Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 March 2023.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

106.I (Stan Dolan)

Find all non-square positive integers n such that the number of positive integer solutions of $x^2 - 4y^2 = n$ is odd.

106.J (David Bevan)

Given $n \in \mathbb{N}$ and $\varphi \in \mathbb{R}$, let $S = S(\varphi)$ be the set of n equally distributed points on the unit circle that contain the point $(\cos\varphi, \sin\varphi)$. Let R be the smallest axis-parallel rectangle enclosing S. For each $n > 1$, determine how many values of $\varphi \in [0, 2\pi/n)$ there are for which R is a square.

106.K (K. S. Bhanu and M. N. Deshpande)

Prove that $\displaystyle\sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n-1}{k}} = 1.$

What is the corresponding result for $\displaystyle\sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n}{k}}$?

106.L (Ankush Kumar Parcha and Toyesh Prakash Sharma)

Find the values of the following infinite nested radicals

(a) $\sqrt[3]{2 + 2\sqrt[3]{3 + 6\sqrt[3]{4 + 12\sqrt[3]{5 + 20\sqrt[3]{6 + 30\sqrt[3]{7 + \ldots}}}}}}$,

(b) $\sqrt[3]{-1 + 7\sqrt[3]{-1 + 13\sqrt[3]{-1 + 21\sqrt[3]{-1 + 31\sqrt[3]{-1 + 43\sqrt[3]{-1 + \ldots}}}}}}$.

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Solutions and comments on **106.A**, **106.B**, **106.C**, **106.D** (March 2022).

106.A (K. S. Bhanu and M. N. Deshpande)

A set of three positive integers (a, b, c) with $a < b < c$ is a *Diophantine triple* if there are integers x, y, z with $ab + 1 = x^2$, $ac + 1 = y^2$, $bc + 1 = z^2$.

Show that there are an infinite number of Diophantine triples such that x, y, z are in arithmetic progression.

Almost all the solutions received went along the following lines.

If $ab + 1 = x^2$, then $(a, b, a + b + 2x)$ is a Diophantine triple because

$$a(a + b + 2x) + 1 = a^2 + 2ax + ab + 1 = a^2 + 2ax + x^2 = (a + x)^2$$

and $$b(a + b + 2x) + 1 = b^2 + 2bx + ab + 1 = (b + x)^2.$$

We require $x, a + x, b + x$ to be in arithmetic progression so $2(a + x) = x + (b + x)$ or $b = 2a$. From $ab + 1 = x^2$ we thus obtain $x^2 - 2a^2 = 1$. This Pell equation has an infinite number of positive integer solutions (x, a) for which the associated Diophantine triple $(a, 2a, 3a + 2x)$ has $x, y = x + a, z = x + 2a$ in arithmetic progression.

Mark Hennings and Didier Pinchon gave the general solution (x_n, a_n) of $x^2 - 2a^2 = 1$ in the forms

$$x_n + a_n\sqrt{2} = \left(3 + 2\sqrt{2}\right)^n, \qquad n \geqslant 0$$

or
$$\begin{cases} a_0 = 0, \ a_1 = 2, \ a_n = 6a_{n-1} - a_{n-2}, \ n \geqslant 2, \\ x_0 = 1, \ x_1 = 3, \ x_n = 6x_{n-1} - x_{n-2}, \ n \geqslant 2. \end{cases}$$

The proposers, K. S. Bhanu and M. N. Deshpande, noted that $x = 2t + 1$ is necessarily odd and $a = 2s$ necessarily even, so that $x^2 - 2a^2 = 1$ is equivalent to the 'square-triangular number' condition $s^2 = \frac{1}{2}t(t + 1) = T_t$. That there are infinitely many such numbers does not need the full theory of the Pell equation: we can just iterate the observation that if $T_t = s^2$, then $T_{8s^2} = 4s^2\left(8s^2 + 1\right) = 4s^2(2t + 1)^2$.

Soham Bhadra's solution was based on the Pell equation $n^2 - 3m^2 = 1$. This has an infinite number of positive integer solutions for which $a = 2m - n$, $b = 2m$, $c = 2m + n$ produces the Diophantine triple $ab + 1 = (m - n)^2$, $ac + 1 = m^2$, $bc + 1 = (m + n)^2$: here both a, b, c and $x = m - n$, $y = m$, $z = m + n$ are in arithmetic progression (with $x < 0$). James Mundie's analysis gave this Pell equation and the previous one and indicated that no others will be involved.

Correct solutions were received from: R. G. Bardelang, S. Bhadra, S. Dolan, M. G. Elliott, M. Hennings, J. A. Mundie, D. Pinchon and the proposers K. S. Bhanu and M. N. Deshpande.

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106.B (Michael Fox)

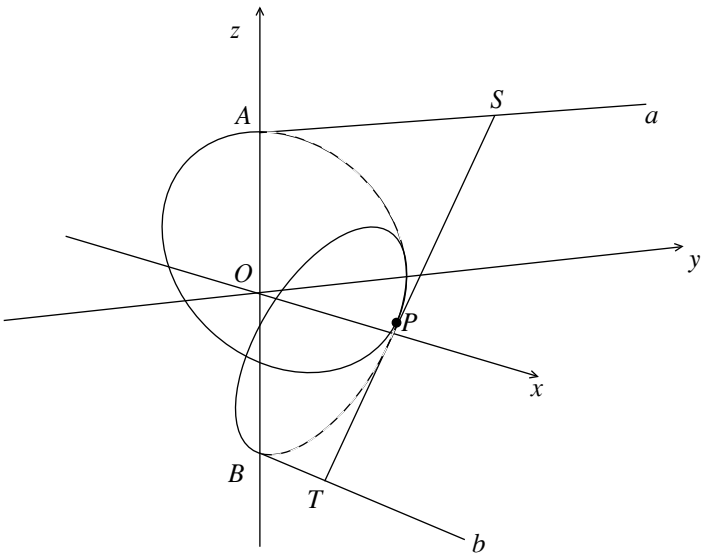
The half-line a has endpoint $A(0, 0, 1)$ and passes through $\alpha(0, 1, 1)$. The half-line b has endpoint $B(0, 0, -1)$ and passes through $\beta(1, 0, -1)$. They are smoothly joined by a curve APB comprising circular arcs AP and BP with equal radii. There is no discontinuity of direction at any point.

Show that there are two such curves, and find the ratio of their lengths, longer to shorter.

Answer: The ratio of lengths is $2 : 1$.

There were five very clear and carefully argued solutions to this problem. Mark Hennings and Didier Pinchon used vectors and coordinate geometry; James Mundie used rotation matrices. M. G. Elliott and the proposer Michael Fox, whose solution follows, used more geometrical arguments.

The figure fixes the notation used in finding the first curve.



Two circles not in the same plane, touching at a point P where they have a common tangent there, lie on a unique sphere. The intersection of their axes, which are coplanar, gives the centre. In our problem, the centre of this sphere is on the plane through A perpendicular to a, i.e. $y = 0$. It is also on the plane through B perpendicular to b, i.e. $x = 0$, and is equidistant from A and B, i.e. on $z = 0$. The sphere is therefore centred on the origin and has unit radius.

The circles whose arcs we require touch at P on this sphere, and have a common tangent there. Because this line and a are two tangents to the arc AP, they must meet at S, $(0, s, 1)$, say, with $SP = SA$. Similarly the common tangent meets the line b at T, $(t, 0, -1)$, with $TP = TB$.

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We have then, $SA + TB = ST$, i.e. $(SA + TB)^2 = ST^2$. Therefore $(s + t)^2 = s^2 + t^2 + 4$. Thus $st = 2$, so s and t have the same sign.

Since the arcs have equal radii, their planes are equidistant from O. This will be so if $AS = BT$, i.e. $s = t$. Consequently $s = t = \pm\sqrt{2}$.

If $s = t = +\sqrt{2}$ then $S = (0, \sqrt{2}, 1)$, and $T = (\sqrt{2}, 0, -1)$. Because $s = t$, the point P is the midpoint of ST, so is $(\sqrt{2}/2, \sqrt{2}/2, 0)$.

Now in triangle ASP, the lengths AS and SP are both $\sqrt{2}$. Also $AP^2 = (\sqrt{2}/2)^2 + (\sqrt{2}/2)^2 + 1^2 = 2$. Thus $AP = \sqrt{2}$, and ASP is equilateral. Since the angle between the tangents at A and P is 60°, the angle subtended by the minor arc AP at its centre is 120°. But this arc meets a to form a cusp: for a smooth join we need the major arc AP, which subtends 240° at its centre.

By symmetry about the radius OP, we need the major are BP, which also subtends 240° at its centre. It we rotate the whole diagram through 180° about the z-axis, the half-lines a and b become their complements a' and b', and P maps onto P', the point given when $s = t = -\sqrt{2}$. The minor arcs AP and BP become AP' and BP', which make cusps with a' and b'. Therefore they join the given half-lines a and b smoothly. This is the second solution.

Each minor arc subtends 120° at its centre, thus each major arc is twice as long as a minor arc. So the longer curve, APB, is exactly twice as long as the shorter curve, $AP'B$, and the ratio is 2 : 1.

M. G. Elliott and James Mundie observed that the planes of the four circles involved in the two curves intersect to form a regular tetrahedron. Michael Fox notes that **106.B** is a special case of the problem of joining two general half-lines smoothly with a curve comprising two circular arcs with equal radii which he looked at in his article 'A three-dimensional pipe-joining problem', *Math. Gaz.* **53** (May 1969) pp. 142-147.

Correct solutions were received from: M. G. Elliott, M. Hennings, J. A. Mundie, D. Pinchon and the proposer Michael Fox.

106.C (Dorin Marghidanu)

If a, b, c, x, y, z are positive real numbers and $n > 1$, prove that

$$(ax + by + cz)^n + (ay + bz + cx)^n + (az + bx + cy)^n \leqslant (a + b + c)^n (x^n + y^n + z^n).$$

Almost all solutions received used the following convexity argument.

Let $\alpha = \dfrac{a}{a + b + c}$, $\beta = \dfrac{b}{a + b + c}$, $\gamma = \dfrac{c}{a + b + c}$ so that $\alpha + \beta + \gamma = 1$.

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The given inequality is then equivalent to

$$(\alpha x + \beta y + \gamma z)^n + (\alpha y + \beta z + \gamma x)^n + (\alpha z + \beta x + \gamma y)^n \leqslant x^n + y^n + z^n. \quad (*)$$

Jenson's inequality, applied to the function x^n $(n > 1)$ which is convex on $(0, \infty)$ gives

$$(\alpha x + \beta y + \gamma z)^n \leqslant \alpha x^n + \beta y^n + \gamma z^n,$$

$$(\alpha y + \beta z + \gamma x)^n \leqslant \alpha y^n + \beta z^n + \gamma x^n,$$

$$(\alpha z + \beta x + \gamma y)^n \leqslant \alpha z^n + \beta x^v + \gamma y^n.$$

Adding both sides, bearing in mind that $\alpha + \beta + \gamma = 1$, then shows that $(*)$ holds, as required. Moreover, there is equality in Jensen and throughout if, and only if, $x = y = z$.

Peter Johnson noted that the result continues to hold for $n < 0$ (since x^n is convex), but that the inequality reverses for $0 < n < 1$ (when x^n is concave).

And, more generally, for any convex function $f(x)$ and $\alpha_1, \ldots, \alpha_k > 0$ with $\alpha_1 + \ldots + \alpha_k = 1$, we have

$$\sum_{\substack{\text{cyclic} \\ x_1,\ldots,x_k}} f(\alpha_1 x_1 + \ldots + \alpha_k x_k) \leqslant f(x_1) + \ldots + f(x_k)$$

by the same argument; in the context of **106.C**, this extends the result to

$$\sum_{\substack{\text{cyclic} \\ x_1,\ldots,x_k}} (\alpha_1 x_1 + \ldots + \alpha_k x_k)^n \leqslant (\alpha_1 + \ldots + \alpha_k)^n (x_1^n + \ldots + x_k^n).$$

There were a few 'non-Jensen' solutions.

Ángel Plaza invoked the weighted power mean inequality to derive $\alpha x + \beta y + \gamma z \leqslant (\alpha x^n + \beta y^n + \gamma z^n)^{1/n}$, while Soham Bhadra went via Hölder's inequality, adding together the final inequality below for the three terms on the left-hand side of **106.C**:

$$ax + by + cz = a^{\frac{n-1}{n}} a^{\frac{1}{n}} x + b^{\frac{n-1}{n}} b^{\frac{1}{n}} y + c^{\frac{n-1}{n}} c^{\frac{1}{n}} z$$

$$\leqslant (a + b + c)^{\frac{n-1}{n}} (ax^n + by^n + cz^n)^{\frac{1}{n}}$$

$$\text{or } (ax + by + cz)^n \leqslant (a + b + c)^{n-1} (ax^n + by^n + cz^n).$$

Correct solutions were received from: S. Bhadra, N. Curwen, P. De, S. Dolan, M. Hennings, P. F. Johnson, D. Pinchon, Á. Plaza, H. Ricardo, S. Riccarelli and the proposer Dorin Marghidanu.

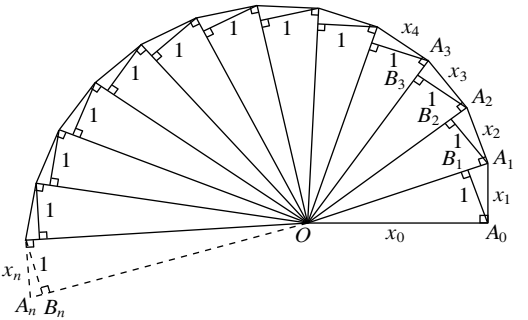
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106.D (Toyesh Prakash Sharma)

The polygonal spiral $A_0A_1A_2\ldots$ with $OA_0 = x_0 > 1$ and $A_0A_1 = x_1$, $A_1A_2 = x_2$, … is constructed from right-angled triangles of height 1 as shown in the diagram.

(a) Show that $OB_n = x_n\left(x_0x_1\ldots x_{n-1} - \dfrac{1}{x_0x_1\ldots x_{n-1}}\right)$, for $n \geqslant 1$.

(b) Does the spiral go around O an infinite number of times?

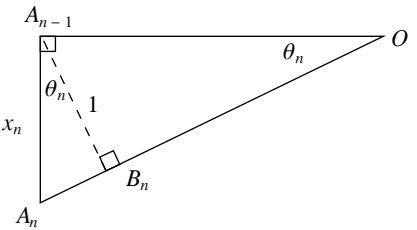


Answer: (b) Yes.

Here, the solutions for part (a) showed considerably less divergence that those for part (b) where I have collated ideas from those received.

(a) From the similar triangles $A_nA_{n-1}B_n$ and A_nOA_{n-1} in the figure, we have $\dfrac{x_1}{1} = \dfrac{OA_n}{OA_{n-1}}$ from which

$$OA_n = OA_{n-1}x_n = OA_{n-2}x_{n-1}x_n = \ldots = x_0x_1\ldots x_{n-1}x_n. \qquad (*)$$



We also have, from triangles $A_nA_{n-1}B_n$ and $A_{n-1}OB_n$, $\dfrac{x_n}{A_nB_n} = \dfrac{OA_{n-1}}{1}$ so that, using $(*)$,

$$OB_n = OA_n - A_nB_n = x_nOA_{n-1} - \frac{x_n}{OA_{n-1}}$$

$$= x_n\left(x_0x_1\ldots x_{n-1} - \frac{1}{x_0x_1\ldots x_{n-1}}\right),$$

as required.

(b) From the figure, we see that θ_n is half the angle subtended by the tangents from O to a unit circle. centre A_{n-1}. Since $OA_n > OA_{n-1}$, we then

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have $\theta_{n+1} < \theta_n$ and, because $x_n = \sec\theta_n$. $x_1 > x_2 > \ldots$. By repeated use of Pythagoras' theorem,

$$OA_{n-1}^2 = x_0^2 + x_1^2 + \ldots + x_{n-1}^2 \leqslant x_0^2 + (n-1)x_1^2, \text{ for } n \geqslant 1.$$

Then, since $\theta < \sin^{-1}\theta$ for $0 < \theta \leqslant 1$, we have

$$\theta_n = \sin^{-1}\left(\frac{1}{OA_{n-1}}\right) > \frac{1}{OA_{n-1}} \geqslant \frac{1}{\sqrt{x_0^2 + (n-1)x_1^2}}.$$

Thus $\sum \theta_n$ diverges by comparison with $\sum \frac{1}{\sqrt{n}}$, so the spiral goes around O an infinite number of times.

The proposer, Toyesh Prakash Sharma, noted that the 'sum = product' relationship

$$x_0^2 x_1^2 \ldots x_n^2 = OA_n^2 = x_0^2 + x_1^2 + \ldots + x_n^2$$

featured in the *Mathematical Spectrum* problem, [1] with a follow-up letter. [2]. on finding rational solutions.

References

1. S. Jafri, A polygon spiral, *Math. Spectrum* **35**(3) (2002/2003) p. 58.
2. B. Bertuello, Letter to the Editor, *Math. Spectrum* **36**(3) (2003/2004) p. 66.

[Both available on-line at:
applied probability.org/publications/mathematical-spectrum/]

Correct solutions were received from: M. V. Channakeshava, N. Curwen, S. Dolan, M. Hennings, K-W. Lau, J. A. Mundie, D. Pinchon, C. Starr and the proposer Toyesh Prakash Sharma.

There have been several recent instances of items being sent to the incorrect sub-Editor. So here is a quick reminder that the respective details are on the back page of the *Gazette* as well as at the start of each section. Also, please bear in mind that we can only respond if contact details are supplied by the sender.

N.J.L.

Looking ahead — 19 down . . . . . .

In July 2023, Nick Lord will have completed 20 years as Problems Editor of Problem Corner and he feels that this would be an appropriate time to hand over to a new Problems Editor. Anyone interested in taking over should in the first instance contact the Editor, Gerry Leversha.

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