Ergod. Th. & Dynam. Sys. (1997), **17**, 1027–1042
Printed in the United Kingdom © *1997 Cambridge University Press*

Differentiable structures of central Cantor sets

RODRIGO BAMÓN†‖, CARLOS G. MOREIRA‡, SERGIO PLAZA§∗∗
and JAIME VERA¶††

† *Depto. de Matemática, Universidad de Chile, Casilla 653 Santiago, Chile*
(e-mail: rbamon@abello.dic.uchile.cl)
‡ *IMPA, Estrada Dona Castorina 110, Jardim Botánico 22460-320,*
Rio de Janeiro, Brasil
(e-mail: gugu@impa.br)
§ *Depto. de Matemática, Facultad de Ciencias, Universidad de Santiago de Chile,*
Casilla 307-Correo 2, Santiago, Chile
(e-mail: splaza@fermat.usach.cl)
¶ *Depto. de Matemática, Facultad de Ciencias, Universidad Católica del Norte,*
Avda. Angamos 0610, Antofagasta, Chile
(e-mail: jvera@socompa.cecun.ucn.cl)

(*Received* 6 *March* 1995 *and revised* 9 *October* 1996)

Abstract. Central Cantor sets form a class of symmetric Cantor sets of the real line. Here we give a complete characterization of the $C^{k+\alpha}$ regularity of these Cantor sets. We also give a classification of central Cantor sets up to global and local diffeomorphisms. Examples of central Cantor sets with special dynamical and measure-theoretical properties are also provided. Finally, we calculate the fractal dimensions of an arbitrary central Cantor set.

0. *Introduction*

Central Cantor sets in the real line form a particular but interesting class of Cantor sets. They are constructed in a simple way, and are useful in providing examples in several mathematical contexts. For instance, in analysis they can be used to prove the sharpness of some recent results concerning the classical Morse–Sard theorem [**M2**]. Even more abundant are their applications in dynamical systems. In particular, they were used by Bowen [**Bo**] to construct a horseshoe of class C^1 with positive Lebesgue measure (see also our Example 4.1), and by Sannami [**Sa**] and by Bamón *et al* [**BPV**] to give examples of regular Cantor sets whose self-arithmetic difference is a Cantor set with positive measure. Here we discuss a general class of such examples (see Example 4.2). This is related to a conjecture of Palis (see [**P2**]).

‖ Partially supported by Fondecyt Grant # 1930863.
∗∗ Partially supported by Fondecyt Grant # 1941080 and Dicyt Grant # 9433 P.S.
†† Partially supported by Fondecyt Grant #1941080 and UCANORTE Grant.

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

Regular or dynamically defined Cantor sets constitute an important class of subsets of the real line that naturally appear in dynamics (see [**PT**, **PT1**, **PT2**, **P1**, **P2**, **M1**, **N**]). In this paper we give a complete classification, for any integer $k \geq 1$ and $\alpha \in [0, 1]$, of central Cantor sets that admit a structure of regular Cantor sets of class $C^{k+\alpha}$ (Theorem A), as well as a classification of central Cantor sets up to global and local diffeomorphisms (Theorem B). The present work, although self-contained, is a natural continuation of [**BPV**].

We are able to find the explicit and simple conditions for the characterization and classification of $C^{k+\alpha}$ regularity of central Cantor sets, given in Theorems A and B, because of the simple and symmetrical structure of this class of Cantor set. This contrasts with the conditions for regularity of more general classes of Cantor sets in the line (see [**S**, **BF**, **PR**, **PrT**]).

1. *Basic definitions and statement of results*

Let $\underline{\lambda} = (\lambda_k)_{k\in\mathbb{N}^*}$ be a sequence of real numbers satisfying $0 < \lambda_k < \frac{1}{2}$, for all $k \in \mathbb{N}^*$, and let $I \subset \mathbb{R}$ be a closed interval. We define *the central Cantor set $K_{\underline{\lambda},I}$ associated to the sequence $\underline{\lambda}$ with support interval I* as follows. Let $P^0_{\underline{\lambda},I} = I$. For each $i \in \mathbb{N}^*$, $P^i_{\underline{\lambda},I}$ is the set consisting of 2^i closed disjoint intervals obtained from $P^{i-1}_{\underline{\lambda},I}$ by removing from each of its intervals the central open interval of proportion $1 - 2\lambda_i$. We call these removed intervals the *gaps of the ith step* of the construction of $K_{\underline{\lambda},I}$, and the intervals of $P^i_{\underline{\lambda},I}$ the *fundamental intervals* of the ith step of the construction of $K_{\underline{\lambda},I}$. Finally, define $K_{\underline{\lambda},I} = \cap_{i=0}^{\infty} P^i_{\underline{\lambda},I}$.

Let $a = |I|$ be the length of the interval I. Then the length of any nth step fundamental interval is $a \cdot \pi_n$, where $\pi_n = \pi_n(\underline{\lambda}) = \Pi_{i=1}^n \lambda_i$, and the length of any nth step gap is $a \cdot g_n$, where $g_n = g_n(\underline{\lambda}) = (1 - 2\lambda_n)\pi_{n-1}$.

When we do not state the interval I we will assume $I = [0, 1]$ and we will use the notations $K_{\underline{\lambda}}$ and $P^i_{\underline{\lambda}}$ instead of $K_{\underline{\lambda},I}$ and $P^i_{\underline{\lambda},I}$.

Now let us define an important class of Cantor sets which naturally appear in dynamical systems. We say that a Cantor set $K \subset \mathbb{R}$ is *dynamically defined or regular* if:

(i) there exist compact disjoint intervals $\overline{K}_1, \overline{K}_2, \ldots, \overline{K}_r$ such that $K \subset \overline{K}_1 \cup \overline{K}_2 \cup \cdots \cup \overline{K}_r$ and the boundary of each $\overline{K}_i$ is contained in K;

(ii) there is an expanding map $\psi : \overline{K}_1 \cup \overline{K}_2 \cup \cdots \cup \overline{K}_r \to \mathbb{R}$ such that:

  (a) for each $i = 1, \ldots, r$, $\psi(\overline{K}_i)$ is the convex hull of the union of some intervals $\overline{K}_j$'s;

  (b) for each $i = 1, \ldots, r$, and n large enough, $\psi^n(K \cap \overline{K}_i) = K$;

  (c) $K = \cap_{n=0}^{\infty} \psi^{-n}(\overline{K}_1 \cup \overline{K}_2 \cup \cdots \cup \overline{K}_r)$.

We call $\{\overline{K}_1, \overline{K}_2, \ldots, \overline{K}_r\}$ a *Markov partition* for K and $D = \overline{K}_1 \cup \overline{K}_2 \cup \cdots \cup \overline{K}_r$ a *Markov domain of K*.

We say that a Cantor set is *regular of class C^k (respectively $C^{k+\alpha}$)* if ψ is a C^k (respectively $C^{k+\alpha}$) function. In many applications we restrict our attention to regular Cantor sets which are at least $C^{1+\varepsilon}$ for some $\varepsilon > 0$.

The main goal of this work is to characterize the central Cantor sets which are regular of a given class of differentiability. We also classify central Cantor sets up to global and

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

local diffeomorphisms. We have the following results.

THEOREM A. *Let K be a central Cantor set defined by the sequence $\underline{\lambda} = (\lambda_i)_{i\in\mathbb{N}^*}$. Then K admits a structure of a regular Cantor set (of class at least C^1) if and only if there is $r \in \mathbb{N}^*$ such that the following conditions hold:*

(a) $\lim_{n\to\infty} \dfrac{\lambda_{n+r}}{\lambda_n} = 1;$

(b) $\lim_{n\to\infty} \dfrac{1-2\lambda_{n+r}}{1-2\lambda_n} = 1;$

(c) *the sequence $(\lambda_{n+1}\cdot\lambda_{n+2}\cdots\lambda_{n+r})_{n\in\mathbb{N}}$ converges to some $\lambda \neq 0$.*

Let $r_0 = \min\{r \in \mathbb{N}^* :$ conditions (a), (b) *and* (c) *above hold$\}$, and let $k \in \mathbb{N}^*$. Then K admits a structure of a regular Cantor set of class C^k if and only if the following equivalent conditions hold:*

(i) $\lim_{n\to\infty} \dfrac{|\lambda_{n+1}\cdot\lambda_{n+2}\cdots\lambda_{n+r_0} - \lambda|}{\pi_n^{k-1}} = 0,$ *where* $\lambda = \lim_{n\to\infty} \lambda_{n+1}\cdots\lambda_{n+r_0}$;

(ii) $\lim_{n\to\infty} \dfrac{|\lambda_{n+1}\cdot\lambda_{n+2}\cdots\lambda_{n+r_0} - \lambda|}{\lambda^{(k-1)n/r_0}} = 0.$

If $k > 1$ then these conditions are also equivalent to the following:

(iii) *for each $j = 1, 2, \ldots, r_0$ the sequence $(\lambda_{r_0n+j})_{n\in\mathbb{N}}$ converges to some $\mu_j \in (0, \frac{1}{2})$ such that*

$$\lim_{n\to\infty} \frac{|\lambda_{r_0n+j} - \mu_j|}{\lambda^{(k-1)n}} = 0, \quad j = 1, 2, \ldots, r_0.$$

We have $\lambda = \mu_1\cdot\mu_2\cdots\mu_{r_0}$.

Let $k \in \mathbb{N}^*$ and $\alpha \in (0, 1]$. Then K admits a structure of a regular Cantor set of class $C^{k+\alpha}$ if and only if the following equivalent conditions hold:

(i′) $\limsup_{n\to\infty} \dfrac{|\lambda_{n+1}\cdots\lambda_{n+r_0} - \lambda|}{\pi_n^{k-1+\alpha}} < \infty;$

(ii′) $\limsup_{n\to\infty} \dfrac{|\lambda_{n+1}\cdots\lambda_{n+r_0} - \lambda|}{\lambda^{(k-1+\alpha)n/r_0}} < \infty;$

(iii′) $\limsup_{n\to\infty} \dfrac{|\lambda_{r_0n+j} - \mu_j|}{\lambda^{(k-1+\alpha)n}} < \infty, \quad j = 1, 2, \ldots, r_0.$

Remarks.

- Condition (iii) above does not necessarily hold when $k = 1$. In fact, we will show that if $0 < a \le b \le \frac{1}{2}$ then there is a sequence $\underline{\lambda} = (\lambda_n)_{n\in\mathbb{N}}$ dense in $[a, b]$ such that $K_{\underline{\lambda}}$ is C^1 (see Example 4.3).
- It follows from the above that $\mu_j \neq 0$, $j = 1, 2, \ldots, r_0$, since $\mu_1\cdot\mu_2\cdots\mu_{r_0} = \lambda$. Moreover, if K is regular of class C^q, $q = k$ or $q = k + \alpha$, with $q > 1$, then $\mu_j < \frac{1}{2}$, $j = 1, 2, \ldots, r_0$.
- The condition in order for K to admit a structure of a regular Cantor set of class C^∞ is that the equivalent conditions (i)–(iii) must be satisfied for each $k \in \mathbb{N}^*$.
- We will use Theorem A to provide examples, for each $q \ge 1$, $q = k$ or $q = k + \alpha$ and $c \in [0, 2)$, of central Cantor sets $K_{\underline{\lambda}}$ which are of class C^q but not of class $C^{q+\varepsilon}$, for all $\varepsilon > 0$, such that $K_{\underline{\lambda}} - K_{\underline{\lambda}} = \{x - y : x, y \in K_{\underline{\lambda}}\}$ is a Cantor set of c Lebesgue measure.

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

- To endow K with a regular structure we need a Markov partition with at least 2^{r_0} intervals. For example, the central Cantor set $\{\frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \ldots\}$ is C^r for every $r \in \mathbb{N} \cup \{\omega\}$ with a Markov partition of four intervals: $P^2_{\underline{\lambda}}$.

THEOREM B. *Let $K_{\underline{\lambda},I}$ and $K_{\underline{\tilde{\lambda}},J}$ be two central Cantor sets. Then there is a C^1 diffeomorphism $\varphi : I \to J$ such that $\varphi(K_{\underline{\lambda},I}) = K_{\underline{\tilde{\lambda}},J}$ if and only if the following conditions hold:*

(i) $\displaystyle\lim_{n\to\infty} \frac{1-2\tilde{\lambda}_n}{1-2\lambda_n} = 1;$

(ii) $\displaystyle\lim_{n\to\infty} \frac{\pi_n(\underline{\tilde{\lambda}})}{\pi_n(\underline{\lambda})} = a \neq 0.$

In this case there is such a diffeomorphism satisfying $\varphi'(x) = a(|J|/|I|)$, for all $x \in K_{\underline{\lambda},I}$, and $\varphi(P^r_{\underline{\lambda},I}) = P^r_{\underline{\tilde{\lambda}},J}$, for all $r \in \mathbb{N}$, i.e. φ preserves the structure of central Cantor sets.

Let $k \in \mathbb{N}^*$. There is such a diffeomorphism of class C^k if and only if

(iii) $\displaystyle\lim_{n\to\infty} \frac{|g_n(\underline{\tilde{\lambda}}) - a g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^k} = 0.$

If $\alpha \in (0,1]$ then there is such a diffeomorphism of class $C^{k+\alpha}$ if and only if

(iii′) $\displaystyle\limsup_{n\to\infty} \frac{|g_n(\underline{\tilde{\lambda}}) - a g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^{k+\alpha}} < \infty.$

Necessary and sufficient conditions for the existence of a local diffeomorphism (respectively a local diffeomorphism of class C^q, $q = k$ or $q = k+\alpha$, $k \in \mathbb{N}^*$, $\alpha \in (0,1]$), $\varphi : U \to V$ such that $\varphi(U \cap K_{\underline{\lambda},I}) = V \cap K_{\underline{\tilde{\lambda}},J}$, where U and V are open intervals and $U \cap K_{\underline{\lambda},I} \neq \emptyset$, are that the corresponding conditions above should be verified for some $\underline{\tilde{\lambda}}^{(p)}$, $p \in \mathbb{Z}$, instead of $\underline{\tilde{\lambda}}$, where $\underline{\tilde{\lambda}}^{(p)} = \underline{\tilde{\lambda}}_{n+p}$ if $n + p \geq 1$ and $\tilde{\lambda}^{(p)}_n = 1/3$ otherwise.

As a consequence of Theorem B, we conclude that if $K_{\underline{\lambda}}$ has the structure of a C^q regular Cantor set, $q = k$ or $q = k + \alpha$, $k \in \mathbb{N}^*$, $\alpha \in (0,1]$, and if $q > 1$ then there is a C^q diffeomorphism $\varphi : [0,1] \to [0,1]$ such that $\varphi(K_{\underline{\lambda}}) = K_{\underline{\tilde{\lambda}}}$, where $\underline{\tilde{\lambda}}$ is the periodic sequence defined by $\tilde{\lambda}_n = \mu_{\overline{n}}$, where $\overline{n} \in \{1, 2, \ldots, r_0\}$ satisfy $n \equiv \overline{n} \pmod{r_0}$, $\mu_{\overline{n}}$ being as in Theorem A. Note that $K_{\underline{\tilde{\lambda}}}$ is an affine Cantor set in the sense of [**PT**].

Finally, we compute the fractal dimensions of a central Cantor set. For each interval I and sequence $\underline{\lambda} = (\lambda_n)_{n\in\mathbb{N}}$:

- $\tau(K_{\underline{\lambda},I}) = \tau(K_{\underline{\lambda}}) = \displaystyle\inf_{n\in\mathbb{N}} \frac{\lambda_n}{1-2\lambda_n};$
- $\tau_{\mathrm{loc}}(K_{\underline{\lambda},I}) = \tau_{\mathrm{loc}}(K_{\underline{\lambda}}) = \displaystyle\liminf_{n\to\infty} \frac{\lambda_n}{1-2\lambda_n};$
- $d(K_{\underline{\lambda},I}) = d(K_{\underline{\lambda}}) = \displaystyle\limsup_{n\to\infty} \frac{-n\log 2}{\log \pi_n(\underline{\lambda})};$
- $HD(K_{\underline{\lambda},I}) = HD(K_{\underline{\lambda}}) = \displaystyle\liminf_{n\to\infty} \frac{-n\log 2}{\log \pi_n(\underline{\lambda})}.$

2. *The structure of diffeomorphisms of central Cantor sets*

In this section we shall prove results that combinatorially characterize diffeomorphisms and local diffeomorphisms of central Cantor sets.

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

PROPOSITION 1. *Let $K_1 = K_{\underline{\lambda},J_1}$ and $K_2 = K_{\tilde{\underline{\lambda}},J_2}$ be two central Cantor sets, and let U, V be two intervals such that $\partial U \cap K_1 = \emptyset$ and $\partial V \cap K_2 = \emptyset$. Let $\varphi : U \to V$ be a C^1 diffeomorphism such that $\varphi(U \cap K_1) = V \cap K_2$. Then for any $\varepsilon > 0$ there exist $m, n_0 \in \mathbb{N}$ such that:*

- *$U \cap K_1 = (\cup_{r=1}^{m} I_r) \cap K_1$, where I_r is a fundamental interval of the n_0th step of the construction of K_1, $r = 1, 2, \ldots, m$;*
- *$|I_1| = |I_2| = \cdots = |I_m| < \varepsilon$;*
- *$\varphi(I_r)$ is a fundamental interval of the construction of K_2, $r = 1, 2, \ldots, m$.*

Remark. We can substitute the hypotheses $\partial U \cap K_1 = \emptyset$ and $\partial V \cap K_2 = \emptyset$ by $U \cap K_1$ and $V \cap K_2$ are open and closed in K_1 and K_2, respectively.

Proof. First we give some definitions. Let K be a Cantor set. A *gap* of K is a connected component of its complement, perhaps unbounded. We say that an interval J between two gaps U_1, U_2 of K is a *bridge* if $|U_1| \geq \frac{6}{5}|U_2|$ or $|U_2| \geq \frac{6}{5}|U_1|$, and there are no gaps U contained in J with $|U| \geq \frac{5}{6}\min(|U_1|, |U_2|)$. Note that a bridge of a central Cantor set must necessarily be a fundamental interval of some step of its construction. In fact, a bridge must lie between two gaps of different steps U_1, U_2, where U_1, say, corresponds to a more advanced step than U_2, and if the interval between U_1 and U_2 is not a fundamental interval then it must contain a gap of the same step of U_1 and hence with the same length of U_1, thus it cannot be a bridge.

We say that the nth step of the construction of a central Cantor set $K_{\underline{\lambda},I}$ is a *good step* if $g_n(\underline{\lambda}) \geq \frac{3}{2}g_m(\underline{\lambda})$, for all $m > n$. In this case, if $g_k(\underline{\lambda}) \geq \frac{6}{5}g_n(\underline{\lambda})$ then any fundamental interval bounded by a gap of the kth step and a gap of the nth step of the construction of $K_{\underline{\lambda},I}$ must be a bridge. Also, note that there are arbitrarily advanced good steps for any $K_{\underline{\lambda},I}$. In fact, if this is not the case, then there exist sufficiently large n_0 such that for $n \geq n_0$ there is $m > n$ with $g_m(\underline{\lambda}) > \frac{2}{3}g_n(\underline{\lambda})$, but this implies that for a positive constant (fixed) C and infinitely many $n \in \mathbb{N}$, $g_n(\underline{\lambda}) > C \cdot (\frac{2}{3})^n$, which contradicts the fact that $g_n(\underline{\lambda}) < \pi_{n-1}(\underline{\lambda}) < (\frac{1}{2})^{n-1}$.

Now we can decompose $U = \widetilde{U}_1 \cup \widetilde{I}_1 \cup \widetilde{U}_2 \cup \cdots \cup \widetilde{I}_\ell \cup \widetilde{U}_{\ell+1}$, for some $\ell \in \mathbb{N}$, in such a way that for each i, $\widetilde{U}_i$ is a gap of K_1. More precisely, $\widetilde{U}_1$ and $\widetilde{U}_{\ell+1}$ are intersections of gaps of K_1 with U, and $\widetilde{I}_j$ are fundamental intervals of K_1 such that

$$\sup_{x,y \in S\widetilde{I}_j} \frac{|\varphi'(y)|}{|\varphi'(x)|} < \frac{10}{9}.$$

Here, if $\widetilde{I}_j = [a, a + \delta]$, $S\widetilde{I}_j$ denotes the interval $[a - 2\delta, a + 3\delta]$.

Let $n_0 \in \mathbb{N}$ be such that the n_0th step of the construction of K_1 is a good step, the fundamental intervals of this n_0th step have length less than ε and each of the gaps of this step have length less than $\frac{1}{2}\min_{1 \leq i \leq \ell+1}(|\widetilde{U}_1|)$. We will prove that the fundamental intervals of this n_0th step contained in U, say $I_1, I_2, \ldots, I_r$, satisfies our statements.

We must prove that $\varphi(I_j)$ is a fundamental interval of K_2, $1 \leq j \leq r$. Let us consider the decomposition of one of these intervals $\widetilde{I}_r$, corresponding to the n_0th step of the construction of K_1, say $\widetilde{I}_r = I_m \cup U_1 \cup I_{m+1} \cup \cdots \cup U_{2^s-1} \cup I_{m+2^s-1}$, where U_i are gaps, and the decomposition is given in increasing order of the intervals. Then, by the above construction, I_m lies between $\widetilde{U}_r$ and U_1, where $|\widetilde{U}_r| > 2|U_1|$, and any gap contained in I_m

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

has length less than or equal to $\frac{2}{3}|U_1|$. So, since $\sup_{x,y\in S\widetilde{I}_r}|\varphi'(y)|/|\varphi'(x)| < \frac{10}{9}$ (estimated distortion of φ in $S\widetilde{I}_r$), $\varphi(I_m)$ lies between $\varphi(\widetilde{U}_r)$ and $\varphi(U_1)$, where $|\varphi(\widetilde{U}_r)| > \frac{9}{5}|\varphi(U_1)|$ and any gap contained in $\varphi(I_m)$ has length less than or equal to $\frac{20}{27}|\varphi(U_1)| < \frac{5}{6}|\varphi(U_1)|$, hence $\varphi(I_m)$ is a bridge and therefore a fundamental interval.

Now let us consider the decomposition of $\widetilde{I}_r$ corresponding to a good step of the construction of K_1 such that the gaps of this step have length less than $\frac{1}{2}\min_{1\le i\le 2^s}(|U_i|)$, say $\widetilde{I}_r = I'_1 \cup U'_1 \cup I'_2 \cup \cdots \cup U'_{2^p-1} \cup I'_{2^p}$. Note that each interval I_ℓ is decomposed into a union of gaps and 2^{p-s} of the intervals I'_j, $m \le \ell \le m + 2^s - 1$. More precisely, I_{m+r} is the convex hull of $\cup_{i=1}^{2^{p-s}} I'_{2^{p-s}r+i}$. We know that $\varphi(I_m)$ is a fundamental interval of K_2 and, by the same reason, $\varphi(I'_1)$ is also a fundamental interval of K_2, say of the n_1th step.

Now we will prove that φ send the gaps $U'_1, U'_3, \ldots, U'_{2^p-1}$ of the same step of the fundamental intervals I'_j into the gaps of the n_1th step contained in $\varphi(\widetilde{I}_r)$. First of all, $\varphi(U'_1)$ is the first gap after $\varphi(I'_1)$, which is the first gap of the n_1th step contained in $\varphi(\widetilde{I}_r)$. Suppose by induction, that $\varphi(U'_{2k-1})$ is the kth gap of the n_1th step contained in $\varphi(\widetilde{I}_r)$.

We have $|U'_{2k+1}| = |U'_{2k-1}|$, and the left end of U'_{2k+1} is at least $2\cdot|I'_{2k}|$ after the right end of U'_{2k-1}. Then, because of the distortion estimate of φ, $|\varphi(U'_{2k+1})| > \frac{5}{6}|\varphi(U'_{2k-1})|$, and the left end of $\varphi(U'_{2k+1})$ is at least $\frac{5}{3}|\varphi(I'_{2k})| \ge \frac{3}{2}|\varphi(I'_1)|$ after the right end of $\varphi(U'_{2k-1})$. Thus, either $\varphi(U'_{2k+1}) = V_{k+1}$, the $(k+1)$th gap of the n_1th step contained in $\varphi(\widetilde{I}_r)$, or $\varphi(U'_{2k+1})$ is after V_{k+1}. The last case cannot occur, because $|V_{k+1}| = |V_k| = |\varphi(U'_{2k-1})|$, and the left end of V_{k+1} is at least $2|\varphi(I'_1)|$ after the right end of $\varphi(U'_{2k-1})$, and so, by the same argument applied with φ^{-1} instead of φ, $\varphi^{-1}(V_{k+1})$ would be a gap of K_1 between U'_{2k-1} and U'_{2k+1}, which lies at least $\frac{3}{2}|I'_1| = \frac{3}{2}|I'_{2k}|$ after the right end of U'_{2k-1}, satisfying $|\varphi^{-1}(V_{k+1})| \ge \frac{3}{4}|U'_{2k-1}|$, which is absurd. Thus our claim is proved.

Since $U'_{2^{p-s}r} = U_r$, $1 \le r \le 2^s - 1$, we have, by the construction of I'_j, that $U'_{2^{p-s}r}$ is a gap that lies between $U'_{2^{p-s}r-1}$ and $U'_{2^{p-s}r+1}$, and $|U'_{2^{p-s}r}| \ge 2|U_{2^{p-s}r-1}|$, so $\varphi(U'_{2^{p-s}r})$ is a gap that lies between $V_{2^{p-s-1}r}$ and $V_{2^{p-s-1}r+1}$ and $|\varphi(U'_{2^{p-s}r})| \ge \frac{3}{2}|V_{2^{p-s-1}r}|$, so $\varphi(U'_{2^{p-s}r})$ must be the unique gap of the older step between $V_{2^{p-s-1}r}$ and $V_{2^{p-s-1}r+1}$. This proves that $\varphi(I_{m+r})$ is the convex hull of the union of 2^{p-s} consecutive fundamental intervals of the n_1th step of the construction of K_2, $0 \le r \le 2^s - 1$, and since $\varphi(I_m)$ is itself a fundamental interval of the construction of K_2, all the $\varphi(I_{m+r})$, $0 \le r \le 2^s - 1$, must also be fundamental intervals of the same step of the construction of K_2. □

COROLLARY 1. *Let $K_{\underline{\lambda},I}$ and $K_{\underline{\tilde{\lambda}},J}$ be two central Cantor sets, and let $\varphi : I \to J$ be a C^1 diffeomorphism such that* $\sup_{x,y\in I}|\varphi'(y)|/|\varphi'(x)| \le 4$ *and $\varphi(K_{\underline{\lambda},I}) = K_{\underline{\tilde{\lambda}},J}$. Then φ maps the decomposition of the mth step of $K_{\underline{\lambda},I}$ into the decomposition of the mth step of $K_{\underline{\tilde{\lambda}},J}$, i.e. $\varphi(P^m_{\underline{\lambda},I}) = P^m_{\underline{\tilde{\lambda}},J}$, for all $m \in \mathbb{N}$.*

Proof. Let $m \in \mathbb{N}$, and consider the decomposition $I = I_1 \cup U_1 \cup \cdots \cup U_{2^m-1} \cup I_{2^m}$ of the mth step of the construction of $K_{\underline{\lambda},I}$. By Proposition 1, there exists $n_0 \in \mathbb{N}$ such that if $I = I'_1 \cup U'_1 \cup \cdots \cup U'_{2^{n_0}-1} \cup I'_{2^{n_0}}$ is the n_0th step of the construction of $K_{\underline{\lambda},I}$ then $|I'_1| = |I'_2| = \cdots = |I'_{2^{n_0}}| < |I_1|$, and $\varphi(I'_r)$ is a fundamental interval of the construction of K_2, $r = 1, 2, \ldots, 2^{n_0}$. But then all $\varphi(I'_r)$ must be fundamental intervals of the n_0th step of the construction of K_2, otherwise there will be fundamental intervals $\varphi(I'_{r_1})$ of a step

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

before the n_0th and $\varphi(I'_{r_2})$ of a step after the n_0th, and we will have $|\varphi(I'_{r_1})|/|\varphi(I'_{r_2})| > 4$, which is a contradiction since $|I'_{r_1}| = |I'_{r_2}|$ and $\sup_{x,y\in I} |\varphi'(y)|/|\varphi'(x)| \leq 4$. This proves our result for the n_0th step of the construction of K_1 and *a fortiori* for the mth step, which is less advanced than the n_0th. □

COROLLARY 2. *Under the conditions of Proposition 1, φ maps decompositions of I_r into fundamental intervals and gaps into the corresponding ones of $\varphi(I_r)$, $1 \leq r \leq m$, provided the I_r are conveniently chosen. In particular, φ maps sufficiently small fundamental intervals of K_1 into fundamental intervals of K_2.*

Proof. This is an immediate consequence of Corollary 1, since if K is a central Cantor set and I is a fundamental interval of K, then $I \cap K$ is itself a central Cantor set, and we have $\sup_{x,y\in I_r} |\varphi'(y)|/|\varphi'(x)| < \frac{10}{9} < 4$, $1 \leq r \leq m$, for I_r obtained as in the proof of Proposition 1. □

Example 2.1. Let $\lambda \in (0, \frac{1}{2})$, and $K_\lambda = K_{\underline{\lambda}}$, where $\lambda_n = \lambda$, for all $n \in \mathbb{N}$. Then there are C^∞ diffeomorphisms $\varphi : [0,1] \to [0,1]$ such that $\varphi(K_\lambda) = K_\lambda$, whose distortion $D\varphi = \sup_{x,y\in[0,1]} |\varphi'(y)|/|\varphi'(x)|$ is arbitrarily close to $1/\lambda^2$. In fact, define

$$\varphi(x) = \begin{cases} \dfrac{x}{\lambda} & x \in [0, \lambda^2] \\ x + 1 - 2\lambda + \lambda^2 & x \in [\lambda(1-\lambda), \lambda] \\ \lambda x + 1 - \lambda & x \in [1-\lambda, 1]. \end{cases}$$

Note that $[0, \lambda]$ is a fundamental interval of K_λ but $\varphi([0,\lambda]) = [0, 1 - \lambda + \lambda^2]$ is not. Letting λ tend to $\frac{1}{2}$, we see that the constant 4 in Corollary 1 is the best possible.

PROPOSITION 2. *Let $K_1 = K_{\underline{\lambda},I}$ and $K_2 = K_{\underline{\tilde{\lambda}},J}$ be two central Cantor sets, and let $\varphi : U \to V$ be a C^1 diffeomorphism, where U and V are intervals such that* $\mathrm{int}(U) \cap K_1 \neq \emptyset$ *and $\varphi(U \cap K_1) = V \cap K_2$. Then there exist $p \in \mathbb{Z}$ such that*

$$\lim_{n\to\infty} \frac{1 - 2\tilde{\lambda}_{n+p}}{1 - 2\lambda_n} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \frac{\pi_{n+p}(\underline{\tilde{\lambda}})}{\pi_n(\underline{\lambda})} = a,$$

for some $a \neq 0$. We also have that $\varphi'|_{K_1\cap U}$ is locally constant. Let $k \in \mathbb{N}^*$. If φ is C^k, then there exist $p \in \mathbb{Z}$ as before such that

$$\lim_{n\to\infty} \frac{|g_{n+p}(\underline{\tilde{\lambda}}) - a g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^k} = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{|\pi_{n+p}(\underline{\tilde{\lambda}}) - a\pi_n(\underline{\lambda})|}{(\pi_n(\underline{\lambda}))^k} = 0.$$

If $\alpha \in (0,1]$ and $\varphi \in C^{k+\alpha}$ then there exist $p \in \mathbb{Z}$ as before such that

$$\limsup_{n\to\infty} \frac{|g_{n+p}(\underline{\tilde{\lambda}}) - a g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^{k+\alpha}} < \infty \quad \textit{and} \quad \limsup_{n\to\infty} \frac{|\pi_{n+p}(\underline{\tilde{\lambda}}) - a\pi_n(\underline{\lambda})|}{(\pi_n(\underline{\lambda}))^{k+\alpha}} < \infty.$$

Proof. We can, if necessary, reduce the interval U to satisfy the hypothesis of Proposition 1. We will restrict our attention to $\varphi|_{I_j}$, where I_j is one of the fundamental intervals obtained in Proposition 1, Corollary 2. I_j is a fundamental interval of the n_0th step of the construction of K_1, for some $n_0 \in \mathbb{N}$, and $\varphi(I_j)$ is a fundamental interval of the $(n_0 + p)$th step of the construction of K_2, for some $p \in \mathbb{Z}$, such that, for $r \in \mathbb{N}$,

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

φ maps fundamental intervals and gaps of the (n_0+r)th step of the construction of K_1 contained in I_j in fundamental intervals and gaps of the (n_0+p+r)th step of the construction of K_1. In particular, if $x \in K_1 \cap I_j$, and $I_n(x)$ is the fundamental interval of the nth step containing x, then

$$\varphi'(x) = \lim_{n\to\infty} \frac{|\varphi(I_n(x))|}{|I_n(x)|} = \frac{|J|}{|I|} \cdot \lim_{n\to\infty} \frac{\pi_{n+p}(\tilde{\underline{\lambda}})}{\pi_n(\underline{\lambda})},$$

so $\lim_{n\to\infty} \pi_{n+p}(\tilde{\underline{\lambda}})/\pi_n(\underline{\lambda})$ exists and it is equal to some $a \neq 0$ such that $\varphi'(x) = a(|J|/|I|)$, for all $x \in K_1 \cap I_j$. If $U_n(x)$ is the gap of the nth step of the construction of K_1 contained in $I_{n-1}(x)$ then

$$\begin{aligned}\lim_{n\to\infty} \frac{1-2\tilde{\lambda}_{n+p}}{1-2\lambda_n} &= \lim_{n\to\infty} \frac{|\varphi(U_n(x))|/|\varphi(I_{n-1}(x))|}{|U_n(x)|/|I_{n-1}(x)|} \\ &= \frac{\lim_{n\to\infty} |\varphi(U_n(x))|/|U_n(x)|}{\lim_{n\to\infty} |\varphi(I_{n-1}(x))|/|I_{n-1}(x)|} = \frac{\varphi'(x)}{\varphi'(x)} = 1.\end{aligned}$$

Let $k \in \mathbb{N}^*$ and suppose φ is C^k. If $k=1$ the results are proved. If $k>1$, since $\varphi'|_{K_1\cap I_j}$ is constant, then $\varphi^{(r)}|_{K_1\cap I_j}$ is identically 0, for $r = 2, 3, \ldots, k$. Then

$$\left| \frac{\pi_{n+p}(\tilde{\underline{\lambda}})}{\pi_n(\underline{\lambda})} - a \right| = \frac{|I|}{|J|} \cdot \left| \frac{|\varphi(I_n(x))|}{|I_n(x)|} - \varphi'(x) \right| = \frac{|I|}{|J|} |\varphi'(c(n,x)) - \varphi'(x)|,$$

for some $c(n,x) \in I_n(x)$. Since $\varphi^{(r)}(x) = 0$, $2 \le r \le k$, we have $\lim_{y\to x} |\varphi'(y) - \varphi'(x)|/|y-x|^{k-1} = 0$, and since $|c(n,x) - x| \le |I_n(x)| = |I| \cdot \pi_n(\underline{\lambda})$, we have $\lim_{n\to\infty} |\varphi'(c(n,x)) - \varphi'(x)|/(\pi_n(\underline{\lambda}))^{k-1} = 0$, and then $\lim_{n\to\infty} |\pi_{n+p}(\tilde{\underline{\lambda}}) - a\pi_n(\underline{\lambda})|/(\pi_n(\underline{\lambda}))^k = 0$. Let (U_n) be a sequence of gaps of the nth step of the construction of K_1 contained in I_j, and let x_n be the left end of U_n. Then

$$\left| \frac{g_{n+p}(\tilde{\underline{\lambda}})}{g_n(\underline{\lambda})} - a \right| = \frac{|I|}{|J|} \left| \frac{|\varphi(U_n)|}{|U_n|} - \varphi'(x_n) \right| = \frac{|I|}{|J|} |\varphi'(c_n) - \varphi'(x_n)|,$$

for some $c_n \in U_n$, and our result about the g_n's follows from $\lim_{n\to\infty} |\varphi'(c_n) - \varphi'(x_n)|/|c_n - x_n|^{k-1} = 0$, which follow from $\varphi'(c_n) = \varphi'(x_n) + (\varphi^{(k)}(\xi_n)/(k-1)!)(c_n - x_n)^{k-1}$, for some $\xi_n \in (x_n, c_n) \subset U_n$. We have $\lim_{n\to\infty} \varphi^{(k)}(\xi_n) = 0$, since φ is C^k. The claims in the case when φ is $C^{k+\alpha}$ follows exactly as before from $|\varphi'(y) - \varphi'(x)| \le M|x-y|^{k+\alpha-1}$, for all $y \in I_j$, and all $x \in K_1 \cap I_j$, where M is a constant. This follows from $\varphi'(y) - \varphi'(x) = (\varphi^{(k)}(c)/(k-1)!)(y-x)^{k-1}$, where c is between x and y, and $|\varphi^{(k)}(c)| = |\varphi^{(k)}(c) - \varphi^{(k)}(x)| \le M|c-x|^\alpha \le M|y-x|^\alpha$, for $k \ge 2$, since φ is $C^{k+\alpha}$, and, for $\varphi \in C^{1+\alpha}$, $|\varphi'(y) - \varphi'(x)| \le M|y-x|^\alpha$. □

Remark. The conclusions of Proposition 2 hold for any p, such that there is a fundamental interval I_j of the n_0th step of the construction of K_1 as in Proposition 1, Corollary 2 such that $\varphi(I_j)$ is a fundamental interval of the (n_0+p)th step of the construction of K_2. In this case we have $\varphi'(x) = a(|J|/|I|)$, for all $x \in K_1 \cap I_j$, where $a = \lim_{n\to\infty} \pi_{n+p}(\tilde{\underline{\lambda}})/\pi_n(\underline{\lambda})$.

COROLLARY. *If $K_{\underline{\lambda}}$ admits the structure of a regular Cantor set (at least of class C^1), then there is $r \in \mathbb{N}^*$ such that*

$$\lim_{n\to\infty} \frac{\lambda_{n+r}}{\lambda_n} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \frac{1-2\lambda_{n+r}}{1-2\lambda_n} = 1,$$

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

and the sequence $(\lambda_{n+1}\cdot\lambda_{n+2}\cdots\lambda_{n+r})_{n\in\mathbb{N}}$ *converges to some* $\lambda\neq 0$. *If* $K_{\underline{\lambda}}$ *admits the structure of a regular Cantor set of class* C^k *(respectively* $C^{k+\alpha}$*) then there exists* r *as before such that*

$$\lim_{n\to\infty}\frac{|\lambda_{n+1}\cdot\lambda_{n+2}\cdots\lambda_{n+r}-\lambda|}{\pi_n^{k-1}}=0$$
$$\left(\textit{respectively } \limsup_{n\to\infty}\frac{|\lambda_{n+1}\cdot\lambda_{n+2}\cdots\lambda_{n+r}-\lambda|}{\pi_n^{k+\alpha-1}}<\infty\right).$$

Proof. If $K_{\underline{\lambda}}$ is a regular Cantor set defined by ψ as defined in §1, this is a direct consequence of Proposition 2 (and the above remark) applied to $K_1=K_2=K_{\underline{\lambda}}$, and V being some connected component of the domain of ψ, $U=\psi(V)$ and $\varphi=(\psi|_V)^{-1}$. □

3. *Construction of diffeomorphisms between central Cantor sets*

PROPOSITION 3. *Let* $K_1=K_{\underline{\lambda},I}$ *and* $K_2=K_{\underline{\tilde{\lambda}},J}$ *be two central Cantor sets such that*

$$\lim_{n\to\infty}\frac{1-2\tilde{\lambda}_n}{1-2\lambda_n}=1 \quad \textit{and} \quad \lim_{n\to\infty}\frac{\pi_n(\underline{\tilde{\lambda}})}{\pi_n(\underline{\lambda})}=\alpha\neq 0.$$

Then there is a C^1 *diffeomorphism* $\varphi: I\to J$ *such that* $\varphi(K_{\underline{\lambda},I})=K_{\underline{\tilde{\lambda}},J}$ *and* $\varphi(P^r_{\underline{\lambda},I})=P^r_{\underline{\tilde{\lambda}},J}$, *for all* $r\in\mathbb{N}$, *and thus* $\varphi'(x)=\alpha(|J|/|I|)$, *for all* $x\in K_{\underline{\lambda},I}$.

Let $k\in\mathbb{N}^*$. *If*

$$\lim_{n\to\infty}\frac{|g_n(\underline{\tilde{\lambda}})-\alpha g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^k}=0,$$

then we can obtain φ *of class* C^k.

Let $\beta\in(0,1]$. *If*

$$\limsup_{n\to\infty}\frac{|g_n(\underline{\tilde{\lambda}})-\alpha g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^{k+\beta}}<\infty,$$

then we obtain φ *of class* $C^{k+\beta}$.

Proof. We will define φ in the gaps of $K_{\underline{\lambda},I}$, the definition of φ in $K_{\underline{\lambda},I}$ being performed by continuity. Let $n\in\mathbb{N}$, $I=I_1^{(n)}\cup U_1^{(n)}\cup\cdots\cup U_{2^n-1}^{(n)}\cup I_{2^n}^{(n)}$ and $J=J_1^{(n)}\cup V_1^{(n)}\cup\cdots\cup V_{2^n-1}^{(n)}\cup J_{2^n}^{(n)}$ be the corresponding decompositions of the nth step of the constructions of K_1 and K_2, respectively. We will have $\varphi(I_r^{(n)})=J_r^{(n)}$ *and* $\varphi(U_r^{(n)})=V_r^{(n)}$, for each r,n. In particular, if $x\in K_1$,

$$\varphi'(x)=\lim_{n\to\infty}\frac{|J_r^{(n)}|}{|I_r^{(n)}|}=\lim_{n\to\infty}\frac{|J|\cdot\pi_n(\underline{\tilde{\lambda}})}{|I|\cdot\pi_n(\underline{\lambda})},$$

since $|I_r^{(n)}|=|I|\cdot\pi_n(\underline{\lambda})$ and $|J_r^{(n)}|=|J|\cdot\pi_n(\underline{\tilde{\lambda}})$, for all r, and so $\varphi'(x)=\alpha(|J|/|I|)$, for all $x\in K_{\underline{\lambda},I}$.

Let $\psi:\mathbb{R}\to\mathbb{R}$ be a C^∞ function such that $\psi(x)\geq 0$, for all $x\in\mathbb{R}$, and $\psi(x)=0$, $x\notin[0,1]$, and $\int_0^1\psi(x)\,dx=1$. Say

$$\psi(x)=\frac{\tilde{\psi}(x)}{\int_0^1\tilde{\psi}(t)\,dt},\quad \text{where } \tilde{\psi}(x)=\begin{cases}e^{1/x(x-1)} & x\in[0,1]\\ 0 & x\notin[0,1].\end{cases}$$

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

Let $U = (a, b)$ be a gap of the nth step of $K_{\underline{\lambda},I}$ and let $V = (c, d)$ be the corresponding gap of the nth step of $K_{\tilde{\underline{\lambda}},J}$. We have $|U| = |I| \cdot g_n(\underline{\lambda})$ and $|V| = |J| \cdot g_n(\tilde{\underline{\lambda}})$. We define, for n large enough, $\varphi|_U$ as follows: $\varphi(a) = c$ and for $x \in [a, b]$,

$$\varphi'(x) = \frac{|J|}{|I|}\left(\alpha + \left(\frac{g_n(\tilde{\underline{\lambda}})}{g_n(\underline{\lambda})} - \alpha\right)\psi\left(\frac{x-a}{b-a}\right)\right).$$

Since $\lim_{n\to\infty} g_n(\tilde{\underline{\lambda}})/g_n(\underline{\lambda}) = \alpha$, this expression for $\varphi'(x)$ is strictly positive for $n \geq n_0$, where n_0 is a sufficiently large positive integer. For the gaps (a, b) of the steps before the n_0th, we define φ in $[a, b]$ as follows: where (c, d) is the gap of K_2 corresponding to (a, b), set $\varphi(a) = c$, and for $x \in [a, b]$, set

$$\varphi'(x) = \alpha\frac{|J|}{|I|} \cdot \exp\left(\rho\psi\left(\frac{x-a}{b-a}\right)\right),$$

where $\rho \in \mathbb{R}$ is chosen so that $\int_a^b \varphi'(x)\,dx = d - c$.

The function φ just constructed satisfies $\varphi(K_{\underline{\lambda},I}) = K_{\tilde{\underline{\lambda}},J}$, and is a C^1 diffeomorphism since $\varphi'|_{I-K_{\underline{\lambda},I}}$ is strictly positive and tends to $\alpha(|J|/|I|)$ in the gaps of the nth step uniformly when $n \to \infty$ and $\lim_{n\to\infty} |\varphi(I_r^{(n)})|/|I_r^{(n)}| = \alpha(|J|/|I|)$ uniformly, and so $\varphi'|_{K_{\underline{\lambda},I}} \equiv \alpha(|J|/|I|)$, and φ' is continuous in I. Suppose we have $\lim_{n\to\infty} |g_n(\tilde{\underline{\lambda}}) - \alpha g_n(\underline{\lambda})|/(g_n(\underline{\lambda}))^k = 0$. Then φ is of class C^k, the derivative of order k of φ in a gap (a, b) of the nth step are given, for $k \geq 2$, by

$$\varphi^{(k)}(x) = \frac{|J|}{|I|^k}\frac{(g_n(\tilde{\underline{\lambda}}) - \alpha g_n(\underline{\lambda}))}{(g_n(\underline{\lambda}))^k} \cdot \psi^{(k)}\left(\frac{x-a}{b-a}\right),$$

which tends to 0 uniformly when $n \to \infty$, and $\varphi^{(k)}|_{K_{\underline{\lambda},I}} \equiv 0$. So $\varphi^{(k)}$ is a continuous function in I.

Now suppose $\beta \in (0, 1]$ and $\limsup_{n\to\infty} |g_n(\tilde{\underline{\lambda}}) - \alpha g_n(\underline{\lambda})|/(g_n(\underline{\lambda}))^{k+\beta} < \infty$. Let $x, y \in I$. Since $\varphi^{(k)}$ is constant in $K_{\underline{\lambda},I}$, for all $k \in \mathbb{N}^*$, we have $|\varphi^{(k)}(x) - \varphi^{(k)}(y)| \leq 2\max(|\varphi^{(k)}(x) - \varphi^{(k)}(\tilde{a})|,\ |\varphi^{(k)}(y) - \varphi^{(k)}(\tilde{b})|)$, where $\tilde{a} = x$ (respectively $\tilde{b} = y$) if $x \in K_{\underline{\lambda},I}$ (respectively $y \in K_{\tilde{\underline{\lambda}},J}$) and $\tilde{a}$ (respectively $\tilde{b}$) is an end of the gap containing x (respectively y). Thus we can suppose that x and y are in the closure of the same gap (a, b), and so

$$|\varphi^{(k)}(x) - \varphi^{(k)}(y)| = \frac{|J|}{|I|^k}\frac{|g_n(\tilde{\underline{\lambda}}) - \alpha g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^k} \cdot |\psi^{k+1}(\xi)| \cdot \frac{|x-y|}{b-a},$$

for some $\xi \in [0, 1]$, where (a, b) is a gap of the nth step of $K_{\underline{\lambda},I}$, and so

$$\frac{|\varphi^{(k)}(x) - \varphi^{(k)}(y)|}{|x-y|^\beta} = \frac{|J|}{|I|^{k+\beta}} \cdot \frac{|g_n(\tilde{\underline{\lambda}}) - \alpha g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^{k+\beta}} \cdot |\varphi^{(k+1)}(\xi)|\left(\frac{|x-y|}{b-a}\right)^{1-\beta},$$

which is bounded (since $\varphi^{(k+1)}$ is bounded and $|x - y| \leq b - a$) by

$$\frac{|J|}{|I|^{k+\beta}} \cdot \sup_{n\in\mathbb{N}} \frac{|g_n(\tilde{\underline{\lambda}}) - \alpha g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^{k+\beta}} \cdot \sup_{x\in[0,1]} |\psi^{(k+1)}(x)|.$$

So φ is of class $C^{k+\beta}$. □

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

COROLLARY. *Let $K_1 = K_{\underline{\lambda},I}$ and $K_2 = K_{\underline{\tilde{\lambda}},J}$ be two central Cantor sets, and let $p \in \mathbb{Z}$ be such that*

$$\lim_{n\to\infty} \frac{1-2\tilde{\lambda}_{n+p}}{1-2\lambda_n} = 1 \quad \textit{and} \quad \lim_{n\to\infty} \frac{\pi_{n+p}(\underline{\tilde{\lambda}})}{\pi_n(\underline{\lambda})} = \alpha \neq 0.$$

Then, if $m \in \mathbb{N}^*$, $m+p \in \mathbb{N}^*$, $\tilde{I}$ is a fundamental interval of the mth step of K_1 and $\tilde{J}$ is a fundamental interval of the $(m+p)$th step of K_2, then there exists a C^1 diffeomorphism $\varphi : \tilde{I} \to \tilde{J}$ such that $\varphi(K_1 \cap \tilde{I}) = K_2 \cap \tilde{J}$, $\varphi(P^r_{\underline{\lambda},I} \cap \tilde{I}) = P^{r+p}_{\underline{\tilde{\lambda}},J} \cap \tilde{J}$, for $r \geq m$, and thus $\varphi'(x) = \alpha(|J|/|I|)$, for all $x \in K_1 \cap \tilde{I}$. Let $k \in \mathbb{N}$ and $\beta \in (0,1]$. Then, if

$$\lim_{n\to\infty} \frac{|g_{n+p}(\underline{\tilde{\lambda}}) - \alpha g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^k} = 0 \quad \left(\textit{respectively } \limsup_{n\to\infty} \frac{|g_{n+p}(\underline{\tilde{\lambda}}) - \alpha g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^{k+\beta}} < \infty\right)$$

then we can obtain φ of class C^k (respectively of class $C^{k+\beta}$).

Remark. If $\alpha = 0$ the result is still valid, except that φ may not be a diffeomorphism, but is a homeomorphism with $\varphi'(x) = 0 \Leftrightarrow x \in K_1$. This is used in [**M2**] to give examples related to critical situations of the Morse–Sard theorem.

4. *Proof of theorems*

Proof of Theorem A. Let us first prove the stated equivalences. If $k = 1$ then conditions (i) and (ii) are the same. Let $P_n = \lambda_{n+1} \cdot \lambda_{n+2} \cdots \lambda_{n+r_0}$. If $q > 1$, ($q = k$ or $q = k + \alpha$) then each of the conditions (i), (ii), (i′), (ii′) implies that P_n converges exponentially to λ, i.e. there are constants $M > 0$ and $C \in (0,1)$ such that $|P_n - \lambda| \leq M \cdot C^n$, for all $n \in \mathbb{N}$. So $\Pi_{n=1}^{\infty}(P_{r_0 n}/\lambda)$ converges exponentially, so $\pi_n^{q-1} = O(\lambda^{(q-1)n/r_0})$ and $\lambda^{(q-1)n/r_0} = O(\pi_n^{q-1})$. (We are using the notation $a_n = O(b_n) \Leftrightarrow \limsup_{n\to\infty} a_n/b_n < \infty$, and $a_n = o(b_n) \Leftrightarrow \lim_{n\to\infty} a_n/b_n = 0$.)

So (i) $\Leftrightarrow$ (ii) and (i′) $\Leftrightarrow$ (ii′). If $k > 1$ (respectively $k + \alpha > 1$),

$$\frac{\lambda_{r_0(n+1)+j}}{\lambda_{r_0 n+j}} = \frac{P_{r_0 n+j}}{P_{r_0 n+j-1}} = \frac{\lambda + o(\lambda^{(k-1)n})}{\lambda + o(\lambda^{(k-1)n})},$$

$$\left(\text{respectively } \frac{\lambda + O(\lambda^{(k+\alpha-1)n})}{\lambda + O(\lambda^{(k+\alpha-1)n})} = 1 + o(\lambda^{(k-1)n})\right)$$

$$(\text{respectively } 1 + O(\lambda^{(k+\alpha-1)n})).$$

So $\lambda_{r_0(n+1)+j} = \lambda_{r_0 n+j} + o(\lambda^{(k-1)n})$ (respectively $\lambda_{r_0 n+j} + O(\lambda^{(k+\alpha-1)n})$), and thus $(\lambda_{r_0 n+j})$ converges exponentially to some μ_j, and

$$|\lambda_{r_0 n+j} - \mu_j| = \sum_{m=n}^{\infty} |\lambda_{r_0 m+j} - \lambda_{r_0(m+1)+j}| = \sum_{m=n}^{\infty} o(\lambda^{(k-1)m})$$

$$\left(\text{respectively } \sum_{m=n}^{\infty} O(\lambda^{(k+\alpha-1)m}) = o(\lambda^{(k-1)n})\right)$$

$$(\text{respectively } O(\lambda^{(k+\alpha-1)n})).$$

If some μ_j is equal to $1/2$ then $(\lambda_{r_0 n+j})$ converges exponentially to $1/2$, contradicting $\lim_{n\to\infty}(1-2\lambda_{n+r_0})/(1-2\lambda_n) = 1$. Moreover, $\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_{r_0} = \lim_{n\to\infty} P_{r_0 n} = \lambda \neq 0$,

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

so $\mu_j \neq 0$, for all j. Thus we have proved $\mu_j \in (0, 1/2)$, for all j. Reciprocally, if $\lambda_{r_0 n+j} = \mu_j + o(\lambda^{(k-1)n})$ (respectively $\lambda_{r_0 n+j} = \mu_j + O(\lambda^{(k+\alpha-1)n})$) then $P_n = \lambda + o(\lambda^{(k-1)n/r_0})$ (respectively $P_n = \lambda + \lambda^{(k+\alpha-1)n/r_0}$). Then (i′) $\Leftrightarrow$ (ii′) $\Leftrightarrow$ (iii′) and, if $k > 1$, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). The corollary to Proposition 2 shows that if K is C^k (respectively K is $C^{k+\alpha}$) then conditions (a)–(c) and condition (i) (respectively (i′)) hold for some r instead of r_0. We will show that this holds also for $r = r_0$. In the case C^1 there is nothing to prove. If $k > 1$ (respectively $k+\alpha > 1$), $\lim_{n\to\infty} \lambda_{n+r_0}/\lambda_n = 1$ implies $\mu_i = \mu_j$ if $i \equiv j \pmod{r_0}$, so $\lim_{n\to\infty} |\lambda_{rn+j} - \mu_j|/\lambda^{(k-1)n} = 0$, $1 \leq j \leq r$, if and only if $\lim_{n\to\infty} |\lambda_n - \mu_{\overline{n}_{(r)}}|/\pi_n^{k-1} = 0$ (respectively $\limsup_{n\to\infty} |\lambda_{rn+j} - \mu_j|/\lambda^{(k-1)n} < \infty$, $1 \leq j \leq r$, if and only if $\limsup_{n\to\infty} |\lambda_n - \mu_{\overline{n}_{(r)}}|/\pi_n^{k+\alpha-1} < \infty$), where $\overline{n}_{(p)} \in \{1, 2, \ldots, p\}$ satisfies $n \equiv \overline{n}_{(p)} \pmod{p}$. However, since $\mu_{\overline{n}_{(r)}} = \mu_{\overline{n}_{(r_0)}}$ (because $r_0 = \min\{p \in G/p > 0\}$, where $G = \{p \in \mathbb{Z}/ \lim_{n\to\infty} \lambda_{n+p}/\lambda_n = \lim_{n\to\infty}(1 - 2\lambda_{n+p})/(1 - 2\lambda_n) = 1\}$ is a subgroup of $\mathbb{Z}$, so $G = r_0\mathbb{Z}$ and $r_0|r$, thus $\overline{n}_{(r)} \equiv \overline{n}_{(r_0)} \pmod{r_0}$), we see that the validity of (i) (respectively of (i′)) for r and for r_0 are equivalent. This shows the necessity of the conditions.

To prove the sufficiency of the conditions, note that

$$\lim_{n\to\infty} \frac{|g_{n+r_0}(\underline{\lambda}) - \lambda g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^k} = 0 \quad \left(\text{respectively} \limsup_{n\to\infty} \frac{|g_{n+n_0}(\underline{\lambda}) - \lambda g_n(\underline{\lambda})|}{(g_n(\underline{\lambda}))^{k+\alpha}} < \infty\right),$$

since

$$\begin{aligned} \frac{g_{n+r_0}(\underline{\lambda})}{g_n(\underline{\lambda})} &= \frac{1 - 2\lambda_{n+r_0}}{1 - 2\lambda_n} \cdot \frac{\pi_{n+r_0-1}(\underline{\lambda})}{\pi_{n-1}(\underline{\lambda})} = (1 + o(\pi_n^{k-1}))(1 + o(\pi_n^{k-1})) \\ &= \lambda + O(\pi_n^{k-1}) \end{aligned}$$

$$(\text{respectively } (1 + O(\pi_n^{k+\alpha-1}))(\lambda + O(\pi_n^{k+\alpha-1})) = \lambda + \mathcal{O}(\pi_n^{k+\alpha-1})),$$

and $\pi_n(\underline{\lambda}) = \mathcal{O}(g_n(\underline{\lambda}))$, provided $k > 1$ (respectively $k + \alpha > 1$), since $\limsup_{n\to\infty} \lambda_n < 1/2$. The case of class C^1 is trivial. Now we can use the corollary to Proposition 3 in the following way. If $I = I_1 \cup U_1 \cup I_2 \cup \cdots \cup U_{2^{r_0}-1} \cup I_{2^{r_0}}$ is the decomposition of the r_0th step of the construction of K then there are diffeomorphisms $\varphi_j : I \to I_j$ of class C^k (respectively $C^{k+\alpha}$) such that $\varphi_j(K) = K \cap I_j$ and φ_j maps decompositions of the construction of K into corresponding decompositions of the construction of $K \cap I_j$ (so $\varphi_j'|_K \equiv \lim_{n\to\infty} \lambda_{n+1} \cdot \lambda_{n+2} \cdots \lambda_{n+r_0} = \lambda$). Let $\psi : P_{\underline{\lambda},I}^{r_0} \to I$ be defined by $\psi|_{I_j} = \varphi_j^{-1}$. Then $\psi'|_K \equiv \lambda^{-1} > 1$. Then there is $k_0 \in \mathbb{N}$ and $\sigma > 1$ such that $\psi'(x) > \sigma$, for all $x \in P_{\underline{\lambda},I}^{k_0 r_0}$. By the construction we have $\psi(P_{\underline{\lambda},I}^{(k+1)r_0}) = P_{\underline{\lambda},I}^{kr_0}$. So, if $\delta = \inf\{(\psi^{k_0})'(x) \mid x \in P_{\underline{\lambda},I}^{k_0 r_0}\}$ and $N \in \mathbb{N}$ satisfy $\delta \cdot \sigma^N > 1$, then $\psi^{N+k_0} : P_{\underline{\lambda},I}^{(N+k_0)r_0} \to I$ is an expansive function of class C^k (respectively $C^{k+\alpha}$) that makes K a regular Cantor set of class C^k (respectively $C^{k+\alpha}$), because $(\psi^{N+k_0})'(x) = (\psi^{k_0})'(\psi^N(x)) \cdot \Pi_{k=0}^{N-1} \psi'(\psi^k(x)) \geq \delta \cdot \sigma^N$, since $\psi^k(x) \in P_{\underline{\lambda},I}^{k_0 r_0}$, for all $i \leq N$. □

Example 4.1. The Lebesgue measure of a central Cantor set $K_{\underline{\lambda}}$ is $\lim_{n\to\infty} 2^n \pi_n(\underline{\lambda})$. Given $a \in (0, 1)$ there is a C^1 Cantor set $K_{\underline{\lambda}}$ with Lebesgue measure equal to a. For this we take

$$\lambda_n = \frac{1}{2} \cdot \frac{1 - ((1-a)/n)}{1 - ((1-a)/(n+1))}.$$

If K is a $C^{1+\varepsilon}$ Cantor set then its Lebesgue measure is 0.

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

Example 4.2. If $\lambda_n < 1/3$ for all $n \in \mathbb{N}$, then $K_{\underline{\lambda}} - K_{\underline{\lambda}} = \{x - y / \ x, y \in K_{\underline{\lambda}}\}$ is a Cantor set with measure $2 \cdot \Pi_{n=1}^{\infty}(3\lambda_n)$ (see [**Sa**] or [**BPV**]). Given $q \geq 1$ ($q = k$ or $q = k + \alpha$) and $a \in (0, 2)$ we can give an example of a central Cantor set $K_{\underline{\lambda}}$ of class C^q but not of class $C^{q+\varepsilon}$, for all $\varepsilon > 0$, such that $m(K_{\underline{\lambda}} - K_{\underline{\lambda}}) = a$, where m denotes the Lebesgue measure in $\mathbb{R}$. For this we take

$$\lambda_n = \frac{1}{3} \cdot \frac{1 - (3^q(2-a)/(2 \cdot 3^{\sqrt{n}+(q-1)n}))}{1 - (3^q(2-a)/(2 \cdot 3^{\sqrt{n+1}+(q-1)(n+1)}))}$$

if $q < \infty$, and

$$\lambda_n = \frac{1}{3} \cdot \frac{1 - (3(2-a)/(2 \cdot 3^{n^2}))}{1 - (3(2-a)/(2 \cdot 3^{(n+1)^2}))},$$

if $q = \infty$.

We remark that if $\lambda_n < 1/3$, for all n, $\lim_{n\to\infty} \lambda_n = 1/3$ and $K_{\underline{\lambda}}$ is C^q, $q > 1$, then $K_{\underline{\lambda}} - K_{\underline{\lambda}}$ is a Cantor set with positive Lebesgue measure, because $\underline{\lambda} = (\lambda_n)_{n\in\mathbb{N}}$ converges exponentially to $1/3$ and so $\Pi_{n=1}^{\infty}(3\lambda_n) > 0$.

Example 4.3. Let $0 \leq a < b \leq 1/2$. The K_λ is of class C^1, where

$$\lambda_{2n-1} = \frac{a+b}{2} + \left(\frac{b-a}{2}\right)\left(1 - \frac{1}{\sqrt{n}}\right)\sin(\log n), \quad \lambda_{2n} = \frac{ab}{\lambda_{2n-1}}$$

is dense in $[a, b]$.

Proof of Theorem B. The necessity and sufficiency of the conditions for local diffeomorphisms follow from Proposition 2 and the corollary to Proposition 3, respectively. The sufficiency of the conditions for diffeomorphisms (i) and (ii) follow from Proposition 3. We need only prove that these conditions are also necessary. By Proposition 1, Corollary 2, we can decompose $I = I_1 \cup U_1 \cup I_2 \cup \cdots \cup U_{2^{n_0}-1} \cup I_{2^{n_0}}$, where I_j are the fundamental intervals of the n_0th step of the construction of $K_{\underline{\lambda}}, I$, such that $\varphi(I_j)$ is a fundamental interval of the construction of $K_{\tilde{\underline{\lambda}},J}$, say of the $n(j)$th step, for $j = 1, 2, \ldots, 2^{n_0}$. Moreover, φ maps decompositions of I_j into fundamental intervals and gaps in the corresponding decompositions of $\varphi(I_j)$. A diffeomorphism between two central Cantor sets with this last property is called a *good diffeomorphism*. Let $A = \{p \in \mathbb{Z} /$ for sufficiently large $n \in \mathbb{N}$, given fundamental intervals $\tilde{I}$ of the nth step of $K_{\tilde{\underline{\lambda}},I}$ and $\tilde{J}$ of the $(n+p)$th step of $K_{\tilde{\underline{\lambda}},J}$ there is a good diffeomorphism $\psi : \tilde{I} \to \tilde{J}$ at least as differentiable as φ such that $\psi(\tilde{I} \cap K_{\tilde{\underline{\lambda}},I}) = \tilde{J} \cap K_{\tilde{\underline{\lambda}},J}\}$, and $B = \{p \in \mathbb{Z} /$ for sufficiently large $n \in \mathbb{N}$, given fundamental intervals $\tilde{I}$ of the nth step of $K_{\tilde{\underline{\lambda}},J}$ and $\tilde{J}$ of the $(n+p)$th step of $K_{\tilde{\underline{\lambda}},J}$ there is a good diffeomorphism $\psi : \tilde{I} \to \tilde{J}$ at least as differentiable as φ such that $\psi(\tilde{I} \cap K_{\tilde{\underline{\lambda}},J}) = \tilde{J} \cap K_{\tilde{\underline{\lambda}},J}\}$. Then:

- $n(j) - n_0 \in A$, $j = 1, 2, \ldots, 2^{n_0}$ (take restrictions of $\varphi|_{I_j}$);
- $p, q \in A \Rightarrow p - q \in B$, and $p, q \in B \Rightarrow p - q \in B$ (being diffeomorphic by a good diffeomorphism is an equivalence relation). In particular, $n(i) - n(j) \in B$, for all i, j.

This implies that $B = d \cdot \mathbb{Z}$ for some $d \in \mathbb{Z}$, and $A = p + d\mathbb{Z}$ for all $p \in A$. If $0 \notin A$ then $d > 1$ and $X \equiv p \pmod{d}$ for all $X \in A$, where $d \nmid p$, so we can take $p \in \{1, 2, \ldots, d-1\}$ such that $n(j) \equiv n_0 + p \pmod{d}$, for all j. Let $N = \max(n(j)+p)$.

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

Then J contains 2^N fundamental intervals of the Nth step of $K_{\tilde{\underline{\lambda}},J}$ and $\varphi(I_j)$ contains $2^{N-n(j)}$ of these intervals, so $\sum_{j=1}^{2^{n_0}} 2^{N-n(j)} = 2^N$ but $n(j) \equiv n_0 + p \pmod d$, so $2^{N-n(j)} \equiv 2^{N-n_0-p} \pmod{2^d-1}$, and thus $2^N = \sum_{j=1}^{2^{n_0}} 2^{N-n(j)} \equiv 2^{n_0} 2^{N-n_0-p} = 2^{N-p} \pmod{2^d-1}$, so $(2^d-1) | 2^{N-p}(2^p-1)$, a contradiction. So $0 \in A$, and the result follows from Proposition 2. □

Example 4.4. The central Cantor sets defined by the sequences $(\frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \ldots)$ and $(\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{3}, \ldots)$ are not diffeomorphic but are locally diffeomorphic.

5. *Calculation of numerical invariants of central Cantor sets*

5.1. *Thickness.* Given a Cantor set K and a gap U of K, let I_U be the interval bounded by U and the closest gap V that is larger than U. We define $\tau(U) = |I_U|/|U|$ and $\tau(K) = \inf\{\tau(U)/U \text{ is a bounded gap of } K\}$, the *thickness* of K.

If $K = K_{\underline{\lambda}}$ is a central Cantor set, then the interval I_U in the above definition must be a fundamental interval of either U or V so, if I_U is a fundamental interval of the nth step, we have

$$\tau(U) = \frac{|I_U|}{|U|} = \frac{|I_U|}{\min(|U|, |V|)} \geq \frac{\pi_n(\underline{\lambda})}{g_n(\underline{\lambda})} = \frac{\lambda_n}{1-2\lambda_n}.$$

On the other hand, if U is the first gap of the nth step,

$$\tau(U) = \frac{\pi_n(\underline{\lambda})}{g_n(\underline{\lambda})} = \frac{\lambda_n}{1-2\lambda_n}.$$

Therefore, $\tau(K) = \inf_{n\in\mathbb{N}} \lambda_n/(1-2\lambda_n)$.

Given $p \in K$ we define $\tau_{\text{loc}}(k, p) = \limsup_{\varepsilon_1,\varepsilon_2\to 0} \tau(\overline{K \cap (p-\varepsilon_2, p+\varepsilon_2)})$. By the above arguments we can conclude that $\tau_{\text{loc}}(K_{\underline{\lambda}}, p) = \liminf_{n\to\infty} \lambda_n(1-2\lambda_n)$, for all $p \in K_{\underline{\lambda}}$.

5.2. *Limit capacity.* Let K be a Cantor set and $\varepsilon > 0$. We define $N(K, \varepsilon)$ as the minimum number of intervals of length ε needed to cover K, and the *limit capacity* $d(K)$ of K is defined by

$$d(K) = \limsup_{\varepsilon\to 0} -\frac{\log N(K, \varepsilon)}{\log \varepsilon}.$$

If $K = K_{\underline{\lambda}}$ is a central Cantor set then we can cover $K_{\underline{\lambda}}$ by 2^{n+1} intervals of length ε for each $\varepsilon > 0$ satisfying $\pi_{n+1}(K_{\underline{\lambda}}) \leq \varepsilon < \pi_n(\underline{\lambda})$, and so

$$d(K_{\underline{\lambda}}) \leq \limsup_{n\to\infty} -(n+1)\frac{\log 2}{\log \pi_n(\underline{\lambda})} = \limsup_{n\to\infty} -n\frac{\log 2}{\log \pi_n(\underline{\lambda})}.$$

On the other hand, if $a_1 < a_2 < \cdots < a_{2^n}$ are the left ends of the fundamental intervals of the nth step of $K_{\underline{\lambda}}$, we have $a_{i+1} - a_i > \pi_n(\underline{\lambda})$, for each i. So, if $\pi_{n+1}(\underline{\lambda}) \leq \varepsilon < \pi_n(\underline{\lambda})$, then intervals of length ε contain at most one of the a_i's and so we must have at least 2^n intervals of length ε to cover $K_{\underline{\lambda}}$, since $a_i \in K_{\underline{\lambda}}$, for all i. Thus $N(K_{\underline{\lambda}}, \varepsilon) \geq 2^n$ and

$$-\frac{\log N(K_{\underline{\lambda}}, \varepsilon)}{\log \varepsilon} \geq -\frac{n \log 2}{\log \pi_n(\underline{\lambda})}$$

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

and so

$$d(K_{\underline{\lambda}}) = \limsup_{n\to\infty} -\frac{n\log 2}{\log \pi_n(\underline{\lambda})}.$$

5.3. *Hausdorff dimension.* Given a Cantor set $K \subset \mathbb{R}$ and a finite cover of K by open intervals $\mathcal{U} = \{U_i\}_{i\in I}$, we define the diameter of $\mathcal{U}$ by $\operatorname{diam}\mathcal{U} = \max\{|U_i|, i \in I\}$. Given $\alpha \in \mathbb{R}$, we define $H_\alpha(\mathcal{U}) = \sum_{i\in I} |U_i|^\alpha$. The Hausdorff α-measure of K is $m_\alpha(K) = \lim_{\varepsilon\to 0}(\inf\{H_\alpha(\mathcal{U})/\mathcal{U}$ is a covering of K with $\operatorname{diam}(\mathcal{U}) < \varepsilon\})$. The Hausdorff dimension of K is defined as being the unique real number $HD(K)$ such that

$$\begin{cases} m_\alpha(K) = +\infty & \alpha < HD(K) \\ m_\alpha(K) = 0 & \alpha > HD(K). \end{cases}$$

If $K = K_{\underline{\lambda}}$, K can be covered by 2^n intervals of length $\pi_n(\underline{\lambda})$, so, if $\alpha > \liminf_{n\to\infty}(-n\log 2/\log \pi_n(\underline{\lambda}))$, there are infinitely many $n \in \mathbb{N}$ such that $2^n(\pi_n(\underline{\lambda}))^\alpha < 1$, and thus $HD(K_{\underline{\lambda}}) \le \alpha$.

Let $\mathcal{U}$ be a covering of $K_{\underline{\lambda}}$ by m intervals of small length and let $\alpha \in \mathbb{R}$. Let $n \in \mathbb{N}$ be such that $m < 2^{n-1}$. Then the set C of fundamental intervals of the nth step which are strictly contained in some interval of $\mathcal{U}$ has at least 2^{n-1} elements. Now, if $U_i \in \mathcal{U}$, let $n(U_i) = \#\{I \subset C/I \subset U_i\}$, and choose a U_{i_0} that minimizes $|U_i|^\alpha/n(U_i)$. Since $\sum n(U_i) \ge 2^{n-1}$ and $\sum |U_i|^\alpha = H_\alpha(\mathcal{U})$, we have $|U_{i_0}|/n(U_{i_0}) \le H_\alpha(\mathcal{U})/2^{n-1}$. If $2^k \le n(U_{i_0}) < 2^{k+1}$, then U_{i_0} contains 2^k consecutive fundamental intervals of the nth step, and so it contains a whole fundamental interval of the $(n-k+1)$th step. Thus $|U_{i_0}| \ge \pi_{n-k+1}(\underline{\lambda})$ and

$$\frac{H_\alpha(\mathcal{U})}{2^{n-1}} \ge \frac{|U_{i_0}|^\alpha}{n(U_{i_0})} > \frac{(\pi_{n-k+1}(\underline{\lambda}))^\alpha}{2^{n+1}},$$

thus $H_\alpha(\mathcal{U}) > 2^{n-k-2}(\pi_{n-k+1}(\underline{\lambda}))^\alpha$. This proves that for each covering $\mathcal{U}$ of $K_{\underline{\lambda}}$ there is $N \in \mathbb{N}$ such that

$$\pi_N(\underline{\lambda}) \le \operatorname{diam}(\mathcal{U}) \quad \text{and} \quad 2^N \cdot (\pi_N(\underline{\lambda}))^\alpha < 8 \cdot H_\alpha(\mathcal{U}). \tag{$*$}$$

In particular, if $\alpha < \liminf_{n\to\infty}(-n\log 2/\log \pi_n(\underline{\lambda}))$, we have $\lim_{n\to\infty} 2^n(\pi_n(\underline{\lambda}))^\alpha = +\infty$ implies $H_\alpha(\mathcal{U}) = +\infty$, by $(*)$. So we have $HD(K_{\underline{\lambda}}) \ge \alpha$, which proves that $HD(K_{\underline{\lambda}}) = \liminf_{n\to\infty}(-n\log 2/\log \pi_n(\underline{\lambda}))$.

Using the same arguments we can conclude that for each $\alpha \in \mathbb{R}$, $\alpha > 0$,

$$\tfrac{1}{8}\liminf_{n\to\infty}\ 2^n \cdot (\pi_n(\underline{\lambda}))^\alpha \le m_\alpha(K_{\underline{\lambda}}) \le \liminf_{n\to\infty}\ 2^n \cdot (\pi_n(\underline{\lambda}))^\alpha.$$

In summary, the above results show that for each interval I and sequence $\underline{\lambda} = (\lambda_n)_{n\in\mathbb{N}}$:

- $\tau(K_{\underline{\lambda},I}) = \tau(K_{\underline{\lambda}}) = \inf_{n\in\mathbb{N}} \dfrac{\lambda_n}{1-2\lambda_n}$;
- $\tau_{\mathrm{loc}}(K_{\underline{\lambda},I}) = \tau_{\mathrm{loc}}(K_{\underline{\lambda}}) = \liminf_{n\to\infty} \dfrac{\lambda_n}{1-2\lambda_n}$;
- $d(K_{\underline{\lambda},I}) = d(K_{\underline{\lambda}}) = \limsup_{n\to\infty} \dfrac{-n\log 2}{\log \pi_n(\underline{\lambda})}$;
- $HD(K_{\underline{\lambda},I}) = HD(K_{\underline{\lambda}}) = \liminf_{n\to\infty} \dfrac{-n\log 2}{\log \pi_n(\underline{\lambda})}$.

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press

Moreover, for each $\alpha > 0$, $m_\alpha(K_{\underline{\lambda},I}) = |I|^\alpha \cdot m_\alpha(K_{\underline{\lambda}})$ and $m_\alpha(K_{\underline{\lambda}})$ satisfies

$$\tfrac{1}{8} \liminf_{n\to\infty} 2^n \cdot (\pi_n(\underline{\lambda}))^\alpha \leq m_\alpha(K_{\underline{\lambda}}) \leq \liminf_{n\to\infty} 2^n \cdot (\pi_n(\underline{\lambda}))^\alpha.$$

Acknowledgements. This paper is a result of research initiated during the visit of C. G. Moreira to Universidad de Santiago de Chile (June, 1994). (The author is grateful to this institution for financial support (grants DICYT # 9433P.S. and # 9433L.B.).) and completed during the visit of S. Plaza to IMPA and C. G. Moreira to Université de Paris-Sud, France. The authors are grateful to IMPA/CNPq for the financial support and stimulating working conditions while we prepared the final version of this work. Finally, we thank Jacob Palis for valuable suggestions for the final version and also thank Juan Rivera for his valuable suggestion during the preparation of the final version.

REFERENCES

[BPV] R. Bamón, S. Plaza and J. Vera. On central Cantor sets with self-arithmetic difference of positive Lebesgue measure. *J. London Math. Soc.* **52** (1995), 137–146.

[BF] T. Bedford and A. Fisher. Ratio geometry, rigidity and the scenery process for hyperbolic Cantor sets. *Ergod. Th. & Dynam. Sys.* **17** (1997), 531–564.

[Bo] R. Bowen. A horseshoe with positive measure. *Inv. Math.* **29** (1975), 203–204.

[M1] C. G. Moreira. Stable intersections of Cantor sets and homoclinic bifurcations. *Analyse Institute Henri Poincare* **XIII** No 6 (1996), 741–781.

[M2] C. G. Moreira. Hausdorff measures and Morse–Sard theorem. Submitted to *Proc. Amer. Math. Soc.*, 1994.

[N] S. Newhouse. The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms. *Publ. Math. I.H.E.S.* **50** (1979), 101–151.

[P1] J. Palis. Homoclinic bifurcations, sensitive-chaotic dynamics and strange attractors. *Dynamical Systems and Related Topics*. World Scientific, 1991, pp. 446–472.

[P2] J. Palis. Homoclinic orbits, hyperbolic dynamics and dimension of Cantor set. *Contemporary Math. Part III* (1987), 204–216.

[PR] A. Pinto and D. Rand. A classification of C^{1+} structures on embeddeed trees. *Ergod. Th. & Dynam. Sys.* **15** (1995), 969–992.

[PT] J. Palis and F. Takens. *Hyperbolicity and Chaotic Dynamics at Homoclinic Bifurcations: Fractal Dimensions and Infinitely Many Attractors*. Cambridge University Press, 1993.

[PT1] J. Palis and F. Takens. Hyperbolicity and the creation of homoclinic orbits. *Annals of Math.* **125** (1987), 337–374.

[PT2] J. Palis and F. Takens. Cycles and measure of bifurcation sets for two-dimensional diffeomorphisms. *Inv. Math.* **82** (1985), 379–442.

[PrT] F. Przytycki and F. Tangerman. Cantor sets in the line: scaling functions and the smoothness of the shift-map. *Nonlinearity* **9** (1996).

[S] D. Sullivan. Differentiable structure on fractal-like sets determined by intrinsic scaling functions on dual Cantor sets. *Proc. Symp. Pure Math.* **48** (1987), 15–23.

[Sa] A. Sannami. An example of regular Cantor set whose difference is a Cantor set with positive Lebesgue measure. *Hokkaido Math. J.* **XXI** (1992), 7–24.

https://doi.org/10.1017/S014338579708629X Published online by Cambridge University Press