Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 March 2016.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

99.I (Stan Dolan)

If x and y are integers such that $x^4 + x^2y^2 - y^4$ is a square, then prove that $x^4 + x^2y^2 - y^4$ is a fourth power.

99.J (M. N. Deshpande)

Triangle ABC is isosceles with $AB = AC$ and centroid G. Let D be the point (other than G) on the circumcircle of triangle BGC such that $BD = CD$. Show that the centroid of triangle BDC lies on the circumcircle of triangle ABC.

99.K (Michael Fox)

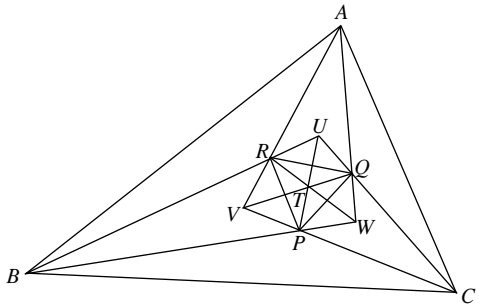
The vertices B and C of a given triangle ABC are the centres of two rectangular hyperbolas whose respective asymptotes bisect the angles $\angle ABC$ and $\angle ACB$. Given that the curves touch at the point D and intersect at E and F prove that:

(i) D is concyclic with A, B and C;

(ii) EF is parallel to the Simson line of D with respect to triangle ABC.

99.L (Isaac Sofair)

In triangle ABC, the lines ARV and AQW trisect $\angle A$, the lines BPW and BRU trisect $\angle B$ and the lines CQU and CPV trisect $\angle C$, as shown in the Figure. It is well known that PQR is an equilateral triangle (the *Morley triangle*); it is also known that PU, QV and RW are concurrent at T.



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Show that:

(i) Area UVW : Area PQR =

$$\frac{1}{8}+\frac{1}{\sqrt{3}}\sin\frac{\pi+A}{3}\sin\frac{\pi+B}{3}\sin\frac{\pi+C}{3} : \cos\frac{\pi-A}{3}\cos\frac{\pi-B}{3}\cos\frac{\pi-C}{3};$$

(ii) $$UV : VW : WU = \cos\frac{\pi-C}{3}\sqrt{1+8\cos\frac{C}{3}\sin\frac{\pi+A}{3}\sin\frac{\pi+B}{3}} :$$

$$\cos\frac{\pi-A}{3}\sqrt{1+8\cos\frac{A}{3}\sin\frac{\pi+B}{3}\sin\frac{\pi+C}{3}} : \cos\frac{\pi-B}{3}\sqrt{1+8\cos\frac{B}{3}\sin\frac{\pi+C}{3}\sin\frac{\pi+A}{3}}.$$

Solutions and comments on **99.A**, **99.B**, **99.C**, **99.D** (March 2015).

99.A (Tom Moore)

The sequence (P_n) of pentagonal numbers begins 1, 5, 12, 22, 35, 51, … and is generated by the formula $P_n = \frac{1}{2}n(3n-1)$ for $n \geqslant 1$. Prove that infinitely many square numbers can be written in the form $P_a + P_bP_c$ for suitable positive integers a, b, c.

This popular, open-ended problem produced an interesting range of possible families of solutions.

The most common family coincided with the proposer's choice of $a = 1, b = n, c = n+4$ which gives

$$P_1 + P_nP_{n+4} = 1 + \tfrac{1}{2}n(3n-1)\cdot\tfrac{1}{2}(n+4)(3n+11)$$

$$= \left[\tfrac{1}{2}(3n^2+11n-2)\right]^2 = (P_{n+2}-6)^2.$$

Neil Curwen, M. G. Elliott, James Mundie and Geoff Strickland generalised this family by using the fact that there are infinitely many square pentagonal numbers because solutions to $P_a = m^2$ correspond to solutions of the Pell equation $(6a-1)^2 - 24m^2 = 1$. For any solution (a, m), we have $P_a + P_nP_{n+4m} = (P_{n+2m} - 6m^2)^2$. The fundamental solution $a = 1$, $m = 1$ is the one used above; the next smallest is $a = 81, m = 99$.

Stan Dolan made the choice $a = c$, $b = 5$ with $P_5 = 35$. This gives $P_a + P_5P_a = 36P_a$ which is square whenever P_a is square.

R. F. Tindall made the choice $a = b = 1$ for which $P_1 + P_1P_n = 1 + P_n = m^2$ if $(6n-1)^2 = 24m^2 - 23$. Any (fixed) solution of this Pell-like equation may be composed with the infinite set of solutions of $x^2 - 24y^2 = 1$ to generate an infinite family of solutions.

More generally, M. V. Channakeshava noted that, for fixed a and $b = 1$, $P_a + P_1P_n = m^2$ produces the Pell-like equation $(6n-1)^2 - 24m^2 = 1 - 24P_a$ and he gave infinite families of solutions when $a = 5$ and $a = 6$.

Sadly, Professor Tom Moore, the proposer of this attractive problem, died last year. He was an inspirational teacher of mathematics who spent

https://doi.org/10.1017/mag.2015.107 Published online by Cambridge University Press

the whole of his career at Bridgewater State University, Massachusetts. He composed over 100 problems for a range of mathematical journals.

Correct solutions were received from: R. G. Bardelang, M. V. Channakeshava, N. Curwen, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, A. P. Harrison, G. Howlett, F. Hunt, L. Ma Li & A. Plaza, J. A. Mundie, G. Strickland, K. B. Subramaniam, I. Timmins, R. F. Tindall and the proposer T. Moore.

99.B (Michael Fox)

Find the smallest triangle whose sides are consecutive integers and whose area is an integer greater than 10^9.

Answer: The smallest triangle has sides 140 451, 140 452, 140 453 with area 8 541 939 510.

Almost all respondents used standard methods to analyse the Pell equation underlying this problem.

If the sides of the triangle are $N - 1, N, N + 1$, then its area A is given by $A = \frac{1}{4}\sqrt{3N^2(N^2 - 4)}$ so that $16A^2 = 3N^2(N^2 - 4)$. It follows that $N = 2x$ is even and that $x^2 - 1 = 3y^2$ with $A = 3xy$. The solutions of $x^2 - 3y^2 = 1$ are generated from the fundamental solution $x_1 = 2, y_1 = 1$ by the recurrence relations

$$x_{n+1} = 2x_n + 3y_n, \qquad y_{n+1} = x_n + 2y_n \tag{A}$$

$$\text{or} \qquad x_{n+1} = 4x_n - x_{n-1}, \qquad y_{n+1} = 4y_n - y_{n-1} \tag{B}$$

which solve to give

$$x_n = \frac{1}{2}\left[(2+\sqrt{3})^n + (2-\sqrt{3})^n\right], y_n = \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})^n - (2-\sqrt{3})^n\right] \tag{C}$$

which may also be written as

$$x_n = \cosh n\theta,\ y_n = \frac{1}{\sqrt{3}}\sinh n\theta \tag{D}$$

where $\theta = \ln(2 + \sqrt{3})$. Any of (A) — (D) may be used to hunt down the required solution $n = 9$ with $x_9 = 70226$. $y_9 = 40545$ giving the triangle in the answer above. The proposer, Michael Fox, noted that the difference in area between this triangle and an equilateral triangle of side 140452 is almost exactly $\frac{\sqrt{3}}{2}$.

Stan Dolan commented that, in the literature, triangles with side-lengths $N - 1$, N, $N + 1$ are sometimes known as 'almost equilateral': a useful reference is the *Wikipedia* entry on Heronian triangles.

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https://doi.org/10.1017/mag.2015.107 Published online by Cambridge University Press

99.C (Abdilkadir Altintaş)

In the triangle ABC, the medians from A, B, C meet the sides BC, AC, AB at A_1, B_1, C_1. Also, the internal angle bisectors of angles A, B, C meet the sides BC, AC, AB at A_2, B_2, C_2. Show that the area of triangle $A_2B_2C_2$ is never greater than the area of triangle $A_1B_1C_1$.

Almost all solvers of this popular problem on geometrical inequalities argued as follows. If $\Delta = [ABC]$ denotes the area of triangle ABC, then $[A_1B_1C_1] = \frac{1}{4}\Delta$. In Figure 1, the angle bisector theorem shows that $AB_2 = \dfrac{bc}{a+c}$ and $AC_2 = \dfrac{bc}{a+b}$ so that $\Delta_A = [AB_2C_2]$ is given by $\Delta_A = \dfrac{1}{2}\dfrac{b^2c^2 \sin A}{(a+c)(a+b)} = \dfrac{bc}{(a+c)(a+b)}\Delta$ with similar expressions for Δ_B and Δ_C.

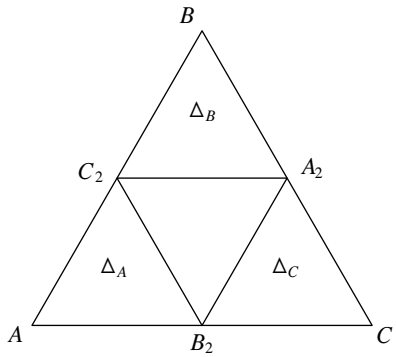


FIGURE 1

Then $[A_2B_2C_2] = \Delta - \Delta_A - \Delta_B - \Delta_C = \dfrac{2abc}{(a+b)(a+c)(b+c)}\Delta$ on substituting for Δ_A, Δ_B, Δ_C and simplifying.

But $\dfrac{2abc}{(a+b)(a+c)(b+c)} \leqslant \dfrac{1}{4}$, either by using the AM-GM inequality, $a + b \geqslant 2\sqrt{ab}$, etc. or by algebraic rearrangement:

$$\frac{1}{4} - \frac{2abc}{(a+b)(a+c)(b+c)} = \frac{a(b-c)^2 + b(a-c)^2 + c(a-b)^2}{4(a+b)(a+c)(b+c)} \geqslant 0.$$

Thus $[A_2B_2C_2] \leqslant \frac{1}{4}\Delta = [A_1B_1C_1]$ with equality if, and only if, $a = b = c$.

Triangle $A_2B_2C_2$ corresponds to the incentre of triangle ABC and the following proved the same result as **99.C** for the orthocentre (Martin Lukarevski) and the Gergonne point (R. F. Tindall). But the definitive generalisation was given by the GCHQ Problem Solving Group and Peter Nüesch. Consider the triangle $A'B'C'$ formed by the arbitrary concurrent Cevians shown in Figure 2 with $B'A : B'C = \lambda : 1 - \lambda$, $A'C : A'B = \mu : 1 - \mu$ and $C'B : C'A = \nu : 1 - \nu$.

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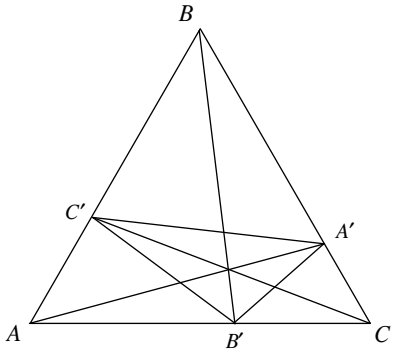


FIGURE 2

Then

$$[A'B'C'] = [1 - \mu(1 - \lambda) - \nu(1 - \mu) - \lambda(1 - \nu)]\,\Delta$$
$$= (1 - \Sigma\lambda + \Sigma\lambda\mu)\,\Delta.$$

But, by Ceva's theorem, $\lambda\mu\nu = (1 - \lambda)(1 - \mu)(1 - \nu)$ or $1 - \Sigma\lambda + \Sigma\lambda\mu = 2\lambda\mu\nu$.

So

$$[A'B'C'] = 2\lambda\mu\nu\Delta = 2\sqrt{\lambda(1 - \lambda)\mu(1 - \mu)\nu(1 - \nu)}\,\Delta \leqslant \tfrac{1}{4}\Delta,$$

since $\lambda(1 - \lambda) \leqslant \frac{1}{4}$, etc. There is equality if, and only if, $\lambda = \mu = \nu = \frac{1}{2}$ when $A'B'C'$ coincides with $A_1B_1C_1$.

Correct solutions were received from: R. G. Bardelang, M. Bataille, M. V. Channakeshava, N. Curwen, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, A. P. Harrison, G. Howlett, M. Lukarevski, J. A. Mundie, Peter Nüesch, G. Strickland, K. B. Subramaniam, E. Swylan, A. Tee, I. Timmins, R. F. Tindall, G. B. Trustrum and the proposer A. Altintaş.

99.D (John D. Mahony)

The triangle ABC (labelled anti-clockwise) has a right-angle at A and side-lengths $a\,(= BC)$, $b\,(= CA)$ and $c\,(= AB)$ where $b < c < a$. Initially, three insects are at rest, one at each vertex of ABC. At the same instant, they start to chase each other in an anti-clockwise direction around the sides of the triangle, each moving the same relative distance $\alpha\,(< 1)$ along their respective pursuit sides before pausing to review their situations. Thus the insect at C stops at point P on CA where $CP = \alpha b$; points Q on AB and R on BC are similarly defined.

(a) If triangle PQR is right-angled at Q show that it is, in fact, similar to triangle ABC.

The insects then start moving again, this time in a clockwise direction along the sides of the right-angled triangle PQR, each moving the same relative distance α along their respective pursuit sides before pausing. The chase continues forever in this manner, alternating between clockwise and anti-clockwise directions of pursuit.

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(b) At what point of triangle ABC do the insects eventually meet and how far has each insect travelled to reach that point?

(c) What happens if the right-angles at A and Q are replaced by 'the largest angle A in a scalene triangle ABC with $b < c < a$'?

Most respondents went straight for the general case here, so we shall follow suit in the following write-up which merges (a), (b), (c) and cherry-picks from the range of methods used.

Let the position vectors of A, B, C with respect to the centroid G of triangle ABC be $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ so that $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.

Consider the first two stages of the chase shown in Figure 1, where the insect at A travels to Q and then to A', etc.

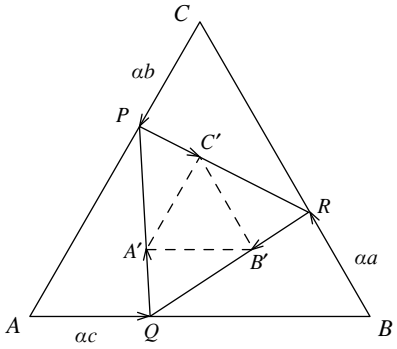


FIGURE 1

Then P, Q, R have position vectors $\mathbf{p} = \alpha\mathbf{u} + \beta\mathbf{w}$, $\mathbf{q} = \alpha\mathbf{v} + \beta\mathbf{u}$, $\mathbf{r} = \alpha\mathbf{w} + \beta\mathbf{v}$ where $\beta = 1 - \alpha$. And A' has position vector

$$\begin{aligned}\mathbf{a}' = \alpha\mathbf{p} + \beta\mathbf{q} &= \alpha(\alpha\mathbf{u} + \beta\mathbf{w}) + \beta(\alpha\mathbf{v} + \beta\mathbf{u}) \\ &= (\alpha^2 + \beta^2)\mathbf{u} + \alpha\beta(\mathbf{v} + \mathbf{w}) \\ &= (\alpha^2 + \beta^2 - \alpha\beta)\mathbf{u} \text{ since } \mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0} \\ &= (1 - 3\alpha + 3\alpha^2)\mathbf{u}.\end{aligned}$$

Similarly $\mathbf{b}' = (1 - 3\alpha + 3\alpha^2)\mathbf{v}$ and $\mathbf{c}' = (1 - 3\alpha + 3\alpha^2)\mathbf{w}$. Thus triangle $A'B'C'$ is an enlargement of triangle ABC, centre G, with scale factor $1 - 3\alpha + 3\alpha^2 < 1$; it follows that the insects eventually meet at G, the common point of the nested, shrinking triangles formed at each stage of the chase.

The hypothesis about angles ($\angle A = \angle Q$) is needed to force the similarity of triangles ABC and QPR. The indicated angles in Figure 2 are equal. The sine rule in triangles APQ and BRQ gives $\dfrac{PQ}{\sin A} = \dfrac{\alpha c}{\sin\theta}$ and $\dfrac{QR}{\sin B} = \dfrac{\alpha a}{\sin\theta}$ from which $\dfrac{PQ}{QR} = \dfrac{c \sin A}{a \sin B} = \dfrac{c}{b}$ so that triangles ABC and QPR are similar with scale factor $\lambda = \sqrt{1 - 3\alpha + 3\alpha^2}$; notice that $0 < \alpha < 1$ implies that $\frac{1}{2} < \lambda < 1$.

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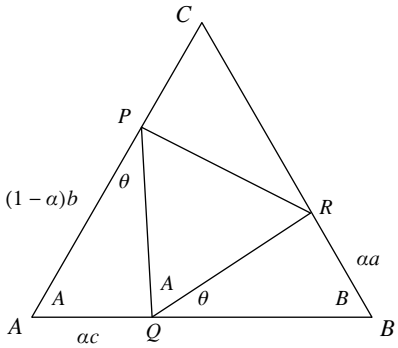


FIGURE 2

The geometry of Figure 2 also means that we can express α (and hence λ) in terms of a, b, c. The cosine rule in triangles APQ and ABC gives

$$\cos A = \frac{\alpha^2c^2 + (1-\alpha)^2b^2 - c^2(1 - 3\alpha + 3\alpha^2)}{2\alpha(1-\alpha)bc} = \frac{b^2 + c^2 - a^2}{2bc}$$

which rearranges to give $\alpha = \dfrac{c^2 - b^2}{a^2 + c^2 - 2b^2}$ as the solution with $0 < \alpha < 1$ since $b < c, a$. Then $\lambda = \sqrt{1 - 3\alpha + 3\alpha^2} = \dfrac{\sqrt{\Sigma a^4 - \Sigma a^2b^2}}{a^2 + c^2 - 2b^2}$. Finally, let d_A, d_B, d_C denote the respective total distances travelled by the insects from A, B, C to G. From Figure 1 and similarity, we have

$$d_A = (AQ + QA')(1 + \lambda^2 + \lambda^4 + \dots) = \frac{\alpha c + \alpha\lambda c}{1 - \lambda^2} = \frac{\alpha c}{1 - \lambda},$$

$$d_B = (BR + RB')(1 + \lambda^2 + \lambda^4 + \dots) = \frac{\alpha a + \alpha\lambda b}{1 - \lambda^2} = \frac{a + \lambda b}{3(1 - \alpha)},$$

$$d_A = (CP + PC')(1 + \lambda^2 + \lambda^4 + \dots) = \frac{\alpha b + \alpha\lambda a}{1 - \lambda^2} = \frac{b + \lambda a}{3(1 - \alpha)}$$

on using $\lambda^2 = 1 - 3\alpha + 3\alpha^2$.

The total distance $d_A + d_B + d_C = \dfrac{\alpha}{1 - \lambda}(a + b + c)$ where, perhaps surprisingly, $\dfrac{\alpha}{1 - \lambda} \to \dfrac{2}{3}$ as $\alpha \to 0$.

If triangle ABC is right-angled at A, some simplification is possible in the formulae since we can eliminate a via $a^2 = b^2 + c^2$ so that $\alpha = \dfrac{c^2 - b^2}{2c^2 - b^2}$, etc.

Correct solutions were received from: N. Curwen, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, A. P. Harrison, G. Howlett, P. F. Johnson, J. A. Mundie, G. Strickland, A. Tee and the proposer J. D. Mahony.

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Readers will have been saddened to read the news of the death of Douglas Quadling in March 2015. Tributes to his immense contribution to the work of The Mathematical Association were featured in the June 2015 issue of *MA News* and the July 2015 issue of the *Gazette*. He was a regular contributor to Problem Corner whose superb expository skills and vast knowledge of mathematics shone through every solution he submitted. His wise, authoritative and enthusiastic support will be greatly missed.

doi:10.1017/mag.2015.107 N.J.L.

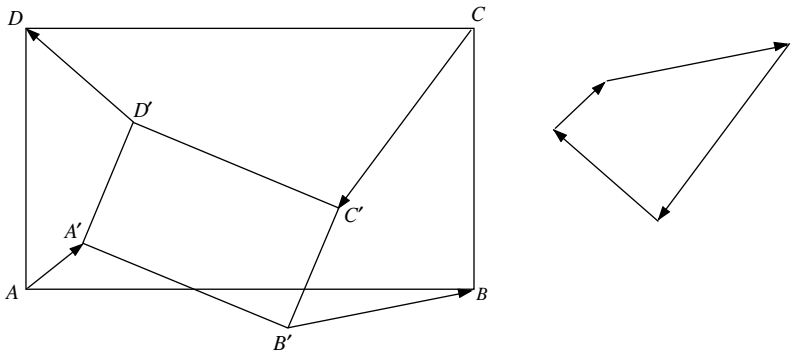
Student Problems

Students up to the age of 19 are invited to send solutions to either of the following problems to Stan Dolan, 126A Harpenden Road, St. Albans, Herts AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by January 20th 2016. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award may be made. Please print your own name clearly! The names of all successful solvers will be published in the March 2016 edition.

Problem 2015.5 (Dao Thanh Oai)

Draw any two similar rectangles $ABCD$ and $A'B'C'D'$ with vertices corresponding in the natural way.



Prove that, as shown above, the vectors $\overrightarrow{AA'}$, $\overrightarrow{B'B}$, $\overrightarrow{CC'}$ and $\overrightarrow{D'D}$ always form a closed quadrilateral. What can you determine about the diagonals of this quadrilateral?

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