$, there exist $N > 0$ and $\delta > 0$ such that

$$\bigcup_{n \geq N} \{H(P, \lambda, \vec{i}) : \vec{i} \in \mathcal{I}_n^\epsilon\} \cap \mathcal{S}_P^\delta = \emptyset, \tag{2.2}$$

where $\mathcal{S}_P^\delta$ denotes the δ-neighbourhood of $\mathcal{S}_P \cap K_\epsilon$. Now let $z \in \mathcal{S}_{P,\lambda}^{n+1} \cap K_\epsilon \setminus \mathcal{S}_{P,\lambda}^n$. Denote by $\vec{i}$ the itinerary of order n of z and set $x := H(P, \lambda, \vec{i})$. Then $|T_{P,\lambda}^n(z) - x| < a_\epsilon^n r_\epsilon$. Taking $n > N$ sufficiently large, we conclude that $x \in \mathcal{S}_P^\delta$. This contradicts (2.2). Thus, $\mathcal{S}_{P,\lambda}^{n+1} \cap K_\epsilon = \mathcal{S}_{P,\lambda}^n \cap K_\epsilon$ for $n \geq 1$ sufficiently large. This means that $T_{P,\lambda}^n$, restricted to K_ϵ, maps each domain of continuity strictly inside another domain of continuity. A contraction fixed point argument finishes the proof. □

3. *Singular connections*

Let $(P, \lambda) \in \mathcal{P}_d \times (0, 1)$. A *singular connection of order* $n \geq 2$ corresponds to a line segment $\ell \subset \mathcal{S}_P$ which is mapped by $T_{P,\lambda}^n$ back to $\mathcal{S}_P$. The precise definition is as follows.

Definition 3.1. A *singular connection of order* $n \geq 2$ is a $T_{P,\lambda}$-orbit segment $x_1, \ldots, x_n$ such that:

(1) $x_1, \ldots, x_{n-1} \notin \mathcal{S}_P$;
(2) $x_n \in \mathcal{S}_P$;
(3) the line segment $[x_1, x_n]$ contains a side of P.

We say that $T_{P,\lambda}$ has *no singular connections* if there are no singular connections of any order.

LEMMA 3.1. $T_{P,\lambda}$ *has no singular connections for almost every* (P, λ).

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Proof. The set of pairs (P, λ) such that $T_{P,\lambda}$ has a singular connection of order $n \geq 2$ has zero Lebesgue measure. Indeed, a singular connection $x_1, \ldots, x_n$ of order n is determined by an admissible itinerary $\{i_1, \ldots, i_{n-1}\}$ of order $n-1$ such that

$$x_n - (-\lambda)^{n-1} x_1 = (1+\lambda) \sum_{j=1}^{n-1} (-\lambda)^{n-1-j} v_{i_j}$$

and the condition that the segment $[x_1, x_n]$ contains the kth side of P for some k. Identifying the complex plane with $\mathbb{R}^2$, this last condition can be written as

$$\det(x_n - x_1,\, v_{k+1} - v_k) = 0,$$

where v_k and v_{k+1} are the vertices of P defining the kth side. Thus,

$$\det\left((1+\lambda) \sum_{j=1}^{n-1} (-\lambda)^{n-1-j} v_{i_j} - (1 - (-\lambda)^{n-1}) v_k,\, v_{k+1} - v_k \right) = 0,$$

which defines a positive-co-dimension algebraic set on the product space $\mathcal{P}_d \times (0, 1)$, since every admissible itinerary of order $n \geq 2$ has at least two distinct symbols (Lemma 2.4). So, the union of all these sets has zero Lebesgue measure. □

PROPOSITION 3.2. *If $T_{P,\lambda}$ has no singular connections, then*

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \#\mathcal{I}_n^{\epsilon}(P, \lambda) = 0.$$

Proof. Let $\rho > 1$ and take an integer $m > 0$ sufficiently large so that $(2d)^{1/m} < \rho$. Since $T_{P,\lambda}$ is a homothety, the number of singular lines of $\mathcal{S}_P^m$ that meet at a common point is bounded from above by d. Let $\tau = \tau(m) > 0$ be the smallest distance between any two intersection points of the lines of $\mathcal{S}_P^m$. Any point in a disc whose diameter does not exceed τ will have at most $2d$ distinct itineraries of size m. Because $T_{P,\lambda}$ has no singular connections, this property holds for every $(P', \lambda') \in \Delta_\epsilon(P, \lambda)$ for some $\epsilon > 0$ sufficiently small. Moreover, choosing ϵ smaller if necessary, we can guarantee that the set of admissible itineraries of order m of $T_{P',\lambda'}$ for any $(P', \lambda') \in \Delta_\epsilon(P, \lambda)$ coincides with the set of admissible itineraries of order m of $T_{P,\lambda}$.

Now choose $N > 0$ (depending on m and ϵ) large enough so that

$$\operatorname{diam}(T_{P',\lambda'}^N(A)) < \tau$$

for every $(P', \lambda') \in \Delta_\epsilon(P, \lambda)$ and any continuity domain A of $T_{P',\lambda'}^N$.

Define $a_n := \#\mathcal{I}_n^{\epsilon}(P, \lambda)$. By the previous observations, $a_{n+m} \leq 2d\, a_n$ for every $n \geq N$. So, $a_{N+im} \leq d^N (2d)^i$ for every $i \geq 0$. In other words,

$$a_n \leq d^N (2d)^{(n-N)/m} \quad \text{for all } n \geq N.$$

Define $C := d^N (2d)^{-N/m}$. Taking into account the choice of m, we get $a_n \leq C\rho^n$ for every $n \geq N$. Hence, $\log a_n \leq \log C + n \log \rho$. Since $\rho > 1$ was arbitrary and $\lim_n (1/n) \log a_n \leq \log \rho$, the claim follows. □

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Remark 3.3. The growth rate of the number of admissible itineraries is known as the *singular entropy*. For a certain class of piecewise affine maps, which contains the polygonal outer billiards with contraction, the singular entropy equals the topological entropy (defined appropriately), which is known to be zero [**22**, Corollary 2]. Because we are considering a larger set of itineraries, i.e., the set of admissible itineraries of all nearby polygonal outer billiards, Proposition 3.2 does not follow directly from the results of [**22**] and we had to impose a generic condition on (P, λ) (see Definition 3.1).

4. *Transversality of power series*

Consider the following class of power series:

$$\mathcal{F}_\alpha = \left\{ p(x) = 1 + \sum_{n=1}^{\infty} a_n x^n \colon a_n \in [-\alpha, \alpha] \right\}, \quad \alpha > 0$$

and define, for each $k \geq 0$,

$$r_\alpha(k) := \inf\{x > 0 \colon \exists p \in \mathcal{F}_\alpha,\ p^{(j)}(x) = 0,\ j = 0, \ldots, k\}.$$

Because $\mathcal{F}_\alpha$ is compact in the topology of uniform convergence in compacta, the infimum is achieved on $\mathcal{F}_\alpha$. Moreover, $r_\alpha(k) \geq r_\alpha(0) = 1/(1+\alpha)$, where $r_\alpha(0)$ is computed using the power series $1 - \alpha \sum_{n=1}^{\infty} x^n = 1 - \alpha x/(1-x)$.

The following theorem gives a lower and an upper bound for $r_\alpha(k)$.

THEOREM 4.1. [**3**] *For every $k \geq 0$,*

$$r_\alpha(k) \leq \left(1 - \frac{1}{k+2}\right)^{\min\{\alpha/9, 1\}}$$

and

$$r_\alpha(k) \geq \left(1 + \frac{1}{k+1}\right)^{-1/2} (\alpha^2(k+1) + 1)^{-1/(2(k+1))}.$$

In particular, this theorem shows that

$$r_\alpha(k) < 1 \quad \text{and} \quad \lim_{k \to \infty} r_\alpha(k) = 1.$$

The problem of estimating $r_1(1)$ was considered in [**30**]. A fine lower bound for $r_1(1)$ was needed in order to ensure a *δ-transversality* property for power series in $\mathcal{F}_1$ when restricted to a suitable interval. The following lemma extends this notion to (δ, k)-transversality.

LEMMA 4.2. *For every $k \geq 1$ and $\tau > 0$, there exists $\delta \in (0, 1)$ such that for every $p \in \mathcal{F}_\alpha$, the following holds. If $x \in [0, r_\alpha(k) - \tau]$ and $|p(x)| < \delta$, then*

$$\max_{1 \leq j \leq k} |p^{(j)}(x)| \geq \delta.$$

Proof. We prove the claim by contradiction. Suppose that there are sequences $p_n \in \mathcal{F}_\alpha$ and $x_n \in [0, r_\alpha(k) - \tau]$ such that $p_n^{(j)}(x_n) \to 0$ as $n \to \infty$ for every $j = 0, \ldots, k$. Taking a subsequence, we may assume that $x_n \to x \in [0, r_\alpha(k) - \tau]$ and $p_n \to q \in \mathcal{F}_\alpha$ in the topology of uniform convergence. Hence, $q^{(j)}(x) = 0$ for every $j = 0, \ldots, k$. But $x < r_\alpha(k)$, which contradicts the definition of $r_\alpha(k)$. □

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Remark 4.3. This lemma, together with the fact $\lim_k r_\alpha(k) = 1$, shows that (δ, k)-transversality, i.e., the conclusion of the lemma, holds for points which get closer to 1 as $k \to \infty$.

The following lemma is a metric Łojasiewicz-type inequality for polynomials of one variable. Its proof follows the same lines of [**20**, Lemma 5.3], which in turn is based on [**11**, Lemma 3].

LEMMA 4.4. *Let $d \in \mathbb{N}$, $0 < \delta < 1$ and $\epsilon > 0$. If $p \in \mathbb{R}[x]$ is a polynomial of degree $n \geq d$ such that for every $x \in [-1, 1]$, we have*

$$|p(x)| < \epsilon \quad \Rightarrow \quad \max_{1 \leq j \leq d} |p^{(j)}(x)| \geq \delta,$$

then

$$\mathrm{Leb}\{x \in [-1, 1] \colon |p(x)| < \epsilon\} \leq C\epsilon^{1/d},$$

where

$$C := \frac{2^{d+3}}{\delta^2}\Big(4n^{2(d+1)} \max_{x \in [-1,1]} |p(x)| + 1\Big).$$

Proof. By Markov's inequality [**4**, Theorem 5.1.8], we have for $j = 0, \ldots, d$ that

$$\max_{x \in [-1,1]} |p^{(j+1)}(x)| \leq (n - j)^2 \max_{x \in [-1,1]} |p^{(j)}(x)|.$$

Iterating this inequality gives

$$\begin{aligned} A &:= \max_{0 \leq j \leq d+1} \max_{x \in [-1,1]} |p^{(j)}(x)| \\ &\leq (n-d)^2(n-d+1)^2 \cdots n^2 \max_{x \in [-1,1]} |p(x)| \\ &\leq n^{2(d+1)} \max_{x \in [-1,1]} |p(x)|. \end{aligned}$$

Now suppose that there is $x_0 \in [-1, 1]$ such that $|p(x_0)| < \epsilon$. Otherwise there is nothing to prove. By hypothesis, $|p^{(j)}(x_0)| \geq \delta$ for some $1 \leq j \leq d$. Let us suppose that $j = d$, since this is the worst case. Then, by the mean value theorem, for every $x \in I_0 := \{x \in [-1, 1] \colon |x - x_0| \leq \delta/2A\}$, there is $\xi \in I_0$ such that

$$\begin{aligned} |p^{(d)}(x_0)| - |p^{(d)}(x)| &\leq |p^{(d)}(x) - p^{(d)}(x_0)| \\ &= |p^{(d+1)}(\xi)||x - x_0| \\ &\leq A|x - x_0| \leq \frac{\delta}{2}. \end{aligned}$$

Thus,

$$|p^{(d)}(x)| \geq \frac{\delta}{2} \quad \text{for all } x \in I_0.$$

Now we consider $p^{(d-1)}$ on the interval I_0. If there is $x_1 \in I_0$ such that $|p^{(d-1)}(x_1)| < \epsilon^{1/d}$, then, again by the mean value theorem, for every $x \in I_0 \setminus I_1$ where $I_1 := \{x \in I_0 \colon |x - x_1| \leq 4\epsilon^{1/d}/\delta\}$ there exists $\xi \in I_0$ such that

$$\begin{aligned} |p^{(d-1)}(x)| + |p^{(d-1)}(x_1)| &\geq |p^{(d-1)}(x) - p^{(d-1)}(x_1)| \\ &= |p^{(d)}(\xi)||x - x_1| \\ &\geq \frac{\delta}{2}|x - x_1| \geq 2\epsilon^{1/d}. \end{aligned}$$

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Therefore,

$$|p^{(d-1)}(x)| \geq \epsilon^{1/d} \quad \text{for all } x \in I_0 \setminus I_1.$$

Notice that I_1 may be empty. Next consider $p^{(d-2)}$ on $I_0 \setminus I_1$. Clearly, $I_0 \setminus I_1$ consists of two intervals, which we denote by J_2 and J_3. On J_2, if there is $x_2 \in J_2$ such that $|p^{(d-2)}(x_2)| < \epsilon^{2/d}$, then, as before,

$$|p^{(p-2)}(x)| \geq \epsilon^{2/d}$$

for every $x \in J_2 \setminus I_2$, where $I_2 := \{x \in J_2 \colon |x - x_2| \leq 2\epsilon^{1/d}\}$. Similarly, there is an interval $I_3 \subset J_3$ having length at most $4\epsilon^{1/d}$ such that

$$|p^{(d-2)}(x)| \geq \epsilon^{2/d} \quad \text{for all } x \in I_0 \setminus (I_1 \cup I_2 \cup I_3).$$

Continuing this process, we obtain $2^d - 1$ subintervals I_i, $i = 1, \ldots, 2^d - 1$, of the interval I_0 having length at most $4\epsilon^{1/d}/\delta$ such that

$$|p(x)| \geq \epsilon \quad \text{for all } x \in I_0 \setminus (I_1 \cup \cdots \cup I_{2^d-1}).$$

Since the interval $[-1, 1]$ can be partitioned into at most $4A/\delta$ intervals having the size of I_0, we conclude that on the whole interval we obtain at most

$$2^d\left(\frac{4A}{\delta} + 1\right)$$

intervals of length at most $8\epsilon^{1/d}/\delta$, where $|p(x)| < \epsilon$. Therefore,

$$\begin{aligned}\text{Leb}\{x \in [-1, 1] \colon |p(x)|\} &\leq 2^{d+3}\frac{4A+1}{\delta^2}\epsilon^{1/d} \\ &\leq \frac{2^{d+3}}{\delta^2}\Big(4n^{2(d+1)} \max_{x\in[-1,1]} |p(x)| + 1\Big)\epsilon^{1/d}. \qquad \square\end{aligned}$$

Remark 4.5. We can apply Lemma 4.4 to any polynomial defined in an interval $I = [a, b] \subset [0, 1]$ for which the hypothesis holds on I. Indeed, let g denote the affine transformation (orientation-preserving) that maps $[-1, 1]$ to I and define $\hat{p} := p \circ g$. Then

$$\text{Leb}\{x \in I \colon |p(x)| < \epsilon\} = \frac{b-a}{2}\,\text{Leb}\{x \in [-1, 1] \colon |\hat{p}(x)| < \epsilon\}.$$

Now we see that the polynomial $\hat{p}$ satisfies the hypothesis of the lemma. For any $x \in [-1, 1]$ such that $|\hat{p}(x)| < \epsilon$, we have

$$\begin{aligned}\max_{1\leq j\leq d} |\hat{p}^{(j)}(x)| &= \max_{1\leq j\leq d}\left(\frac{b-a}{2}\right)^j |p^{(j)}(g(x))| \\ &\geq \left(\frac{b-a}{2}\right)^d \max_{1\leq j\leq d} |p^{(j)}(g(x))| \\ &\geq \left(\frac{b-a}{2}\right)^d \delta.\end{aligned}$$

Therefore,

$$\text{Leb}\{x \in I \colon |p(x)| < \epsilon\} \leq C'\epsilon^{1/d},$$

where

$$C' := \left(\frac{2}{b-a}\right)^{2d-1}\frac{2^{d+3}}{\delta^2}\Big(4n^{2(d+1)} \max_{x\in I} |p(x)| + 1\Big).$$

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Denote by $\mathcal{F}_\alpha^{(n)} \subset \mathcal{F}_\alpha$ the set of polynomials with coefficients in $[-\alpha, \alpha]$ and having degree n. The following theorem is the main result of this section.

THEOREM 4.6. *Let $I = [a, b]$ with $0 \le a < b < 1$. There exist $k \ge 1$ and $0 < \delta < 1$ such that for every $0 < \epsilon < \delta$ and every polynomial $p \in \bigcup_{n\ge 0} \mathcal{F}_\alpha^{(n)}$,*

$$\mathrm{Leb}\{x \in I : |p(x)| < \epsilon\} \le C\epsilon^{1/k},$$

where

$$C := \frac{2^{3k+5}(1+\alpha)^{2k}\deg(p)^{2(k+1)}}{\delta^2(1 - r_\alpha(k))}.$$

Proof. Let $0 < \tau < \min\{1 - b, r_\alpha(1) - r_\alpha(0)\}$. Recall that $r_\alpha(0) = 1/(1+\alpha)$. Define

$$k := \inf\left\{m \in \mathbb{N} : \frac{1}{b+\tau} > \left(1 + \frac{1}{m+1}\right)^{1/2} (\alpha^2(m+1)+1)^{1/2(m+1)}\right\}.$$

By Theorem 4.1, $b \le r_\alpha(k) - \tau$. Thus, $I = [a, b] \subset [0, r_\alpha(k) - \tau]$. Moreover, it follows from Lemma 4.2 that there is $0 < \delta < 1$, depending only on k and τ, such for every $p \in \bigcup_{n\ge 0} \mathcal{F}_\alpha^{(n)}$ and every $x \in [0, r_\alpha(k) - \tau]$, we have

$$|p(x)| < \delta \quad \Rightarrow \quad \max_{1\le j\le k} |p^{(j)}(x)| \ge \delta.$$

Therefore, by Lemma 4.4 and Remark 4.5,

$$\begin{aligned}\mathrm{Leb}\{x \in I : |p(x)| < \epsilon\} &\le \mathrm{Leb}\{x \in [0, r_\alpha(k) - \tau] : |p(x)| < \epsilon\} \\ &\le C\epsilon^{1/k},\end{aligned}$$

where

$$\begin{aligned}C &:= \left(\frac{2}{r_\alpha(k) - \tau}\right)^{2k-1} \frac{2^{k+3}}{\delta^2}\left(4\deg(p)^{2(k+1)} \max_{x\in I} |p(x)| + 1\right) \\ &\le \frac{2^{3k+2}(1+\alpha)^{2k-1}}{\delta^2}\left(\frac{4(1+\alpha)\deg(p)^{2(k+1)}}{1 - r_\alpha(k)} + 1\right) \\ &\le \frac{2^{3k+5}(1+\alpha)^{2k}\deg(p)^{2(k+1)}}{\delta^2(1 - r_\alpha(k))},\end{aligned}$$

where we have used the inequalities

$$\max_{x\in I} |p(x)| \le 1 + \alpha \max_{x\in[0, r_\alpha(k)]} \sum_{i=1}^{\infty} x^i \le \frac{1+\alpha}{1 - r_\alpha(k)}$$

and

$$r_\alpha(k) - \tau > \frac{1}{1+\alpha}.$$

□

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5. *Proof of Theorem 1.1*

We want to show that the set $\mathcal{Z}$ of pairs $(P, \lambda) \in \mathcal{P}_d \times (0, 1)$ such that $T_{P,\lambda}$ is not asymptotically periodic has zero Lebesgue measure. Given any polygon $P \in \mathcal{P}_d$, denote by $\mathcal{E}(P)$ the set of lines extending the sides and diagonals of P. We say that a polygon $P \in \mathcal{P}_d$ is in *general position* if every pair of distinct lines in $\mathcal{E}(P)$ intersect at a single point. The set of polygons in general position is denoted by $\hat{\mathcal{P}}_d$.

LEMMA 5.1. *Almost every polygon in $\mathcal{P}_d$ is in general position.*

Proof. A polygon $P \in \mathcal{P}_d$ is not in general position if it has two distinct lines in $\mathcal{E}(P)$ which are parallel. This imposes an algebraic condition on the set of vertices of P which has positive co-dimension in $\mathcal{P}_d$. □

By Lemma 3.1, we know that for almost every (P, λ) the dissipative outer billiard map $T_{P,\lambda}$ has no singular connections. Denote by $\mathcal{R}$ this full Lebesgue measure set. Therefore, it is sufficient to show that the set

$$\mathcal{Z}' := \mathcal{Z} \cap \mathcal{R} \cap (\hat{\mathcal{P}}_d \times (0, 1))$$

has zero Lebesgue measure.

Let $(P', \lambda') \in \mathcal{Z}'$ and take $\epsilon > 0$ small to be chosen during the proof. We recall the definition of the ϵ-neighbourhood Δ_ϵ of (P', λ'),

$$\Delta_\epsilon = \mathcal{B}_\epsilon(P') \times I_\epsilon, \quad I_\epsilon := (\lambda' - \epsilon, \lambda' + \epsilon).$$

Also recall the definition of a_ϵ and b_ϵ in (2.1). By Lemma 2.5,

$$\begin{aligned} \mathcal{Z}' \cap \Delta_\epsilon &\subset \{(P, \lambda) \in \Delta_\epsilon : T_{P,\lambda} \text{ is not asymptotically periodic}\} \\ &\subset \{(P, \lambda) \in \Delta_\epsilon : \Lambda_\epsilon(P, \lambda) \cap \mathcal{S}_P \neq \emptyset\}. \end{aligned}$$

Moreover, by Lemma 2.3, for every $n \geq 1$ and every $(P, \lambda) \in \Delta_\epsilon$, the limit set $\Lambda_\epsilon(P, \lambda)$ can be covered by at most $\#\mathcal{I}_n^\epsilon$ discs of radius $\rho_n := 2r_\epsilon a_\epsilon^n$ centred at the points $H(P, \lambda, \vec{i})$, where $\vec{i} \in \mathcal{I}_n^\epsilon$. Therefore,

$$\mathrm{Leb}(\mathcal{Z}' \cap \Delta_\epsilon) \leq \sum_{\vec{i} \in \mathcal{I}_n^\epsilon} \mathrm{Leb}\{(P, \lambda) \in \Delta_\epsilon : \mathbb{D}_{\rho_n}(H(P, \lambda, \vec{i})) \cap \mathcal{S}_P \neq \emptyset\},$$

where $\mathbb{D}_{\rho_n}(H(P, \lambda, \vec{i}))$ is the disc in the complex plane with radius ρ_n and centred at $H(P, \lambda, \vec{i})$. Notice that

$$S_P \subset \bigcup_{j=1}^{d} L_j(P),$$

where $L_j(P) \in \mathcal{E}(P)$ denotes the supporting line of P containing the jth side, i.e., joining the vertices v_j and v_{j+1} of P. Thus,

$$\mathrm{Leb}(\mathcal{Z}' \cap \Delta_\epsilon) \leq \sum_{\vec{i} \in \mathcal{I}_n^\epsilon} \sum_{j=1}^{d} \mathrm{Leb}\, \Omega_\epsilon(n, \vec{i}, j),$$

where

$$\Omega_\epsilon(n, \vec{i}, j) := \{(P, \lambda) \in \Delta_\epsilon : H(P, \lambda, \vec{i}) \in L_j^{\rho_n}(P)\},$$

and $L_j^{\rho_n}(P)$ is the ρ_n-neighbourhood of the line $L_j(P)$.

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LEMMA 5.2. *There exist $\epsilon > 0$ and $k \geq 1$, $N > 0$ and $C > 0$ depending only on ϵ such that for every $n > N$, $\vec{i} \in \mathcal{I}_n^\epsilon$ and $j \in \{1, \ldots, d\}$,*

$$\text{Leb}\, \Omega_\epsilon(n, \vec{i}, j) \leq C\rho_n^{1/k}.$$

Proof. Let $(P, \lambda) \in \Delta_\epsilon$ and $\vec{i} \in \mathcal{I}_n^\epsilon$. Identifying the complex plane with $\mathbb{R}^2$, the condition $H(P, \lambda, \vec{i}) \in L_j^{\rho_n}(P)$ can be written in the following way:

$$|\langle H(P, \lambda, \vec{i}) - v_j, \eta_j \rangle| < \rho_n,$$

where η_j is a unit vector perpendicular to $L_j(P)$. Using the definition of $H(P, \lambda, \vec{i})$, a simple computation shows that

$$\begin{aligned} h_j(P, \lambda, \vec{i}) &:= \langle H(P, \lambda, \vec{i}) - v_j, \eta_j \rangle \\ &= c_0 - c_1\lambda + c_2\lambda^2 + \cdots + c_n(-\lambda)^n, \end{aligned}$$

where

$$\begin{aligned} c_0 &= \langle v_{i_{n-1}} - v_j, \eta_j \rangle, \\ c_\ell &= \langle v_{i_{n-\ell-1}} - v_{i_{n-\ell}}, \eta_j \rangle, \quad \ell = 1, \ldots, n-1, \\ c_n &= -\langle v_{i_0}, \eta_j \rangle. \end{aligned}$$

Notice that $|c_\ell| \leq 2\|P\| \leq 2b_\epsilon$ for every $0 \leq \ell \leq n$. Hence, $h_j(P, \cdot, \vec{i})$ is a polynomial of degree at most n with coefficients belonging to the interval $[-2b_\epsilon, 2b_\epsilon]$.

Since P' is in general position, any polygon $P \in \mathcal{B}_\epsilon(P')$ is also in general position for $\epsilon > 0$ sufficiently small. This implies that there is a positive constant $\tau_\epsilon = \tau_\epsilon(P') > 0$, which can be explicitly computed from the angles of the sides and diagonals of P', such that

$$c_\ell \neq 0 \quad \Rightarrow \quad |c_\ell| \geq \tau_\epsilon, \quad 0 \leq \ell \leq n-1.$$

Taking $\epsilon > 0$ smaller if necessary, Lemma 2.4 gives the existence of an integer $N > 0$ such that every itinerary of $\mathcal{I}_N^\epsilon$ contains at least three distinct vertices. This implies that for every $n > N$ and $\vec{i} \in \mathcal{I}_n^\epsilon$, some coefficient c_ℓ of $h_j(P, \lambda, \vec{i})$ is non-zero for $0 \leq \ell \leq N$. Assuming that c_ℓ is the first non-zero coefficient of $h_j(P, \lambda, \vec{i})$, we have

$$h_j(P, \lambda, \vec{i}) = c_\ell(-\lambda)^\ell \hat{h}_j(P, \lambda, \vec{i}),$$

where $\hat{h}_j(P, 0, \vec{i}) = 1$, $|c_\ell| \geq \tau_\epsilon$ and the coefficients of the polynomial $\hat{h}_j(P, \cdot, \vec{i})$ belong to the interval $[-\alpha_\epsilon, \alpha_\epsilon]$ with $\alpha_\epsilon := 2b_\epsilon/\tau_\epsilon$. This means that $\hat{h}_j(P, \cdot, \vec{i}) \in \mathcal{F}_{\alpha_\epsilon}^{(n)}$.

Now, for every $n > N$ and $\vec{i} \in \mathcal{I}_n^\epsilon$, if $|h_j(P, \lambda, \vec{i})| < \rho_n$, then

$$|\hat{h}_j(P, \lambda, \vec{i})| = \frac{|h_j(P, \lambda, \vec{i})|}{|c_\ell|\lambda^\ell} < \frac{\rho_n}{\tau_\epsilon \epsilon^N}.$$

Applying Theorem 4.6, there are $k \geq 1$ and $0 < \delta < 1$ depending only on I_ϵ such that taking $N' > N$ sufficiently large so that $\rho_{N'}/\tau_\epsilon\epsilon^N < \delta$, we get

$$\text{Leb}\{\lambda \in I_\epsilon : |h_j(P, \lambda, \vec{i})| < \rho_n\} \leq C\, n^{2(k+1)} \rho_n^{1/k} \quad \text{for all } n > N', \vec{i} \in \mathcal{I}_n^\epsilon,$$

where

$$C := \frac{2^{3k+5}(1 + \alpha_\epsilon)^{2k}}{(\tau_\epsilon \epsilon^N)^{1/k} \rho_{N'}^2 (1 - r_{\alpha_\epsilon}(k))}.$$

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Since ρ_n decreases exponentially with n, we can remove the term $n^{2(k+1)}$ from the upper bound at the expense of increasing k and changing the constant C. Finally, in order to complete the proof of the lemma we can use Fubini's theorem and obtain the desired estimate for Leb $\Omega_\epsilon(n, \vec{i}, j)$. □

Applying this lemma, there are $\epsilon > 0$, $N > 0$ and $k \geq 1$ such that

$$\text{Leb}(\mathcal{Z}' \cap \Delta_\epsilon) \leq dC\rho_n^{1/k} \# \mathcal{I}_n^\epsilon \quad \text{for all } n > N,$$

where the constant $C > 0$ is independent of n. Hence, for $\epsilon > 0$ sufficiently small, we conclude by Proposition 3.2 that

$$\text{Leb}(\mathcal{Z}' \cap \Delta_\epsilon) \leq dC \lim_{n\to\infty} \rho_n^{1/k} \# \mathcal{I}_n^\epsilon = 0,$$

because ρ_n decreases exponentially at rate $a_\epsilon < 1$. Since $(P', \lambda') \in \mathcal{Z}'$ was arbitrary, this concludes the proof of Theorem 1.1. □

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