Probability in the Engineering and Informational Sciences, **14**, 2000, 353–373. Printed in the U.S.A.

NONHOMOGENEOUS POISSON PROCESSES AND LOGCONCAVITY

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In this article, we identify conditions under which the epoch times and the interepoch intervals of a nonhomogeneous Poisson process have logconcave densities. The results are extended to relevation counting processes. We also study discrete-time counting processes and find conditions under which the epoch times and the interepoch intervals of these discrete-time processes have logconcave discrete probability densities. The results are interpreted in terms of minimal repair and record values. Several examples illustrate the theory.

1. INTRODUCTION

The nonhomogeneous Poisson process arises naturally in reliability theory as the process that records the epochs and the number of repairs that a minimally repaired item or system goes through (see, for example, Ascher and Feingold [1], Block et al. [6], or Beichelt [5]). In other applications, the nonhomogeneous Poisson process is the process that records the number and the values of records associated with a sequence of independent and identically distributed random lifetimes (i.e., non-

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https://doi.org/10.1017/S0269964800143062 Published online by Cambridge University Press

negative random variables). Many authors have studied various facets of the nonhomogeneous Poisson process. A recent search in MathSciNet for matches of "record value" yielded 178 references, most of which are studies of various properties of the nonhomogeneous Poisson process. We cannot list all these references here. Some recent references that contain further bibliography are the articles by Baxter et al. [4], Feuerverger and Hall [9], Kochar [20], Kuo and Yang [21], Huang and Li [12], Kirmani and Gupta [17], and Gupta and Kirmani [11].

The nonhomogeneous Poisson process is useful also in areas other than reliability theory. In some data-recording contexts, the values that are of the greatest interest are the extremes, and sometimes these particular data are recorded in a very accessible form (Feuerverger and Hall [9]); thus, stochastic properties of these record values may be useful to have for the purpose of improving current statistical inference methods based on record values. The nonhomogeneous Poisson process is also a common model in information management science for software reliability and reliability growth (Kuo and Yang [21]); thus, stochastic properties of this process can guide the software engineer in finding a proper stochastic model for testing new software.

In this article, we focus on some stochastic properties of the jump times (epochs) and of the interepochs intervals of a nonhomogeneous Poisson process. In the context of minimal repair, the times of epochs are the calendar times at which minimal repair occurs. It is of interest also to study the interepoch intervals, which are the periods during which the item or the system functions without interruptions.

A nonhomogeneous Poisson process is parameterized by its intensity (or rate) function, which we denote by r. We assume that

$$\int_t^\infty r(u)\,du = \infty \quad \text{for all } t < \infty; \tag{1.1}$$

this ensures that with probability 1, the process has a jump after any time point t. A non-negative function r which satisfies (1.1) can be interpreted as the hazard rate function of a lifetime of an item. More explicitly, if r satisfies (1.1) and we define f by

$$f(t) = r(t)e^{-\int_0^t r(u)\,du} = r(t)e^{-R(t)}, \quad t \geq 0, \tag{1.2}$$

where $R(t) \equiv \int_0^t r(u)\,du$, then f is a probability density function of a lifetime; in fact, f is the probability density function of the time of the first epoch of the underlying nonhomogeneous Poisson process. The corresponding survival function $\bar{F}$ and distribution function F are given by

$$\bar{F}(t) = \int_t^\infty f(u)\,du = e^{-R(t)}, \qquad F(t) = 1 - \bar{F}(t), \quad t \geq 0.$$

Various authors have studied properties of f that are inherited by the epoch times of the underlying nonhomogeneous Poisson process. Baxter [3], Gupta and Kirmani [11], and Kochar [19] proved that if f has the *increasing failure rate* (IFR) property, then all the epoch times of the underlying nonhomogeneous Poisson pro-

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cess have this property. Gupta and Kirmani [11] have also proven that if f has the *increasing failure rate average* (IFRA) [respectively, *new better than used* (NBU)] property, then all the epoch times of the underlying nonhomogeneous Poisson process have this property.

In this article, we study an aging notion that is stronger than the IFR, IFRA, and NBU properties. This notion corresponds to the *logconcavity* of the density function of the lifetime of interest. It is well known that the logconcavity of the density function implies the notions of IFR, IFRA, and NBU (Barlow and Proschan [2]). Shaked and Shanthikumar [25] have studied this kind of logconcavity as an aging notion. Shaked and Shanthikumar [25] pointed out some of the benefits, in the areas of reliability theory and statistics, that a researcher can obtain from the knowledge that a density function is logconcave. Keilson and Gerber [16] have noted the interest of strong unimodality (i.e., logconcavity of the density function) in connection with optimization and mathematical programming.

In Section 2, it is shown that the logconcavity of f does not suffice to imply the logconcavity of the density functions of all the epoch times of the underlying nonhomogeneous Poisson process. However, in the main result of Section 2, it is shown that the logconcavity of f, together with the logconcavity of the cumulative hazard function R are necessary and sufficient conditions for the logconcavity of the density functions of all the epoch times of the underlying nonhomogeneous Poisson process.

Turning to interepoch intervals, in the main result of Section 3 it is shown that the logconcavity of r implies the aging property of logconcavity of all the interepoch intervals of the underlying nonhomogeneous Poisson process. This is to be contrasted with a result of Gupta and Kirmani [11] which shows that if f has the antiaging *decreasing failure rate* (DFR) property, then all the epoch times of the underlying nonhomogeneous Poisson process have this property.

The nonhomogeneous Poisson process can be generalized to what we call a *relevation counting process.* A relevation counting process arises when each failed item is replaced by a working item of the same age, but the replacing item is not identical with the item it replaces. Some of our results are extended to relevation counting processes in Sections 2 and 3. We also have studied some properties of a discrete-time analog of the nonhomogeneous Poisson process, and the results of our study are described in Section 4.

In order to prove our main results, we derive in the Appendix some preliminary results which describe the interrelationships among the logconcavity properties of f, R, and r; these may be of independent interest. The Appendix also contains some examples to some of which the main results in Sections 2 and 3 apply.

In this article, "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing," respectively.

2. LOGCONCAVITY OF EPOCH TIMES

Consider a nonhomogeneous Poisson process with intensity function r which satisfies (1.1). As in the Appendix, for convenience we define $r(t) = 0$ for $t < 0$, and if

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$r(t_0) = \infty$ for some t_0, then we define $r(t) = \infty$ for $t \geq t_0$. The corresponding cumulative intensity function R is defined by $R(t) \equiv \int_0^t r(u)\,du$, $t \geq 0$.

Let $0 \equiv T_0 \leq T_1 \leq T_2 \leq \cdots$ be the epoch times of the nonhomogeneous Poisson process. Denote by f_n the density function of T_n, $n \geq 1$. Then,

$$f_n(t) = f(t)\,\frac{(R(t))^{n-1}}{(n-1)!}, \quad t \geq 0,\ n \geq 1; \tag{2.1}$$

this is Eq. (3) in Baxter [3]. Note, in particular, that $f_1 \equiv f$.

We have sought conditions, as weak as we could find, under which f_n is logconcave. A complete solution is given in Theorem 2.1. Clearly, if (A.1) and (A.2) hold, then f_n is logconcave for each $n \geq 1$. If we want f_n to be logconcave *for each* $n \geq 1$, then (A.1) and (A.2) are also necessary; this is shown in the following result.

THEOREM 2.1: *The random variables* T_n, $n \geq 1$, *all have logconcave densities if and only if* (*A*.1) *and* (*A*.2) *hold*.

PROOF: The sufficiency part is immediate from (2.1). In order to prove the necessity part, suppose that all the f_n's are logconcave. Then, in particular, $f_1 = f$ is logconcave; that is, (A.1) holds. Assume that (A.2) does not hold; that is, R is not logconcave. This means that there exist some $t_1, t_2 \in [0,\infty)$ and an $\alpha \in (0,1)$ such that

$$\alpha \log R(t_1) + \bar{\alpha} \log R(t_2) - \log R(\alpha t_1 + \bar{\alpha} t_2) > 0,$$

where $\bar{\alpha} \equiv 1 - \alpha$. Now, from (2.1), we have

$$\log f_n(t) = \log f(t) + (n-1)\log R(t) - \log((n-1)!), \quad t \geq 0,\ n \geq 1.$$

Therefore,

$$\begin{aligned} &\alpha \log f_n(t_1) + \bar{\alpha} \log f_n(t_2) - \log f_n(\alpha t_1 + \bar{\alpha} t_2) \\ &\quad = \alpha \log f(t_1) + \bar{\alpha} \log f(t_2) - \log f(\alpha t_1 + \bar{\alpha} t_2) \\ &\qquad + (n-1)[\alpha \log R(t_1) + \bar{\alpha} \log R(t_2) - \log R(\alpha t_1 + \bar{\alpha} t_2)]. \end{aligned} \tag{2.2}$$

Thus, for large enough n, we see from (2.2) that $\alpha \log f_n(t_1) + \bar{\alpha} \log f_n(t_2) - \log f_n(\alpha t_1 + \bar{\alpha} t_2) > 0$—a contradiction to the assumption that all the f_n's are logconcave. ■

Baxter [3], Gupta and Kirmani [11], and Kochar [19] proved that if X (with density function f) is IFR, then all the T_n's are IFR. Gupta and Kirmani [11] have also proven that if X is IFRA [respectively, NBU], then all the T_n's are IFRA [respectively, NBU]. Here, in Theorem 2.1 we assume more (because logconcavity of f implies that f is IFR, IFRA, and NBU) and we conclude more.

From Propositions A.2 and A.4 and Theorem 2.1 we obtain

COROLLARY 2.2: *If* r *is logconcave, then the random variables* T_n, $n \geq 1$, *all have logconcave densities*.

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2.1. Epoch Times of Relevation Counting Processes

One interpretation of the epoch times $T_1 \le T_2 \le \cdots$ of a nonhomogeneous Poisson process with intensity function r is the following. Suppose a new item with hazard rate function r starts to function at time 0. It fails at time T_1 and is then replaced by a similar item selected from a pool of working items of age T_1 (this is the case when an item is supported by a large number of identical warm standby items). This procedure repeats each time an item fails—T_n then is the time of the nth replacement. A common generalization of the above interpretation of minimal repair is the so-called "relevation transform" (see Baxter [3], Johnson and Kotz [13], Shanthikumar and Baxter [28], Lau and Rao [22], and some more recent references in Fosam and Shanbhag [10]). In this model, an item with life Y_1 and distribution function K_1 is replaced upon its failure at time T_1 (here, $T_1 = Y_1$) by another item with life Y_2 and distribution function K_2, but the replacement item is then of age T_1. Intuitively, the relevation of Y_1 with Y_2 is the time that the replacement item fails. Formally, the relevation of Y_1 with Y_2 (or of K_1 with K_2) is defined as the random variable with survival function $\bar{H}$ defined by

$$\bar{H}(t) = \bar{K}_1(t) + \int_0^t \frac{\bar{K}_2(t)}{\bar{K}_2(u)}\, dK_1(u), \quad t \ge 0.$$

The corresponding density function h is given by

$$h(t) = k_2(t) \int_0^t \frac{k_1(u)}{\bar{K}_2(u)}\, du, \quad t \ge 0.$$

For example, the distribution function of the second epoch time T_2 of a nonhomogeneous Poisson process with intensity function r is the relevation of F (with hazard rate function r) with itself. More generally, the distribution function of the nth epoch time T_n of that nonhomogeneous Poisson process is the n-fold relevation of F (with hazard rate function r) with itself.

Let us generalize the nonhomogeneous Poisson process to what we will call a *relevation counting process*. Let $\{Y_n, n = 1,2,\ldots\}$ be a sequence of independent absolutely continuous non-negative random variables with density, distribution, and hazard rate functions k_n, K_n, and r_n, respectively. Define

$$T_1 =_{\text{st}} Y_1, \tag{2.3}$$

$$T_n =_{\text{st}} [Y_n | Y_n > T_{n-1}], \quad n \ge 2, \tag{2.4}$$

where $=_{\text{st}}$ denotes equality in law, and for any event A, the notation $[Y_n|A]$ stands for any random variable whose distribution is the conditional distribution of Y_n given A. The corresponding relevation counting process is the process $\{N(t), t \ge 0\}$, with the T_n's as epoch times, defined by

$$N(t) = \sup\{n\colon T_n \le t\}, \quad t \ge 0.$$

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A relevation counting process is parameterized by a sequence $\{K_n, n = 1,2,\ldots\}$ of life distributions. When all the K_n's are identical, the relevation counting process reduces to a nonhomogeneous Poisson process.

Let h_n denote the probability density function of T_n defined in (2.3) and (2.4). Then,

$$h_1(t) = k_1(t), \quad t \geq 0, \tag{2.5}$$

$$h_n(t) = k_n(t) \int_0^t \frac{h_{n-1}(u)}{\bar{K}_n(u)}\, du, \quad t \geq 0,\ n \geq 2. \tag{2.6}$$

Let us denote

$$U_n(t) = \int_0^t \frac{h_{n-1}(u)}{\bar{K}_n(u)}\, du, \quad t \geq 0,\ n \geq 2.$$

Then,

$$h_n(t) = k_n(t) U_n(t), \quad t \geq 0,\ n \geq 2. \tag{2.7}$$

Thus, if we find conditions which imply that k_n and U_n are logconcave, then it follows that h_n is logconcave. The following lemma is the first step in that direction.

LEMMA 2.3: *If r_n and k_n/k_{n+1} are logconcave for all $n \geq 1$, then U_n is logconcave for all $n \geq 2$.*

PROOF: Recall that a function, say r_n, is logconcave if and only if $r_n(t-s)$ is totally positive of order 2 (TP_2) (Karlin [14]); here and below, we let $r_n(t) = 0$, $k_n(t) = 0$, $U_n(t) = 0$, and $h_n(t) = 0$ for $t < 0$. The proof is by induction on $n \geq 2$. First, write

$$U_2(t-s) = \int_{-\infty}^{\infty} I_{(-\infty,t]}(u)\, \frac{k_1(u-s)}{k_2(u-s)}\, r_2(u-s)\, du,$$

where I denotes the indicator function. The kernel $I_{(-\infty,t]}(u)$ is TP_2 in (u,t), and the kernel $[k_1(u-s)/k_2(u-s)]r_2(u-s)$ is TP_2 in (u,s) because k_1/k_2 and r_2 are logconcave. It follows, by the Basic Composition Formula (Karlin [14]), that $U_2(t-s)$ is TP_2 in (t,s); that is, U_2 is logconcave.

Let now $n \geq 3$. Write

$$U_n(t-s) = \int_{-\infty}^{\infty} I_{(-\infty,t]}(u)\, \frac{k_{n-1}(u-s)}{k_n(u-s)}\, U_{n-1}(u-s) r_n(u-s)\, du.$$

Again, $I_{(-\infty,t]}(u)$ is TP_2 in (u,t) and $[k_{n-1}(u-s)/k_n(u-s)]U_{n-1}(u-s)r_n(u-s)$ is TP_2 in (u,s) (by the assumptions and by the induction hypothesis); therefore, by the Basic Composition Formula, it follows that $U_n(t-s)$ is TP_2 in (t,s). ■

The next result is an extension of Corollary 2.2.

THEOREM 2.4: *If r_n and k_n/k_{n+1} are logconcave for all $n \geq 1$, then T_n [defined in* (2.3) *and* (2.4)*] has a logconcave density function, $n \geq 1$.*

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PROOF: From Proposition A.2 we see that the logconcavity of r_n implies the logconcavity of k_n. By Lemma 2.3, U_n is logconcave. The stated result then follows from (2.7). ■

The following examples point out some instances in which the conditions of Theorem 2.4 hold and, therefore, the corresponding T_n's have logconcave density functions.

Example 2.5 (Weibull distributions): If

$$k_n(t) = \beta_n \alpha t^{\alpha-1} e^{-\beta_n t^\alpha}, \quad t \geq 0,$$

where $\alpha \geq 1$, $\beta_n > 0$, and β_n is increasing in n, then (with the aid of Example A.5) it can be verified that the conditions of Theorem 2.4 hold. In particular, the epochs times of a pure birth process (i.e., when $\alpha = 1$) with intensities β_n, increasing in n, have logconcave densities. (Note, however, that this result, involving the epoch times of a pure birth process, is true even if the intensities β_n do not increase in n; this follows from the lack of memory property of the exponential distribution and from the well-known fact that convolutions of logconcave densities are logconcave.)

Example 2.6 (Truncated extreme value distributions): If

$$k_n(t) = \beta_n e^{\beta_n t + 1 - e^{\beta_n t}}, \quad t \geq 0,$$

where $\beta_n > 0$ and β_n is increasing in n, then (with the aid of Example A.8) it can be verified that the conditions of Theorem 2.4 hold.

3. LOGCONCAVITY OF INTEREPOCH INTERVALS

In Section 2 (Theorem 2.1 and Corollary 2.2), we gave conditions under which the epoch times of a nonhomogeneous Poisson process have logconcave densities. In this section, we turn our attention to the interepoch intervals.

Let $T_1 \leq T_2 \leq \cdots$ be the epoch times of a nonhomogeneous Poisson process with rate function r (as at the beginning of Section 2). Let $X_n = T_n - T_{n-1}$, $n \geq 1$, be the interepoch intervals. Denote by g_n the density function of X_n, $n \geq 1$. Then, $g_1 = f$ and

$$g_n(t) = \int_0^\infty r(s) \frac{R^{n-2}(s)}{(n-2)!} f(s+t)\, ds, \quad t \geq 0,\ n \geq 2; \tag{3.1}$$

this is Eq. (7) of Baxter [3].

THEOREM 3.1: *If r is logconcave, then the random variables X_n, $n \geq 1$, all have logconcave densities.*

PROOF: Rewrite (3.1) as

$$g_n(t-s) = \int_{-\infty}^\infty r(u-t) \frac{R^{n-2}(u-t)}{(n-2)!} f(u-s)\, du, \quad t, s \in (-\infty, \infty) \tag{3.2}$$

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[here, we use the convention that $r(s) = 0$ for $s < 0$ and also the fact that $f(s) = 0$ for $s < 0$]. Because r is logconcave, it follows by Proposition A.4 that R is logconcave. Therefore, $r(\cdot)R^{n-2}(\cdot)/(n-2)!$ is logconcave. By Proposition A.2, the logconcavity of r implies the logconcavity of f. By the theory of total positivity, it follows from (3.2), using the logconcavity of f, that g_n is logconcave. ■

Gupta and Kirmani [11] noticed that if X (with density function f) is DFR, then each X_n is DFR. At first glance, this fact seems to contradict Theorem 3.1 because logconcavity of g_n is a notion of aging, whereas DFR is a notion of antiaging. However, there is no contradiction between these two facts. In Theorem 3.1, it is assumed that r is logconcave, and, therefore, by Lemma A.1, it follows that r is increasing. Thus, under the conditions of Theorem 3.1, X (with density function f) cannot be DFR unless it is an exponential random variable. However, if X is exponential, then the underlying nonhomogeneous Poisson process is actually a homogeneous Poisson process and, then, each X_n is exponential (thus, X_n has both properties: it is DFR and it has a logconcave hazard rate function).

3.1. Interepoch Intervals of Relevation Counting Processes

Consider a relevation counting process as defined in Subsection 2.1, and let $T_1 \leq T_2 \leq \cdots$ be the associated epoch times as defined in (2.3) and (2.4). Let X_n, $n \geq 1$, be the corresponding interepoch intervals; that is,

$$X_1 =_{\text{st}} Y_1, \tag{3.3}$$

$$X_n =_{\text{st}} [Y_n - T_{n-1} | Y_n > T_{n-1}], \quad n \geq 2. \tag{3.4}$$

The probability density function g_n of X_n is given by

$$g_1(t) = k_1(t), \quad t \geq 0,$$

$$g_n(t) = \int_0^t k_n(u+t) \frac{h_{n-1}(u)}{\bar{K}_n(u)}\, du, \quad t \geq 0,\ n \geq 2,$$

where the h_n's are defined in (2.5) and (2.6).

The next result is an extension of Theorem 3.1.

THEOREM 3.2: *If r_n and k_n/k_{n+1} are logconcave for all $n \geq 1$, then X_n [defined in* (3.3) *and* (3.4)*] has a logconcave density function for all $n \geq 1$.*

PROOF: Write

$$\begin{aligned} g_n(t-s) &= \int_{-\infty}^{\infty} k_n(u-s) \frac{h_{n-1}(u-t)}{\bar{K}_n(u-t)}\, du \\ &= \int_{-\infty}^{\infty} k_n(u-s) \frac{h_{n-1}(u-t)}{k_n(u-t)} \frac{k_n(u-t)}{\bar{K}_n(u-t)}\, du \\ &= \int_{-\infty}^{\infty} k_n(u-s) \frac{k_{n-1}(u-t)}{k_n(u-t)} U_{n-1}(u-t) r_n(u-t)\, du. \end{aligned}$$

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From Proposition A.2, we see that the logconcavity of r_n implies the logconcavity of k_n; that is, $k_n(u-s)$ is TP_2 in (u,s). By Lemma 2.3, we see that U_{n-1} is logconcave. Therefore, $[k_{n-1}(u-t)/k_n(u-t)]U_{n-1}(u-t)r_n(u-t)$ is TP_2 in (u,t). The stated result now follows from the Basic Composition Formula. ■

Examples 2.5 and 2.6 point out some instances in which the conditions of Theorem 3.2 hold and, therefore, the corresponding X_n's have logconcave density functions.

4. LOGCONCAVITY IN DISCRETE-TIME PROCESSES

Discrete lifetimes arise in several common situations in reliability theory where clock time is not the best scale on which to describe lifetime. For example, in weapons reliability, the number of rounds fired until failure is more important than age in failure. This is the case also when a piece of equipment operates in cycles and the observation is the number of cycles successfully completed prior to failure. In other situations, a device is monitored only once per time period and the observation then is the number of time periods successfully completed prior to the failure of the device (Shaked et al. [26,27]). Further applications of discrete lifetimes in space exploration, repair facilities, and rescue missions are described in Rocha-Martínez and Shaked [23].

In this section, we study a discrete-time process that, at any time i, either has a jump of size one or has no jump. The probability of a jump, denoted by r_i, may depend on the calendar time i, but not on the past of the process before that time. Formally, let B_i be independent random variables such that

$$\begin{aligned} P(B_i = 1) &= r_i, \\ P(B_i = 0) &= 1 - r_i, \end{aligned} \qquad i = 1,2,\ldots,$$

and define $Z_j = \sum_{i=1}^{j} B_i$, $j = 1,2,\ldots$. Let

$$T_n = \min\{j\colon Z_j = n\}, \quad n = 1,2,\ldots \tag{4.1}$$

($T_0 \equiv 0$); that is, T_n is the discrete time of the nth jump. We will assume that $\Pi_{j=i}^{\infty}(1 - r_j) = 0$, $i = 1,2,\ldots$ (this ensures that T_n is finite with probability 1, $n = 1,2,\ldots$). From Knopp [18, p. 96], it follows that for each $i = 1,2,\ldots$, we have

$$\prod_{j=i}^{\infty}(1 - r_j) = 0 \Leftrightarrow \sum_{j=i}^{\infty} r_j = \infty. \tag{4.2}$$

The discrete density function of T_1 is easily seen to be given by

$$p_i \equiv P(T_1 = i) = r_i \prod_{j=1}^{i-1}(1 - r_j), \quad i = 1,2,\ldots. \tag{4.3}$$

A straightforward computation gives $\sum_{i=1}^{k} p_i = 1 - \Pi_{j=1}^{k}(1 - r_j)$. Thus, indeed, assumption (4.2) ensures that p_i is a proper discrete density function. A relationship between the p_i's and the r_i's is

$$r_i = \frac{p_i}{\sum_{j=i}^{\infty} p_j}, \quad i = 1,2,\ldots;$$

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that is, the r_i's can be thought of as discrete-time hazard rates of T_1 (see, for example, Shaked et al. [26,27], Rocha-Martínez and Shaked [23], and references therein).

Denote $p_n(i) = P(T_n = i)$. Then, $p_1(i) = p_i$ and

$$p_2(i) = r_i \sum_{j=1}^{i-1} r_j \prod_{k\neq j}^{i-1} (1 - r_k) = r_i \left(\prod_{k=1}^{i-1} (1 - r_k) \right) \sum_{j=1}^{i-1} \frac{r_j}{1 - r_j} = p_i \sum_{j=1}^{i-1} \frac{r_j}{1 - r_j}, \quad i \geq 2.$$

Denoting

$$s_i = \frac{r_i}{1 - r_i} \quad \left(\text{then } r_i = \frac{s_i}{1 + s_i} \right), \tag{4.4}$$

we thus get

$$p_2(i) = p_i \sum_{j=1}^{i-1} s_j, \quad i \geq 2.$$

In order to obtain an expression for $p_n(i)$ when $n > 2$, denote

$$R_1(i) = \sum_{j=1}^{i} s_j, \quad i \geq 1,$$

and, by induction,

$$R_n(i) = \begin{cases} 0 & \text{if } i < n, \\ \sum_{j=1}^{i} s_j R_{n-1}(j-1) & \text{if } i \geq n, \end{cases} \quad n \geq 2. \tag{4.5}$$

Then, the discrete density of T_n is given by

$$p_n(i) = p_i R_{n-1}(i-1), \quad i \geq n, n \geq 3. \tag{4.6}$$

Let $X_n = T_n - T_{n-1}$, $n \geq 1$, be the interepoch intervals. Denote

$$q_n(i) = P(X_n = i), \quad i \geq 1.$$

Then,

$$q_1(i) = p_i, \quad i \geq 1.$$

A straightforward computation yields

$$q_2(i) = \sum_{j=1}^{\infty} p_{j+i} s_j, \quad i \geq 1,$$

and, recursively, one obtains

$$q_n(i) = \sum_{j=1}^{\infty} p_{j+i} s_j R_{n-1}(j-1), \quad i \geq 1, n \geq 3.$$

Remark 4.1: Before we proceed to derive some properties of the T_n's and of the X_n's, let us interpret these in some practical terms.

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One way to view an item with a discrete lifetime T_1, with discrete-time hazard rates r_i, is to imagine that it is associated with a list of tasks, numbered $1,2,\ldots$. Associated with any task i, there is a probability, r_i, of failure of the item during the execution of that task. The tasks are executed in sequence until the first failure. Thus, the probability r_i is really the conditional probability of failure to execute task i, given that the previous tasks $1,2,\ldots,i-1$ have been successfully executed. It follows that the lifetime T_1 of the item can be interpreted as the number of the task at which the item fails (see Rocha-Martínez and Shaked [23,24]).

In these terms, the stochastic model of the present section corresponds to the following situation. An item, associated with the failure probabilities (i.e., discrete-time hazard rates) r_i, executes sequentially its tasks until its failure at, say, task i_1 [i.e., $T_1 = i_1$, where T_1 is defined in (4.1)]. Upon failure, the item is replaced by a similar live item, which then proceeds to execute task $i_1 + 1$ and the following tasks. Suppose that this second item fails for the first time at task i_2 [i.e., $T_2 = i_2$, where T_2 is defined in (4.1)]. It is then replaced by a live item which then proceeds to execute task $i_2 + 1$ and the following tasks, and so forth.

Note that in this model, a task at which an item fails (e.g., tasks i_1 or i_2 above, etc.) is skipped and is never perfectly executed! This may be a reasonable model in some situations (e.g., when the item is an electric breaker which is checked only at the end of each month, and if it found to be defective at the end of the month i, say, then it is replaced by another breaker which is ready to work in month $i+1$ onward). However, the present model may not be a proper one when all tasks must be perfectly performed in their particular order (e.g., when the item is a repair facility with a fixed list of jobs that must be executed in a particular order). In order to model the latter situation, an item which fails at time i must be replaced by a similar item which then starts from the execution of task i (rather than task $i+1$) onward. Of course, the replacement item also has a probability r_i of failure to execute task i, and in case of such a failure, it is replaced once more by a working item in order to eventually perfectly execute task i. When the particular application is of this kind, the stochastic model of the present article does not apply to it.

We now proceed to derive some logconcavity properties of $p_n(i)$ and $q_n(i)$. Such properties have been studied in Keilson [15] and Keilson and Gerber [16]. For example, Keilson and Gerber [16] have noticed the equivalence of discrete strong unimodality and discrete logconcavity. They pointed out the theoretical and practical importance of discrete logconcavity in statistics and in optimization theory.

We first need to obtain some technical preliminaries. First, note that the s_i's and the p_i's determine each other:

$$p_i = \frac{s_i}{\prod_{j=1}^{i}(1+s_j)}, \qquad s_i = \frac{p_i}{\sum_{j=i+1}^{\infty} p_j}, \quad i \geq 1. \tag{4.8}$$

We remark that the above relations show that s_i is close to, but is not, the common hazard rate. However, it has some properties similar to r_i. The next lemma is an instance of this assertion—it will be needed later in the sequel.

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LEMMA 4.2: *Let $r_i \in [0,1)$ and $s_i \geq 0$, $i = 1,2,\ldots$, be related as in* (4.4). *Then, for every $i = 1,2,\ldots$, we have that $\sum_{j=i}^{\infty} r_j = \infty$ if and only if $\sum_{j=i}^{\infty} s_j = \infty$.*

PROOF: Because for every $t \geq 0$ we have

$$\frac{1}{2}\min(1,t) \leq \frac{t}{1+t} \leq \min(1,t),$$

it follows that the series $\sum_{j=i}^{\infty} s_j$ [see (4.4)] diverges if and only if the series $\sum_{j=i}^{\infty} \min(1,r_j)$ diverges. The latter series diverges if and only if the series $\sum_{j=i}^{\infty} r_j$ diverges. ■

Consider the following conditions:

$$p_i \text{ is logconcave in } i \geq 1, \tag{4.9}$$

$$R_n(i) \text{ is logconcave in } i \geq 1 \text{ for all } n \geq 1, \tag{4.10}$$

$$s_i \text{ is logconcave in } i \geq 1, \tag{4.11}$$

$$s_i \text{ is increasing in } i \geq 1. \tag{4.12}$$

We will show below the following interrelationships among these conditions:

$$\begin{array}{ccccc} & & (4.9) & & \\ & \nearrow & & \searrow & \\ (4.11) & & & & (4.12). \\ & \searrow & & & \\ & & (4.10) & & \end{array}$$

LEMMA 4.3: (4.11) $\Rightarrow$ (4.12).

PROOF: Suppose that s_i is logconcave and that s_i is not increasing. Then, for some i_0, we have $s_{i_0} - s_{i_0-1} = \alpha < 0$. Therefore,

$$1 - \frac{s_{i_0-1}}{s_{i_0}} = \frac{\alpha}{s_{i_0}} < 0.$$

By logconcavity, for $i \geq i_0$ we have

$$\frac{s_{i-1}}{s_i} \geq \frac{s_{i_0-1}}{s_{i_0}} = 1 - \frac{\alpha}{s_{i_0}} > 1.$$

Denoting $\beta = 1 - \alpha/s_{i_0}$, we have

$$s_i \leq \beta^{-1} s_{i-1} \leq \cdots \leq \beta^{-(i-i_0)} s_{i_0}, \quad i \geq i_0.$$

Note that $\beta^{-1} < 1$. Therefore,

$$\sum_{i=i_0}^{\infty} s_i \leq s_{i_0} \sum_{i=i_0}^{\infty} \beta^{-(i-i_0)} < \infty.$$

By Lemma 4.2, the last inequality contradicts (4.2). ■

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Using Lemma 4.3, we obtain the following implication.

Proposition 4.4: (4.11) $\Rightarrow$ (4.9).

PROOF: Note that p_i is logconcave in i if and only if $p_i p_{i+2} \le p_{i+1}^2$ for $i \ge 1$. In order to obtain this inequality, we compute, using (4.8),

$$\begin{aligned} p_i p_{i+2} &= \frac{s_i}{\prod_{j=1}^{i}(1+s_j)} \frac{s_{i+2}}{\prod_{j=1}^{i+2}(1+s_j)} \\ &= \left(\prod_{j=1}^{i}(1+s_j) \right)^{-2} \frac{s_i s_{i+2}}{(1+s_{i+1})(1+s_{i+2})} \\ &\le \left(\prod_{j=1}^{i}(1+s_j) \right)^{-2} \frac{s_{i+1}^2}{(1+s_{i+1})(1+s_{i+1})} \\ &= p_{i+1}^2, \end{aligned}$$

where the inequality follows from the logconcavity of s_i ($s_i s_{i+2} \le s_{i+1}^2$) and from the monotonicity of s_i ($s_{i+1} \le s_{i+2}$) which follows from Lemma 4.3. ■

Proposition 4.5: (4.9) $\Rightarrow$ (4.12).

PROOF: By the logconcavity of p_i, we have

$$\frac{p_j}{p_i} \ge \frac{p_{j+1}}{p_{i+1}} \quad \text{for } j \ge i.$$

Therefore,

$$\sum_{j=i+1}^{\infty} \frac{p_j}{p_i} \ge \sum_{j=i+1}^{\infty} \frac{p_{j+1}}{p_{i+1}} = \sum_{j=i+2}^{\infty} \frac{p_j}{p_{i+1}}, \quad i \ge 1.$$

So, by (4.8),

$$s_i = \frac{p_i}{\sum_{j=i+1}^{\infty} p_j} \le \frac{p_{i+1}}{\sum_{j=i+2}^{\infty} p_j} = s_{i+1}.$$ ■

Proposition 4.6: (4.11) $\Rightarrow$ (4.10).

PROOF: We prove that $R_n(i)$ is logconcave in i by induction on n. Using the theory of total positivity, it is seen from (4.11) that $R_1(i)$ is logconcave in i. Suppose that $R_{n-1}(i)$ is logconcave in i. Then, $s_i R_{n-1}(i-1)$ is logconcave in i, and from (4.5) and the theory of total positivity, we get that $R_n(i)$ is logconcave in i. ■

We are now ready to state and prove the main results of this section. A discrete-time version of Corollary 2.2 is the following theorem.

THEOREM 4.7: *If s_i is logconcave, then $p_n(i)$ is logconcave in i for each n.*

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PROOF: From Proposition 4.4, we see that p_i is logconcave in i. From Proposition 4.6, we see that $R_{n-1}(i-1)$ is logconcave in i. The stated result thus follows from (4.6). ■

A discrete-time version of Theorem 3.1 is the following result.

THEOREM 4.8: *If s_i is logconcave, then $q_n(i)$ is logconcave in i for each n.*

PROOF: Rewrite (4.7) as

$$q_n(i-k) = \sum_{j=-\infty}^{\infty} p_{j-k} s_{j-i} R_{n-1}(j-i-1).$$

By Proposition 4.4, p_i is logconcave. By assumption, s_i is logconcave. Finally, by Proposition 4.6, $R_{n-1}(i-1)$ is logconcave in i. By the theory of total positivity, it follows that $q_n(i)$ is logconcave in i. ■

Remark 4.9: It is seen from Theorems 4.7 and 4.8 that the logconcavity of s_i is a crucial assumption in the present study. One may wonder whether the logconcavity of r_i implies logconcavity of s_i or vice versa. It turns out that the logconcavity of s_i implies the logconcavity of r_i. In order to see it, note that the logconcavity of s_i implies $s_{i+1} \le s_i^2/s_{i-1}$, and because $s_{i+1}/(1+s_{i+1})$ is increasing in s_{i+1}, we have

$$\frac{s_{i+1}}{1+s_{i+1}} \le \frac{s_i^2/s_{i-1}}{1+s_i^2/s_{i-1}} = \frac{s_i^2}{s_{i-1}+s_i^2}. \tag{4.13}$$

Thus,

$$r_{i-1}r_{i+1} = \frac{s_{i-1}}{1+s_{i-1}} \frac{s_{i+1}}{1+s_{i+1}} \le \frac{s_{i-1}}{1+s_{i-1}} \frac{s_i^2}{s_{i-1}+s_i^2} \le \frac{s_i^2}{(1+s_i)^2} = r_i^2,$$

where the first inequality follows from (4.13), and the second by a simple straightforward computation.

On the other hand, the logconcavity of r_i does not necessarily imply the logconcavity of s_i. Suppose $r_1 = \frac{1}{7}$, $r_2 = \frac{2}{7}$, and $r_i = \frac{4}{7}$ for $i \ge 3$. Then, r_i is logconcave, but $s_1 s_3 > s_2^2$.

In the following example, we describe a random variable with associated s_i's that satisfy the assumptions of Theorems 4.7 and 4.8.

Example 4.10: For a fixed $\theta \in (0,1)$, define

$$r_i = 1 - \theta^i, \quad i = 1,2,\ldots.$$

Let T_1 be a discrete random variable with the hazard rates r_i. Its discrete probability density function, computed via (4.3), is given by

$$p_i \equiv P(T_1 = i) = \theta^{i(i-1)/2}(1-\theta^i), \quad i = 1,2,\ldots.$$

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It is easily seen that the above r_i's satisfy (4.2) and, therefore, T_1 is finite with probability 1. The corresponding sequence of s_i's is given by

$$s_i = \frac{r_i}{1 - r_i} = \frac{1}{\theta^i} - 1, \quad i = 1,2,\ldots.$$

Because

$$s_{i-1}s_{i+1} = \frac{(1-\theta^{i-1})(1-\theta^{i+1})}{\theta^{2i}} \leq \frac{(1-\theta^i)^2}{\theta^{2i}} = s_i^2,$$

it is seen that s_i is logconcave. Therefore, by Theorems 4.7 and 4.8, with the interpretation described in Remark 4.1, the associated discrete times of minimal repairs $T_1, T_2, \ldots$ [defined in (4.1)] and the associated discrete times between minimal repairs $X_1, X_2, \ldots$ [defined before (4.6)] have logconcave discrete density functions.

Acknowledgment

We thank a referee for a careful reading of, and useful comments on, a previous version of this article.

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APPENDIX: RELATIONS AMONG LOGCONCAVITY PROPERTIES

Consider a hazard rate function r {i.e., r: $[0,\infty) \to [0,\infty]$ and (1.1) holds} [for convenience, we define $r(t) = 0$ for $t < 0$, and if $r(t_0) = \infty$ for some t_0, then we define $r(t) = \infty$ for $t \geq t_0$]. The corresponding cumulative hazard function R is defined by $R(t) \equiv \int_0^t r(u)\, du$, $t \geq 0$, and the corresponding probability density function is given in (1.2). Finally, the corresponding survival function $\bar{F}$ and distribution function F are given by

$$\bar{F}(t) = \int_t^\infty f(s)\, ds = e^{-R(t)}, \qquad F(t) = 1 - \bar{F}(t), \quad t \geq 0.$$

Consider the following conditions:

f is logconcave, **(A.1)**

R is logconcave, **(A.2)**

r is logconcave, **(A.3)**

r is increasing. **(A.4)**

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We will show below the following interrelationships among these conditions:

$$\begin{array}{ccccc} & & (\text{A.1}) & & \\ & \nearrow & & \searrow & \\ (\text{A.3}) & & \Downarrow & & (\text{A.4}). \\ & \searrow & & \nearrow & \\ & & (\text{A.2}) & & \end{array}$$

LEMMA A.1: $(A.3) \Rightarrow (A.4)$.

PROOF: First, suppose that r is differentiable, that it is logconcave, and that it is not increasing on $[0,\infty)$. Then, r is strictly decreasing at some point t_0 with slope $\alpha < 0$, say. Therefore, $\log r$ is strictly decreasing at t_0 with slope $\alpha/r(t_0)$. Note that $r(t_0) < \infty$; otherwise, the support of F is a subset of $[0, t_0]$ and, thus, r is not strictly decreasing at t_0. From the concavity of $\log r$ it follows that $\log r$ is strictly decreasing on $[t_0,\infty)$. In fact,

$$\frac{d}{dt} \log r(t) \leq \frac{\alpha}{r(t_0)}, \quad t \geq t_0;$$

that is,

$$\frac{r'(t)}{r(t)} \leq \frac{\alpha}{r(t_0)}, \quad t \geq t_0,$$

or

$$r(t) \leq \frac{r(t_0)}{\alpha} r'(t), \quad t \geq t_0 \text{ (because } \alpha < 0).$$

Thus,

$$\int_{t_0}^{\infty} r(t)\, dt \leq \frac{r(t_0)}{\alpha} \int_{t_0}^{\infty} r'(t)\, dt = \frac{r(t_0)}{\alpha} [r(\infty) - r(t_0)],$$

where $r(\infty) = \lim_{t\to\infty} r(t)$; the latter limit exists because $\log r$ is strictly decreasing on $[t_0,\infty)$. Now, because $r(t_0)$ is finite and r is decreasing on $[t_0,\infty)$, it follows that $r(\infty)$ is also finite and, therefore, $\int_{t_0}^{\infty} r(t)\, dt < \infty$. This contradicts (1.1).

Now suppose that r is not differentiable everywhere, but that it is logconcave and that it is not increasing on $[0,\infty)$. Since r is not increasing it follows by logconcavity that it is first increasing and then decreasing, and hence it is bounded. Now convolve r with a sequence of Gaussian densities with means 0 and with variances going to 0. The convolutions are logconcave because r and the Gaussian densities are logconcave, and logconcavity is preserved under convolutions. Also, the convolutions are differentiable. By Fubini's Theorem the integrals of the convolutions are infinite. By the previous paragraph each of these convolutions is increasing (otherwise the integral of a convolution that is not increasing is finite—a contradiction). Since r is continuous (by (A.3)), the limit of these convolutions is r. Hence r is increasing; that is, (A.4) holds.

Proposition A.2. (A.3) $\Rightarrow$ (A.1).

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PROOF: The proof uses Lemma A.1 above and the ideas in Lemma 2.1 of Di Crescenzo and Pellerey [8]. Rewrite (1.2) as

$$\log f(t) = \log r(t) - \int_0^t r(u)\,du.$$

From Lemma A.1, it follows that r is increasing; therefore, $-\int_0^t r(u)\,du$ is concave in t. Also, $\log r(t)$ is concave by assumption. Thus, $\log f(t)$ is concave. ■

The following proposition is a well-known result; see Lemma 5.8 of Barlow and Proschan [2].

Proposition A.3: (A.1) $\Rightarrow$ (A.4).

Note that at a first glance, it looks as if Lemma A.1 is a corollary of Propositions A.2 and A.3. However, this is not the case because Lemma A.1 is used in the proof of Proposition A.2.

The following result follows from the theory of total positivity (Karlin [14]).

Proposition A.4: (A.3) $\Rightarrow$ (A.2).

We complete the analysis of the conditions (A.1)–(A.4) by listing some common life distributions, pointing out the values of the parameters under which (A.1), (A.2), (A.3), or (A.4) hold. Some of these examples can be used to construct models (through the corresponding nonhomogeneous Poisson process) to which the results of Sections 2 and 3 apply. We also give below two counterexamples which prove the counterimplications noted earlier.

Example A.5 (Weibull and exponential distributions): For $\alpha > 0$, the function R_α, defined by

$$R_\alpha(t) = t^\alpha, \quad t \geq 0,$$

is the cumulative hazard function of the Weibull distribution. The corresponding hazard rate and density functions are given by

$$r_\alpha(t) = \alpha t^{\alpha-1}, \quad t \geq 0,$$

$$f_\alpha(t) = \alpha t^{\alpha-1} e^{-t^\alpha}, \quad t \geq 0.$$

Now, R_α is logconcave if and only if $\alpha > 0$. On the other hand, each of (A.1), (A.3), and (A.4) holds if and only if $\alpha \geq 1$. Thus, when $\alpha \in (0,1)$, we see that (A.2) $\not\Rightarrow$ (A.4) [and, hence, also for (A.2) $\not\Rightarrow$ (A.1)]. When $\alpha = 1$, the Weibull distribution reduces to the exponential distribution for which (A.1)–(A.4) hold.

Example A.6 (Pareto distribution): The function $\bar{F}$ defined by

$$\bar{F}(t) = (1+t)^{-1}, \quad t \geq 0,$$

is the survival function associated with the Pareto distribution. The corresponding cumulative hazard, hazard rate, and density functions respectively are given by

$$R(t) = \log(1+t), \quad t \geq 0,$$

$$r(t) = (1+t)^{-1}, \quad t \geq 0,$$

$$f(t) = (1+t)^{-2}, \quad t \geq 0.$$

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Clearly, R is logconcave, but (A.1), (A.3), and (A.4) do not hold. This provides another example in which (A.2) $\not\Rightarrow$ (A.4).

The following example shows that (A.1) $\not\Rightarrow$ (A.2).

Example A.7 (Uniform distribution): The density function of the uniform distribution on [0,1] is given by

$$f(t) = 1, \quad t \in [0,1].$$

The corresponding hazard rate and cumulative hazard functions are given by

$$r(t) = (1-t)^{-1}, \quad t \in [0,1],$$
$$R(t) = -\log(1-t), \quad t \in [0,1].$$

Clearly, f is logconcave, so (A.1) [and, therefore, also (A.4)] hold. On the other hand, R is not logconcave, so (A.2) [and, therefore, also (A.3)] do not hold.

Example A.8 (Truncated extreme value distribution): The function R defined by

$$R(t) = e^t - 1, \quad t \geq 0,$$

is the cumulative hazard function of the truncated extreme value distribution. The corresponding hazard rate and density functions are given by

$$r(t) = e^t, \quad t \geq 0,$$
$$f(t) = e^{1+t-e^t}, \quad t \geq 0.$$

Clearly, r is logconcave and, therefore, (A.1)–(A.4) hold.

Example A.9 (Truncated logistic distribution): The function f defined by

$$f(t) = 2e^t(1+e^t)^{-2}, \quad t \geq 0,$$

is the density function associated with the truncated logistic distribution. The corresponding hazard rate and cumulative hazard functions are given by

$$r(t) = e^t(1+e^t)^{-1}, \quad t \geq 0,$$
$$R(t) = \log(1+e^t) - \log 2, \quad t \geq 0.$$

It is easy to verify that the derivative of $\log r$ is decreasing; that is, r is logconcave and, therefore, (A.1)–(A.4) hold.

Example A.10: The functions r_1 and r_2 defined by

$$r_1(t) = t(1+t)^{-1} + \log(1+t), \quad t \geq 0,$$
$$r_2(t) = 1 + \tanh t, \quad t \geq 0,$$

are hazard rate functions discussed in Block and Joe [7]. It is easy to verify that the derivatives of $\log r_1$ and $\log r_2$ are decreasing; that is, r_1 and r_2 are logconcave.

The following counterexample shows that (A.1) $\not\Rightarrow$ (A.2) even if the underlying support is $[0,\infty)$.

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Counterexample A.11: Let

$$r(t) = te^{t^2/2}, \quad t \geq 0.$$

Then,

$$R(t) = e^{t^2/2} - 1, \quad t \geq 0,$$

$$f(t) = te^{t^2/2}e^{-e^{t^2/2}+1}, \quad t \geq 0.$$

Now,

$$\frac{d^2}{dt^2}\log f(t) = -\frac{1}{t^2} + 1 - (1+t^2)e^{t^2/2} \leq -\frac{1}{t^2} + 1 - (1+t^2) \leq 0, \quad t \geq 0.$$

Thus, f is logconcave.

Now, let us check the logconcavity of R. We compute

$$\frac{d}{dt}\log R(t) = \frac{te^{t^2/2}}{e^{t^2/2}-1} = \frac{t}{1-e^{-t^2/2}}, \quad t \geq 0.$$

Note that, using L'Hôpital's rule, we have

$$\lim_{t\to 0}\frac{d}{dt}\log R(t) = \lim_{t\to 0}\frac{t}{1-e^{-t^2/2}} = \lim_{t\to 0}\frac{1}{te^{-t^2/2}} = \infty$$

and

$$\lim_{t\to\infty}\frac{d}{dt}\log R(t) = \lim_{t\to\infty}\frac{t}{1-e^{-t^2/2}} = \infty.$$

However, $d/dt \log R(t) = t/(1 - e^{-t^2}/2)$ is finite for any $t \in (0,\infty)$. Therefore, $d/dt \log R$ is not monotone, and, hence, R is not logconcave.

The above examples and counterexample show that (A.1) $\not\Rightarrow$ (A.3) and (A.2) $\not\Rightarrow$ (A.3). The following counterexample shows that even if both (A.1) and (A.2) hold, it still does not follow that (A.3) holds.

Counterexample A.12: Let

$$r(t) = \frac{t^2}{2} + t + 2, \quad t \geq 0.$$

Then,

$$\frac{d}{dt}\log r(t) = \frac{t+1}{t^2/2 + t + 2}, \quad t \geq 0,$$

$$\frac{d^2}{dt^2}\log r(t) = \frac{-t^2/2 - t + 1}{(t^2/2 + t + 2)^2}, \quad t \geq 0.$$

The latter expression is positive for small t and, therefore, r is not logconcave.

https://doi.org/10.1017/S0269964800143062 Published online by Cambridge University Press

Next,

$$R(t) = \frac{t^3}{6} + \frac{t^2}{2} + 2t, \quad t \geq 0.$$

Thus,

$$\frac{d}{dt} \log R(t) = \frac{3t^2 + 6t + 12}{t^3 + 3t^2 + 12t}, \quad t \geq 0,$$

$$\frac{d^2}{dt^2} \log R(t) = \frac{-3t^4 - 12t^3 - 18t^2 - 72t - 144}{(t^3 + 3t^2 + 12t)^2} \leq 0, \quad t \geq 0.$$

Therefore, R is logconcave.

Finally,

$$\log f(t) = \log\left(\frac{t^2}{2} + t + 2\right) - \left(\frac{t^3}{6} + \frac{t^2}{2} + 2t\right), \quad t \geq 0,$$

$$\frac{d}{dt} \log f(t) = \frac{t+1}{t^2/2 + t + 2} - \left(\frac{t^2}{2} + t + 2\right), \quad t \geq 0,$$

$$\frac{d^2}{dt^2} \log f(t) = \frac{-t^2/2 - t - [(t+1)(t^2/2 + t + 2)^2 - 1]}{(t^2/2 + t + 2)^2} \leq 0, \quad t \geq 0.$$

Therefore, f is logconcave.

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