Proceedings of the Royal Society of Edinburgh, **155**, 1646–1660, *2025*
DOI:10.1017/prm.2024.7

Common valuations of division polynomials

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(Received 3 April 2023; accepted 18 January 2024)

In this note, we prove a formula for the cancellation exponent $k_{v,n}$ between division polynomials ψ_n and ϕ_n associated with a sequence $\{nP\}_{n\in\mathbb{N}}$ of points on an elliptic curve E defined over a discrete valuation field K. The formula greatly generalizes the previously known special cases and treats also the case of non-standard Kodaira types for non-perfect residue fields.

Keywords: Elliptic curves; Néron models; division polynomials; height functions; discrete valuation rings

2020 *Mathematics Subject Classification:* 14H52; 11G05; 11G07; 11G50

1. Introduction

Let P be a point on an elliptic curve E defined over a discrete valuation field K. In this paper, we study some properties of the sequence $\{nP\}_{n\in\mathbb{N}}$ of the multiples of the point P. It is known that this sequence has some remarkable properties, in particular when a Weierstrass model is fixed

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

the elements $nP = (\phi_n(P)/\psi_n^2(P), \omega_n(P)/\psi_n^3(P))$ for $P \in E(K)$ satisfy certain recurrent formulas (see properties (A), (B), (C), p. 4 or [**20**, Exercise 3.7]).

When the curve E is defined over a number field the sequence of polynomials $(\psi_n(x,y))_{n\in\mathbb{N}}$ has been studied as an example of a divisibility sequence, a notion which was studied classically for Lucas sequences and by Ward for higher degree recurrences. Many results are known, including [**27**] and [**18**].

Over function fields, in particular over the function field of a smooth projective curve C over an algebraically closed field k, one can study the set of effective divisors $\{D_{nP}\}$ where D_{nP} is the divisor of poles of the element $x(nP)$ for a fixed n, cf. [**9**], [**14**], [**15**], [**4**].

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The question which we investigate in this paper is to what extent, for a fixed discrete valuation v in the field K, the numbers $v(\psi_n(P))$ and $v(\phi_n(P))$ can be both positive and what is the maximum exponent r for the power π^r of the local at v uniformizing element $\pi \in K$ which divides both $\psi_n(P)$ and $\phi_n(P)$. Such a situation only happens when the point P has bad reduction at the place v and we produce a complete formula that deals with all Kodaira reduction types.

Our main result shows that the main theorem of Yabuta and Voutier [**26**], which applies for finite extensions of $\mathbb{Q}_p$, holds for more general discrete valuation fields. Our proof applied to the case of $\mathbb{Q}_p$ is much shorter and depends exclusively on the elementary properties of the Néron local heights. Up to some standard facts from [**10**, **11**], and [**19**] our proof of Equations (1.2) and (1.3) is essentially self-contained. The calculations that reveal the exact form of $k_{v,n}$ for a given reduction type of place v are calculated using explicit arithmetic models, cf. [**6**, **24**], and [**22**].

Let n be a positive integer and define

$$k_{v,n} = \min\left\{v\left(\psi_n^2(P)\right), v\left(\phi_n(P)\right)\right\}. \tag{1.1}$$

THEOREM 1.1. *Let R be a discrete valuation ring with quotient field K. Let v be the valuation of K. Let E/K be an elliptic curve defined by a Weierstrass model in minimal form with respect to v, let $P = (x, y) \in E(K)$ and let n be a positive integer.*

If $P \neq O$ is non-singular modulo v, then

$$k_{v,n} = \min\left(0, n^2 v(x(P))\right). \tag{1.2}$$

If $P \neq O$ is singular modulo v, then

$$k_{v,n} = n^2 c_v(P) - c_v(nP). \tag{1.3}$$

The function $c_v(nP)$ will be defined in definition 2.3 and it is periodic in n. The value of $n^2 c_v(P) - c_v(nP)$ is computed in Table 1, for the case of standard reduction types at v, and in Table 2 for the case of non-standard reduction types. The constants m_P and a_P will be defined in the next section.

If the residue field κ is perfect, then the Kodaira symbol of the curve is one of those of Table 1 and it can be computed using [**20**, Table C.15.1]. If κ is not perfect, one can compute the Kodaira symbol of the curve using [**22**, **23**]. If the Kodaira symbol is not one of the symbols in Table 1 and P is singular modulo v, then the Kodaira symbol is one of those in Table 2.

We also emphasize that the formula for the cancellation $k_{v,n}$ is a quadratic polynomial in n with rational coefficients which depend only on the correction terms $c_v(P)$. When K is a function field of a curve over an algebraically closed field, the quantity $c_v(P)$ was defined by other means in Cox–Zucker [**5**] and used extensively by [**16**] in his theory of Mordell–Weil lattices. There are two key facts that make the proof ultimately very short.

1. The Néron local height is a sum of two ingredients: local intersection pairing with the zero section and the 'correction part' which depends only on the component which the given point hits at the place v, cf. [**10**], [**12**],[**2**] [**7**].

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Table 1. *Tables of the common valuation $k_{v,n}$ for standard Kodaira types*

Kodaira symbol	m_P	$k_{v,n} = n^2 c_v(P) - c_v(nP)$	
III^*	2	$3n^2/2$	if $n \equiv 0 \bmod m_P$
		$3(n^2-1)/2$	if $n \not\equiv 0 \bmod m_P$
IV^*	3	$4n^2/3$	if $n \equiv 0 \bmod m_P$
		$4(n^2-1)/3$	if $n \not\equiv 0 \bmod m_P$
III	2	$n^2/2$	if $n \equiv 0 \bmod m_P$
		$(n^2-1)/2$	if $n \not\equiv 0 \bmod m_P$
IV	3	$2n^2/3$	if $n \equiv 0 \bmod m_P$
		$2(n^2-1)/3$	if $n \not\equiv 0 \bmod m_P$
I_m^*	2	n^2	if $n \equiv 0 \bmod m_P$
		n^2-1	if $n \not\equiv 0 \bmod m_P$
I_m^*	4	$n^2(m+4)/4$	if $n \equiv 0 \bmod m_P$
		$(n^2-1)(m+4)/4$	if $n \equiv 1,3 \bmod m_P$
		$(n^2(m+4)/4)-1$	if $n \equiv 2 \bmod m_P$
I_m	$\frac{m}{\gcd(a_P,m)}$	$n^2\frac{a_P(m-a_P)}{m} - \frac{n'(m-n')}{m}$	$n' := a_P n (\bmod\ m)$

Table 2. *Tables of the common valuation $k_{v,n}$ for non-standard Kodaira types*

Kodaira symbol	m_P	$k_{v,n} = n^2 c_v(P) - c_v(nP)$	
X_2	2	n^2	if $n \equiv 0 \bmod m_P$
		n^2-1	if $n \not\equiv 0 \bmod m_P$
K_{2m}	2	$\frac{mn^2}{2}$	if $n \equiv 0 \bmod m_P$
		$\frac{m(n^2-1)}{2}$	if $n \not\equiv 0 \bmod m_P$
T_m	2	n^2	if $n \equiv 0 \bmod m_P$
		n^2-1	if $n \not\equiv 0 \bmod m_P$

2. The valuation $v(\psi_n(P))$ is a quadratic polynomial in n whose coefficients depend on the local Néron height at arguments nP and P, respectively, and on the valuation of the minimal discriminant Δ of the elliptic curve E, cf. lemmas 3.3,**3.6**. In the case of finite extensions of $\mathbb{Q}_p$, this was already proven in [**21**].

On the way we reprove in our setting some results of [**1**] and obtain a much simpler analysis of the "troublemaker sequences' used by Stange [**21**] and [**26**] in their proofs. In the number field case, the sequence $k_{v,n}$ received considerable attention during the last 30 years, with [**1**, **3**, **8**] attempting to understand the shape of $k_{v,n}$. In this case, [**26**] established the precise value of $k_{v,n}$. The goal of this paper is to extend their result to the most general setting. One can easily check that our Table 1 agrees with [**26**, Table 1.1].

The study of the sequence $k_{v,n}$ has some interesting applications. For example, knowing the behaviour of $k_{v,n}$ was necessary to show that every elliptic divisibility

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sequence satisfies a recurrence relation in the rational case, cf. [**25**]. In § 4 we will show how theorem 1.1 helps to generalize that result. Another application can be found in [**21**] and [**8**], where the study of $k_{v,n}$ was necessary to study the problem of understanding when a multiple of P is integral.

2. First definitions

Let R be a discrete valuation ring with quotient field K and residue field κ. Let v be the valuation of K and assume that $v(K^*) = \mathbb{Z}$. Let E be an elliptic curve over K which is a generic fibre of the regular proper model $\pi : \mathcal{E} \to \operatorname{Spec} R = \{o, s\}$. The fibre $E_s = \pi^{-1}(s)$ is the special fibre (singular or not). We denote by $\widetilde{\mathcal{E}} \to \operatorname{Spec} R = \{o, s\}$ the associated Néron model which is the subscheme of smooth points over R. Assume that Weierstrass model of E is in minimal form with respect to v and let $P = (x, y) \in E(K)$. We denote by Δ the discriminant of the minimal model E.

The point $P \in E(K)$ extends to a section $\sigma_P : \operatorname{Spec} R \to \overline{\mathcal{E}}$ by the Néron model property. Let Φ_v denote the group of components of the special fibre E_s. We have a natural homomorphism $comp_v : E(K) \to \Phi_v$ which sends the point P to the element of the component group. We say that the point P is non-singular modulo v if the image $comp_v(P)$ is the identity element, otherwise the point P is singular modulo v.

Let $E_0(K) \subset E(K)$ be the kernel of $comp_v$, i.e. the group of points that reduce to non-singular points modulo v. Every element $comp_v(P)$ belongs to a cyclic component of Φ_v. Let a_P be the order of this component and let m_P be the order of $comp_v(P)$ in Φ_v.

In the proof of theorem 1.1, without loss of generality, K may be replaced by its completion. So, we will assume that K is complete (with respect to v).

Definition 2.1. *Let $(P.O)_v$ denote the local intersection number of the point $P \in E(K) \setminus \{O\}$ with the zero point O at v defined by the formula*

$$(P.O)_v = \max\{0, -v(x(P))\}$$

for the minimal model E. In particular, $(P.O)_v > 0$ if and only if P reduces to the identity modulo v and then P is non-singular. For more details on local intersection, see [19, Section IV.7].

Remark 2.2. For the behaviour of the sequence of integers $\{(nP.O)_v\}_n$, see [**14**, Section 7 and Lemma 8.2] in the function field case and [**21**, Lemma 5.1] in the number field case.

Definition 2.3. *With a pair (P, v) we associate a rational number $c_v(P)$ which is called the correction term. Let $\widehat{\lambda_v}(P)$ be the local Néron height as defined in [**10**, Theorem III.4.1]. Define*

$$c_v(P) = (P.O)_v + v(\Delta)/6 - 2\widehat{\lambda_v}(P). \tag{2.1}$$

Remark 2.4. Observe that $c_v(P)$ depends only on $comp_v(P)$. This follows from [**11**, Theorem 11.5.1] when P is singular modulo v and from [**19**, Theorem VI.4.1]

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when P is non-singular modulo v. In particular, [19, Theorem VI.4.1] shows that $c_v(P) = 0$ when P is non-singular modulo v. In lemma 3.4 we will show how to explicitly compute $c_v(P)$. Finally, note that the definition of local Néron height in [10, Theorem III.4.1] is given for discrete valuation fields, as is noted by Lang in [10, p. 66].

The division polynomials are defined in [20, Exercise 3.7]. We recall the properties that we will use:

(A) If nP is not equal to the identity O of the curve, then

$$x(nP) = \frac{\phi_n(P)}{\psi_n^2(P)};$$

(B) The polynomials have integer coefficients and depends only on $x(P)$. Seen as polynomials in $x(P)$, $\psi_n^2(x)$ has degree $n^2 - 1$ and $\phi_n(x)$ is monic and has degree n^2;

(C) For all $n \geqslant 1$,

$$\phi_n(P) = x(P)\psi_n^2(P) - \psi_{n-1}(P)\psi_{n+1}(P).$$

(D) For all $n \geqslant m \geqslant r$,

$$\psi_{n+m}(P)\psi_{n-m}(P)\psi_r^2(P) = \psi_{n+r}(P)\psi_{n-r}(P)\psi_m^2(P) - \psi_{m+r}(P)\psi_{m-r}(P)\psi_n^2.$$

Remark 2.5. Our proof clearly works also for number fields. In that setting the result was known [26], but the two proofs are completely different. Indeed, the proof of Yabuta and Voutier is almost completely arithmetical, while ours is more geometrical.

Remark 2.6. We will work assuming that $v(0) = \infty$. Under this assumption, our theorem works also in the case when $nP = O$. In this case $\psi_n^2(P) = 0$ and then $k_{v,n}(P) = v(\phi_n(P))$.

3. Proof of the main theorem

We start this section by recalling some classical facts on the local Néron height.

Lemma 3.1. *The following hold:*

- *For all $R, Q \in E(K)$ with $R, Q, R \pm Q \neq O$,*

$$\widehat{\lambda_v}(R+Q) + \widehat{\lambda_v}(R-Q) = 2\widehat{\lambda_v}(R) + 2\widehat{\lambda_v}(Q) + v(x(R) - x(Q)) - v(\Delta)/6. \tag{3.1}$$

- *The Néron local height does not change if we replace K with a finite extension. Moreover, $\widehat{\lambda_v}(\cdot)$ does not depend on the choice of the minimal Weierstrass model defining the curve.*

- *If $comp_v(P) = comp_v(P')$, then $\widehat{\lambda_v}(P) = \widehat{\lambda_v}(P')$.*

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- *If $P \notin E_0(K)$, then $\widehat{\lambda_v}(P)$ just depends on the image of P in $E(K)/E_0(K)$.*

Proof. For the first two statements, see [**10**, Theorem III.4.1 and Page 63]. For the last two, see [**11**, Theorem 11.5.1]. □

LEMMA 3.2. *If P has non-singular reduction, then*

$$\widehat{\lambda_v}(P) = v(\Delta)/12 + (P.O)_v/2.$$

Proof. See [**19**, Theorem VI.4.1] for the case of perfect fields and [**10**, Chapter III, §4, §5] for the general case. Observe that $\widehat{\lambda_v}(P)$ is denote by $\lambda_v(P)$ in both [**10**, Chapter III, §4, §5] and [**19**]. □

LEMMA 3.3. *If $nP \neq O$, then*

$$\widehat{\lambda_v}(nP) = n^2\widehat{\lambda_v}(P) + v(\psi_n(P)) - \frac{n^2-1}{12}v(\Delta), \tag{3.2}$$

where Δ is the discriminant of the curve.

Proof. This lemma is stated as an exercise in [**19**, Exercise 6.4 (e)]. First, assume that P is a non-torsion point. We prove the lemma by induction. If $n = 1$, it is obvious since $\psi_1(P) = 1$. If $n = 2$, then it follows from [**19**, Theorem VI.1.1]. Note that, in [**19**], it is assumed that κ is perfect, but the proof of the theorem works in the exact same way without that requirement, cf. [**10**, Chapter III, §4, §5]. So, we assume that the lemma is true for $i \leqslant n$ and we show that it holds also for $n + 1 \geqslant 3$. Put $R = nP$ and $Q = P$. So, $R, Q, R \pm Q \neq O$. Then, by lemma 3.1,

$$\widehat{\lambda_v}((n+1)P) = -\widehat{\lambda_v}((n-1)P) + 2\widehat{\lambda_v}(nP) + 2\widehat{\lambda_v}(P) + v(x(nP) - x(P)) - \frac{v(\Delta)}{6}. \tag{3.3}$$

For the definition of division polynomials we have

$$\begin{aligned} x(nP) - x(P) &= \frac{\phi_n(P)}{\psi_n^2(P)} - x(P) \\ &= x(P) + \frac{\psi_{n+1}(P)\psi_{n-1}(P)}{\psi_n^2(P)} - x(P) \\ &= \frac{\psi_{n+1}(P)\psi_{n-1}(P)}{\psi_n^2(P)}. \end{aligned} \tag{3.4}$$

Put $d_n = \widehat{\lambda_v}(nP) - v(\psi_n(P))$ and then combining (3.3) and (3.4) we have

$$d_{n+1} = 2d_n - d_{n-1} + 2d_1 - \frac{v(\Delta)}{6}.$$

By induction, we have $d_i = i^2 d_1 - (i^2 - 1)v(\Delta)/12$ for $i \leqslant n$ and then

$$\begin{aligned} d_{n+1} &= (2n^2 - (n-1)^2 + 2)d_1 - v(\Delta)\left(\frac{2(n^2-1) - ((n-1)^2 - 1) + 2}{12}\right) \\ &= (n+1)^2 d_1 - v(\Delta)\left(\frac{(n+1)^2 - 1}{12}\right). \end{aligned}$$

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Therefore,

$$\widehat{\lambda_v}((n+1)P) = v(\psi_{n+1}(P)) + d_{n+1}$$
$$= v(\psi_{n+1}(P)) + (n+1)^2\widehat{\lambda_v}(P) - v(\Delta)\left(\frac{(n+1)^2-1}{12}\right).$$

Now, we need to prove the lemma in the case when P is a torsion point. Fix $n \in \mathbb{N}$. First, we can assume that K is complete since clearly the statement does not change. In a finite extension of K we can find a non-torsion point P' such that $(P - P'.O)_v$ is very large. The choice of P' will depend on n. Thanks to lemma 3.1, the Néron local height does not change if we replace K with a finite extension, so we can assume $P' \in E(K)$. Since $(P - P'.O)_v$ is very large, $(nP.O)_v = (nP'.O)_v$ and $(P.O)_v = (P'.O)_v$ (this is certainly true if $(P - P'.O)_v > (nP.O)_v$). Moreover, $comp_v(P) = comp_v(P')$ and then, from lemma 3.1, $\widehat{\lambda_v}(P) = \widehat{\lambda_v}(P')$ and $\widehat{\lambda_v}(nP) = \widehat{\lambda_v}(nP')$. Furthermore, $v(\psi_n(P)) = v(\psi_n(P'))$ since ψ_n is a continuous function and v is discrete. We know that the lemma holds for P' since it is a non-torsion point and then

$$\widehat{\lambda_v}(nP) = \widehat{\lambda_v}(nP') = n^2\widehat{\lambda_v}(P') + v(\psi_n(P')) - \frac{n^2-1}{12}v(\Delta)$$
$$= n^2\widehat{\lambda_v}(P) + v(\psi_n(P)) - \frac{n^2-1}{12}v(\Delta). \qquad \square$$

Recall that, in the case κ perfect, the Kodaira symbol of an elliptic curve can be computed using [**20**, Table C.15.1]. The analogous reference for the non-standard Kodaira symbols is [**22**, **23**]. Recall that a_P and m_P are defined at the beginning of § 2.

LEMMA 3.4. *The value of $c_v(P)$ is provided as a function of the Kodaira symbol and m_P in the following table.*

Kodaira symbol	m_P	$c_v(P)$
Any	1	0
III	2	1/2
III^*	2	3/2
I_m^*	2	1
X_2	2	1
K_{2m}	2	$m/2$
T_m	2	1
IV^*	3	4/3
IV	3	2/3
I_m^*	4	$(m+4)/4$
I_m	$\frac{m}{\gcd(a_P, m)}$	$\frac{a_P(m-a_P)}{m}$

In the last line, we are assuming that $0 < a_P < m$.

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Proof. Recall that, by definition 2.3,

$$c_v(P) = (P.O)_v + v(\Delta)/6 - 2\widehat{\lambda_v}(P).$$

Assume that P has non-singular reduction. By lemma 3.2, we have $\widehat{\lambda_v}(P) = v(\Delta)/12 + (P.O)_v/2$. Hence, $c_v(P) = 0$.

If P has singular reduction, then $(P.O)_v = 0$ since the identity is a non-singular point. The calculation of the values of $c_v(P)$ is based on the information obtained from the resolution of singularities performed on Weierstrass minimal model. By lemma 3.1, $\widehat{\lambda_v}(P)$ does not depend on the choice of the minimal Weierstrass model defining the curve. So, $c_v(P)$ does not depend on the choice of the minimal Weierstrass model.

If the residue field κ is perfect, Tate algorithm [**24**] proves the existence of the flat proper regular model of an elliptic curve. The procedure of Tate was extended in the work of Michael Szydlo [**22**, **23**] for any elliptic curve over DVR, in particular to the case when κ is not a perfect field. We will refer to this general procedure as GTA (Generalized Tate's algorithm).

When $\operatorname{char}\kappa \neq 2, 3$ the values of $c_v(P)$ can be determined from the data obtained by the procedure of Tate. In fact, one follows the calculations in [**6**, Proposition 6] and notes that the valuation of the minimal model and the valuation of the (x, y) coordinates of the point which reduces to the singularity are uniform across all fields with $\operatorname{char}\kappa \neq 2, 3$, cf. [**23**, §4.1] and in particular [**23**, Table 1].

We will explain now in detail the computations for residue fields of characteristic 2 and 3, when new reduction types arise and some care must be taken. We essentially follow the strategy laid out above and supplement it with the data about valuations of the coefficients provided in [**22**, **23**]. When the fibre at v has reduction type I_0, II, II^* or any of the following non-standard Kodaira types: $Z_1, Z_2, X_1, Y_1, Y_2, Y_3, K'_{2n}, K_{2n+1}$, the group of components is trivial (see [**23**, Equation (20)]).

Case $m_P = 1$**.** In such a case P has non-singular reduction and then, as we pointed out at the beginning of the proof, $c_v(P) = 0$.

Case $m_P = 2$**.** Since $comp_v$ is a group homomorphism and $\Phi_v \cong \mathbb{Z}/2\mathbb{Z}$ it follows that if $P \notin E_0(K_v)$, then $c_v(2P) = 0$ and $c_v(P) = c_v(3P)$. Moreover, $(P.O)_v = (3P.O)_v = 0$, hence $\widehat{\lambda_v}(P) = \widehat{\lambda_v}(3P)$. We get from lemma 3.3 for $n = 3$ that

$$8\widehat{\lambda_v}(P) = 2v(\Delta)/3 - v(\psi_3(P)).$$

From the definition of $c_v(P)$ (see equation (2.3)) we get that

$$c_v(P) = v(\psi_3(P))/4.$$

For notational convenience, put $x = x(P)$. Since $\psi_3(P) = 3x^4 + b_2x^3 + 3b_4x^2 + 3b_6x + b_8$ it remains to calculate the valuation estimates for the suitable quantities. For their definition, see [**20**, Section III.1]. By [**23**, Equation (20)], the only non-standard Kodaira types that we have to consider are X_2, K_{2m}, and T_m.

Types III, III^*, I_m^*. We follow the argument in [**6**, Proposition 6] combined with the valuation tables in [**23**] for the non-perfect field cases. Notice that, for these

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types, Tate's algorithm works in the exact same way in the perfect and non-perfect field case (see in particular [23, Tables 1, 4, 5]).

Type X_2. It follows from [22, p.41] that we can construct a minimal model where $v(a_1) \geqslant 1$, $v(a_2) \geqslant 2$, $v(a_3) \geqslant 2$, $v(a_4) = 2$, $v(a_6) \geqslant 4$ and the sequence of blowups at the singular points resolve the model in such a way that the point $P = (x, y) \notin E_0(K_v)$ has $v(x) \geqslant 2$, cf.[23, Table 5]. It follows that $v(b_2) \geqslant 2$, $v(b_4) \geqslant 3$, $v(b_6) \geqslant 4$ and finally $v(b_8) = 4$. Thus, we have $v(\psi_3(P)) = 4$ and then $c_v(P) = 1$.

Type K_{2m}. It follows from [23, Table 6] that $v(b_2) \geqslant 2$, $v(b_4) \geqslant 2$, $v(b_6) \geqslant 2\,m$, and $v(b_8) = 2\,m$. The resolution procedure in [22, §6.12] implies that for the minimal model with the previous assumptions we have $v(x) > m$. Hence, we obtain $v(\psi_3(P)) = 2\,m$, proving that $c_v(P) = m/2$.

Type T_m. In the resolution [22, pp. 48-51] the valuation of x is equal to 1, cf. [22, p. 51]. It follows from [23, Table 7] that all terms of $\psi_3(P)$ but $3x^4$ have valuation at least 5. Hence $v(\psi_3(P)) = 4$ and thus $c_v(P) = 1$.

Case $m_P = 3$**.** Type IV, IV^*. First of all, observe that $\widehat{\lambda_v}(P) = \widehat{\lambda_v}(-P)$. Indeed, if we put $\widehat{\lambda'_v}(P) \coloneqq \widehat{\lambda_v}(-P)$, then $\widehat{\lambda'_v}$ satisfies (3.1). Thanks to [10, Theorem III.4.1], we have $\widehat{\lambda'_v} = \widehat{\lambda_v}$ since $\widehat{\lambda_v}$ is unique. So, $\widehat{\lambda_v}(2P) = \widehat{\lambda_v}(-P) = \widehat{\lambda_v}(P)$. Using lemma 3.3 with $n = 2$, we have

$$3\widehat{\lambda_v}(P) = \frac{v(\Delta)}{4} - v(\psi_2(P))$$

and, by (2.1),

$$c_v(P) = (P.O)_v + \frac{v(\Delta)}{6} - 2\widehat{\lambda_v}(P) = \frac{v(\psi_2^2(P))}{3}.$$

From the definition of $\psi_2^2(P)$ we have $\psi_2^2(P) = 4x(P)^3 + b_2x(P)^2 + 2b_4x(P) + b_6$. Since $\widehat{\lambda_v}(P)$ does not depend on the choice of the minimal Weierstrass equation defining the curve, we can assume that the coefficients satisfy GTA. In the case for the type IV, we have $v(b_2) > 0$, $v(b_4) > 1$, and $v(b_6) = 2$. Moreover, $v(x(P)) > 0$ holds due to the resolution construction. Hence,

$$2 = v(b_6) = v(4x(P)^3 + b_2x(P)^2 + 2b_4x(P) + b_6) = v(\psi_2^2(P)) = 3c_v(P)$$

and we are done. A very analogous calculation reveals that for the type IV^* we have $c_v(P) = 4/3$.

Case $m_P > 3$**.** We obtain for every residue field, even non-perfect the values of $c_v(P)$ are subject to more complicated counting rule. We basically follow the argument described in [6], enhanced by the classification of fibres by Szydlo [23].

Type I_m^* $(m_P = 4)$. Following [17, p.353] we note that if $P \notin E_0(K_v)$, then $comp_v(P) = comp_v(-3P) = comp_v(3P)$ since $comp_v$ is a homomorphism and $\widehat{\lambda}_v$ is unique, cf. case $m_P = 3$. Therefore, we can apply again lemma 3.3 with $n = 3$ to conclude that $c_v(P) = v(\psi_3(P))/4$. The explicit formulas follow from the inspections of Tables in [23].

Type I_m. The correction formula for the multiplicative type can be explained in several ways. The down-to-earth approach is to invoke the theory of the Tate curve [10, III §5] which is independent of the residue field. We extract $c_v(P) =$

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$k(m-k)/m$ for the intersection of P with the k-th cyclic component from [**10**, Thm. 3.5.1], cf. [**19**, Thm. 6.4.2(b)] and [**6**, Propositions 5,6]. □

REMARK 3.5. A more conceptual approach to the proof of lemma 3.4 follows from abstract intersection theory developed by Néron, cf. [**12**]. This is an analogue of the calculations one can do in the theory of elliptic surfaces, cf. [**19**, III §8]. This is a very general approach that would in principle allow us to calculate all correction terms using just the suitable fibral divisor. In fact, none of the references treats properly the case of non-perfect fields so we refrain from giving here all the fine details of this argument.

LEMMA 3.6. *Let P be a point on the elliptic curve E as above. For every n such that $nP \neq O$, we have*

$$v(\psi_n^2(P)) = (nP.O)_v - n^2(P.O)_v + n^2 c_v(P) - c_v(nP).$$

In particular, $n^2 c_v(P) - c_v(nP)$ is an integer.

Proof. From (3.2) we have

$$v(\psi_n(P)) = \widehat{\lambda_v}(nP) + \frac{n^2-1}{12}v(\Delta) - n^2\widehat{\lambda_v}(P).$$

From definition 2.3 we have

$$\widehat{\lambda_v}(P) = \frac{v(\Delta)}{12} - \frac{c_v(P)}{2} + \frac{(P.O)_v}{2}.$$

Hence,

$$\begin{aligned} v(\psi_n(P)) &= \widehat{\lambda_v}(nP) + \frac{n^2-1}{12}v(\Delta) - n^2\widehat{\lambda_v}(P) \\ &= \frac{v(\Delta)}{12} - \frac{c_v(nP)}{2} + \frac{(nP.O)_v}{2} + \frac{n^2-1}{12}v(\Delta) \\ &- n^2\frac{v(\Delta)}{12} + n^2\frac{c_v(P)}{2} - \frac{n^2(P.O)_v}{2} \\ &= \frac{(nP.O)_v}{2} - \frac{n^2(P.O)_v}{2} + \frac{1}{2}n^2 c_v(P) - \frac{1}{2}c_v(nP). \end{aligned}$$ □

REMARK 3.7. Thanks to the previous lemma, one can easily show that the valuation $v(\psi_n(P))$ is independent of the choice of the minimal Weierstrass model.

Put n_P as the smallest positive integer such that $n_P P$ reduces to the identity modulo v. Observe that n_P does not necessarily exist.

LEMMA 3.8. *Let k be an integer. If $(kP.O)_v > 0$, then $(mkP.O)_v \geqslant (kP.O)_v$ for all $m \geqslant 1$. Moreover, if n_P exists, then $(kP.O)_v > 0$ if and only if $n_P \mid k$.*

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Proof. Put $P' := kP$ and by assumption we have $v(x(P')) < 0$. Hence, $v(\phi_m(P')) = m^2 v(x(P'))$ (recall that $\phi_m(P')$, seen as a polynomial in $x(P')$, is monic) and $v(\psi_m^2(P')) \geqslant (m^2-1)v(x(P'))$. So,

$$v(x(mkP)) = v(\phi_m(P')) - v(\psi_m^2(P')) \leqslant v(x(P')) = v(x(kP)) < 0.$$

Now, we prove the second part of the lemma. If $n_P \mid k$, then $(kP.O)_v > 0$. If $n_P \nmid k$, then $k = qn_P + r$ with $0 < r < n_P$. Recall that the identity is a non-singular point of the curve reduced modulo v. Hence, if $(kP.O)_v > 0$, then kP reduces to the identity modulo v and then also the point $rP = kP - q(n_P P)$ reduces to the identity. This is absurd from the definition of n_P. □

Proposition 3.9. *Assume that P has non-singular reduction and $(P.O)_v = 0$. Then, $k_{v,n}(P) = 0$ for all $n \geqslant 1$.*

Remark 3.10. This proposition was proved for number fields by Ayad [**1**]. Their proof works also in our case, but it is very different from ours.

Proof. Since P has non-singular reduction, then $c_v(kP) = 0$ for any k (see lemma 3.4). Hence, from lemma 3.6, $v(\psi_k(P)) = (kP.O)_v$ for all k such that $kP \neq O$.

Assume $nP \neq O$. If $(nP.O)_v > 0$, then due to lemma 3.8 the numbers $((n-1)P.O)_v$ and $((n+1)P.O)_v$ are equal to zero. Therefore, $v(\psi_n(P)^2) = (nP.O)_v > 0$, $v(\psi_{n-1}(P)\psi_{n+1}(P)) = 0$ and $v(\phi_n(P)) = 0$ from the properties (A), (B), (C), proving $k_{v,n}(P) = 0$.

If $(nP.O)_v = 0$, then $v(\psi_n(P)^2) = 0$ and $v(\phi_n(P)) \geqslant 0$, hence again $k_{v,n}(P) = 0$. Here we are using that the polynomials ψ_n^2 and ϕ_n have integer coefficients.

Assume now that $nP = O$. Then $\psi_n^2(P) = 0$ and so $\phi_n(P) = \psi_{n-1}(P)\psi_{n+1}(P)$. Observe that $((n \pm 1)P.O)_v = 0$ since otherwise $(P.O)_v$ would be strictly positive. Hence, $k_{v,n\pm1}(P) = v(\psi_{n\pm1}^2(P))$. For the first part of the lemma, $k_{v,n\pm1}(P) = 0$ and then

$$v(\phi_n(P)) = v(\psi_{n-1}(P)\psi_{n+1}(P)) = \frac{k_{v,n-1}(P) + k_{v,n+1}(P)}{2} = 0. \qquad \square$$

Now, we are ready to prove our main result.

Proof of theorem 1.1. Assume that P has non-singular reduction. If $v(x(P)) < 0$, then, using that ψ_n^2 (resp. ϕ_n) has integer coefficients and degree n^2-1 (resp. n^2), we have $v(\psi_n^2(P)) \geqslant (n^2-1)v(x(P))$ and $v(\phi_n(P)) = n^2 v(x(P))$ (recall that ϕ_n is monic). So, $k_{v,n}(P) = n^2 v(x(P))$. If $v(x(P)) \geqslant 0$, then $(P.O)_v = 0$ and we conclude thanks to the previous proposition.

Assume now that P has singular reduction, that n_P exists, and that $n_P \mid n$. Then, $v(x(nP)) < 0$. Therefore, $v(\phi_n(P)) < v(\psi_n^2(P))$ and so $k_{v,n} = v(\phi_n(P))$. Since P is singular we have $v(x(P)) \geqslant 0$. Indeed, if $v(x(P)) < 0$ then P is the identity modulo v and the identity is a non-singular point. Therefore, $v(\phi_n(P)) < v(x(P)\psi_n^2(P))$.

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Recall that $\phi_n(P) = x(P)\psi_n^2(P) + \psi_{n-1}(P)\psi_{n+1}(P)$. So,

$$v(\phi_n(P)) = v(\psi_{n+1}(P)) + v(\psi_{n-1}(P)).$$

Observe that $(n \pm 1)P \neq O$ and then in particular $((n \pm 1)P.O)_v = 0$. Recall that $c_v(P)$ depends only on $comp_v(P)$ and observe that $c_v(P) = c_v(-P)$. Thus, $c_v((n \pm 1)P) = c_v(P)$ since $m_P \mid n_P \mid n$. Hence, we can apply lemma 3.6 and we have

$$v(\psi_{n\pm1}(P)) = \frac{(n \pm 1)^2 c_v(P) - c_v((n \pm 1)P)}{2} = \frac{((n \pm 1)^2 - 1)c_v(P)}{2}.$$

Thus,

$$k_{v,n} = v(\phi_n(P)) = v(\psi_{n-1}(P)\psi_{n+1}(P)) = n^2 c_v(P) = n^2 c_v(P) - c_v(nP).$$

Here we are using that $c_n(nP) = 0$ since nP is non-singular.

It remains to prove the case when P has singular reduction and that n_P does not exist or it exists but $n_P \nmid n$. In this case $v(x(nP)) \geqslant 0$ and so $k_{v,n} = v(\psi_n^2(P)) \geqslant 0$. Observe that $(P.O)_v = (nP.O)_v = 0$ and then from lemma 3.6,

$$k_{v,n} = v(\psi_n^2(P)) = n^2 c_v(P) - c_v(nP).$$

To conclude the proof, we want to show how to compute the values $n^2 c_v(P) - c_v(nP)$, as we did in Tables 1 and 2. If the Kodaira symbol is not I_m, then the value can be computed directly using lemma 3.4. If the Kodaira symbol is I_m, then one can easily prove that $a_{nP} \equiv a_P n \mod m$. Therefore,

$$n^2 c_v(P) - c_v(nP) = n^2 \frac{a_P\,(m - a_P)}{m} - \frac{n'\,(m - n')}{m}$$

with n' the smallest positive integer such that $a_P n \equiv n' \bmod m$. □

4. A corollary and an example

In this section, we want to show an application of the main theorem and an example. Let E be an elliptic curve defined by a Weierstrass equation with coefficients in the principal ideal domain R with fraction field K. Given a point $P \in E(K)$, one can define a sequence of integral ideals B_n that represents the square root of the denominator of the fractional ideal $(x(nP))R$ (the fact that the denominator is a square follows from the fact that E is defined by an equation with integer coefficients). The sequence of the B_n is a so-called *elliptic divisibility sequence*. If R is a principal ideal domain, we consider a choice of β_n such that $B_n = (\beta_n)$ and then we have a sequence $\{\beta_n\}_{n\in\mathbb{N}}$ of elements in R such that β_n generates the square root of the denominator of $(x(nP))R$. The choice of β_n is clearly not unique. In [**25**], it is proved that, if $K = \mathbb{Q}$ and we choose the β_n in an appropriate way, the sequence $\{\beta_n\}_{n\in\mathbb{N}}$ satisfies a recurrence relation. The two main ingredients of the proof of this fact are the study of the behaviour of the sequence $k_{v,n}$ for all finite places in $\mathbb{Q}$ and a recurrence relation that is satisfied by the sequence $\{\psi_n(P)\}_{n\in\mathbb{N}}$. Thanks to the main theorem of this paper, we know the behaviour of $k_{v,n}$ in a very

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general case and then we can generalize the result of [25] to the case when K is not $\mathbb{Q}$. Indeed, we can prove the following.

COROLLARY 4.1. *Let R be a principal ideal domain with field of fractions K. Let E be an elliptic curve with a minimal Weierstrass model over R and let $P \in E(K)$. Define $M(P)$ as the smallest positive integer such that $[M(P)]P$ is non-singular modulo every maximal ideal of R.*

We can define a sequence $\{\beta_n\}_{n\in\mathbb{N}}$ of elements in R such that:

- *β_n generates the square root of the ideal generated by the denominator of $x(nP)$;*
- *For all triples (n, m, r) of positive integers such that $n \geqslant m \geqslant r$ and any two of them are multiples of $M(P)$, we have*

$$\beta_{n+m}\beta_{n-m}\beta_r^2 = \beta_{n+r}\beta_{n-r}\beta_m^2 - \beta_{m+r}\beta_{m-r}\beta_n^2.$$

Proof. The proof follows easily from the proof of [25, Theorem 1.9] and theorem 1.1, so we just sketch it. We define β_n such that $\beta_n^2 = \psi_n^2(P)/\gcd(\phi_n(P), \psi_n^2(P))$. We can define the gcd since R is a PID. We have

$$\psi_{n+m}\psi_{n-m}\psi_r^2 = \psi_{n+r}\psi_{n-r}\psi_m^2 - \psi_{m+r}\psi_{m-r}\psi_n^2 \quad \text{for all } n \geqslant m \geqslant r.$$

We divide this equation by

$$\gcd(\phi_n(P), \psi_n^2(P)) \cdot \gcd(\phi_m(P), \psi_m^2(P)) \cdot \gcd(\phi_r(P), \psi_r^2(P))$$

and we obtain

$$L_{m,n}\beta_{n+m}\beta_{n-m}\beta_r^2 = L_{n,r}\beta_{n+r}\beta_{n-r}\beta_m^2 - L_{m,r}\beta_{m+r}\beta_{m-r}\beta_n^2$$

where

$$L_{m_1,m_2} = \frac{\gcd(\phi_{m_2+m_1}(P), \psi_{m_2+m_1}^2(P))\gcd(\phi_{m_2-m_1}(P), \psi_{m_2-m_1}^2(P))}{\gcd(\phi_{m_2}(P), \psi_{m_2}^2(P))^2 \gcd(\phi_{m_2}(P), \psi_{m_2}^2(P))^2}.$$

Note that

$$v(L_{m_1,m_2}) = k_{v,m_1+m_2} + k_{v,m_1-m_2} - 2k_{v,m_1} - 2k_{v,m_2}.$$

If m_1 or m_2 is a multiple of $M(P)$, then $v(L_{m_1,m_2}) = 0$ for each v thanks to theorem 1.1. Therefore, $L_{m_1,m_2} = 1$ and, since two of n, m, r are multiples of $M(P)$, we have

$$\beta_{n+m}\beta_{n-m}\beta_r^2 = \beta_{n+r}\beta_{n-r}\beta_m^2 - \beta_{m+r}\beta_{m-r}\beta_n^2. \qquad \square$$

Our original goal was to compute $k_{v,n}$ when K is a function field since, as we explained in the introduction, $k_{v,n}$ is related to elliptic divisibility sequences and elliptic divisibility sequences over function fields received much attention in the last years. Hence, we show an example where we explicitly compute $k_{v,n}$ in the function field case. For more details on the following examples, see [13].

EXAMPLE 4.2. Let $\kappa = \mathbb{C}$, $C = \mathbb{P}^1_\kappa$, and $K = \kappa(C)$. Then, $K = \mathbb{C}(t)$. Consider the elliptic curve $y^2 = x(x - f^2)(x - g^2)$ defined over K where $(f, g, h) = (t^2 -$

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$1, 2t, t^2+1$). Notice that $f^2+g^2=h^2$. Let $P=((f-h)(g-h),(f+g)(f-h)(g-h))\in E(K)$. We denote by v_1 the place such that $v_1(t-1)=1$. One can easily show that E is singular modulo v_1 since $v_1(f)=1$. Moreover $v_1(g-h)=2$ and then $v_1(x(P))>0$ and $v_1(y(P))>0$. So, P is singular modulo v. By direct computation, $v_1(\Delta_E)=4$ and $v_1(j(E))=-4$. Then, E is in minimal form over K_{v_1} and E/K_{v_1} has Kodaira symbol I_4. By direct computation, $2P=(t^4+2t^2+1,-2t^5+2t)$ and then $2P$ is non-singular modulo v_1, observing that $v_1(x(2P))=0$. So, $a_{P,v_1}=2$ and $m_{P,v_1}=2$. Using lemma 3.4, we have

$$c_{v_1}(P)=\frac{2(4-2)}{4}=1.$$

Therefore, thanks to theorem 1.1,

$$k_{v_1,n}(P)=\begin{cases} n^2-1 \text{ if } n \text{ is odd},\\ n^2 \text{ if } n \text{ is even}.\end{cases}$$

This agrees with the direct computation of some cases for n small. If $n=2$, then

$$\psi_2^2(P)=4(x(P)(x(P)-f^2)(x(P)-g^2))=16(t^2+2t-1)^2(t^2-2t+1)^2$$

and $v_1(\psi_2^2(P))=4$. If $n=3$, then

$$\psi_3^2(P)=256(t^4+4t-1)^2(t^2+2t-1)^4(t-1)^8$$

and so $v_1(\psi_3^2(P))=8$. Moreover, $v_1(\psi_4^2(P))=18$ and $v_1(x(P))=2$. Therefore, using $\phi_n(P)=x(P)\psi_n^2(P)-\psi_{n-1}(P)\psi_{n+1}(P)$ we obtain

$$\begin{aligned}\phi_2(P)=&\left(8(t^2+2t-1)(t^2-2t+1)\right)^2(2(t-1)^2)\\&-16(t^4+4t-1)(t^2+2t-1)^2(t-1)^4\end{aligned}$$

and then $v(\phi_2(P))=4$. In the same way $v_1(\phi_3(P))=10$. Hence, we conclude that $k_{v_1,2}=4$ and $k_{v_1,3}=8$.

Acknowledgements

We want to thank Stefan Barańczuk, Matija Kazalicki, Joseph Silverman, and Paul Voutier for the comments on the earlier version of this paper. The first author acknowledges the support by Dioscuri programme initiated by the Max Planck Society, jointly managed with the National Science Centre (Poland), and mutually funded by the Polish Ministry of Science and Higher Education and the German Federal Ministry of Education and Research. The second author has been supported by MIUR (Italy) through PRIN 2017 'Geometric, algebraic and analytic methods in arithmetic' and has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No. 101034413.

References

1 M. Ayad. Points S-entiers des courbes elliptiques. *Manuscr. Math.* **76** (1992), 305–324.

https://doi.org/10.1017/prm.2024.7 Published online by Cambridge University Press

2 V. Busch and J. S. Müller. Local heights on elliptic curves and intersection multiplicities. *Int. J. Num. Theory* **8** (2012), 1477–1484.

3 J. H. Cheon and S. G. Hahn. Explicit valuations of division polynomials of an elliptic curve. *Manuscr. Math.* **97** (1998), 319–328.

4 G. Cornelissen and J. Reynolds. The perfect power problem for elliptic curves over function fields. *New York J. Math.* **22** (2016), 95–114.

5 D. A. Cox and S. Zucker. Intersection numbers of sections of elliptic surfaces. *Invent. Mathe.* **53** (1979), 1–44.

6 J. E. Cremona, M. Prickett and S. Siksek. Height difference bounds for elliptic curves over number fields. *J. Number. Theory.* **116** (2006), 42–68.

7 D. Holmes (2012) *Neron-Tate heights on the Jacobians of high-genus hyperelliptic curves.* ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–University of Warwick (United Kingdom).

8 P. Ingram. Multiples of integral points on elliptic curves. *J. Num. Theory* **129** (2009), 182–208.

9 P. Ingram, V. Mahé, J. H. Silverman, K. E. Stange and M. Streng. Algebraic divisibility sequences over function fields. *J. Austr. Math. Soc.* **92** (2012), 99–126.

10 S. Lang (1978) *Elliptic curves: Diophantine analysis*, volume 231 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. (Springer-Verlag, Berlin-New York).

11 S. Lang. *Fundamentals of Diophantine geometry* (Springer-Verlag, New York, 1983).

12 J. S. Müller. Computing canonical heights using arithmetic intersection theory. *Math. Comp.* **83** (2014), 311–336.

13 B. Naskręcki. Distribution of Mordell–Weil ranks of families of elliptic curves. *Banach Center Publicat.* **108** (2016a), 201–229.

14 B. Naskręcki. Divisibility sequences of polynomials and heights estimates. *New York J. Math.* **22** (2016b), 989–1020.

15 B. Naskręcki and M. Streng. Primitive divisors of elliptic divisibility sequences over function fields with constant j-invariant. *J. Num. Theory* **213** (2020), 152–186.

16 T. Shioda. On the Mordell–Weil lattices. *Rikkyo Daigaku sugaku zasshi* **39** (1990), 211–240.

17 J. H. Silverman. Computing heights on elliptic curves. *Math. Comput.* **51** (1988a), 339–358.

18 J. H. Silverman. Wieferich's criterion and the *abc*-conjecture. *J. Num. Theory* **30** (1988b), 226–237.

19 J. H. Silverman (1994) *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. (Springer-Verlag, New York).

20 J. H. Silverman (2009) *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. 2nd ed. (Springer, Dordrecht).

21 K. E. Stange. Integral points on elliptic curves and explicit valuations of division polynomials. *Canad. J. Math.* **68** (2016), 1120–1158.

22 M. G. Szydlo (1999) *Flat regular models of elliptic schemes.* ProQuest LLC, Ann Arbor, MI. Thesis (Ph.D.)–Harvard University.

23 M. G. Szydlo. Elliptic fibers over non-perfect residue fields. *J. Number. Theory.* **104** (2004), 75–99.

24 J. Tate (1975) Algorithm for determining the type of a singular fiber in an elliptic pencil. In *Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, volume Vol. 476 of *Lecture Notes in Math.* (Springer, Berlin-New York), pp. 33–52.

25 M. Verzobio. A recurrence relation for elliptic divisibility sequences. *Rivista di Matematica della Universitá di Parma* **13** (2022), 223–242.

26 P. Voutier and M. Yabuta. The greatest common valuation of ϕ_n and ψ_n^2 at points on elliptic curves. *J. Number. Theory.* **229** (2021), 16–38.

27 M. Ward. Memoir on elliptic divisibility sequences. *Amer. J. Math.* **70** (1948), 31–74.

https://doi.org/10.1017/prm.2024.7 Published online by Cambridge University Press