Ergod. Th. & Dynam. Sys. (1999), **19**, 1085–1091
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Bounded recurrent sets for planar homeomorphisms

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(*Received* 9 *July* 1997 *and accepted in revised form* 4 *April* 1998)

Abstract. We answer a question proposed by Barge and Franks in 1993 by showing that a bounded connected set in the plane which is closed or open and meets an infinite number of its iterates under an orientation-preserving fixed point free homeomorphism must meet all of them.

1. *Introduction*

A *Brouwer homeomorphism* is a planar, orientation-preserving, fixed point free homeomorphism. Let h be a Brouwer homeomorphism and X a connected subset of the plane which is open or closed. Let $\mathbb{Z}$ be the set of integers, and $\mathbb{Z}^+$ the set of positive integers; we define:

$$E(X,h) = \{n > 0 \mid h^n(X) \cap X = \emptyset\}$$

and

$$F(X,h) = \mathbb{Z}^+ \setminus E(X,h).$$

In [**1**], Barge and Franks have shown that the set $E(X,h)$ is closed under addition. Conversely, given any set E of positive integers closed under addition, they have constructed an open topological disk D such that $E(D,\tau) = E$, where τ is the translation $(x,y) \mapsto (x+1,y)$. Moreover, if $\mathbb{Z}^+ \setminus E$ is finite, D may be taken to be bounded. This leads them to ask the following question: if h is a Brouwer homeomorphism and D is an open topological disk such that both $E(D,h)$ and $F(D,h)$ are infinite, can D be bounded? We answer this question negatively by proving the following.

THEOREM 1. *Let h be a Brouwer homeomorphism, and X a bounded connected set in the plane which is open or closed. If the set $E(X,h)$ is not empty, then its complementary set $F(X,h)$ is finite.*

https://doi.org/10.1017/S0143385799141646 Published online by Cambridge University Press

Thus, for example, $E(D, h)$ cannot be the set of positive even integers if D is a bounded topological disk. As a consequence of Theorem 1, we get a new characterization of the homeomorphisms which are conjugate to the translation τ, and we also characterize the compact connected sets which are contained in a translation domain.

Throughout this article, h is a Brouwer homeomorphism. The plane is denoted by $\mathbb{R}^2$. A subset X of $\mathbb{R}^2$ is *free* for h if $h(X) \cap X = \emptyset$. A *closed (or open) topological disk* is a subset of $\mathbb{R}^2$ homeomorphic to a closed (or open) disk. An *iterate* of a set X is any set $h^p(X)$ for some integer p. We will use the following classical lemma.

LEMMA 2. (Free disks lemma) *Let h be a Brouwer homeomorphism, and D a closed topological disk; if D is free for h, then $D \cap h^n(D) = \emptyset$ for all $n \neq 0$.*

Proof. See, for example, [**2**]. This is obviously part of Barge and Franks's result, so one can also refer to [**1**]. □

COROLLARY 3. *The free disks lemma holds when D is an open connected set.*

Proof. This is again contained in the results of [**1**]; we will show how it can be deduced from the free disks lemma. Suppose h is a Brouwer homeomorphism, D an open connected set which is free for h and n a positive integer such that $h^n(D) \cap D \neq \emptyset$. Let x be a point in D such that $h^{-n}(x)$ is also in D; since D is open and connected, we can find an arc in D with endpoints x and $h^{-n}(x)$. Making use of the Schoënflies theorem, we can thicken this arc to get a closed topological disk in D containing both x and $h^{-n}(x)$. This is a free disk meeting one of its iterates, contradicting Lemma 2. □

Recall that a point p is a *wandering point* if there is a neighbourhood V of p such that all the iterates of V under h are disjoint from V. The following corollary follows immediately from the free disks lemma.

COROLLARY 4. *Under the action of a Brouwer homeomorphism, all the points of the plane are wandering points. In particular, a Brouwer homeomorphism has no periodic point.*

2. *Various versions of the 'key lemma'*

We give three versions of a lemma which is the crucial tool for the proof of Theorem 1. In order to quote the first version, we have to recall a little bit of vocabulary concerning Brouwer homeomorphisms. A *translation arc* is a closed simple arc γ with endpoints p and $h(p)$ such that $h(\gamma) \cap \gamma = \{h(p)\}$. The *translation line generated by* γ is the set $\cup_{n\in\mathbb{Z}} h^n(\gamma)$; it is a continuous injective image of the real line $\mathbb{R}$ (see [**2**, Théorème 3.3]).

LEMMA 5. ('Lemme-clé' of [**2**]) *Let Γ be a translation line and α an arc whose endpoints are on Γ; let β be the arc on Γ with the same endpoints as α. Suppose that $\beta \cap h(\beta)$ contains more than one point (i.e. β properly contains a translation arc), and that $\alpha \cup \beta$ is a Jordan curve. Then α is not free.*

Proof. See [**2**]. □

Here are two equivalent formulations of this lemma.

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LEMMA 6. *Let γ be a translation arc and α a free arc; then the set*

$$\{n \in \mathbb{Z} \mid h^n(\gamma) \cap \alpha \neq \emptyset\}$$

is an interval of $\mathbb{Z}$.

LEMMA 7. *Let D_1 and D_2 be two free disks; then the set*

$$\{n \in \mathbb{Z} \mid h^n(D_1) \cap D_2 \neq \emptyset\}$$

is an interval of $\mathbb{Z}$.

Remark. The set of integers involved in Lemma 7 can be *any* interval of $\mathbb{Z}$ (an easy example where it is equal to $\mathbb{Z}$ can be found in [**4**, Chapitre 8]).

Proof of Lemma 6. Under the hypothesis of the lemma, suppose there exist three integers $n_1 < n_2 < n_3$ such that $h^{n_1}(\gamma) \cap \alpha \neq \emptyset$, $h^{n_2}(\gamma) \cap \alpha = \emptyset$ and $h^{n_3}(\gamma) \cap \alpha \neq \emptyset$. Let Γ be the translation line generated by γ; Γ is oriented by the action of h. We call a the last point of α met by Γ before $h^{n_2}(\gamma)$, and b the first point of α met by Γ after $h^{n_2}(\gamma)$. If we replace α by the subarc of α with endpoints a and b, we get a situation contradicting the Key-Lemma 5. □

For the proof of the third version, we need to define another term. A *critical disk* is a closed topological disk D such that $h(D) \cap D$ is not empty but has empty interior. Recall that Corollary 4 asserts that for all x, $h^{-1}(x) \neq h(x)$; it follows that if D is a critical disk and x a point of $h(D) \cap D$, any simple arc γ with endpoints $h^{-1}(x)$ and x such that $\gamma \setminus \{h^{-1}(x), x\}$ is contained in the interior of D is a translation arc (see [**2**]).

Proof of Lemma 7. Let D_1 and D_2 be two free disks, and suppose there exist three integers $n_1 < n_2 < n_3$ such that $h^{n_1}(D_1) \cap D_2 \neq \emptyset$, $h^{n_2}(D_1) \cap D_2 = \emptyset$ and $h^{n_3}(D_1) \cap D_2 \neq \emptyset$. Let us choose a point $h^{n_1}(a)$ in $h^{n_1}(D_1) \cap D_2$ and a point $h^{n_3}(b)$ in $h^{n_3}(D_1) \cap D_2$. We pick an arc α in D_2 with endpoints $h^{n_1}(a)$ and $h^{n_3}(b)$; α is a free arc.

Let x be any point in $h^{n_2+1}(D_1) \setminus \alpha$. Since the sets $h^{n_2}(D_1)$, α and $\{x\}$ are pairwise disjoint, making use of the Schoënflies theorem, we can suppose:

(i) $h^{n_2}(D_1) = B_1(0)$;

(ii) $\alpha = [3, 4] \times \{0\}$;

(iii) $x = (2, 0)$,

where $B_r(0)$ stands for the closed disk with centre at zero and radius r. Let $R = \mathrm{Sup}\{r > 0 \mid B_r(0) \text{ is free}\}$; $D = B_R(0)$ is a critical disk containing $h^{n_2}(D_1)$. Furthermore, because $x \in h(h^{n_2}(D_1))$, the disk $B_2(0)$ is not free, so $R \leq 2$; thus D and α do not meet. The disk D contains a translation arc γ, and we can choose γ such that the points $h^{n_2}(a)$ and $h^{n_2}(b)$ are on γ: thus we have $h^{-(n_2-n_1)}(\gamma) \cap \alpha \neq \emptyset$, $\gamma \cap \alpha = \emptyset$ and $h^{(n_3-n_2)}(\gamma) \cap \alpha \neq \emptyset$, getting a situation that contradicts the previous lemma. □

3. *Recurrent sets: the compact connected case*

We prove the theorem when X is a compact connected set. Let n_0 be a positive integer such that $h^{n_0}(X) \cap X = \emptyset$; we aim to show that $F(X, h) = \{n > 0 \mid h^n(X) \cap X \neq \emptyset\}$ is a finite set.

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Proof. Since $h^{n_0}(X) \cap X = \emptyset$, we can find an open connected set V containing X such that $h^{n_0}(V) \cap V = \emptyset$. According to Corollary 4, $g = h^{n_0}$ is again a Brouwer homeomorphism; and Corollary 3 applied to g and V tells us that $h^{pn_0}(V) \cap V = \emptyset$ holds for every $p > 0$.

For all x in X there exists a free closed disk containing x and contained in V. Since X is compact, we have $X \subset D_1 \cup \cdots \cup D_k$ where each D_i is a free closed disk in V. For each couple (i, j), $1 \leq i, j \leq k$, we define:

$$F_{i,j} = \{n > 0 \mid h^n(D_i) \cap D_j \neq \emptyset\}.$$

Note that for all $n > 0$,

$$h^n(X) \cap X \subset \bigcup_{i,j} (h^n(D_i) \cap D_j)$$

so that $F(X, h) \subset \cup_{i,j} F_{i,j}$. Because of Lemma 7, each set $F_{i,j}$ is an interval of $\mathbb{Z}^+$; but since the disks D_i and D_j are subsets of V, the set $F_{i,j}$ does not contain any multiple of n_0, thus it must be finite. This shows that $F(X, h)$ is a finite set, and the proof is complete. □

4. *Recurrent sets: the open bounded connected case*

We now deal with the open case. Let X be a bounded open connected set, and n_0 an integer such that $h^{n_0}(X) \cap X = \emptyset$. Let $g = h^{n_0}$; recall that $g^p(X) \cap X = \emptyset$ for all $p > 0$ (Corollary 3). Let $\overline{X}$ stands for the closure of X, and ∂X for its boundary.

Proof. First we will show the existence of an integer $p > 0$ such that $g^p(\overline{X}) \cap \overline{X} = \emptyset$.

We argue by contradiction. Suppose that $g^p(\overline{X}) \cap \overline{X} \neq \emptyset$ for all $p > 0$. Since $g^p(X) \cap X = \emptyset$ for all $p > 0$, we have also $g^p(\partial X) \cap \partial X \neq \emptyset$ for all $p > 0$. Let $K_p = g^p(\partial X) \cap \partial X$; K_p is a nonempty compact set. We show that for all $p > 0$, $K_{p+1} \subset K_p$.

Suppose on the contrary that there exists an integer $p > 0$ and a point x in $K_{p+1} \setminus K_p$. Then $x \notin g^p(\partial X)$; moreover $x \in \partial X$, $g^p(X) \cap X = \emptyset$ and X is open, so $x \notin g^p(X)$; therefore $x \notin g^p(\overline{X})$. Thus we can choose a closed topological free disk D_1 containing x in its interior and such that D_1 and $g^p(X)$ are disjoint.

On the other hand, D_1 meets X and $g^{p+1}(X)$; let D_2 be another closed topological disk included in X such that D_2 meets D_1 and $g^{-(p+1)}(D_1)$ (the existence of D_2 follows from the arcwise connectivity of X and the Schoënflies theorem, as in the proof of Corollary 3). Because $D_2 \subset X$ and X is free for g, D_2 is also free; the following holds:

(i) $D_1 \cap D_2 \neq \emptyset$;
(ii) $D_1 \cap g^p(D_2) = \emptyset$;
(iii) $D_1 \cap g^{p+1}(D_2) \neq \emptyset$;

and this situation contradicts Lemma 7.

Thus $\{K_p\}_{p \geq 0}$ is a decreasing sequence of nonempty compact sets. We define $K = \bigcap_{p>0} K_p$; K is therefore a nonempty compact set. But $h^{-1}(K) \subset K$, and this is not compatible with the fact that the points of K should be wandering under the action of h^{-1} (Corollary 4).

So there exists an integer $p > 0$ such that $g^p(\overline{X}) \cap \overline{X} = \emptyset$. We have proved the compact connected case in the previous section, so we can conclude that $F(\overline{X}, h)$ is a finite set. Then the proof is complete since $F(X, h) \subset F(\overline{X}, h)$. □

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5. *Homeomorphisms which are conjugate to the translation*

Recall that τ is the Brouwer homeomorphism $(x, y) \mapsto (x + 1, y)$. The following proposition is the classical way to characterize those homeomorphisms that are conjugate to τ.

PROPOSITION 8. *Let h be a plane homeomorphism; then h is conjugate to τ if and only if for all K compact set of the plane, $F(K, h)$ is a finite set, i.e. only a finite number of iterates of K meet K.*

Proof. Clearly, if h is conjugate to τ then $F(K, h)$ is a finite set.

Now let h be any Brouwer homeomorphism, and let A be the space obtained by quotienting the plane by the action of h. Because of the free disks lemma, A is locally homeomorphic to the plane. We can see that the set $F(K, h)$ being finite for any compact set K implies A is Hausdorff; then A is a surface with fundamental group $\mathbb{Z}$, so it is homeomorphic to $S^1 \times \mathbb{R}$, and from this it follows that h is conjugate to τ. □

Theorem 1 allows us to enhance the previous proposition to get this new one.

COROLLARY 9. *Let h be a plane homeomorphism; then h is conjugate to τ if and only if for all K, compact sets of the plane, there exists at least* one *iterate of K avoiding K (i.e. $E(K, h)$ is not empty).*

Proof. Let h be a Brouwer homeomorphism such that for all compact sets K, $E(K, h) \neq \emptyset$. We show that h satisfies the criterion of Proposition 8. It suffices to show that for any disk D the set $F(D, h)$ is finite. But Theorem 1 claims that this is equivalent to $E(D, h)$ not being empty, so the proof is complete. □

COROLLARY 10. *The set $\mathcal{B}_\tau$ of homeomorphisms which are conjugate to τ is a G_δ set (i.e. a countable intersection of open sets) in the space $\mathcal{B}$ of Brouwer homeomorphisms equipped with the compact open topology.*

Proof. Recall that $B_r(0)$ denotes the closed disk with centre at zero and radius r. Corollary 9 implies that $\mathcal{B}_\tau = \bigcap_n \bigcup_p O_{n,p}$ where

$$O_{n,p} = \{h \in \mathcal{B} \mid h^p(B_n(0)) \cap B_n(0) = \emptyset\}.$$

Therefore the result follows from the fact that each set $O_{n,p}$ is open in $\mathcal{B}$. □

Note that the last corollary is also a consequence of the 'semi-continuity of translation domains' proved at the end of Chapter 6 in [**4**].

6. *Compact connected sets and translation domains*

A *domain* is a connected open subset of the plane. A simply connected invariant domain Δ is a *translation domain* for h if $h_{|\Delta}$ is conjugate to the translation τ. This amounts to saying that each compact set in Δ is moved off itself by a power of h (see Corollary 9). Note that this definition is weaker than the one in [**2**].

The following statement helps to clarify Theorem 1.

THEOREM 11. *Let K be a compact connected subset of the plane; then K is in a translation domain for h if and only if there exists an integer n such that $h^n(K) \cap K = \emptyset$.*

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In particular, this theorem obviously implies the closed case of Theorem 1.

Proof. If X is any connected subset of the plane, we define $\mathrm{Fill}(X)$ as the union of X and the connected components of $\mathbb{R}^2 \setminus X$ which are bounded. Observe that the set $\mathrm{Fill}(X)$ is connected; if X is bounded, $\mathrm{Fill}(X)$ is also bounded, and its complementary set is connected. If X is open then $\mathrm{Fill}(X)$ is a simply connected open set; actually, it is the simply connected open set containing X which is minimal for inclusion among the sets having these properties.

Let h be a Brouwer homeomorphism, K a compact connected set and n an integer such that $h^n(K) \cap K = \emptyset$. We first prove the following.

LEMMA 12. *In this situation,* $h^n(\mathrm{Fill}(K)) \cap \mathrm{Fill}(K) = \emptyset$ *also holds.*

Proof of Lemma 12. For otherwise, because $h^n(K) \cap K = \emptyset$, one can check that $h^n(\mathrm{Fill}(K)) \subset \mathrm{Fill}(K)$ or $\mathrm{Fill}(K) \subset h^n(\mathrm{Fill}(K))$, and since $\mathrm{Fill}(K)$ is a compact set in both cases we get a contradiction with Corollary 4 (the points are wandering). □

Any compact connected set in the plane whose complementary set is also connected is a decreasing intersection of topological disks: this can be shown directly; or one can argue that the complementary set in the 2-sphere $\mathbb{R}^2 \cup \{\infty\}$ is a simply connected open set, hence it is homeomorphic to an open topological disk, which makes the statement obvious. Thus we can find a closed topological disk D_1 containing $\mathrm{Fill}(K)$ such that $h^n(D_1) \cap D_1 = \emptyset$; and it suffices to show that D_1 is contained in a translation domain.

The free disks lemma implies that $h^{2n}(D_1) \cap D_1 = \emptyset$, so in case $n = 1$ we can replace n by $2n$, and thus we may assume that $n > 1$. Since D_1 is free for h^n it is contained in a critical disk D_2 for h^n. The disk D_2 cannot be free for h; that it can neither be critical for h is a consequence of the following lemma, which will be proved at the end of the text.

LEMMA 13. *A critical disk for* h *is free for* h^p *for all* $p > 1$.

Thus the interior of D_2 (denoted by $\mathrm{Int}(D_2)$) is not free, and the set

$$U = \bigcup_{n \in \mathbb{Z}} h^n(\mathrm{Int}(D_2))$$

is connected. We define $\Delta = \mathrm{Fill}(U)$; it is an open simply connected set which is invariant for h. We will complete the proof by showing that $h_{|\Delta}$ is conjugate to τ. According to Proposition 8, it suffices to prove that for any compact connected set C contained in Δ, $F(C, h)$ is a finite set. We will need the following lemma (see the proof at the end).

LEMMA 14. *Let* U *be a connected open subset of the plane. Then:*

(i) $\mathrm{Fill}(U) = \bigcup \{B \mid B$ *is a closed topological disk and* $\partial B \subset U\}$*;*

(ii) *for all compact sets* $C \subset \mathrm{Fill}(U)$, C *is contained in a closed topological disk* B *whose boundary lies in* U.

Now let B be a closed topological disk such that $C \subset B$ and $\partial B \subset U$; then $F(C, h) \subset F(B, h)$. Furthermore, for any $p \neq 0$, the cases $h^p(B) \subset B$ and $B \subset h^p(B)$ are forbidden by Corollary 4; therefore each time an iterate $h^p(B)$ meets B, the iterate $h^p(\partial B)$ meets ∂B, so that $F(B, h) = F(\partial B, h)$. Thus it remains to prove that $F(\partial B, h)$ is a finite set.

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The compact set ∂B is contained in U. It follows that there exists a positive integer n_0 such that

$$\partial B \subset \bigcup_{-n_0 \le k \le n_0} h^k(\mathrm{Int}(D_2)).$$

Since D_2 is a critical disk for h^n, $\mathrm{Int}(D_2)$ is free for h^n, and Theorem 1 implies that $\mathrm{Int}(D_2)$ meets only a finite number of its iterates under h. Thus this is also true for ∂B, and the proof is complete. □

Here are the proofs of the lemmas.

Proof of Lemma 13. Let D be a critical disk for h, and let $\mathrm{Int}(D)$ be the interior of D. Now suppose there exists a point $x \in D$ and an integer $p > 1$ such that $h^{-p}(x) \in D$. Let α be an arc with end-points x and $h^{-p}(x)$ such that $\alpha - \{x, h^{-p}(x)\} \subset \mathrm{Int}(D)$. Since $\mathrm{Int}(D)$ is free and x is not a periodic point (Corollary 4), we see that α is free; thus we can fatten α to get a free topological disk containing x and $h^{-p}(x)$, and this contradicts the free disks lemma. □

Proof of Lemma 14. Let $U' = \bigcup\{B \mid B \text{ is a closed topological disk and } \partial B \subset U\}$; clearly U' is an open set included in $\mathrm{Fill}(U)$, and containing U; therefore it suffices to prove that U' is simply connected. If f maps continuously the circle S^1 in U', $f(S^1)$ is a compact connected set in U' and then we can find disks $B_1, \ldots, B_k$, with $\partial B_i \subset U$ and $B_1 \cup \cdots \cup B_k$ is a connected set containing $f(S^1)$. Kerékjártó [**3**] has proved that $D = \mathrm{Fill}(B_1 \cup \cdots \cup B_k)$ is a closed topological disk whose boundary lies in $\partial B_1 \cup \cdots \cup \partial B_k$. Thus $D \subset U'$, so the map f is nulhomotopic in U', and the proof of the first claim is complete.

Now let $C \subset \mathrm{Fill}(U)$; there exist k disks $B_1, \ldots, B_k$ whose boundaries lie in U and such that $C \subset B_1 \cup \cdots \cup B_k$. Again Kerékjártó tells us $B = \mathrm{Fill}(B_1 \cup \cdots \cup B_k)$ is a closed topological disk; and we obviously have $\partial B \subset U$ and $C \subset B$. □

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