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On one-parameter Koopman groups

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Abstract. We characterize Koopman one-parameter C_0-groups, in the class of all unitary one-parameter C_0-groups on $L_2(X)$, as those that preserve $L_\infty(X)$ and for which the infinitesimal generator is a derivation on the bounded functions in its domain.

1. *Introduction*

Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Moreover, let $T : X \to X$ be an (almost everywhere (a.e.)) invertible, measurable and measure-preserving map: i.e., $\mu(A) = \mu(T^{-1}A)$ for each $A \in \mathcal{B}$. Then T induces on $L_2(X)$ a unitary operator U_T, called a Koopman operator, defined by $U_T f := f \circ T$ for all $f \in L_2(X)$. However, one may ask about the converse: given a unitary operator U on $L_2(X)$, how can it be recognized as a Koopman operator? The classical answer says that if U preserves multiplication of bounded functions, i.e., if

$$U(fg) = U(f)U(g) \tag{1}$$

for all $f, g \in L_\infty(X)$, then U is a Koopman operator by a combination of the multiplication theorem in [**Hal**, p. 45] and [**Kec**, Theorem 15.9]. One may also ask the following question: given a unitary operator U on an abstract Hilbert space, how can it be recognized as being unitarily equivalent to a Koopman operator (see for example, [**CR**, **Cho**, **Rid**, **Den**])?

The problem of which unitary operators can be realized as Koopman operators remains one of the most important and still unsolved problems in ergodic theory (see, e.g., the discussion on this problem in [**KL**, **KT**] and also the survey article [**Lem**]). Up to unitary equivalence, each unitary operator U is determined by the two invariants: the equivalence class $[\sigma]$ of a finite positive Borel measure σ on the circle, called the maximal spectral type σ_U of U; and the (Borel) multiplicity function $M = M_U : \mathbb{T} \to \{1, 2, \ldots\} \cup \{\infty\}$, which is defined σ-a.e. Once a pair $([\sigma], M)$ is given, it is easy to construct, on the abstract level,

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a unitary operator U for which $(\sigma_U, M_U) = ([\sigma], M)$. Nevertheless, it remains an open problem as to whether there exists a (unitary) Koopman operator U such that $(\sigma_U, M_U) = ([\sigma], M)$. (Some restrictions must be imposed on σ: for example, σ must be of symmetric type and its topological support must be full if the construction is sought in the class of U_T with T ergodic.) While some progress has recently been made in the spectral theory of single transformations (cf. [**Lem**]), still little is known about unitary one-parameter groups.

A unitary one-parameter C_0-group $(U_t)_{t\in\mathbb{R}}$ is called a Koopman group if, for all $t \in \mathbb{R}$, there exists a measurable $T_t : X \to X$ such that $U_t f = f \circ T_t$ for all $f \in L_2(X)$. It is clear that a Koopman group must preserve $L_\infty(X)$, but this latter condition is also satisfied for many unitary one-parameter C_0-groups which are not Koopman groups. By the Stone theorem [**Sto**], the generator A of a unitary one-parameter C_0-group is skew-adjoint. Therefore each unitary one-parameter C_0-group is determined up to unitary equivalence by (σ_U, M_U), where $\sigma_U = [\sigma]$ for some finite positive Borel measure on $\mathbb{R}$. In order to characterize those pairs $([\sigma], M)$ which can be realized by Koopman groups, it seems natural to first characterize those generators A for which $(e^{tA})_{t\in\mathbb{R}}$ is equivalent to a Koopman group. However, even the problem of characterizing unitary one-parameter C_0-groups as Koopman groups in terms of their generator seems to be far from obvious. Once such a characterization is done, the question of whether a perturbation of a Koopman representation remains Koopman can be considered. This is of independent interest.

In order to formulate the main results of the paper, first recall that if A is an operator in a function space E and $\mathcal{D} \subset D(A)$ is an algebra, then we say that A is a *derivation on* $\mathcal{D}$ if

$$A(fg) = (Af)g + f(Ag)$$

for all $f, g \in \mathcal{D}$. The main result of the paper is the following theorem.

THEOREM 1.1. *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a unitary one-parameter C_0-group on $L_2(X)$ with generator A. Then the following are equivalent.*

(i) *For all $t \in \mathbb{R}$ there exists an a.e. invertible measurable and measure-preserving map $T_t : X \to X$ such that $U_t f = f \circ T_t$ for all $f \in L_2(X)$.*

(ii) *The space $L_\infty(X)$ is invariant under U. Moreover, the space $D(A) \cap L_\infty(X)$ is an algebra and A is a derivation on $D(A) \cap L_\infty(X)$.*

In Corollary 2.9, we are also able to prove a generalization of the above theorem in which the group U is a C_0-group that is not necessarily unitary and we do not require measure preserving in Condition (i). Moreover, we have a generalization in which the measure μ is merely σ-finite (see Theorem 2.8 below).

A theorem of the same nature as Theorem 1.1 was given by Gallavotti and Pulvirenti, [**GP**, Theorem 4].

THEOREM 1.2. [**GP**] *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a unitary one-parameter C_0-group on $L_2(X)$ with generator A. Let $\mathcal{D} \subset D(A) \subset L_\infty(X)$. Suppose that $\mathcal{D}$ is a core for A, $\mathbb{1} \in \mathcal{D}$, $\mathcal{D}$ is an algebra, $\mathcal{D}$ is self-adjoint (that is if $f \in \mathcal{D}$ then $\overline{f} \in \mathcal{D}$), A is a derivation on $\mathcal{D}$ and $\overline{Af} = A\overline{f}$ for all $f \in \mathcal{D}$. Then (i) in Theorem 1.1 is valid.*

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The theorem of Gallavotti and Pulvirenti does not extend to the case in which A is merely a C_0-group generator, and it is essential in [**GP**] that the measure μ is finite.

Yet another theorem of the same type as Theorem 1.1 is in the paper [**DN**, Theorem 2.5], where the space $L_2(X)$ is replaced by $C(X)$, with X compact.

The main application of Theorems 1.1 and 2.8 is a characterization of those C_0-groups on $L_2(X)$ which are pointwise the product of a Koopman operator and a multiplication operator.

THEOREM 1.3. *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a unitary C_0-group on $L_2(X)$ preserving $L_\infty(X)$. Assume that $\mathbb{1} \in D(A)$ with $A\mathbb{1} \in L_\infty(X)$, where A is the generator of U. Then the following are equivalent.*

(i) *For all $t \in \mathbb{R}$ there exists an a.e. invertible, measurable and measure-preserving map $T_t : X \to X$ and a function $\psi_t : X \to \mathbb{C}$ such that $U_t f = \psi_t \cdot (f \circ T_t)$ for all $f \in L_2(X)$.*

(ii) *For all $t \in \mathbb{R}$ one has $|U_t \mathbb{1}| = 1$ a.e. Moreover, $D(A) \cap L_\infty(X)$ is an algebra and $A - (A\mathbb{1})I$ is a derivation on $D(A) \cap L_\infty(X)$.*

We also extend this theorem to C_0-groups which are not necessarily unitary (see Theorem 3.10). The above result can be viewed as the one-parameter counterpart of the classical Banach–Lamperti theorem, [**Lam**, Theorem 3.1], which classifies all isometries of $L_p(X)$ for all $p \in [1, \infty)\backslash\{2\}$ as being of the form

$$f \mapsto \psi \cdot (f \circ T) \tag{2}$$

for some pointwise map $T : X \to X$ and $\psi : X \to (0, \infty)$. In [**GGM**], the authors also proved that unitary positivity-preserving maps are of the form (2).

In §2 we prove Theorem 1.1 and its extension for general C_0-groups. As a tool and by-product, we prove, in Theorem 2.5, that if $(X, \mathcal{B}, \mu)$ is a finite measure space and S is a C_0-group in $L_2(X)$, then S extends consistently to a C_0-group on $L_1(X)$ if and only if the dual group S^* leaves $L_\infty(X)$ invariant. This is a new result in (semi)group theory. In §3 we prove Theorem 3.10, which characterizes weighted non-singular one-parameter C_0-groups and has Theorem 1.3 as a corollary. It turns out that in Theorem 1.3(i) one has $\psi_t \in L_\infty(X)$ and

$$\psi_{t+s} = \psi_t \cdot (\psi_s \circ T_t) \text{ a.e.}$$

for all $t, s \in \mathbb{R}$. Finally, in §4, we determine the form of such ψ, assuming a differentiability condition.

2. *Derivations*

If $(X, \mathcal{B}, \mu)$ is a measure space and $f, g \in L_2(X)$, then we denote the inner product by $(f, g) = \int_X f\overline{g}\, d\mu$. Moreover, if $f \in L_\infty(X)$ and $g \in L_1(X)$ then we denote the duality by $\langle f, g\rangle = \int_X f\overline{g}\, d\mu$. If the measure space is clear from the context, then we abbreviate to $L_p = L_p(X)$ for all $p \in [1, \infty]$. Further, let $p, q \in [1, \infty]$, let U be a one-parameter (semi)group on $L_p(X)$ and V be a one-parameter (semi)group on $L_q(X)$. We say that U and V are *consistent* if $U_t f = V_t f$ for all $t \in \mathbb{R}$ (or $t \in (0, \infty)$) and $f \in L_p(X) \cap L_q(X)$.

For the proof of Theorem 1.1 we need several lemmas as preparation. The first two seem to be folklore.

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LEMMA 2.1. *Let $(X, \mathcal{B}, \mu)$ be a measure space, $c > 0$ and $E : L_2(X) \to L_2(X)$ be a bounded operator. Suppose that $\|Ef\|_\infty \le c\|f\|_\infty$ for all $f \in L_2(X) \cap L_\infty(X)$. Then there exist unique $\widehat{E} \in \mathcal{B}(L_1(X))$ and $\widetilde{E} \in \mathcal{B}(L_\infty(X))$ such that $\widehat{E}f = E^*f$ for all $f \in L_1(X) \cap L_2(X)$ and $\widetilde{E}f = Ef$ for all $f \in L_\infty(X) \cap L_2(X)$. Moreover, $\|\widetilde{E}\|_{\infty\to\infty} \le c$ and $\widetilde{E} = (\widehat{E})^*$.*

Proof. Let $f \in L_1 \cap L_2$. Then $|(E^*f, g)| = |(f, Eg)| \le \|f\|_1\|Eg\|_\infty \le c\|f\|_1\|g\|_\infty$ for all $g \in L_2 \cap L_\infty$. So $E^*f \in L_1$ and $\|E^*f\|_1 \le c\|f\|_1$. Since $L_1 \cap L_2$ is dense in L_1, it follows that there exists a unique $\widehat{E} \in \mathcal{B}(L_1)$ such that $\widehat{E}f = E^*f$ for all $f \in L_1 \cap L_2$. Choose $\widetilde{E} = (\widehat{E})^*$. Then the existence follows. The uniqueness on L_∞ is a consequence of the w^*-density of $L_2 \cap L_\infty$ in L_∞. □

Therefore one has the next lemma.

LEMMA 2.2. *Let $(X, \mathcal{B}, \mu)$ be a measure space and S a semigroup on $L_2(X)$. Suppose that for all $t \in (0, 1]$ there exists a $c > 0$ such that $\|S_t f\|_\infty \le c\|f\|_\infty$ for all $f \in L_2(X) \cap L_\infty(X)$. Then there exist a unique semigroup $\widetilde{S}$ on $L_\infty(X)$ and a unique semigroup $\widehat{S}$ on $L_1(X)$ such that $\widetilde{S}$ is consistent with S and $\widehat{S}$ is consistent with S^*. Moreover, if there exists a $\tilde{c} \ge 1$ such that $\|S_t f\|_\infty \le \tilde{c}\|f\|_\infty$ for all $t \in (0, 1]$ and $f \in L_2(X) \cap L_\infty(X)$, then $\|\widetilde{S}_t\|_{\infty\to\infty} = \|\widehat{S}_t\|_{1\to1} \le \tilde{c}e^{t \log \tilde{c}}$ for all $t \in (0, \infty)$.*

The next lemma is less well known.

LEMMA 2.3. *Let $(X, \mathcal{B}, \mu)$ be a measure space, $c \ge 1$ and S a C_0-semigroup on $L_2(X)$ with generator A. Suppose that $\|S_t f\|_\infty \le c\|f\|_\infty$ for all $t \in (0, 1]$ and $f \in L_2(X) \cap L_\infty(X)$. Then $D(A) \cap L_\infty(X)$ is dense in $L_2(X)$.*

Proof. Since $L_2 \cap L_\infty$ is dense in L_2, it suffices to show that for all $f \in L_2 \cap L_\infty$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $D(A) \cap L_\infty$ such that $\lim f_n = f$ in L_2. Fix $\varphi \in C_c^\infty(0, \infty)$ with $\int \varphi = 1$. For all $n \in \mathbb{N}$ define $\varphi_n \in C_c^\infty(0, \infty)$ by $\varphi_n(t) = n\varphi(nt)$. Let $f \in L_2 \cap L_\infty$ and $n \in \mathbb{N}$. Define $f_n \in L_2$ by

$$f_n = \int_{(0,\infty)} \varphi_n(t) S_t f \, dt.$$

Then $f_n \in D(A)$. Moreover, $\lim_{n\to\infty} f_n = f$ in L_2, since S is a continuous semigroup. It remains to show that $f_n \in L_\infty$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $g \in L_1 \cap L_2$. Then

$$|\varphi_n(t)(S_t f, g)| \le |\varphi_n(t)|\|S_t f\|_\infty\|g\|_1 \le |\varphi_n(t)|ce^{t \log c}\|f\|_\infty\|g\|_1$$

for all $t \in (0, \infty)$. Hence

$$|(f_n, g)| = \left| \int_{(0,\infty)} \varphi_n(t)(S_t f, g) \, dt \right| \le M_n\|f\|_\infty\|g\|_1,$$

where $M_n = \int_{(0,\infty)} |\varphi_n(t)|ce^{t \log c} \, dt < \infty$. So $f_n \in L_\infty$ as required. □

Remark 2.4. Under the conditions of Lemma 2.3 the space $D(A) \cap L_\infty$ is a core for A, since the space $D(A) \cap L_\infty$ is invariant under S (see [**EN**, Proposition II.1.7]).

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It seems that the next theorem is new. Note that we do not assume a uniform bound of the type (3) in Condition (ii). For the definition of C_0-group we refer to [**ABHN**, Definition 3.1.19].

THEOREM 2.5. *Let $(X, \mathcal{B}, \mu)$ be a finite measure space. Let S be a C_0-group on $L_2(X)$. Then the following are equivalent.*

(i) *The group S extends consistently to a C_0-group on $L_1(X)$.*

(ii) *The space $L_\infty(X)$ is invariant under S^*: that is, $S_t^*(L_\infty(X)) \subset L_\infty(X)$ for all $t \in \mathbb{R}$.*

If these conditions are satisfied, then there exist $M \geq 1$ and $\omega \geq 0$ such that

$$\|S_t^* f\|_\infty \leq M e^{\omega|t|} \|f\|_\infty \tag{3}$$

for all $t \in \mathbb{R}$ and $f \in L_\infty(X)$.

Proof. The implication '(i)⇒(ii)' is trivial (cf. the proof of Lemma 2.1). So it remains to show that '(ii)⇒(i)'.

Let $t \in \mathbb{R}$. It follows from the closed graph theorem that there exists a $c > 0$ such that $\|S_t^* f\|_\infty \leq c\|f\|_\infty$ for all $f \in L_\infty$. Note that we use here that the measure μ is finite. Also, note that c depends on t. Hence, by Lemma 2.2, there exist a one-parameter group $\widehat{S}$ on L_1 and a one-parameter group $\widetilde{S}$ on L_∞ such that $\widehat{S}$ is consistent with S and $\widetilde{S}$ is consistent with S^*. Moreover, $\widetilde{S}_t = (\widehat{S}_t)^*$ for all $t \in \mathbb{R}$.

We shall show that $\{\widetilde{S}_t : t \in [2, 3]\}$ is bounded in $\mathcal{B}(L_\infty)$. By the uniform boundedness principle, if suffices to show that $\{\|\widetilde{S}_t f\|_\infty : t \in [2, 3]\}$ is bounded for all $f \in L_\infty$. For this, we use a similar argument to that used in the first step of the proof of [**ABHN**, Lemma 3.16.4]. Fix $f \in L_\infty$. If $t \in \mathbb{R}$ then

$$\begin{aligned}\|\widetilde{S}_t f\|_\infty &= \sup\{|\langle \widetilde{S}_t f, g\rangle| : g \in L_1 \text{ and } \|g\|_1 \leq 1\}\\ &= \sup\{|\langle \widetilde{S}_t f, g\rangle| : g \in L_2 \text{ and } \|g\|_1 \leq 1\}\\ &= \sup\{|(f, S_t g)| : g \in L_2 \text{ and } \|g\|_1 \leq 1\}.\end{aligned}$$

For each $g \in L_2$ the map $t \mapsto |(f, S_t g)|$ is continuous by the strong continuity of S on L_2. So $t \mapsto \|\widetilde{S}_t f\|_\infty$ is lower semicontinuous and therefore a measurable function on $\mathbb{R}$. This is the key assumption in the first step of the proof of [**ABHN**, Lemma 3.16.4]. In order to have this paper self-contained, we include the proof, with minor modifications. Suppose that $\{\|\widetilde{S}_t f\|_\infty : t \in [2, 3]\}$ is not bounded. Then there are $t_0, t_1, t_2, \ldots \in [2, 3]$ such that $\lim_{n\to\infty} t_n = t_0$ and $\|\widetilde{S}_{t_n} f\|_\infty \geq n$ for all $n \in \mathbb{N}$. Since $t \mapsto \|\widetilde{S}_t f\|_\infty$ is measurable, there are $M > 0$ and a measurable set $F \subset [0, t_0]$ such that $\lambda(F) > 1$ and $\|\widetilde{S}_t f\|_\infty \leq M$ for all $t \in F$, where λ is the Lebesgue measure on $\mathbb{R}$. Let $n \in \mathbb{N}$. Then

$$n \leq \|\widetilde{S}_{t_n} f\|_\infty \leq \|\widetilde{S}_{t_n - t}\| \|\widetilde{S}_t f\|_\infty \leq M\|\widetilde{S}_{t_n - t}\|$$

for all $t \in F$. So $\|\widetilde{S}_s\| \geq M^{-1} n$ for all $s \in E_n$, where

$$E_n = \{t_n - t : t \in F \cap [0, t_n]\}.$$

Note that E_n is measurable and $\lambda(E_n) \geq 1$ if $|t_n - t_0| < \lambda(F) - 1$. Let $E = \limsup_{n\to\infty} E_n = \bigcap_{m=1}^\infty \bigcup_{n=m}^\infty E_n$. Then E is measurable and $\lambda(E) \geq 1$. In particular, $E \neq \emptyset$. Moreover, $\|\widetilde{S}_s\| = \infty$ for all $s \in E$. This is a contradiction.

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Thus $\{\widetilde{S}_t : t \in [2, 3]\}$ is bounded in $\mathcal{B}(L_\infty)$. Since $\widetilde{S}_t = (\widehat{S}_t)^*$, it follows that $\{\widehat{S}_t : t \in [2, 3]\}$ is bounded in $\mathcal{B}(L_1)$. By the group property, the set $\{\widehat{S}_t : t \in [-1, 1]\}$ is also bounded in $\mathcal{B}(L_1)$. Let $c = \sup\{\|\widehat{S}_t\| : t \in [-1, 1]\} < \infty$. Let $g \in L_\infty$. Then $\lim_{t\to 0}\langle g, \widehat{S}_t f\rangle = \lim_{t\to 0}(g, S_t f) = (g, f) = \langle g, f\rangle$ for all $f \in L_2$. Since L_2 is dense in L_1 and $c < \infty$, it follows that $\lim_{t\to 0}\langle g, \widehat{S}_t f\rangle = \langle g, f\rangle$ for all $f \in L_1$. So $\widehat{S}$ is weakly continuous and hence $\widehat{S}$ is a C_0-group. Finally, there are $M \geq 1$ and $\omega \geq 0$ such that $\|\widehat{S}_t\|_{1\to 1} \leq Me^{\omega|t|}$ for all $t \in \mathbb{R}$. Then $\|\widetilde{S}_t\|_{\infty\to\infty} = |\widehat{S}_t\|_{1\to 1} \leq Me^{\omega|t|}$ for all $t \in \mathbb{R}$. □

We now turn to the proof of Theorem 1.1. The implication (i)⇒(ii) in Theorem 1.1 is a special case of the next proposition.

PROPOSITION 2.6. *Let U be a C_0-group on $L_2(X)$ with generator A, where $(X, \mathcal{B}, \mu)$ is a measure space. Suppose that for every $t \in \mathbb{R}$ there exists a measurable map $T_t : X \to X$ such that $U_t f = f \circ T_t$ for all $f \in L_2(X)$. Then U extends consistently to a w*-continuous contraction group on $L_\infty(X)$. Moreover, $D(A) \cap L_\infty(X)$ is an algebra and A is a derivation on $D(A) \cap L_\infty(X)$.*

Proof. Let $t \in \mathbb{R}$. If $f \in L_2 \cap L_\infty$, then $\|U_t f\|_\infty = \|f \circ T_t\|_\infty \leq \|f\|_\infty$. Hence, by Lemma 2.2, there exist a unique group $\widetilde{U}$ on L_∞ and a unique group $\widehat{U}$ on L_1 such that $\widetilde{U}$ is consistent with U and $\widehat{U}$ is consistent with U^*. Moreover, $\|\widetilde{U}_t\|_{\infty\to\infty} = \|\widehat{U}_t\|_{1\to 1} = 1$ for all $t \in \mathbb{R}$. Then $\widehat{U}$ is a C_0-group on L_1 by [**Voi**, Proposition 4]. Therefore $\widetilde{U}$ is a w*-continuous group on L_∞.

Let $f, g \in D(A) \cap L_\infty$. Then $U_t(fg) = (U_t f)(U_t g)$ for all $t \in \mathbb{R}$. Hence

$$\frac{1}{t}(U_t(fg) - fg) = \frac{1}{t}(U_t f - f)U_t g + \frac{1}{t} f(U_t g - g)$$

for all $t > 0$. So $fg \in D(A)$ and $A(fg) = (Af)g + f(Ag)$. □

Under further conditions there is a converse of Proposition 2.6. A first step is the next proposition.

PROPOSITION 2.7. *Let U be a C_0-group on $L_2(X)$ with generator A, where $(X, \mathcal{B}, \mu)$ is a measure space. Suppose that $D(A) \cap L_\infty(X)$ is an algebra and A is a derivation on $D(A) \cap L_\infty(X)$. Moreover, suppose that there exists a $c \geq 1$ such that*

$$\|U_t f\|_\infty \leq c\|f\|_\infty$$

for all $t \in [-1, 1]$ and $f \in L_2(X) \cap L_\infty(X)$. Then there exists a unique one-parameter group $\widetilde{U}$ on $L_\infty(X)$ which is consistent with U. Moreover,

$$\widetilde{U}_t(fg) = (\widetilde{U}_t f)(\widetilde{U}_t g)$$

for all $f, g \in L_\infty$ and $t \in \mathbb{R}$.

Proof. The existence and uniqueness of the one-parameter group $\widetilde{U}$ on L_∞ follows from Lemma 2.2. Then $\|\widetilde{U}_t f\|_\infty \leq ce^{\omega|t|}\|f\|_\infty$ for all $t \in \mathbb{R}$ and $f \in L_\infty$, where $\omega = \log c$.

Clearly, $t \mapsto \langle \widetilde{U}_t f, g\rangle = (U_t f, g)$ is continuous for all $f \in L_\infty \cap L_2$ and $g \in L_1 \cap L_2$. Let $f \in L_\infty \cap L_2$ and $g \in L_1$. Let $t \in \mathbb{R}$ and $\varepsilon > 0$. There exists a $g' \in L_1 \cap L_2$ such that

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$\|g - g'\|_1 \le \varepsilon$. Then

$$\begin{aligned}|\langle \widetilde{U}_{t+k} f, g\rangle - \langle \widetilde{U}_t f, g\rangle| &= |\langle \widetilde{U}_{t+k} f, g - g'\rangle + \langle \widetilde{U}_{t+k} f - \widetilde{U}_t f, g'\rangle + \langle \widetilde{U}_t f, g' - g\rangle| \\ &\le ce^{\omega(|t|+1)}\varepsilon\|f\|_\infty + |\langle \widetilde{U}_{t+k} f - \widetilde{U}_t f, g'\rangle| + ce^{\omega|t|}\varepsilon\|f\|_\infty\end{aligned}$$

for all $k \in [-1, 1]$. Hence

$$\lim_{k\to 0} \langle \widetilde{U}_{t+k} f, g\rangle = \langle \widetilde{U}_t f, g\rangle \tag{4}$$

for all $f \in L_\infty \cap L_2$, $g \in L_1$ and $t \in \mathbb{R}$.

Let $f, g \in D(A) \cap L_\infty$. Define $\alpha : \mathbb{R} \to L_2$ by $\alpha(t) = (U_t f)(U_t g)$. Let $h \in L_2$. We shall show that $t \mapsto (\alpha(t), h)$ is differentiable and that

$$\frac{d}{dt}(\alpha(t), h) = ((AU_t f)(U_t g) + (U_t f)(AU_t g), h) \tag{5}$$

for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$. If $k \in \mathbb{R}\backslash\{0\}$, then

$$\begin{aligned}\frac{1}{k}((\alpha(t+k), h) - (\alpha(t), h)) &= \frac{1}{k}((U_{t+k} f - U_t f)U_{t+k} g, h) \\ &\quad + \frac{1}{k}((U_t f)(U_{t+k} g - U_t g), h).\end{aligned} \tag{6}$$

For the first term, we shall prove that

$$\lim_{k\to 0} \frac{1}{k}((U_{t+k} f - U_t f)U_{t+k} g, h) = ((AU_t f)(U_t g), h).$$

Let $k \in \mathbb{R}\backslash\{0\}$. Then

$$\begin{aligned}&\left|\frac{1}{k}((U_{t+k} f - U_t f)U_{t+k} g, h) - ((AU_t f)(U_t g), h)\right| \\ &= \left|\left(\left(\frac{1}{k}(U_{t+k} f - U_t f) - AU_t f\right)\widetilde{U}_{t+k} g, h\right) + ((AU_t f)(\widetilde{U}_{t+k} g - \widetilde{U}_t g), h)\right| \\ &\le \left\|\frac{1}{k}(U_{t+k} f - U_t f) - AU_t f\right\|_2 \|\widetilde{U}_{t+k} g\|_\infty \|h\|_2 + |\langle \widetilde{U}_{t+k} g - \widetilde{U}_t g, h\overline{(AU_t f)}\rangle|.\end{aligned}$$

Since $\lim_{k\to 0} \|(1/k)(U_{t+k} f - U_t f) - AU_t f\|_2 = 0$, $\sup_{k\in[-1,1]} \|\widetilde{U}_{t+k} g\|_\infty < \infty$ and $h\overline{(AU_t f)} \in L_1$, it follows from (4) that

$$\lim_{k\to 0} \frac{1}{k}((U_{t+k} f - U_t f)U_{t+k} g, h) = ((AU_t f)(U_t g), h).$$

Similarly, one proves for the second term in (6) that

$$\lim_{k\to 0} \frac{1}{k}((U_t f)(U_{t+k} g - U_t g), h) = ((U_t f)(AU_t g), h).$$

Hence $t \mapsto (\alpha(t), h)$ is differentiable and (5) is valid. Thus α is weakly differentiable, with weak derivative

$$\alpha'(t) = (AU_t f)(U_t g) + (U_t f)(AU_t g).$$

But A is a derivation on $D(A) \cap L_\infty$. Therefore

$$\alpha'(t) = (AU_t f)(U_t g) + (U_t f)(AU_t g) = A((U_t f)(U_t g)) = A(\alpha(t))$$

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for all $t \in \mathbb{R}$. Obviously, $\alpha(0) = fg$. The uniqueness of the Cauchy problem yields $\alpha(t) = e^{tA}(fg) = U_t(fg)$ for all $t \in \mathbb{R}$.

Fix $t \in \mathbb{R}$. Let $g \in D(A) \cap L_\infty$. Then

$$U_t(fg) = (U_t f)(U_t g) = (U_t f)(\widetilde{U}_t g)$$

for all $f \in D(A) \cap L_\infty$. By Lemma 2.3, the space $D(A) \cap L_\infty$ is dense in L_2. Hence it follows, by continuity, that $U_t(fg) = (U_t f)(\widetilde{U}_t g)$ is valid for all $f \in L_2$ and, in particular, for all $f \in L_2 \cap L_\infty$. So

$$U_t(fg) = (\widetilde{U}_t f)(U_t g) \tag{7}$$

for all $f \in L_2 \cap L_\infty$ and $g \in D(A) \cap L_\infty$. Since $D(A) \cap L_\infty$ is dense in L_2, it follows that (7) is valid for all $f, g \in L_2 \cap L_\infty$. So

$$\widetilde{U}_t(fg) = (\widetilde{U}_t f)(\widetilde{U}_t g) \tag{8}$$

for all $f, g \in L_2 \cap L_\infty$. Let $f \in L_2 \cap L_\infty$ and $h \in L_1$. Then

$$\langle \widetilde{U}_t(fg), h\rangle = \langle fg, \widehat{U}_t h\rangle = \langle g, \overline{f}\widehat{U}_t h\rangle \tag{9}$$

for all $g \in L_\infty$. Moreover,

$$\langle (\widetilde{U}_t f)(\widetilde{U}_t g), h\rangle = \langle \widetilde{U}_t g, \overline{\widetilde{U}_t f}h\rangle = \langle g, \widehat{U}_t(\overline{\widetilde{U}_t f}h)\rangle \tag{10}$$

for all $g \in L_\infty$. It follows from (8), (9) and (10) that

$$\langle g, \overline{f}\widehat{U}_t h\rangle = \langle g, \widehat{U}_t(\overline{\widetilde{U}_t f}h)\rangle \tag{11}$$

for all $g \in L_2 \cap L_\infty$. But $L_2 \cap L_\infty$ is w*-dense in L_∞. So (11) is valid for all $g \in L_\infty$. Using (9) and (10) again, one deduces that

$$\langle \widetilde{U}_t(fg), h\rangle = \langle (\widetilde{U}_t f)(\widetilde{U}_t g), h\rangle$$

for all $g \in L_\infty$. This is for all $h \in L_1$. So (8) is valid for all $f \in L_2 \cap L_\infty$ and $g \in L_\infty$. Finally, by a similar argument, one establishes that (8) is valid for all $f, g \in L_\infty$. □

THEOREM 2.8. *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space such that $(X, \mathcal{B})$ is a standard Borel space. Let U be a C_0-group on $L_2(X)$ with generator A. Then the following are equivalent.*

(i) *For all $t \in \mathbb{R}$ there exists a measurable map $T_t : X \to X$ such that $U_t f = f \circ T_t$ for all $f \in L_2(X)$.*

(ii) *The space $D(A) \cap L_\infty(X)$ is an algebra and A is a derivation on $D(A) \cap L_\infty(X)$. Moreover, there exists a $c > 0$ such that*

$$\|U_t f\|_\infty \le c\|f\|_\infty$$

for all $t \in [-1, 1]$ and $f \in L_2(X) \cap L_\infty(X)$.

Proof. '(i)⇒(ii)'. This follows from Proposition 2.6.

'(ii)⇒(i)'. By Proposition 2.7, there exists a unique one-parameter group $\widetilde{U}$ on L_∞ which is consistent with U. Moreover,

$$\widetilde{U}_t(fg) = (\widetilde{U}_t f)(\widetilde{U}_t g) \tag{12}$$

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for all $f, g \in L_\infty$ and $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$. Let $\mathcal{I} = \{B \in \mathcal{B} : \mu(B) = 0\}$. Then $\mathcal{I}$ is a σ-ideal in $\mathcal{B}$. Let $B \in \mathcal{B}$. Then $\widetilde{U}_t \mathbb{1}_B = \widetilde{U}_t(\mathbb{1}_B^2) = (\widetilde{U}_t \mathbb{1}_B)^2$ by (12). Therefore there exists a $B' \in \mathcal{B}$ such that $\widetilde{U}_t \mathbb{1}_B = \mathbb{1}_{B'}$. If, also, $B'' \in \mathcal{B}$ is such that $\widetilde{U}_t \mathbb{1}_B = \mathbb{1}_{B''}$, then $B' \triangle B'' \in \mathcal{I}$, where $\triangle$ denotes the symmetric difference. Define $\Phi(B) = B' \triangle \mathcal{I} \in \mathcal{B}/\mathcal{I}$. Then Φ is a map from $\mathcal{B}$ into $\mathcal{B}/\mathcal{I}$.

Clearly, $\Phi(\emptyset) = \emptyset \triangle \mathcal{I}$. Let $B_1, B_2 \in \mathcal{B}$. It follows from (12) that $\Phi(B_1 \cap B_2) = \Phi(B_1) \wedge \Phi(B_2)$. Moreover, if $B_1 \cap B_2 = \emptyset$ and $B_1', B_2' \in \mathcal{B}$ are such that $\widetilde{U}_t \mathbb{1}_{B_1} = \mathbb{1}_{B_1'}$ and $\widetilde{U}_t \mathbb{1}_{B_2} = \mathbb{1}_{B_2'}$, then $\mathbb{1}_{B_1' \cap B_2'} = U_t \mathbb{1}_{B_1 \cap B_2} = 0$, so $\widetilde{U}_t \mathbb{1}_{B_1 \cup B_2} = \widetilde{U}_t(\mathbb{1}_{B_1} + \mathbb{1}_{B_2}) = \mathbb{1}_{B_1'} + \mathbb{1}_{B_2'} = \mathbb{1}_{B_1' \cup B_2'}$. Note that $(\widetilde{U}_t)^{-1} = \widetilde{U}_{-t}$ has the same properties as $\widetilde{U}_t$. Hence there exists a $B \in \mathcal{B}$ such that $\widetilde{U}_{-t} \mathbb{1} = \mathbb{1}_B$. Then $\widetilde{U}_t \mathbb{1}_B = \mathbb{1}$. Consequently,

$$\mathbb{1} = \widetilde{U}_t \mathbb{1}_B = \widetilde{U}_t(\mathbb{1}_B \mathbb{1}) = (\widetilde{U}_t \mathbb{1}_B)(\widetilde{U}_t \mathbb{1}) = \mathbb{1} \widetilde{U}_t \mathbb{1} = \widetilde{U}_t \mathbb{1}.$$

So Φ is a homomorphism. Since $\widetilde{U}_t$ is continuous, it follows that $\widetilde{U}_t$ is a σ-homomorphism of Boolean σ-algebras. By [**Kec**, Theorem 15.9], there exists a measurable map $T_t : X \to X$ such that $\Phi(B) = T_t^{-1}(B) \triangle \mathcal{I}$ for all $B \in \mathcal{B}$. So $U_t \mathbb{1}_B = \widetilde{U}_t \mathbb{1}_B = \mathbb{1}_B \circ T_t$ for all $B \in \mathcal{B}$ with $\mu(B) < \infty$. Using the continuity of U_t and the image measure under T_t, one deduces that $U_t f = f \circ T_t$, first for all $f \in L_1 \cap L_2$ and then for all $f \in L_2$. □

COROLLARY 2.9. *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a C_0-group on $L_2(X)$ with generator A. Then the following are equivalent.*

(i) *For all $t \in \mathbb{R}$ there exists a measurable map $T_t : X \to X$ such that $U_t f = f \circ T_t$ for all $f \in L_2(X)$.*

(ii) *The space $L_\infty(X)$ is invariant under U. Moreover, the space $D(A) \cap L_\infty(X)$ is an algebra and A is a derivation on $D(A) \cap L_\infty(X)$.*

Proof. This is a consequence of Theorems 2.5 and 2.8. □

Note that the map U_t is unitary if and only if the map T_t is measure preserving in Theorem 2.8(i).

Proof of Theorem 1.1. This follows immediately from Corollary 2.9.

Remark 2.10. Note that, in Theorem 1.1, the map $T_t : X \to X$ is measure preserving for all $t \in \mathbb{R}$. Moreover,

$$T_{t_1 + t_2} = T_{t_1} \circ T_{t_2} \text{ a.e.}$$

for all $t_1, t_2 \in \mathbb{R}$. Since the one-parameter group U is strongly continuous, it follows from [**GTW**, p. 307] that the group $(T_t)_{t \in \mathbb{R}}$ enjoys the following measurability property: there exists a Borel map $F : \mathbb{R} \times X \to X$ such that for all $t \in \mathbb{R}$ one has

$$F(t, x) = T_t x \quad \text{for almost every } x \in X.$$

Thus $(T_t)_{t \in \mathbb{R}}$ is a measurable measure-preserving flow.

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3. *Weighted non-singular C_0-groups*

Throughout this section, let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a one-parameter group on $L_2(X)$ with $U_0 = I$. The group U is called *weighted non-singular* if, for each $t \in \mathbb{R}$, there exist a map $T_t : X \to X$ and a function $\psi_t : X \to \mathbb{C}$ such that

$$U_t f = \psi_t \cdot (f \circ T_t) \tag{13}$$

for all $f \in L_2(X)$. By substituting $f = \mathbb{1}$, we obtain that $\psi_t = U_t \mathbb{1}$ for all $t \in \mathbb{R}$ and, in particular, that ψ_t is measurable. Moreover, $\psi_0 = \mathbb{1}$ and the group property of U implies the cocycle identity

$$\psi_{t+t'} = \psi_t \cdot (\psi_{t'} \circ T_t) \tag{14}$$

and the group property

$$T_{t+t'} = T_t \circ T_{t'} \text{ a.e.} \tag{15}$$

for all $t, t' \in \mathbb{R}$. Let $t \in \mathbb{R}$. It follows that $\mathbb{1} = \psi_t \cdot (\psi_{-t} \circ T_t)$, and hence $\psi_t \neq 0$ a.e. and

$$\frac{1}{\psi_t} = \psi_{-t} \circ T_t. \tag{16}$$

Therefore

$$f \circ T_t = \frac{1}{\psi_t} \cdot U_t f$$

for all $f \in L_2(X)$, so T_t is measurable. In general, a measurable map $S : X \to X$ is called *non-singular* if $\mu(S^{-1}(A)) = 0$ for all $A \in \mathcal{B}$ with $\mu(A) = 0$. Now note that T_t is a non-singular map of $(X, \mathcal{B}, \mu)$ and that the measure μ and the image measure $T_{t*}\mu$ are equivalent, where

$$(T_{t*}\mu)(A) := \mu(T_t^{-1} A)$$

for all $A \in \mathcal{B}$. Indeed, if $B \in \mathcal{B}$ and $\mu(B) = 0$, then $0 = U_t \mathbb{1}_B = \psi_t \cdot (\mathbb{1}_B \circ T_t) = \psi_t \cdot \mathbb{1}_{T_t^{-1} B}$. Hence $(T_{t*}\mu)(B) = \mu(T_t^{-1} B) = 0$.

A weighted non-singular one-parameter group U is called a *weighted Koopman group* if T_t is measure preserving for all $t \in \mathbb{R}$.

LEMMA 3.1. *Let U be a weighted non-singular one-parameter group given by (13). Then*

$$\left\| |\psi_t|^2 \cdot \left(\frac{d(T_{t*}\mu)}{d\mu} \circ T_t \right) \right\|_\infty \leq \|U_t\|_{2\to 2}^2$$

for all $t \in \mathbb{R}$.

Proof. If $f \in L_2$, then

$$\int |\psi_t|^2 \cdot |f \circ T_t|^2 \, d\mu = \|U_t f\|_2^2 \leq c \|f\|_2^2,$$

where $c = \|U_t\|_{2\to 2}^2$. Hence

$$\int |\psi_t|^2 \cdot (f \circ T_t) \, d\mu \leq c \|f\|_1$$

for all $0 \leq f \in L_1(X, \mu)$. Equivalently,

$$\int (|\psi_t|^2 \circ T_t^{-1}) \cdot f \cdot \frac{d(T_{t*}\mu)}{d\mu} \, d\mu = \int (|\psi_t|^2 \circ T_t^{-1}) \cdot f \, d(T_{t*}\mu) \leq c \|f\|_1$$

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for all $0 \le f \in L_1(X, \mu)$. Since $(|\psi_t|^2 \circ T_t^{-1}) \cdot (d(T_{t*}\mu))/d\mu \ge 0$, one deduces that

$$\left\| (|\psi_t|^2 \circ T_t^{-1}) \cdot \frac{d(T_{t*}\mu)}{d\mu} \right\|_\infty \le c$$

and the result follows by the non-singularity of T_t. □

LEMMA 3.2. *Let U be a weighted non-singular one-parameter group given by (13). Then the following are equivalent.*

(i) *The representation U preserves $L_\infty(X)$.*
(ii) *$\psi_t = U_t \mathbb{1} \in L_\infty(X)$ for all $t \in \mathbb{R}$.*
(iii) *$(d(T_{t*}\mu))/d\mu \in L_\infty(X)$ for all $t \in \mathbb{R}$.*

Proof. '(i)⇒(ii)' is trivial and '(ii)⇒(i)' follows from (13) and the fact that T_t is non-singular for all $t \in \mathbb{R}$.

'(ii)⇒(iii)'. Lemma 3.1 and (16) imply that

$$\left\| \frac{d(T_{t*}\mu)}{d\mu} \circ T_t \right\|_\infty \le \|U_t\|_{2\to 2}^2 \|\psi_{-t}\|_\infty^2 < \infty$$

for all $t \in \mathbb{R}$.

'(iii)⇒(ii)'. Since $T_t \circ T_{-t} = I$ a.e., it follows that

$$\left(\frac{d(T_{t*}\mu)}{d\mu} \circ T_t \right) \cdot \frac{d(T_{-t*}\mu)}{d\mu} = \mathbb{1}$$

for all $t \in \mathbb{R}$. Then the claim is a consequence of Lemma 3.1. □

Remark 3.3. Let U be a C_0-group which is weighted Koopman and unitary. Let $t \in \mathbb{R}$. Then

$$\int |f|^2 \circ T_t \, d\mu = \int |f|^2 \, d\mu = \|U_t f\|_2^2 = \int |\psi_t|^2 \cdot (|f|^2 \circ T_t) \, d\mu$$

for all $f \in L_2(X)$. Hence $\int (|\psi_t|^2 - 1) \cdot (|f|^2 \circ T_t) \, d\mu = 0$ for all $f \in L_2(X)$ and therefore $|\psi_t| = 1$ a.e.

There are many one-parameter C_0-groups which preserve $L_\infty(X)$, but which are not weighted non-singular.

Example 3.4. Let $B \in \mathcal{B}$ be such that $\mu(B) \ne 0 \ne \mu(X \setminus B)$. Define $A : L_2(X) \to L_2(X)$ by $Af = (f, \mathbb{1}_B)\mathbb{1}_{X\setminus B}$. Then A is bounded, so it generates a C_0-group U. Since $A^2 = 0$, one deduces that $U_t = I + tA$ for all $t \in \mathbb{R}$. Hence, obviously, U leaves $L_\infty(X)$ invariant. Now choose $t = -\mu(B)^{-1}$. Then

$$U_t \mathbb{1} = \mathbb{1} + t(\mathbb{1}, \mathbb{1}_B)\mathbb{1}_{X\setminus B} = \mathbb{1} + t\mu(B)\mathbb{1}_{X\setminus B} = \mathbb{1} - \mathbb{1}_{X\setminus B} = \mathbb{1}_B.$$

Since $\mu(\{x \in X : (U_t\mathbb{1})(x) = 0\}) = \mu(X \setminus B) > 0$, the group U is not weighted non-singular by (16).

We next consider weighted non-singular one-parameter groups which preserve $L_\infty(X)$.

LEMMA 3.5. *Let U be a weighted non-singular one-parameter group given by (13). Assume that U preserves $L_\infty(X)$. Then $f \circ T_t \in L_2(X)$ for all $f \in L_2(X)$ and $t \in \mathbb{R}$. Define $V_t : L_2(X) \to L_2(X)$ by*

$$V_t f = f \circ T_t.$$

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Then one has the following.

(a) $(V_t)_{t\in\mathbb{R}}$ *is a one-parameter group on* $L_2(X)$.

(b) *If* U *is a* C_0*-group, then also* $(V_t)_{t\in\mathbb{R}}$ *is a* C_0*-group.*

Proof. Note that (13) and (16) imply that

$$V_t f = f \circ T_t = (\psi_{-t} \circ T_t) U_t f \in L_2$$

for all $t \in \mathbb{R}$ and $f \in L_2$. Then Statement (a) is a consequence of (15).

'(b)'. By Theorem 2.5 there exist $M \geq 1$ and $\omega \geq 0$ such that $\|U_t f\|_\infty \leq M e^{\omega|t|} \|f\|_\infty$ for all $t \in \mathbb{R}$ and $f \in L_\infty$.

Fix $f \in L_\infty$. Let $t \in (0, 1)$. Then (16) gives

$$\begin{aligned} V_t f - f &= \frac{1}{U_t \mathbb{1}}((U_t f - f) + (\mathbb{1} - U_t \mathbb{1}) f) \\ &= ((U_{-t}\mathbb{1}) \circ T_t)((U_t f - f) + (\mathbb{1} - U_t \mathbb{1}) f). \end{aligned} \tag{17}$$

Therefore

$$\begin{aligned} \|V_t f - f\|_2 &\leq \|(U_{-t}\mathbb{1}) \circ T_t\|_\infty (\|U_t f - f\|_2 + \|\mathbb{1} - U_t \mathbb{1}\|_2 \|f\|_\infty) \\ &\leq M e^{\omega} (\|U_t f - f\|_2 + \|\mathbb{1} - U_t \mathbb{1}\|_2 \|f\|_\infty) \end{aligned} \tag{18}$$

and $\lim_{t\downarrow 0} V_t f = f$. The result follows since L_∞ is dense in L_2. □

PROPOSITION 3.6. *Let* $(X, \mathcal{B}, \mu)$ *be a standard Borel probability space. Let* U *be a* C_0*-group on* $L_2(X)$ *preserving* $L_\infty(X)$*. Then the following are equivalent.*

(i) *The representation* U *is weighted non-singular.*

(ii) *For all* $t \in \mathbb{R}$ *one has* $U_t \mathbb{1} \neq 0$ *a.e. and* $1/(U_t\mathbb{1}) \in L_\infty(X)$*. Moreover,* $V = (V_t)_{t\in\mathbb{R}}$ *is a* C_0*-group on* $L_2(X)$*, where*

$$V_t f = \frac{1}{U_t \mathbb{1}} U_t f \tag{19}$$

for all $t \in \mathbb{R}$*. In addition,* $D(B) \cap L_\infty(X)$ *is an algebra and* B *is a derivation on* $D(B) \cap L_\infty(X)$*, where* B *is the generator of* V*.*

Proof. '(i)⇒(ii)'. This follows from (16), Lemma 3.5(b) and Proposition 2.6.

'(ii)⇒(i)'. It follows from (19) that V leaves L_∞ invariant. Then apply Corollary 2.9 to V and the result follows from (19). □

COROLLARY 3.7. *Let* $(X, \mathcal{B}, \mu)$ *be a standard Borel probability space. Let* U *be a unitary* C_0*-group on* $L_2(X)$ *preserving* $L_\infty(X)$*. Then the following are equivalent.*

(i) *The group* U *is a weighted Koopman group.*

(ii) *For all* $t \in \mathbb{R}$*, one has* $|U_t\mathbb{1}| = 1$ *a.e. Moreover,* $V = (V_t)_{t\in\mathbb{R}}$ *is a unitary* C_0*-group on* $L_2(X)$*, where*

$$V_t f = \overline{U_t \mathbb{1}} \cdot U_t f$$

for all $t \in \mathbb{R}$*. In addition,* $D(B) \cap L_\infty(X)$ *is an algebra and* B *is a derivation on* $D(B) \cap L_\infty(X)$*, where* B *is the generator of* V*.*

In order to obtain a relationship between the generators of the two C_0-groups in Lemma 3.5(b), we need the following observation.

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LEMMA 3.8. *Let U be a weighted non-singular one-parameter C_0-group. Let $V = (V_t)_{t\in\mathbb{R}}$ be the group on $L_2(X)$ as in Lemma 3.5. Then*

$$\lim_{t\to 0} \|V_t(U_{-t}\mathbb{1})\cdot g - g\|_2 = 0$$

for all $g \in L_\infty(X)$.

Proof. It follows from Lemma 3.5(b) that V is a C_0-group. Hence $\sup_{t\in[-1,1]} \|V_t\|_{2\to 2} < \infty$. Let $t \in (-1, 1)$. Then

$$\begin{aligned}\|V_t(U_{-t}\mathbb{1})\cdot g - g\|_2 &= \|(V_t(U_{-t}\mathbb{1} - \mathbb{1}) + V_t\mathbb{1} - \mathbb{1})g\|_2\\ &\le (\|V_t\|_{2\to 2}\|U_{-t}\mathbb{1} - \mathbb{1}\|_2 + \|V_t\mathbb{1} - \mathbb{1}\|_2)\|g\|_\infty\end{aligned}$$

and the result follows. □

LEMMA 3.9. *Let U be a weighted non-singular one-parameter C_0-group. Assume that U preserves $L_\infty(X)$. Let $V = (V_t)_{t\in\mathbb{R}}$ be the C_0-group on $L_2(X)$ as in Lemma 3.5. Denote by A and B the generators of U and V, respectively. Assume that*

$$\mathbb{1} \in D(A).$$

Then $D(A) \cap L_\infty(X) = D(B) \cap L_\infty(X)$ and $Bf = Af - f\cdot A\mathbb{1}$ for each $f \in D(A) \cap L_\infty(X)$.

Proof. Let $f \in D(A) \cap L_\infty$. Since $\mathbb{1} \in D(A)$, it follows from (18) that there exists a $c > 0$ such that $\|V_t f - f\|_2 \le ct$ for all $t \in (0, 1)$. Therefore $f \in D(B)$ by [**EN**, Corollary II.5.21]. Hence $D(A) \cap L_\infty \subset D(B) \cap L_\infty$. Let $g \in L_\infty$. Then (17) gives

$$\frac{1}{t}(V_t f - f, g) = \left(\frac{1}{t}(U_t f - f) - f\cdot\frac{1}{t}(U_t\mathbb{1} - \mathbb{1}), \overline{(U_{-t}\mathbb{1})\circ T_t}\cdot g\right)$$

for all $t \in (0, 1)$. Now take the limit $t \to 0$ and use Lemma 3.8. It follows that

$$(Bf, g) = (Af - f\cdot A\mathbb{1}, g).$$

Therefore $Bf = Af - (A\mathbb{1})\cdot f$.

Conversely, let $f \in D(B) \cap L_\infty$. Then $U_t f - f = (U_t\mathbb{1})(V_t f - f) + (U_t\mathbb{1} - \mathbb{1})f$ for all $t \in \mathbb{R}$. The bounds (3) of Theorem 2.5 imply that there exists a $c > 0$ such that

$$\|U_t f - f\|_2 \le \|U_t\mathbb{1}\|_\infty\|V_t f - f\|_2 + \|U_t\mathbb{1} - \mathbb{1}\|_2\|f\|_\infty \le c|t|$$

for all $t \in (0, 1)$. Hence $f \in D(A)$ as before. □

We can now prove the main theorem of this section.

THEOREM 3.10. *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let U be a C_0-group on $L_2(X)$ preserving $L_\infty(X)$. Assume that $\mathbb{1} \in D(A)$ with $A\mathbb{1} \in L_\infty(X)$, where A is the generator of U. Then the following are equivalent.*

(i) *The representation U is weighted non-singular.*

(ii) *The space $D(A) \cap L_\infty(X)$ is an algebra and $A - (A\mathbb{1})I$ is a derivation on $D(A) \cap L_\infty(X)$.*

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Proof. '(i)⇒(ii)'. This follows from Proposition 3.6 and Lemma 3.9. Note that this implication does not require the assumption $A\mathbb{1} \in L_\infty$.

'(ii)⇒(i)'. Consider first U^*, which is a C_0-group on L_2 whose generator is A^*. By Theorem 2.5(ii)⇒(i) the one-parameter group U^* extends consistently to a C_0-group $\widehat{U}$ on L_1. Denote the generator of this group by $\widehat{A}$.

Since $(A\mathbb{1})I$ is a bounded operator, the operator $A - (A\mathbb{1})I$ generates a C_0-group V on L_2 by perturbation theory [**EN**, Theorem III.1.3]. Then $A^* - \overline{(A\mathbb{1})}I$ is the generator of V^*. Moreover, again by perturbation theory, $\widehat{A} - \overline{(A\mathbb{1})}I$ is the generator of a C_0-group $\widehat{V}$ on L_1. Let $t \in \mathbb{R}$. The Trotter–Kato formula [**EN**, Exercise III.5.11(1)] gives

$$V_t^* = \lim_{n\to\infty} (e^{-(t/n)\overline{(A\mathbb{1})}I} U_{t/n}^*)^n \quad \text{strongly in } \mathcal{B}(L_2)$$

and

$$\widehat{V}_t = \lim_{n\to\infty} (e^{-(t/n)\overline{(A\mathbb{1})}I} \widehat{U}_{t/n})^n \quad \text{strongly in } \mathcal{B}(L_1).$$

Let $f \in L_2$. Then $f \in L_1$ and, since U^* and $\widehat{U}$ are consistent, one deduces that

$$(e^{-(t/n)\overline{(A\mathbb{1})}I} U_{t/n}^*)^n f = (e^{-(t/n)\overline{(A\mathbb{1})}I} \widehat{U}_{t/n})^n f \quad \text{a.e.}$$

for all $n \in \mathbb{N}$. Hence $V_t^* f = \widehat{V}_t f$ a.e. and V^* and $\widehat{V}$ are consistent. By Theorem 2.5(i)⇒(ii), applied with $S = V^*$, it follows that V leaves the space L_∞ invariant. By Theorem 2.8, it follows that, for all $t \in \mathbb{R}$, there exists a non-singular measurable map $T_t : X \to X$ such that $V_t f = f \circ T_t$ for all $f \in L_2$.

Note that

$$(V_t \circ e^{s(A\mathbb{1})I}) f = V_t(e^{s(A\mathbb{1})} f) = (e^{s(A\mathbb{1})} f) \circ T_t = (e^{s((A\mathbb{1})\circ T_s)I} \circ V_t) f$$

for all $t, s \in \mathbb{R}$ and $f \in L_2$. Iteration gives

$$\begin{aligned}(V_{t/n} \circ e^{(t/n)(A\mathbb{1})I})^n &= e^{(t/n)((A\mathbb{1})\circ T_{t/n} + \cdots + (A\mathbb{1})\circ T_{(nt)/n})I} \circ (V_{t/n})^n \\ &= e^{(t/n)((A\mathbb{1})\circ T_{t/n} + \cdots + (A\mathbb{1})\circ T_{(nt)/n})I} \circ V_t \end{aligned} \tag{20}$$

for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Since $A = (A - (A\mathbb{1})I) + (A\mathbb{1})I$, one can consider the generator of the C_0-group U as a perturbation of the generator of the C_0-group V. Then the Trotter–Kato formula gives

$$U_t = \lim_{n\to\infty} (V_{t/n} \circ e^{(t/n)(A\mathbb{1})I})^n$$

strongly in $\mathcal{B}(L_2)$. Hence (20) gives $U_t = \psi_t \cdot V_t$ for all $t \in \mathbb{R}$, where

$$\psi_t = e^{\int_0^t (A\mathbb{1})\circ T_r \, dr} \in L_\infty.$$

This completes the proof. □

Clearly Theorem 1.3 is a consequence of Theorem 3.10.

In general, the condition $\mathbb{1} \in D(A)$ is not satisfied. We give a wide class of examples.

Example 3.11. Let $V = (V_t)_{t\in\mathbb{R}}$ be a unitary C_0-group on $L_2(X)$ given by a measure-preserving flow $T = (T_t)_{t\in\mathbb{R}}$ which is ergodic. So $V_t f = f \circ T_t$ for all $t \in \mathbb{R}$ and $f \in L_2(X)$ and the only $f \in L_2(X)$ which are invariant under V_t for all $t \in \mathbb{R}$ are the constants. We will now show that, for all $t \in \mathbb{R}$, we can find a measurable $\psi_t : X \to \mathbb{R}$, bounded and

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bounded away from zero, such that $U = (U_t)_{t\in\mathbb{R}}$ is a continuous C_0-group on $L_2(X)$ for which $\mathbb{1} \notin D(A)$, where

$$U_t f = \psi_t \cdot (f \circ T_t)$$

for all $t \in \mathbb{R}$.

Indeed, by the Ambrose–Kakutani theorem (see for example [**CFS**, Theorem 11.2.1]), we can represent T as a special flow over an ergodic automorphism S of a standard Borel probability space $(Y, \mathcal{C}, \rho)$: i.e., there exist $F : Y \to \mathbb{R}$ and $c > 0$ such that $F > c$, $\int_Y F\, d\rho < \infty$ and

$$X = Y^F := \{(y, s) \in Y \times \mathbb{R} : 0 \le s \le F(y)\}.$$

On Y^F we consider the restriction of the product measurable structure from $Y \times \mathbb{R}$ together with $\rho^F := (\rho \otimes \mathrm{Leb}_{\mathbb{R}})|_{Y^F}$. The flow T acts as $S^F = (S_t^F)_{t\in\mathbb{R}}$, where under the action of S_t^F (with $t > 0$) a point (y, r) moves up vertically with unit speed until it hits the point $(y, f(y))$ which is identified with $(Sy, 0)$, and this movement is continued until time t. In this way, we obtain a unitary C_0-group $V = (V_t)_{t\in\mathbb{R}}$, where $V_t f = f \circ S_t^F$ on $L_2(Y^F, \rho^F)$.

Let $a, b \in \mathbb{R}$ be such that $0 < a < b < c$ and consider the strip $H := Y \times [a, b]$. Then $H \subset Y^F$ and $\rho^F(H) = b - a$. If $t \in \mathbb{R}$ with $|t| < a \wedge (c - b) \wedge (b - a)$, then

$$\rho^F(H \triangle S_t^F(H)) = 2|t|. \tag{21}$$

We claim that $g := \mathbb{1}_H \notin D(B)$, where B is the generator of V. Indeed, for all $t \in \mathbb{R}$ with $|t| < a \wedge (c - b) \wedge (b - a)$, it follows, from (21), that

$$\|g - g \circ S_t^F\|_2 = \left(\int_Y |\mathbb{1}_H - \mathbb{1}_H \circ S_t^F|^2\, d\rho^F\right)^{1/2} = (\rho^F(H \triangle S_t^F(H)))^{1/2} = \sqrt{2|t|}.$$

Therefore there is no constant $\kappa > 0$ such that $\|g - g \circ S_t^F\|_2 \le \kappa|t|$ for all sufficiently small $|t| > 0$ and hence $g \notin D(B)$.

Let $\theta := g + \mathbb{1}$. Then $\theta \notin D(B)$ and $\theta, 1/\theta \in L_\infty(Y^F)$. Set $\psi_t := \theta/(\theta \circ S_t^F)$ for all $t \in \mathbb{R}$. Then $(\psi_t)_{t\in\mathbb{R}}$ satisfies the cocycle identity (14) and, by setting

$$U_t f = \psi_t \cdot (f \circ S_t^F),$$

we obtain a C_0-group $U = (U_t)_{t\in\mathbb{R}}$ on $L_2(Y^F)$. Now

$$\frac{1}{t}(U_t\mathbb{1} - \mathbb{1}) = \frac{1}{t}(\theta - \theta \circ S_t^F) \cdot \frac{1}{\theta \circ S_t^F} = \frac{1}{t}(\theta - V_t\theta) \cdot \frac{1}{\theta \circ S_t^F}$$

and, since V is a C_0-group and $\theta \notin D(B)$, we must have $\mathbb{1} \notin D(A)$, where A is the generator of U.

Remark 3.12. By considering the function $\xi = (-1)^{\mathbb{1}_H} = \mathbb{1}_{X\setminus H} - \mathbb{1}_H$, we obtain a measurable function for which $\xi \notin D(B)$ taking values in $\{-1, 1\}$, and if we set $\psi_t := \xi/(\xi \circ S_t^F)$, then the corresponding group U is weighted Koopman for which $\mathbb{1} \notin D(A)$.

Even if $\mathbb{1} \in D(A)$, then, in general, $A\mathbb{1} \notin L_\infty(X)$. An example is as follows.

Example 3.13. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the torus with normalized Haar measure. Let

$$E = \left\{\eta \in L_2(\mathbb{T}) : \int \eta = 0\right\}.$$

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Then E is a closed subspace of $L_2(\mathbb{T})$. We provide E with the norm of $L_2(\mathbb{T})$. For all $\eta \in L_2(\mathbb{T})$, define $\tilde{\eta} \in L_{2,\mathrm{loc}}(\mathbb{R})$ by $\tilde{\eta}(x) = \eta(e^{ix})$.

Fix $\zeta \in E$. For all $t \in \mathbb{R}$, define $\varphi_t \in C(\mathbb{T})$ by

$$\varphi_t(e^{ix}) = \int_x^{x+t} \tilde{\zeta}.$$

Note that φ_t is well defined. Since $\int_{\mathbb{T}} \zeta = 0$, one deduces that $\|\varphi_t\|_\infty \leq 2\pi \|\zeta\|_1$. If $s, t \in \mathbb{R}$, then

$$\tilde{\varphi}_{t+s}(x) = \int_x^{x+t} \tilde{\zeta} + \int_{x+t}^{x+t+s} \tilde{\zeta} = \tilde{\varphi}_t(x) + \tilde{\varphi}_s(x+t)$$

for all $x \in \mathbb{R}$. For all $t \in \mathbb{R}$, define $\psi_t \in C(\mathbb{T})$ by

$$\psi_t = e^{\varphi_t}$$

and define $U_t : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ by

$$(U_t f)(z) = \psi_t(z) f(e^{it} z).$$

It is easy to verify that $U_t L_\infty(\mathbb{T}) \subset L_\infty(\mathbb{T})$ for all $t \in \mathbb{R}$ and that $U = (U_t)_{t \in \mathbb{R}}$ is a C_0-group. Let A be the generator of U. Clearly, $\psi_t = U_t \mathbb{1}$ for all $t \in \mathbb{R}$. Up to now, everything has worked if $\zeta \in L_1(\mathbb{T})$ with $\int \zeta = 0$, but from now on we will use $\zeta \in L_2(\mathbb{T})$. We shall prove that $\mathbb{1} \in D(A)$ and $A\mathbb{1} = \zeta$.

Let $t \in (0, 1)$. Then

$$\begin{aligned} \left| \frac{1}{t}(U_t \mathbb{1} - \mathbb{1}) - \zeta \right| &\leq \left| \frac{e^{\varphi_t} - \mathbb{1} - \varphi_t}{t} \right| + \left| \frac{1}{t}\varphi_t - \zeta \right| \\ &\leq \frac{1}{t} |\varphi_t|^2 e^{|\varphi_t|} + \left| \frac{1}{t}\varphi_t - \zeta \right| \\ &\leq \frac{1}{t} |\varphi_t|^2 e^{2\pi \|\zeta\|_1} + \left| \frac{1}{t}\varphi_t - \zeta \right|. \end{aligned} \tag{22}$$

We estimate the terms in (22) separately in $L_2(\mathbb{T})$, in the limit $t \downarrow 0$.

We start with the second term. For all $t \in (0, 1)$ define $F_t : E \to C(\mathbb{T})$ by

$$(F_t \eta)(e^{ix}) = \frac{1}{t} \int_x^{x+t} \tilde{\eta}.$$

Note that $F_t(\zeta) = (1/t)\varphi_t$. Let $\eta \in E$ and $\tau \in L_2(\mathbb{T})$. Then Fubini's theorem and the Cauchy–Schwarz inequality give

$$\begin{aligned} |(F_t(\eta), \tau)_{L_2(\mathbb{T})}| &= \frac{1}{t} \left| \int_0^{2\pi} \int_x^{x+t} \tilde{\eta}(s)\, ds \overline{\tilde{\tau}(x)}\, dx \right| \\ &= \frac{1}{t} \left| \int_0^{2\pi} \int_0^{t} \tilde{\eta}(x+s)\, ds \overline{\tilde{\tau}(x)}\, dx \right| \\ &= \frac{1}{t} \left| \int_0^{t} \int_0^{2\pi} \tilde{\eta}(x+s) \overline{\tilde{\tau}(x)}\, dx\, ds \right| \\ &\leq \frac{1}{t} \int_0^t 2\pi \|\eta\|_{L_2(\mathbb{T})} \|\tau\|_{L_2(\mathbb{T})}\, ds \\ &= 2\pi \|\eta\|_{L_2(\mathbb{T})} \|\tau\|_{L_2(\mathbb{T})}. \end{aligned}$$

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So $\|F_t(\eta)\|_{L_2(\mathbb{T})} \leq 2\pi\|\eta\|_{L_2(\mathbb{T})}$ and the set $\{F_t : t \in (0, 1)\}$ is bounded in $\mathcal{B}(E, L_2(\mathbb{T}))$. Clearly, $\lim_{t\downarrow 0} F_t(\eta) = \eta$ in $L_2(\mathbb{T})$ for all $\eta \in C(\mathbb{T})$. Since $E \cap C(\mathbb{T})$ is dense in E, it follows that $\lim_{t\downarrow 0} F_t(\eta) = \eta$ in $L_2(\mathbb{T})$ for all $\eta \in E$. In particular, for ζ, one deduces that

$$\lim_{t\downarrow 0}\left|\frac{1}{t}\varphi_t - \zeta\right| = 0 \tag{23}$$

in $L_2(\mathbb{T})$. This settles the second term in (22).

Now we consider the first term in (22). We shall show that $\lim_{t\downarrow 0}(1/t)|\varphi_t|^2 = 0$ in $L_2(\mathbb{T})$. If $t \in (0, 1)$, then

$$|\varphi_t(e^{ix})| = \left|\int_x^{x+t} \tilde{\zeta}\right| \leq \sqrt{2\pi t}\|\zeta\|_2$$

for all $x \in \mathbb{R}$ by the Cauchy–Schwarz inequality. So $\|(1/t)|\varphi_t|^2\|_\infty \leq 2\pi\|\zeta\|_2^2$ for all $t \in (0, 1)$. Let $t_1, t_2, \ldots \in (0, 1)$ and assume that $\lim_{n\to\infty} t_n = 0$. Then, passing to a subsequence if necessary, it follows from (23) that $\lim_{n\to\infty}(1/t_n)\varphi_{t_n}(z) = \zeta(z)$ for almost every $z \in \mathbb{T}$. Hence

$$\lim_{n\to\infty}\frac{1}{t_n}|\varphi_{t_n}(z)|^2 = \lim_{n\to\infty} t_n\left|\frac{1}{t_n}\varphi_{t_n}(z)\right|^2 = 0$$

for almost every $z \in \mathbb{T}$. Then the bounded convergence theorem of Lebesgue gives $\lim_{n\to\infty}(1/t_n)|\varphi_{t_n}|^2 = 0$ in $L_2(\mathbb{T})$. Hence $\lim_{t\downarrow 0}(1/t)|\varphi_t|^2 = 0$ in $L_2(\mathbb{T})$.

Combining the two estimates, it follows from (22) that $\mathbb{1} \in D(A)$ and $A\mathbb{1} = \zeta$. Finally, if one chooses $\zeta \in E$ such that $\zeta \notin L_\infty(\mathbb{T})$, then $A\mathbb{1} \notin L_\infty(\mathbb{T})$.

4. *Cocycles*

In the previous section, we started with a group U on $L_2(X)$ and, in the case where U was weighted non-singular as in (13), we defined the representation V given by $V_t f = f \circ T_t$. In that case $U_t = \psi_t V_t$. In this section we reverse the order. We start with a representation of the form $V_t f = f \circ T_t$ and wish to construct a representation U of the form (13) which is as general as possible: that is $U_t = \psi_t V_t$ for all $t \in \mathbb{R}$.

Throughout this section, let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. For all $t \in \mathbb{R}$, let $T_t : X \to X$ be a measurable map such that $V = (V_t)_{t\in\mathbb{R}}$ is a C_0-group on $L_2(X)$, where $V_t f := f \circ T_t$ for all $t \in \mathbb{R}$. Let B be the generator of V.

We need a few definitions. A map $\psi : \mathbb{R} \to L_\infty(X)$ is said to be a *cocycle (over V)* if

$$\psi_{t+t'} = \psi_t \cdot (\psi_{t'} \circ T_t) \tag{24}$$

for all $t, t' \in \mathbb{R}$, where we write, for simplicity, $\psi_t = \psi(t)$ for all $t \in \mathbb{R}$. Note that $\psi = 0$ is a cocycle over V. Suppose that ψ is a cocycle. For all $t \in \mathbb{R}$, define $U_t = \psi_t V_t \in \mathcal{B}(L_2(X))$. Clearly, $\|U_t\|_{2\to 2} \leq \|\psi_t\|_\infty\|V_t\|_{2\to 2}$. If $t, t' \in \mathbb{R}$ then

$$U_{t+t'} f = (\psi_t \cdot (\psi_{t'} \circ T_t))V_{t+t'} f = U_t(U_{t'} f) = (U_t \circ U_{t'})f$$

for all $f \in L_2(X)$, so $U = (U_t)_{t\in\mathbb{R}}$ is a one-parameter group on $L_2(X)$, which leaves $L_\infty(X)$ invariant. We call U the *one-parameter group associated with ψ*. Possibly $U_0 = 0$. With a continuity condition, this is not the case.

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LEMMA 4.1. *If* $\lim_{t\to 0}\|\psi_t - \mathbb{1}\|_1 = 0$, *then* $\psi_0 = \mathbb{1}$ *a.e. and* $U_0 = I$.

Proof. Let $B \in \mathcal{B}$ and suppose that $\psi_0|_B = 0$ a.e. Then $\psi_t|_B = 0$ a.e. by (24). Since $\lim_{t\to 0}\|\psi_t - \mathbb{1}\|_1 = 0$, one deduces that $\mu(B) = 0$. So $\psi_0 \neq 0$ a.e. In addition, (24) gives $\psi_0 = \psi_{0+0} = \psi_0^2$. Hence $\psi_0 = \mathbb{1}$ a.e. □

The cocycle ψ is called a *C_0-cocycle (over V)* if U is a C_0-group on $L_2(X)$. If $\theta \in L_\infty(X)$ is such that $\theta \neq 0$ a.e., and $1/\theta \in L_\infty(X)$, then it is easy to verify that $t \mapsto (\theta \circ T_t)/\theta$ is a cocycle. A cocycle ψ is called a *coboundary* if there exists a $\theta \in L_\infty(X)$ such that $\theta \neq 0$ a.e., $1/\theta \in L_\infty(X)$ and

$$\psi_t = \frac{\theta \circ T_t}{\theta}$$

for all $t \in \mathbb{R}$. The function θ is called a *transfer function* of the coboundary. If, in addition, $\theta \in D(B)$ and $B\theta \in L_\infty(X)$, then ψ is called a *coboundary with an L_∞-differentiable transfer function*.

If ψ is a cocycle and $\zeta \in L_2(X)$, then ζ is called the *derivative* of ψ if $\lim_{t\to 0}(1/t)(\psi_t - \mathbb{1}) = \zeta$ in $L_2(X)$. We say that a cocycle ψ is *differentiable* if there exists an $\zeta \in L_2(X)$ such that ζ is the derivative of ψ.

We start with a characterization of C_0-cocycles.

PROPOSITION 4.2. *Let* $\psi : \mathbb{R} \to L_\infty(X)$ *be a cocycle over* V. *Then the following are equivalent.*

(i) ψ *is a C_0-cocycle.*

(ii) $\lim_{t\to 0}\|\psi_t - \mathbb{1}\|_2 = 0$.

Proof. Let U be the one-parameter group associated with ψ. Since $\psi_t = U_t\mathbb{1}$ for all $t \in \mathbb{R}$, the implication (i)⇒(ii) is trivial. So it remains to prove the converse.

Because $\lim_{t\to 0}\|\psi_t - \mathbb{1}\|_1 = 0$ by (ii), it follows, from Lemma 4.1, that $\psi_0 = \mathbb{1}$ a.e. Clearly, $\|U_t\|_{2\to 2} \le \|\psi_t\|_\infty \|V_t\|_{2\to 2}$ for all $t \in \mathbb{R}$. If $f \in L_\infty$ then $\|U_t f\|_\infty \le \|\psi_t\|_\infty \|f\|_\infty < \infty$. Hence the operator $\widetilde{U}_t := U_t|_{L_\infty} : L_\infty \to L_\infty$ is bounded. Obviously, $(\widetilde{U}_t)_{t\in\mathbb{R}}$ is a one-parameter group on L_∞.

Let $f \in L_\infty$. Then

$$U_t f - f = (\psi_t - \mathbb{1})V_t f + V_t f - f$$

for all $t \in \mathbb{R}$, so

$$\|U_t f - f\|_2 \le \|\psi_t - \mathbb{1}\|_2 \|V_t f\|_\infty + \|V_t f - f\|_2 = \|f\|_\infty \|\psi_t - \mathbb{1}\|_2 + \|V_t f - f\|_2$$

and therefore

$$\lim_{t\to 0}\|U_t f - f\|_2 = 0 \tag{25}$$

by assumption. Fix $t_0 \in \mathbb{R}$. Then

$$\|U_{t_0+t} f - U_{t_0} f\|_2 \le \|U_{t_0}\|_{2\to 2}\|U_t f - f\|_2$$

for all $t \in \mathbb{R}$. Hence, $\lim_{t\to t_0} U_t f = U_{t_0} f$ in L_2. So $t \mapsto U_t f$ is continuous from $\mathbb{R}$ into L_2. Hence the map $t \mapsto |(U_t f, g)|$ from $\mathbb{R}$ into $\mathbb{R}$ is continuous for all $g \in L_2$.

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Let $f \in L_\infty$ and $t \in \mathbb{R}$. Then

$$\begin{aligned}\|\widetilde{U}_t f\|_\infty &= \sup\{|\langle \widetilde{U}_t f, g\rangle| : g \in L_1 \text{ and } \|g\|_1 \le 1\},\\ &= \sup\{|\langle \widetilde{U}_t f, g\rangle| : g \in L_2 \text{ and } \|g\|_1 \le 1\},\\ &= \sup\{|(U_t f, g)| : g \in L_2 \text{ and } \|g\|_1 \le 1\}.\end{aligned}$$

Since the map $t \mapsto |(U_t f, g)|$ is continuous for each $g \in L_2$, it follows that the map $t \mapsto \|\widetilde{U}_t f\|_\infty$ is lower semicontinuous, and hence it is measurable on $\mathbb{R}$. By the proof of Theorem 2.5, we deduce that the set $\{\widetilde{U}_t : t \in [2, 3]\}$ is bounded in $\mathcal{B}(L_\infty)$. Since $(\widetilde{U}_t)_{t\in\mathbb{R}}$ is a one-parameter group on L_∞, the set $\{\widetilde{U}_t : t \in [-1, 1]\}$ is also bounded in $\mathcal{B}(L_\infty)$. Let $c := \sup\{\|\widetilde{U}_t\|_{\infty\to\infty} : t \in [-1, 1]\}$. Then $\|\psi_t\|_\infty = \|\widetilde{U}_t \mathbb{1}\|_\infty \le c$ for all $t \in [-1, 1]$. Hence $\sup\{\|U_t\| : t \in [-1, 1]\} < \infty$. Since L_∞ is dense in L_2, it follows from (25) that U is a C_0-group. □

COROLLARY 4.3. *Every differentiable cocycle is a C_0-cocycle. Every coboundary is a C_0-cocycle.*

PROPOSITION 4.4. *Let $\zeta \in L_2(X)$. Then there exists at most one function $\psi : \mathbb{R} \to L_\infty(X)$ such that ψ is a cocycle over V and the cocycle ψ is differentiable with derivative ζ.*

Proof. Let $\psi, \widetilde{\psi} : \mathbb{R} \to L_\infty$ be cocycles over V which are differentiable with derivative ζ. Then $\psi_0 = \mathbb{1} = \widetilde{\psi}_0$ a.e. Let U be the group associated with ψ. Then U is a C_0-group, by Proposition 4.2. Moreover, $\sup\{\|\psi_t\|_\infty : t \in [-1, 1]\} < \infty$ by (3) in Theorem 2.5, or the proof of Proposition 4.2. In addition, $\psi_t \ne 0$ a.e. and $1/\psi_t = \psi_{-t} \circ T_t$ for all $t \in \mathbb{R}$, by (16). Define $\eta : \mathbb{R} \to L_\infty$ by $\eta(t) = \eta_t := \widetilde{\psi}_t/\psi_t$. Then $\eta_{t+t'} = \eta_t \cdot (\eta_{t'} \circ T_t)$ for all $t, t' \in \mathbb{R}$, so η is a cocycle over V. Moreover,

$$\frac{1}{t}(\eta_t - \mathbb{1}) = \frac{1}{t}\frac{\widetilde{\psi}_t - \psi_t}{\psi_t} = \frac{1}{\psi_t}\left(\frac{\widetilde{\psi}_t - \mathbb{1}}{t} - \frac{\psi_t - \mathbb{1}}{t}\right) = (\psi_{-t} \circ T_t)\left(\frac{\widetilde{\psi}_t - \mathbb{1}}{t} - \frac{\psi_t - \mathbb{1}}{t}\right)$$

for all $t \in \mathbb{R}\backslash\{0\}$. Since $\sup\{\|\psi_{-t}\|_\infty : t \in [-1, 1]\} < \infty$, one deduces that the cocycle η is differentiable and $\lim_{t\to 0} (1/t)(\eta_t - \mathbb{1}) = 0$ in L_2.

Let $t \in \mathbb{R}$ and $h \in \mathbb{R}\backslash\{0\}$. Then

$$\begin{aligned}\frac{1}{h}(\eta_{t+h} - \eta_t) &= \frac{1}{h}(\eta_t \cdot (\eta_h \circ T_t) - \eta_t)\\ &= \eta_t \cdot \left(\frac{\eta_h - \mathbb{1}}{h} \circ T_t\right) = \eta_t \cdot V_t\left(\frac{\eta_h - \mathbb{1}}{h}\right).\end{aligned}$$

It follows that $\lim_{h\to 0} (1/h)(\eta_{t+h} - \eta_t) = 0$ in L_2. Therefore η is differentiable from $\mathbb{R}$ into L_2 and $\eta'(t) = 0$ for all $t \in \mathbb{R}$. So η is constant and $\eta(t) = \eta(0) = \mathbb{1}$ for all $t \in \mathbb{R}$. Hence $\widetilde{\psi}_t = \psi_t$ for all $t \in \mathbb{R}$, which completes the proof. □

LEMMA 4.5. *Let $\zeta \in L_\infty(X)$. Define $\psi : \mathbb{R} \to L_\infty(X)$ by $\psi_t := e^{\int_0^t \zeta \circ T_s \, ds}$. Then ψ is a differentiable cocycle with derivative ζ.*

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Proof. We first show that ψ is a cocycle over V. Let $t, t' \in \mathbb{R}$. Then

$$\begin{aligned}\int_0^{t+t'} \zeta \circ T_s \, ds &= \int_0^t \zeta \circ T_s \, ds + \int_t^{t+t'} \zeta \circ T_s \, ds \\ &= \int_0^t \zeta \circ T_s \, ds + \int_0^{t'} \zeta \circ T_{s+t} \, ds \\ &= \int_0^t \zeta \circ T_s \, ds + \Big(\int_0^{t'} \zeta \circ T_s \, ds \Big) \circ T_t .\end{aligned}$$

Hence ψ is a cocycle over V.

Next we show that ψ is differentiable. Recall that $|e^z - 1 - z| \le |z|^2 e^{|z|}$ for all $z \in \mathbb{C}$. Let $t \in [-1, 1] \backslash \{0\}$. Then

$$\begin{aligned}\Big| \frac{\psi_t - \mathbb{1}}{t} - \zeta \Big| &\le \Big| \frac{\psi_t - \mathbb{1} - \int_0^t \zeta \circ T_s \, ds}{t} \Big| + \Big| \frac{1}{t} \int_0^t \zeta \circ T_s \, ds - \zeta \Big| \\ &\le \frac{1}{|t|} \Big(\int_0^t \|\zeta\|_\infty \Big)^2 e^{|t| \|\zeta\|_\infty} + \Big| \frac{1}{t} \int_0^t |\zeta \circ T_s - \zeta| \, ds \Big| \\ &\le |t| \|\zeta\|_\infty^2 e^{\|\zeta\|_\infty} + \Big| \frac{1}{t} \int_0^t |V_s \zeta - \zeta| \, ds \Big| .\end{aligned}$$

Therefore

$$\Big\| \frac{1}{t} (\psi_t - \mathbb{1}) - \zeta \Big\|_2 \le |t| \|\zeta\|_\infty^2 e^{\|\zeta\|_\infty} + \Big| \frac{1}{t} \int_0^t \|V_s \zeta - \zeta\|_2 \, ds \Big| .$$

Since $s \mapsto \|V_s \zeta - \zeta\|_2$ is continuous, one deduces that the cocycle ψ is differentiable with derivative ζ. □

LEMMA 4.6. *Let $\zeta \in L_\infty(X)$ and let $\psi : \mathbb{R} \to L_\infty(X)$ be a cocycle. Then ψ is differentiable with derivative ζ if and only if $\psi_t = e^{\int_0^t \zeta \circ T_s \, ds}$ for all $t \in \mathbb{R}$.*

Proof. This follows immediately from Proposition 4.4 and Lemma 4.5. □

Next we turn to coboundaries.

LEMMA 4.7. *Let ψ be a coboundary with transfer function θ.*

(a) *The coboundary ψ is differentiable if and only if $\theta \in D(B)$. Moreover, if ψ is differentiable, then the derivative is $(B\theta)/\theta$.*

(b) *If $\theta \in D(B)$ and $B\theta \in L_\infty(X)$, then*

$$\psi_t = e^{\int_0^t ((B\theta)/\theta) \circ T_s \, ds}$$

for all $t \in \mathbb{R}$.

Proof. If $t \in \mathbb{R} \backslash \{0\}$, then

$$\frac{\psi_t - \mathbb{1}}{t} = \frac{1}{\theta} \cdot \frac{\theta \circ T_t - \theta}{t} = \frac{1}{\theta} \cdot \frac{V_t \theta - \theta}{t} .$$

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Hence ψ is differentiable if and only if $\theta \in D(B)$. Moreover, if ψ is differentiable, then the derivative is $(1/\theta)B\theta$. This proves Statement (a).

If $\theta \in D(B)$ and $B\theta \in L_\infty$, then $\zeta = (1/\theta)B\theta \in L_\infty$. Now Statement (b) follows from Lemma 4.6. □

Note that Example 3.11 yields a C_0-cocycle (in fact a coboundary) which is not differentiable. It also gives an example of a coboundary which is not a coboundary with an L_∞-differentiable transfer function.

LEMMA 4.8. *Let ψ be a differentiable cocycle with derivative $\zeta \in L_\infty(X)$. Then the following conditions are equivalent.*

(i) *ψ is a coboundary.*

(ii) *ψ is a coboundary with an L_∞-differentiable transfer function.*

(iii) *There exists a $\theta \in D(B) \cap L_\infty(X)$ such that $\theta \neq 0$-a.e., $1/\theta \in L_\infty(X)$, $B\theta \in L_\infty(X)$ and $\zeta = (B\theta)/\theta$.*

Proof. The implication (ii)⇒(i) is trivial.

'(i)⇒(ii)' and '(i)⇒(iii)'. Let θ be a transfer function of ψ. By definition $\theta \in L_\infty$, $\theta \neq 0$-a.e. and $1/\theta \in L_\infty$. Then Lemma 4.7(a) gives $\theta \in D(B)$ and $\zeta = (B\theta)/\theta$. So $B\theta = \zeta\theta \in L_\infty$.

'(iii)⇒(i)'. Define $\widetilde{\psi} : \mathbb{R} \to L_\infty$ by $\widetilde{\psi}_t = (\theta \circ T_t)/\theta$. Then Lemma 4.7(a) implies that the coboundary $\widetilde{\psi}$ is differentiable with derivative ζ. By the uniqueness of Proposition 4.4, one deduces that $\psi = \widetilde{\psi}$. So ψ is a coboundary. □

We now give an example of a differentiable cocycle which is not a coboundary.

Example 4.9. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the torus with normalized Haar measure. For all $t \in \mathbb{R}$, define $T_t : \mathbb{T} \to \mathbb{T}$ by $T_t z = e^{it} z$ and define $V_t : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ by $V_t f = f \circ T_t$. Then $V = (V_t)_{t\in\mathbb{R}}$ is a C_0-group. Fix $\zeta \in L_\infty(\mathbb{T})$ with $\int \zeta \notin i\mathbb{Z}$. Define $\psi : \mathbb{R} \to L_\infty(\mathbb{T})$ by

$$\psi_t = e^{\int_0^t \zeta \circ T_s \, ds}.$$

Then ψ is a differentiable cocycle, by Lemma 4.5. Now suppose that ψ is a coboundary. Let θ be a transfer function. Then $\psi_{2\pi} = (\theta \circ T_{2\pi})/\theta = \mathbb{1}$. Hence $\int_0^{2\pi} \zeta \circ T_s \, ds \in 2\pi i\mathbb{Z}$ a.e. But $\int_0^{2\pi} \zeta \circ T_s \, ds = 2\pi \int_{\mathbb{T}} \zeta$ a.e. So $\int_{\mathbb{T}} \zeta \in i\mathbb{Z}$. This is a contradiction.

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