102.23 A visual proof that $\sqrt{N}$ is irrational

Suppose that $\sqrt{N}$ is rational for the whole number N. Then there is a rectangle with whole number side-lengths a, b with $\sqrt{N} = \frac{a}{b}$. Because $\frac{a}{b} = \frac{Nb}{a}$, the two rectangles shown in Figure 1 are similar.

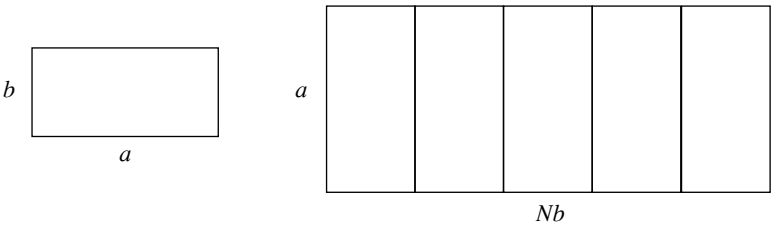


FIGURE 1

We now attempt to tile the diagonal of the large rectangle with copies of the small rectangle. One possibility is that, as in Figure 2(a), we exactly reach the bottom right-hand corner in which case (looking at the shorter sides), b divides a so that $\sqrt{N} = \frac{a}{b}$ is an integer. Alternatively, as in Figure 2(b), we stop short of the bottom right-hand corner when the shaded rectangle left is strictly smaller than the $a \times b$ rectangle, but is similar to it and with integer side-lengths. This is a contradiction – either because the argument may be repeated to give an infinite decreasing sequence of whole numbers or, by starting with a, b, the side-lengths of the smallest rectangle with $\sqrt{N} = \frac{a}{b}$.

FIGURE 2(a)

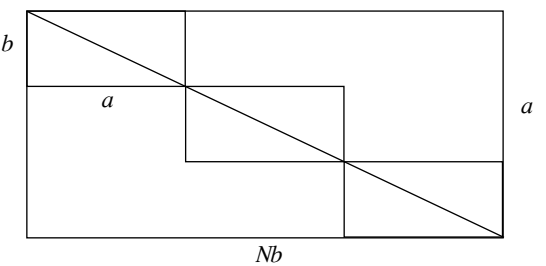
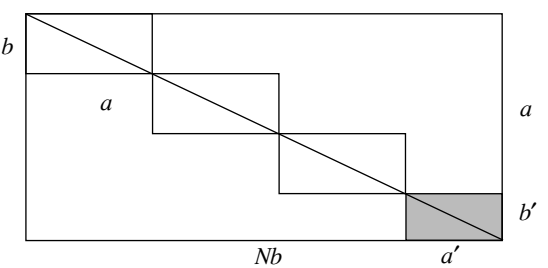


FIGURE 2(b)



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Comments

In Figure 2(b), $k = \lfloor \sqrt{N} \rfloor$ small $a \times b$ rectangles fit on the diagonal of the large rectangle so the shaded $a' \times b'$ rectangle has $a' = Nb - ka$, $b' = a - kb$. Our proof is thus a geometrical presentation of a familiar proof which goes back to Estermann in [1] and was taken up in subsequent *Gazette* articles such as [2, 3, 4]. In the case of $\sqrt{2}$, we get an additional proof using A4-sized paper to those given in [5]. For then the rectangles in Figure 1 correspond to (say) sheets of A5 and A4 paper and, placing the A5 one on the diagonal of the A4 one as in Figure 2(b) gives $a' = 2b - a$, $b' = a - b$ for the remaining shaded rectangle.

Finally, we note that there are other geometrical proofs that $\sqrt{N}$ is irrational including Terence Jackson's using similar triangles in [6], Derek Ball's based on successively cutting squares from rectangles in [7] and Steven Miller's and David Montague's use of tiling with regular polygons for small values of N in [8].

References

1. T. Estermann, The irrationality of $\sqrt{2}$, *Math. Gaz.* **59** (June 1975) p.110.
2. M. Grant and M. Perella, Descending to the irrational, *Math. Gaz.* **83** (July 1999) pp. 263-267.
3. P. Shiu, More on Estermann and Pythagoras, *Math. Gaz.* **83** (July 1999) pp. 267-269.
4. A. L. Brown, The irrationality of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, …, *Math. Gaz.* **87** (March 2003) p. 143.
5. N. Lord, Using A4-sized paper to illustrate that $\sqrt{2}$ is irrational, *Math. Gaz.* **101** (March 2017) pp. 142-145.
6. T. Jackson, Irrational square roots of natural numbers – a geometrical approach, *Math. Gaz.* **95** (July 2011) pp. 327-330.
7. D. G. Ball, Cutting squares from rectangles, *Math. Gaz.* **58** (June 1974) pp. 74-77.
8. S. J. Miller and D. Montague, Picturing irrationality, *Math. Mag.* **85** (2012) pp.110-114.

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102.24 Area of a circle via the Fundamental Theorem of Calculus

In a recent article [1], Denis Bell showed that it is possible to compute the area of a circle using the Second Fundamental Theorem of Calculus and the function $F(x) = \int_{\cos x}^{\sin x} \sqrt{1 - t^2}\, dt$. This avoids the use of trigonometric substitutions, and thus can be taught earlier in the calculus sequence.

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