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Resolving kinematic redundancy of a robot using a quadratically constrained optimization technique

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SUMMARY

The constraints on the physical limit should be considered in a kinematic redundancy resolution problem of a robot. This paper proposes a new optimization scheme to resolve kinematic redundancy of the robot while considering physical constraints. In the proposed scheme, quadratic inequality constraints are used in place of linear inequality constraints, thus a quadratically constrained optimization technique is applied to resolve the redundancy. It is shown that the use of quadratic inequality constraints considerably reduces the number of constraints. Therefore, the proposed method reduces the problem size considerably and makes the problem simple resulting in computational efficiency. A numerical example of a 4-link planar redundant robot is included to demonstrate the efficiency of the proposed optimization technique. In this example, simulation results using the proposed method and another well-known method are compared and discussed.

KEYWORDS: Redundant robot; Redundancy resolution; Optimization technique; Efficiency; Physical constraints.

1. INTRODUCTION

Recently, many researchers have been interested in the area of kinematically redundant robots. Redundant robots have more degrees of freedom than the space they are working in, and thus the arising redundancy should be resolved to accomplish additional objectives in addition to the pre-specified end-effector task. In redundancy resolution problem, kinetic energy,[1–3] torque[3,4] and weighted velocity norm[1,3,5] minimization; physical limit avoidance,[3,6–10] obstacle avoidance,[10,11] cyclic joint motion conservation,[9,11,12] motion stabilization,[4] singularity avoidance,[11–13] etc. have been achieved as the additional objectives.

Consider the differentiated kinematic equation of a robot manipulator with respect to time:

$$\dot{x}=J(\theta)\dot{\theta} \tag{1}$$

where $\dot{x}$ is an $[m\times 1]$ end-effector velocity vector, $J(\theta)$ is an $[m\times n]$ Jacobian matrix, and $\dot{\theta}$ denotes an $[n\times 1]$ joint velocity vector. Because of the relation $m < n$ of redundant manipulators, an infinite number of joint velocity solution $\dot{\theta}$ exist for a given $\dot{x}$ in (1). Resolving redundancy is to determine the solution trajectory $\dot{\theta}(t)$ of (1) for a given end-effector trajectory $\dot{x}(t)$, in order to achieve additional objectives mentioned above.

There are two possible approaches to resolve redundancy: global optimization and local optimization.[1,2] Global optimization schemes solve the problem by optimizing a performance criterion over the complete time interval. Since global schemes are computationally intensive and require complete trajectory information in advance, local optimization schemes, which solve the problem at each point along the end-effector task, are used for real time application.

Various local optimization techniques have been developed to solve the redundancy resolution problem. Conventionally, pseudo-inverse based methods have been proposed, in which the general solution for (1) is obtained as

$$\dot{\theta}=J^{\dagger}\dot{x}+(I_n-J^{\dagger}J)z \tag{2}$$

where $J^{\dagger}$ denotes the $[n\times m]$ pseudo-inverse matrix of J, I_n is the $[n\times n]$ identity matrix, and z is an $[n\times 1]$ arbitrary vector which is to be chosen for proper performance optimization.[6] Hollerbach and Suh[3] proposed several torque optimization schemes using the pseudo-inverse matrix and null space vector with and without weighting matrix. Ma and Nenchev[4] presented a formulation for torque minimization by using pseudo-inverse matrix. Dubey et al.[7] developed a real-time control scheme using the gradient projection method for 7-DOF redundant manipulators. Chan and Dubey[8] proposed weighted least-norm solution scheme for joint limit avoidance and compared it with the gradient projection method. Zhou and Nguyen[9] used a state augmentation method in conjunction with the Pontryagin's maximum principle to avoid joint limits and non-conservation problem. But these methods cannot directly deal with various physical limit constraints such as joint velocity and torque limits, and the capability of real-time application was not mentioned except in references 7 and 8.

In order to deal with various physical limits, several new redundancy resolution methods using *inequality constraints* in their formulation have been proposed recently.[5,10–16] Pin and Tulloch[10] presented an approach utilizing the recently proposed Full Space Parameterization (FSP) method and considering inequality constraints to avoid joint limits and obstacles. Sung et al.[13] proposed a velocity level constrained optimization algorithm with position level objective function using Lagrange multiplier. Yet, as they pointed out, their algorithm have some limitations in particular cases.

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Seraji and Colbaugh[14] used the improved configuration control scheme in which inactive inequality constraints are ignored and only active ones are considered. But this method has difficulties in determining the weighting factors.[13] Abdel-Rahman[15] presented a joint constraint mapping approach to consider joint position, velocity and acceleration limits in resolving the kinematic redundancy. But the computational requirements are high.[12] Park et al.[16] also proposed a formulation of a constrained optimization problem with nonlinear inequality constraints, such as joint limits and obstacles, and applied the gradient projection method.

If a quadratic objective function with linear inequality constraints are used in the formulation on behalf of general objective functions and general inequality constraints, the problem can be solved more easily and efficiently. Carignan[5] formulated an optimization problem to minimize the weighted velocity norm with linear inequality constraints and solved the dual problem instead of the original one. However, it is computationally inefficient since the problem size is not reduced. Cheng et al.[11,12] proposed the Compact QP (quadratic programming) method to efficiently solve the redundancy resolution problem by using Goldfarb and Idnani's efficient QP routine.[17] In their method, the problem is formulated as a QP problem with linear inequality constraints which include joint position, velocity and torque limits.

In this paper, the redundancy resolution problem is formulated as a quadratically constrained quadratic programming (QCQP) problem and an optimization technique applying the duality theory[18,19] is proposed to solve the problem more efficiently. It is shown that the use of quadratic inequality constraints (QICs) makes the problem size smaller than is made when linear inequality constraints are used. The QICs in this scheme are approximated versions of conventional linear inequality constraints.[20–22] The efficiency of the proposed scheme, called the *Dual QCQP method*, is demonstrated in a numerical example of a 4-link planar redundant robot in comparison with the Compact QP method.[12] Kwon and Lee,[21,22] showed that the QCQP formulation with the duality theory could be used for real time force optimization scheme of multiple cooperating robots. The main contribution of this paper is to show that the use of QICs combined with the duality theory can also be successfully applied to real time implementation of resolving redundancy.

This paper is organized as follows: In Section 2, linear and quadratic inequality constraints for physical limits of redundant robots are constructed. The redundancy resolution problem is formulated as a QCQP problem in Section 3 and the Dual QCQP method is proposed and verified to efficiently solve the problem, in Section 4. Finally, a numerical example is presented to show the efficiency of the proposed scheme, followed by conclusions.

2. INEQUALITY CONSTRAINTS

Joint position, velocity and torque limits are taken into account as physical limit constraints of redundant robots, in this paper. These constraints are converted into inequalities in terms of joint velocity, since the variable used in the optimization problem is the joint velocity. Other physical limit constraints which can be represented with respect to joint velocity term can also be considered in the same manner.

The inequality constraints in conventional methods are linear inequalities, since it is more convenient to solve a linearly constrained problem than a nonlinearly constrained one. In this section, these linear inequality constraints are obtained at first, and then it is shown that QICs can be obtained by approximating these linear inequality constraints.

2.1 Linear Inequality Constraints

The joint position, velocity and torque limits are to be described in terms of joint velocity. First, the joint velocity limits are described as

$$\dot{\theta}_{\mathrm{l}} \le \dot{\theta} \le \dot{\theta}_{\mathrm{u}} \tag{3}$$

where $\dot{\theta}_{\mathrm{l}}(\dot{\theta}_{\mathrm{u}})$ is an $[n \times 1]$ vector in which the i-th element denotes the lower (upper) limit of the i-th joint velocity. Hereafter, it is considered that an inequality between two vectors represents inequalities between corresponding individual elements of the two vectors.

The joint position bounds determined by configuration of the robot are

$$\theta_{\mathrm{l}} \le \theta \le \theta_{\mathrm{u}} \tag{4}$$

where θ_{l} and θ_{u} are lower and upper limit vectors of the joint position, respectively. Given the i-th joint position value at time $t - \Delta t$, $\theta_i(t - \Delta t)$, where Δt is the sampling time, the joint velocity value at time t is described by the following first order Taylor expansion,

$$\dot{\theta}_i(t) \approx \frac{\Delta\theta_i}{\Delta t} = \frac{\theta_i(t) - \theta_i(t - \Delta t)}{\Delta t}. \tag{5}$$

Substituting (4) into (5), the joint position constraints are mapped on joint velocity space and the following new joint velocity constraints in linear inequalities are yielded,[15]

$$\frac{\theta_{li} - \theta_i(t - \Delta t)}{\Delta t} \le \dot{\theta}_i(t) \le \frac{\theta_{ui} - \theta_i(t - \Delta t)}{\Delta t} \tag{6}$$

where θ_{li} (θ_{ui}) represents the i-th element of $\theta_{\mathrm{l}}(\theta_{\mathrm{u}})$. Combining the joint velocity constraints (3) and the joint position constraints (6) yields the following inequalities in vector notation as

$$\dot{\theta}_{\min}(t) \le \dot{\theta}(t) \le \dot{\theta}_{\max}(t) \tag{7}$$

where

$$\dot{\theta}_{\min}(t) \triangleq max\left[\dot{\theta}_{\mathrm{l}}, \frac{\theta_{\mathrm{l}} - \theta(t - \Delta t)}{\Delta t}\right]$$

$$\dot{\theta}_{\max}(t) \triangleq min\left[\dot{\theta}_{\mathrm{u}}, \frac{\theta_u - \theta(t - \Delta t)}{\Delta t}\right].$$

If robot dynamics is taken into account, joint torque bounds should be introduced as

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$$\tau_1 \le \tau \le \tau_u \tag{8}$$

where $\tau_1(\tau_u)$ is an $[n \times 1]$ vector meaning lower(upper) limit of the joint torque. Consider the robot dynamics equation which relates joint acceleration to joint torque as

$$\tau = D\ddot{\theta} + \zeta \tag{9}$$

where D is the inertia matrix of the robot and ζ is the nonlinear term that includes Coriolis, centrifugal and gravity terms. Combining (8) with (9) yields

$$\tau_1 - \zeta \le D\ddot{\theta} \le \tau_u - \zeta \tag{10}$$

where D and ζ are evaluated at time $t - \Delta t$. The following first order Taylor expansion of the joint acceleration is also required to convert these torque constraints into inequalities in terms of joint velocity,

$$\ddot{\theta}_i(t) \approx \frac{\Delta \dot{\theta}_i}{\Delta t} = \frac{\dot{\theta}_i(t) - \dot{\theta}_i(t - \Delta t)}{\Delta t}. \tag{11}$$

Combining (10) with (11) leads to new joint velocity constraints[12] in linear inequalities as

$$(\tau_1 - \zeta)\Delta t + D\dot{\theta}(t - \Delta t) \le D\dot{\theta}(t) \le (\tau_u - \zeta)\Delta t + D\dot{\theta}(t - \Delta t) \tag{12}$$

or

$$\phi_1(t) \le D\dot{\theta}(t) \le \phi_u(t) \tag{13}$$

where

$$\phi_1(t) \triangleq (\tau_1 - \zeta)\Delta t + D\dot{\theta}(t - \Delta t)$$
$$\phi_u(t) \triangleq (\tau_u - \zeta)\Delta t + D\dot{\theta}(t - \Delta t).$$

2.2 Quadratic Inequality Constraints

The QICs can be obtained by approximating linear inequalities as follows. Consider the following two linear inequalities leading to 2-dimensional rectangular feasible region as

$$-a_1 \le x_1 \le a_1 \tag{14}$$

$$-a_2 \le x_2 \le a_2 \tag{15}$$

where x_1, x_2 $a_1(>0)$ and a_2 (>0) are scalar values. This 2-dimensional rectangular region can be approximated as a 2-dimensional ellipse as shown in Fig. 1. Thus, the approximated quadratic inequality is described as

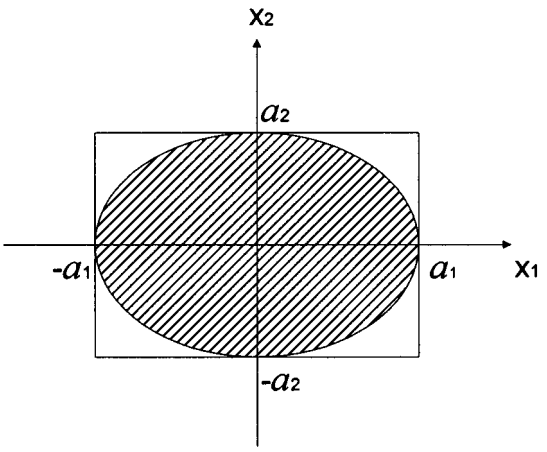


Fig. 1. Approximation of rectangular region.

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} \le 1. \tag{16}$$

The QICs representing joint position, velocity and torque bounds are obtained by approximating the linear inequalities (7) and (13) in the same manner. For simplicity of the notation, the time index t in (7) and (13) is omitted. The linear inequalities (7) are rewritten as

$$\frac{\dot{\theta}_{\min} - \dot{\theta}_{\max}}{2} \le \dot{\theta} - \frac{\dot{\theta}_{\max} + \dot{\theta}_{\min}}{2} \le \frac{\dot{\theta}_{\max} - \dot{\theta}_{\min}}{2} \tag{17}$$

or

$$-\dot{\theta}_{\text{mid}} \le \dot{\theta} - \dot{\theta}_{\text{av}} \le \dot{\theta}_{\text{mid}} \tag{18}$$

where

$$\dot{\theta}_{\text{mid}} \triangleq \frac{\dot{\theta}_{\max} - \dot{\theta}_{\min}}{2} \qquad \dot{\theta}_{\text{av}} \triangleq \frac{\dot{\theta}_{\max} + \dot{\theta}_{\min}}{2}.$$

Then, through the approximation of the n-dimensional rectangloidal region induced by (18) into n-dimensional ellipsoidal region, the approximated QICs are obtained as

$$\dot{\theta}^T A\, \dot{\theta} + b^T \dot{\theta} + c \le 1 \tag{19}$$

where

$$A \triangleq diag\left\{ \frac{1}{(\dot{\theta}_{\text{mid}1})^2}, \ldots, \frac{1}{(\dot{\theta}_{\text{mid}n})^2} \right\}$$

$$b \triangleq -2A\, \dot{\theta}_{\text{av}}$$

$$c \triangleq \dot{\theta}_{\text{av}}^T A\, \dot{\theta}_{\text{av}}$$

in which $\dot{\theta}_{\text{mid}i}$ denotes the i-th individual element of $\dot{\theta}_{\text{mid}}$.

The inequality constraints (13) are rewritten as

$$\frac{\phi_1 - \phi_u}{2} \le D\dot{\theta} - \frac{\phi_u + \phi_1}{2} \le \frac{\phi_u - \phi_1}{2} \tag{20}$$

or

$$-\phi_{\text{mid}} \le D\dot{\theta} - \phi_{\text{av}} \le \phi_{\text{mid}} \tag{21}$$

where

$$\phi_{\text{mid}} \triangleq \frac{\phi_u - \phi_1}{2} \qquad \phi_{\text{av}} \triangleq \frac{\phi_u + \phi_1}{2}.$$

Similarly, the QICs corresponding to these inequalities are obtained as

$$\dot{\theta}^T \tilde{A} \dot{\theta} + \tilde{b}^T \dot{\theta} + \tilde{c} \le 1 \tag{22}$$

where

$$\tilde{A} \triangleq D^T W D$$

$$\tilde{b} \triangleq -2D^T W \phi_{\text{av}}$$

$$\tilde{c} \triangleq \phi_{\text{av}}{}^T W \phi_{\text{av}}$$

$$W \triangleq diag\left\{ \frac{1}{(\phi_{\text{mid}1})^2}, \ldots, \frac{1}{(\phi_{\text{mid}n})^2} \right\}.$$

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in which ϕ_{midi} denotes the i-th individual element of ϕ_{mid}.

Besides the two typical QICs (19) and (22), other QICs can also be obtained from some inequality constraints if they create a feasible region which can be approximated by an ellipsoidal region.

3. PROBLEM FORMULATION

The redundancy resolution problem is formulated as a QCQP problem with a quadratic objective function, an equality constraint and some QICs. The general quadratic objective function is expressed as

$$f(\dot{\theta}) = \dot{\theta}^T P \dot{\theta} + q^T \dot{\theta} \tag{23}$$

where P is an $[n \times n]$ symmetric positive definite matrix. Literature surveys show that many objectives in the redundancy resolution problem can be represented as quadratic functions, such as kinetic energy,[1–3] torque[3,4] and weighted velocity norm[1,3,5] minimization; physical limit avoidance,[3,6] obstacle avoidance,[11] cyclic path repeatability,[11,12] etc. All these objectives can be included in (23) by simply substituting proper matrix and vector for P and q in (23).

The kinematic equation (1) is considered as the equality constraint, and QICs such as (19) and (22) are also employed as the physical constraints. Then, a quadratically constrained optimization problem is formulated as

$$\begin{aligned} &\text{Minimize} && f(\dot{\theta}) = \dot{\theta}^T P \dot{\theta} + q^T \dot{\theta} \\ &\text{subject to} && h(\dot{\theta}) \triangleq J\dot{\theta} - \dot{x} = 0 \\ & && g_1(\dot{\theta}) \triangleq \dot{\theta}^T A_1 \dot{\theta} + b_1{}^T \dot{\theta} + c_1 - 1 \le 0 \\ & && \vdots \\ & && g_l(\dot{\theta}) \triangleq \dot{\theta}^T A_l \dot{\theta} + b_l{}^T \dot{\theta} + c_l - 1 \le 0 \end{aligned} \tag{24}$$

where l is the number of QICs employed and A_i for $i = 1, \ldots, l$ are symmetric positive semidefinite matrices. Note that the number of inequality constraints is reduced to l from $2nl$ of non-approximated problem which has linear inequalities such as (18) and (21), and thus the size of the problem is considerably reduced.

The linear equality constraint $h(\dot{\theta}) = 0$ is merged into the variables by the compact formulation.[12] It is assumed that J has full rank m. Then, J can be decomposed into $[m \times m]$ linear independent block and remaining block by permutation of the columns. The equality constraint $h(\dot{\theta}) = J\dot{\theta} - \dot{x} = 0$ is rewritten as

$$h(\dot{\theta}) = [J_1\ J_2] \begin{bmatrix} \dot{\theta}_b \\ \dot{\varphi} \end{bmatrix} - \dot{x} = 0 \tag{25}$$

where J_1 is $[m \times m]$ linear independent block, J_2 is $[m \times r]$ remaining block, and $\dot{\theta}_b$, $\dot{\varphi}$ are corresponding partial vectors, respectively, and $r \triangleq n - m$ is defined as the degree of redundancy (DOR). The solution of (25) is decomposed as

$$\dot{\theta} = \begin{bmatrix} \dot{\theta}_b \\ \dot{\varphi} \end{bmatrix} = \dot{\theta}_p + N\dot{\varphi} \tag{26}$$

where $\dot{\theta}_p \triangleq \begin{bmatrix} J_1{}^{-1}\dot{x} \\ 0_r \end{bmatrix}$ is the particular solution and $N \triangleq \begin{bmatrix} -J_1{}^{-1}J_2 \\ I_r \end{bmatrix}$ is $[n \times r]$ matrix which maps $\dot{\varphi}$ onto the null space of J, and 0_r and I_r are the $[r \times 1]$ zero vector and the $[r \times r]$ identity matrix, respectively. Substituting (26) into (24) yields the following *primal problem*,

$$\begin{aligned} &\text{Minimize} && \hat{f}(\dot{\varphi}) = \dot{\varphi}^T \hat{P} \dot{\varphi} + \hat{q}^T \dot{\varphi} \\ &\text{subject to} && \hat{g}_1(\dot{\varphi}) \triangleq \dot{\varphi}^T \hat{A}_1 \dot{\varphi} + \hat{b}_1^T \dot{\varphi} + \hat{c}_1 - 1 \le 0 \\ & && \vdots \\ & && \hat{g}_l(\dot{\varphi}) \triangleq \dot{\varphi}^T \hat{A}_l \dot{\varphi} + \hat{b}_l^T \dot{\varphi} + \hat{c}_l - 1 \le 0 \end{aligned} \tag{27}$$

where

$$\begin{aligned} \hat{f}(\dot{\varphi}) &\triangleq f(\dot{\theta}) - q^T\dot{\theta}_p - \dot{\theta}_p^T P \dot{\theta}_p \\ \hat{P} &\triangleq N^T P N \\ \hat{q} &\triangleq N^T(q + 2P\dot{\theta}_p) \\ \hat{A}_i &\triangleq N^T A_i N \\ \hat{b}_i &\triangleq N^T(b_i + 2A_i\dot{\theta}_p) \\ \hat{c}_i &\triangleq c_i + \dot{\theta}_p^T A_i \dot{\theta}_p + b_i^T \dot{\theta}_p \quad \text{for } i = 1, \ldots, l. \end{aligned}$$

In order to efficiently solve this problem, the Dual QCQP method is devised in the following Section.

4. DUAL QCQP METHOD

Usually, it is difficult to solve the primal problem (27) directly, since it has nonlinear constraints. It is shown that this problem can be converted into a dual problem by applying the duality theory and that the dual problem can be solved easily. This method is called the Dual QCQP method.

Given a nonlinear programming (NLP) problem, there exists another NLP problem closely associated with it.[18,19] The former is called the primal problem and the latter is called the dual problem. In the duality theory, the Dual method does not solve the original primal problem directly but instead solves the dual problem, whose unknowns are the Lagrange multiplier of the primal problem.

Let us consider the following *dual problem* which is associated with the primal problem (27),

$$\begin{aligned} &\text{Maximize} && \Psi(\mu) \\ &\text{subject to} && \mu \ge 0 \end{aligned} \tag{28}$$

where $\Psi(\mu)$ is the dual function of the Lagrangian $\hat{l}(\dot{\varphi}) \triangleq \hat{f}(\dot{\varphi}) + \mu^T \hat{g}(\dot{\varphi})$ defined on $\mu(:[l \times 1]) \ge 0$ such that

$$\Psi(\mu) \triangleq \min_{\dot{\varphi}} [\hat{f}(\dot{\varphi}) + \mu^T \hat{g}(\dot{\varphi})] \tag{29}$$

where $\hat{g}(\dot{\varphi}) \triangleq [\hat{g}_1(\dot{\varphi}), \ldots, \hat{g}_l(\dot{\varphi})]^T$.

The constraints of this problem are only l simple non-negativity conditions, which are fairly simplified when compared with $2nl$ linear inequality constraints in conventional non-approximated problem. Besides, only

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l-dimensional search problem need to be solved instead of r-dimensional problem, which is simplified when $l < r$, where r is the dimension of the search space of the primal problem.

In order to solve this problem, some theoretical bases are needed. First, it needs to be proven that the Lagrangian $\hat{l}(\dot{\varphi})$ is a strictly convex function for any $\mu \geq 0$. The Hessian of the objective function, $\nabla^2\hat{f}(\dot{\varphi})$, is immediately obtained as $2\hat{P}$, which is positive definite by definition, and the Hessian of $\hat{g}(\dot{\varphi})$ is $\nabla^2\hat{g}(\dot{\varphi}) = 2[\hat{A}_1^T \ \ldots \ \hat{A}_l^T]^T$. Since $\hat{A}_1, \ldots, \hat{A}_l$ are positive semidefinite by definition, $\mu^T\nabla^2\hat{g}(\dot{\varphi})$ is also positive semidefinite for any $\mu \geq 0$. Thus, the Hessian of the Lagrangian, $\nabla^2\hat{l}(\dot{\varphi}) = \nabla^2\hat{f}(\dot{\varphi}) + \mu^T\nabla^2\hat{g}(\dot{\varphi})$, is positive definite, and so, the Langrian $\hat{l}(\dot{\varphi})$ is strictly convex for any $\mu \geq 0$. Then, the Lagrangian $\hat{l}(\dot{\varphi})$ always has a unique finite minimum value for any $\mu \geq 0$, and so, the dual function $\Psi(\mu)$ is well-defined on $\mu \geq 0$; that is, Ψ assigns exactly one finite value to each point that has nonnegative elements in μ-space. Hence, the formulation of the dual problem (28) can be justified.

It is known that solutions of a primal problem and its dual problem are not the same, in general, as the duality gap shows.[19] But, in this case, the solution of the primal problem (27) can be obtained via the solution of the dual problem (28) by the following proposition.

Proposition 1. *The solution of the primal problem (27) corresponds to the solution of the dual problem (28).*

Proof: Suppose that the primal problem (27) has a local solution at $\dot{\varphi}^*$ with a corresponding minimum value ρ^* and Lagrange multiplier μ^*. Since the Hessian of the Lagrangian is positive definite, the dual problem (28) has a local solution at μ^* with corresponding $\dot{\varphi}^*$ and maximum value ρ^* by the Local Duality Theorem.[18] Since the objective function $\Psi(\mu)$ of the dual problem (28) is known to be concave on $\mu \geq 0$,[18] any local solution of the dual problem (28) is also the global solution. Because the objective function $\hat{f}(\dot{\varphi})$ is strictly convex for all $\dot{\varphi}$ and the feasible region by the QICs is convex, the minimum solution of $\hat{f}(\dot{\varphi})$ is global and unique over the feasible region. Therefore, the primal problem (27) also has a global and unique solution. This completes the proof. □

Proposition 1 means that a QCQP problem can be transformed to a dual problem which has only simple non-negativity constraints. Hence, we have only to solve the dual problem (28) that has a smaller problem size than primal problem. But, in general, it is known to be difficult to solve the dual problem (28) since the evaluation of $\Psi(\mu)$ includes a minimization subproblem of (29). This difficulty is overcome by the following proposition where the explicit expression of global minimum solution $\bar{\varphi}$ of this minimization is obtained in a closed form with respect to μ.

Proposition 2. *The unique global minimum solution* $\bar{\varphi}$ *of* $\hat{f}(\dot{\varphi}) + \mu^T\hat{g}(\dot{\varphi})$ *in* (29) *is described as*

$$\bar{\varphi} = \bar{\varphi}(\mu) = -2\left(\hat{P} + \sum_{i=1}^{l} \mu_i \hat{A}_i\right)^{-1}\left(\hat{q} + \sum_{i=1}^{l} \mu_i \hat{b}_i\right). \quad (30)$$

Proof: The first order necessary condition of the minimum point of the Lagrangian yields

$$\nabla\hat{l}(\dot{\varphi}) = \nabla\hat{f}(\dot{\varphi}) + \mu^T\nabla\hat{g}(\dot{\varphi}) = 0. \quad (31)$$

Since $\hat{l}(\dot{\varphi})$ is strictly convex for any $\mu \geq 0$, (31) is also the sufficient condition of minimum solution of the Lagrangian. Some calculation for this condition leads to

$$\nabla\hat{l}(\dot{\varphi}) = 2\dot{\varphi}^T\hat{P} + \hat{q}^T + \mu^T\begin{bmatrix} 2\dot{\varphi}^T\hat{A}_1 + \hat{b}_1^T \\ \vdots \\ 2\dot{\varphi}^T\hat{A}_l + \hat{b}_l^T \end{bmatrix}. \quad (32)$$

Hence, the solution $\bar{\varphi}$ of $\nabla\hat{l}(\dot{\varphi}) = 0$ can be explicitly expressed as (30). □

By this proposition, the dual function $\Psi(\mu)$ is always evaluated explicitly in terms of μ as

$$\Psi(\mu) = \hat{f}(\bar{\varphi}(\mu)) + \mu^T\hat{g}(\bar{\varphi}(\mu)). \quad (33)$$

Since $\Psi(\mu)$ is a concave function in the region $\mu \geq 0$[18], the global maximum of $\Psi(\mu)$ can be easily obtained by slightly modified unconstrained NLP techniques. We use the projected steepest ascent method,[19] which is a simple NLP search technique based on gradient direction, in order to solve the dual problem (28). The dual function $\Psi(\mu)$ has gradient $\nabla\Psi(\mu) = \hat{g}(\bar{\varphi}(\mu))^T$ which can be proved from the first order necessary condition of the Lagrangian.[18] This fact is of practical importance, since it simplifies the calculation of the gradient by evaluating the corresponding constraints without further calculation.

Until now, we have discussed the Dual QCQP method to solve the primal problem (27) indirectly via the dual problem (28). The key point of this method is that we can reduce the problem size by solving the dual problem instead of the primal problem owing to the employment of QICs. This implies that the dual problem (28) can be more efficiently solved than the primal problem. In the following Section, this fact is demonstrated by a numerical example.

5. NUMERICAL EXAMPLE

The optimization scheme presented in this paper is implemented on a simulation of a 4-link planar redundant robot as depicted in Fig. 2, and compared with an existing efficient

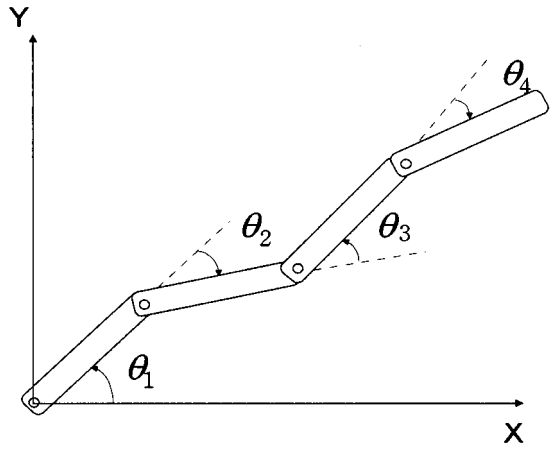


Fig. 2. A 4-link planar robot.

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method – the Compact QP method[12] which utilises conventional linear inequality constraints. The simulation block diagram of this system is illustrated in Fig. 3. The input data for the simulation are the desired Cartesian position of current instant and the joint value of the previous instant, which are denoted as $x_d(t)$ and $\theta(t-\Delta t)$, respectively, in Fig. 3. The value $x_d(t)$ is given by off-line path planning of the tested paths, and $\theta(t-\Delta t)$ is obtained from the optimal solution of the previous instant, $\dot{\theta}^*(t-\Delta t)$.

The length, mass and inertia of each link are assumed to be 1m, 1kg and $\frac{1}{12}$kgm^2, respectively, and the gravity term is ignored in the dynamics equation (9). The center of mass of each link is located on the middle point of the link and the mass distribution is assumed to be uniform.

Let the equally weighted joint velocity norm $\dot{\theta}^T\dot{\theta}$ be the objective function. The Jacobian relation, which is used as the equality constraint, is expressed as $J(\theta(t-\Delta t))\dot{\theta}(t) = \dot{x}_d(t)$ where $\dot{x}_d(t) \triangleq \frac{x_d(t)-x(t-\Delta t)}{\Delta t}$. The joint position, velocity and torque bounds are assumed to be $(\pm\pi, \pm\pi, \pm\pi, \pm\pi)$, $(\pm\pi, \pm\pi, \pm\frac{\pi}{3}, \pm\pi)$, and $(\pm 100, \pm 100, \pm 100, \pm 100)$, respectively. We let the velocity limit of joint 3 be $\frac{1}{3}$ times as the others to make the solution profile reach the boundary value. The obtained optimal solution $\dot{\theta}^*(t)$ is integrated to make joint position which can be used as a command input in the robot controller. In this example, the robot controller is virtual and it only provides the joint position of the previous instant, $\theta(t-\Delta t)$.

The simulated movements are 3 straight line Cartesian paths; denoted as Path 1, 2 and 3 as depicted in Fig. 4. Each line motion has velocity profiles with zero starting and ending velocity with parabolic blends. The initial joint configuration of each path is $(\frac{\pi}{4}, -\frac{\pi}{3}, \frac{\pi}{3}, 0)$. The times taken to move on the line paths are assumed to be 2sec, 1sec, and 2 sec, respectively.

In the formulation of this example, two QICs as (19) and (22) are used for joint position, velocity and torque bounds. The problem size, namely the number of variables and inequality constraints, of the dual problem in the proposed method and that of the Compact QP method are exhibited in Table I. Note that the problem size is considerably reduced in the proposed method. That is, the number of variables of the Dual QCQP method is l which is equal to or smaller than that of the Compact QP method when $l \le n - m(=r)$. And, the number of inequality constraints of the former is l which is also smaller than that of the latter, $2nl$. These facts imply that the computation of the proposed method will be more efficient. On the other hand, when $l > r$, as a robot with only one DOR, it is difficult to predict which method is more efficient, since the superiority of the problem size is ambiguous: The Compact QP method has still more inequality constraints. Therefore, it is noted that the proposed method is well-suited particularly for the case $l \le r$, i.e. redundant manipulators with two or larger DOR and hyper-redundant manipulators.

The computation is performed on Pentium 200MHz CPU with 32MB RAM using C in DOS environment. The mean computation times taken to solve the problem for each path are exhibited in Table II. The results of Table II show that the proposed Dual QCQP method is more efficient in computation time, at least twice the speed, and has the capability of real time application. It is noteworthy that the speed of CPU is getting faster and even the worst case of this example may still be fast enough to enable real time application. This computation time, however, is the result of the presented example only, and in other cases, the computational weight may be so large that real time application could be impossible. Therefore, using QICs to reduce computation time via the reduction of constraints has its own value.

To achieve higher accuracy of the optimal solution, more computation time, that is, more iterations of the algorithm need to be executed. But, since excessive iterations considerably increase the computational weight, proper stopping criteria should be determined. In this paper's simulation, the stopping criteria of the algorithm are determined by iterative tuning of related parameters so as to obtain the solution within predetermined tolerance limit. We choose the stopping criteria considering both computation time and the accuracy of the solution. In addition, since the proposed Dual QCQP method employs the *approximated* QICs in place of the *exact* linear inequality constraints, the optimal solution of the proposed method may also be

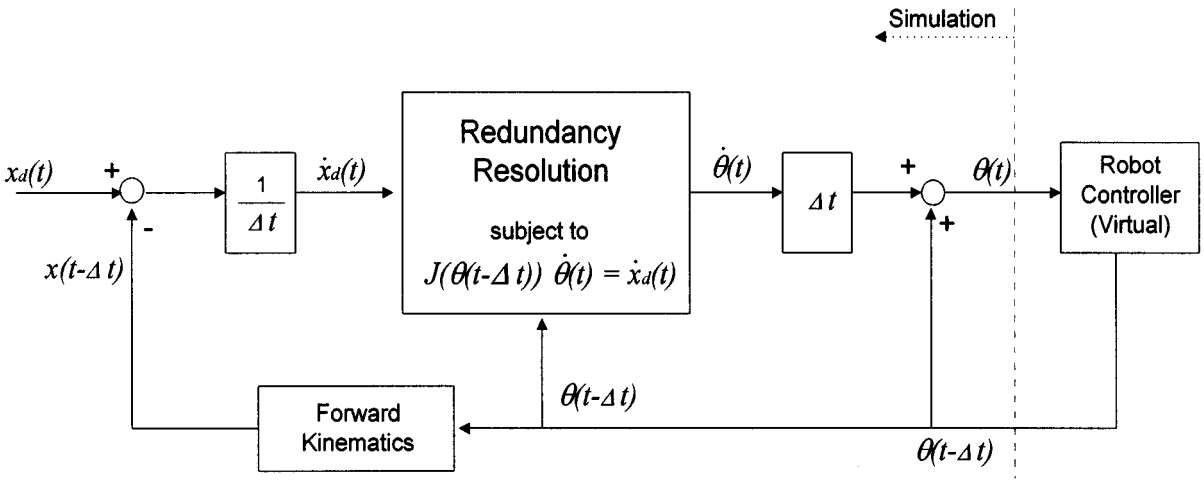


Fig. 3. Simulation block diagram of the system.

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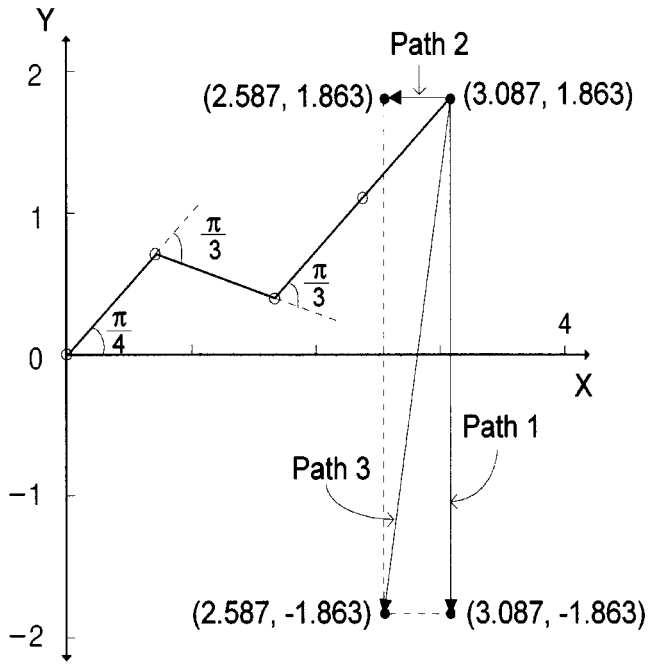


Fig. 4. Tested paths.

different from the solution obtained from linear inequality constraints, due to loss of the feasible region. In our previous work[22] we proposed a measure to represent the loss of the feasible region, AR of QICs, defined as the ratio of the volume of the approximated region to the original region. We leave the application of this measure for reducing the effect of the loss and increasing the accuracy, as a future research topic. In Figs 5 and 6, the resulting optimal joint position, velocity and torque profiles of Path 3 of both methods are illustrated to intuitively estimate the accuracy of the solutions. The mean absolute difference of the joint velocity solution between both methods in this task is about 0.05 rad/sec which is believed to be sufficiently small.

As shown in Fig. 6(b), the optimal velocity of joint 3 reaches the upper bound $\frac{\pi}{3}$ (≈ 1.05) rad/sec in the neighborhood of 0.2 sec in the Compact QP method. Also, as shown in Fig. 6(c), the optimal joint torque of joint 1 in the Compact QP method reaches the bound -100 N in the neighborhood of 0.1 sec. On the other hand, the optimal velocity and torque solutions in the Dual QCQP method do not reach the bound as shown in Figs 5(b) and 5(c), but, instead reach the bound of the QICs in problem (24). This is because the feasible region of the Dual QCQP method is approximated as an ellipsoidal region as shown in Fig. 1.

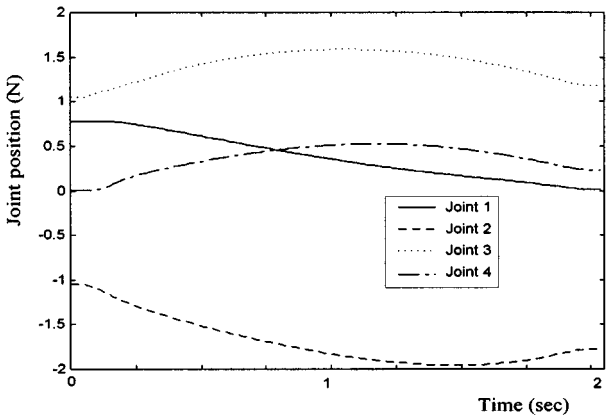
There exist peaks in the profile of joint position and joint torque obtained by the proposed method, as shown in Fig. 5. Also, peaks are found in the graph of the Compact QP method as in Fig. 6. Therefore, it seems that the peaks of the

Table I. Problem size of both methods

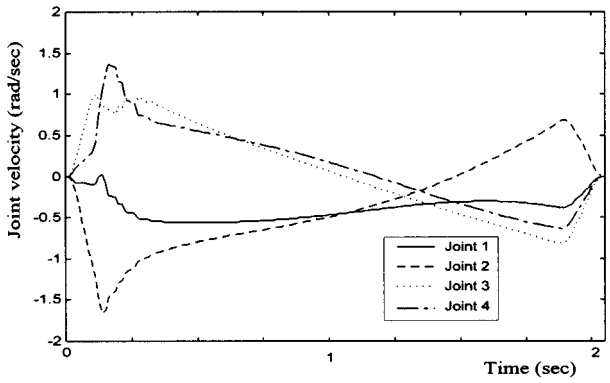
	Dual QCQP method	Compact QP method
Variables	2 (l)	2 (r)
Inequality Constraints	2 (l)	16 ($2nl$)

Table II. Computation time(msec) of both methods for each path

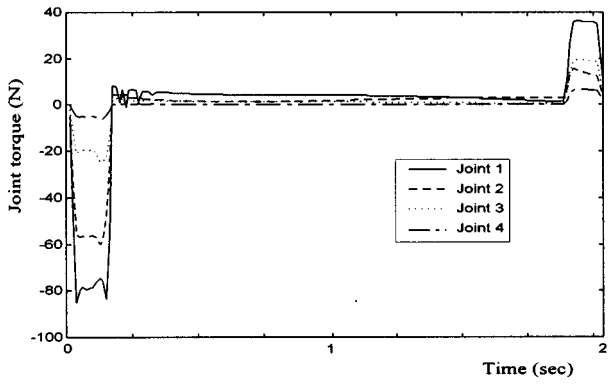
	Dual QCQP method	Compact QP method	speed-up
Path 1	0.18	0.41	2.28
Path 2	0.14	0.39	2.79
Path 3	0.21	0.42	2.00



(a) Joint position



(b) Joint velocity



(c) Joint torque

Fig. 5. Optimal solution of Path 3 by the Dual QCQP method.

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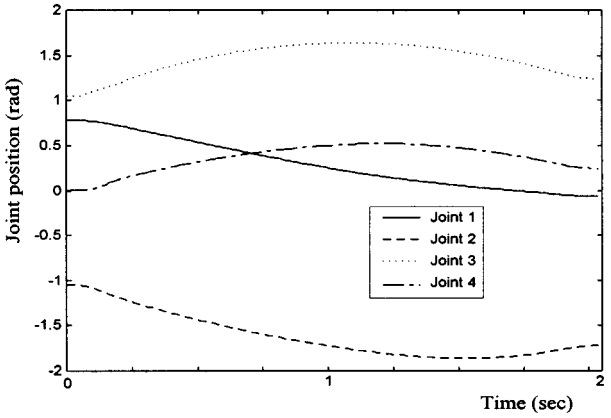
graphs are somewhat unavoidable depending on the given task. But, comparing Fig. 5 with Fig. 6 actually demonstrates that the proposed method is somewhat disadvantageous in the smoothness of the graph. We leave it as a future research topic.

6. CONCLUSIONS

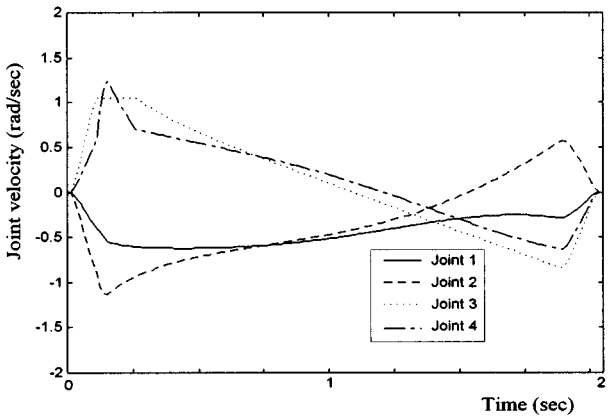
An efficient optimization method, called the Dual QCQP method, was proposed to resolve redundancy of a robot considering physical limit constraints. The quadratic inequality constraints (QICs) were used in this method to approximate linear inequality constraints which represent joint position, velocity and torque bounds. Therefore, the size of the problem was considerably reduced in that the number of constraints was reduced from $2nl$ to l, and that of variables from r to l. The proposed method was particularly well-suited for robots with two or larger DOR.

In the proposed method, the optimization problem was first formulated with a quadratic objective function, linear equality constraints and QICs. Then, the problem was converted into a dual problem by eliminating linear equality constraints and by applying the duality theory, and thus, was solved easily.

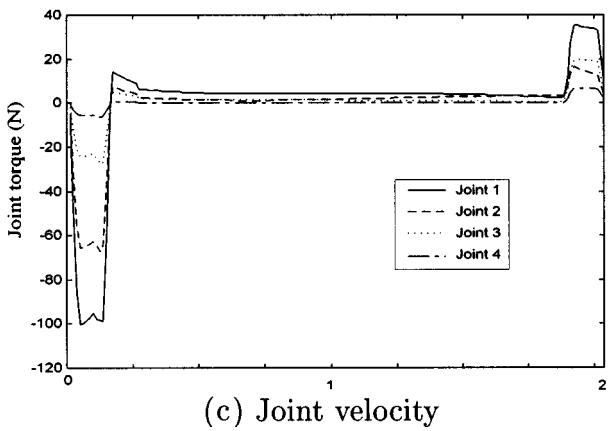
Simulation results demonstrated the efficiency of the proposed method by showing that the computation time was reduced to about $\frac{1}{2}$ times as that of an existing efficient method in the example. This work will provide a motive for the development of fast optimization techniques in redundancy resolution problem with physical constraints.



(a) Joint position



(b) Joint velocity



(c) Joint velocity

Fig. 6. Optimal solution of Path 3 by the Compact QP method.

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