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On the relation between Gibbs and *g*-measures

STEVEN BERGHOUT †, ROBERTO FERNÁNDEZ‡ and EVGENY VERBITSKIY†§

† *Mathematical Institute, Leiden University, Postbus 9512,*
2300 RA Leiden, The Netherlands
(e-mail: s.berghout@math.leidenuniv.nl)
‡ *Department of Mathematics, Utrecht University, PO Box 80010,*
3508 TA Utrecht, The Netherlands
(e-mail: R.Fernandez1@uu.nl)
§ *Department of Mathematics, University of Groningen, PO Box 407,*
9700 AK Groningen, The Netherlands
(e-mail: evgeny@math.leidenuniv.nl)

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Abstract. Thermodynamic formalism, the theory of equilibrium states, is studied both in dynamical systems and probability theory. Various closely related notions have been developed: e.g. Dobrushin–Lanford–Ruelle Gibbs, Bowen–Gibbs and *g*-measures. We discuss the relation between Gibbs and *g*-measures in a one-dimensional context. Often *g*-measures are also Gibbs, but recently an example to the contrary has been presented. In this paper we discuss exactly when a *g*-measure is Gibbs and how this relates to notions such as uniqueness and reversibility of *g*-measures.

1. *Introduction*

Thermodynamic formalism, used in symbolic dynamics, has strong similarities to the study of Dobrushin–Lanford–Ruelle (DLR) Gibbs measures [**8**, **21**] in statistical mechanics. For *g*-measures [**18**], or similar objects, such as chains of complete connections, variable length Markov chains and chains of infinite order, these similarities are particularly pronounced. Via its natural extension a *g*-measure could be a one-dimensional Gibbs measure, or the corresponding counterpart in dynamical systems, a Bowen–Gibbs measure [**4**]. A recent example [**14**] shows that on $\mathbb{Z}$ the notions are not equivalent. The fundamental underlying question is the relation between one-sided conditional probabilities and their two-sided counterparts. In this respect these problems have a strong similarity to the reversibility question for *g*-measures, i.e. when a projection of an extended *g*-measure on the negative integers is a *g*-measure in its own right. Both questions have been addressed in the literature. An early comparison between dynamical systems and

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statistical mechanics, by Sinai [**25**], uses Gibbs measures to study Anosov dynamical systems. Sufficient conditions for a g-measure to be Gibbs can be found in [**12**]. Besides the example in [**14**] it can easily be shown that an example constructed by Walters [**30**], used to show the existence of a non-reversible g-measure, is also non-Gibbsian.

We will extend these results in the literature by:

- presenting a necessary and sufficient condition, Theorem 4.1, for a g-measure to be Gibbs;
- discussing when a g-measure is reversible;
- discussing how well-known classes of g-measures compare to the Gibbs condition;
- showing that there exist g-measures that are Gibbs measures in the non-uniqueness regime;
- giving an example demonstrating that there exists a g-measure with a potential in Bowen's class for which the reverse is not a g-measure.

In particular, we will show that the non-Gibbsian example in [**14**] is a Bowen–Gibbs measure and that it is reversible. However, g-measures with a potential in Walters' class are all Gibbs measures. Furthermore, we will discuss an adaptation of Walters' example [**30**] to construct a reversible g-measure, for which the reverse g-measure has a slower decay of variation. As a preparation for these results and examples we will use the first sections of this paper to define the relevant classes of measures and recall some of their properties. In §5 examples are given to highlight properties of some of the conditions mentioned throughout the paper. A table giving a comparison of some of these conditions is added as well. In the last section we review existing results on when Gibbs measures are g-measures.

2. *Four classes of measures*

2.1. *General setting and notation.* We will restrict ourselves to symbolic systems with a finite alphabet $\mathcal{A}$. The corresponding measurable spaces are the sets $X = \mathcal{A}^{\mathbb{Z}}$, $X_+ = \mathcal{A}^{\mathbb{Z}_+}$ and $X_- = \mathcal{A}^{\mathbb{Z}_-}$, equipped with the product topology and the corresponding σ-algebra of Borel sets. Elements of X will be referred to as two-sided sequences and elements of X_+ and X_- as one-sided sequences. We will use $\omega_i^j = \omega_i\omega_{i+1}\cdots\omega_j$ as a shorthand notation for strings (words) over the alphabet $\mathcal{A}$. Another shorthand notation we will sometimes use is a^n, with $a \in \mathcal{A}$ and $n \in \mathbb{N}$, denoting a sequence of n subsequent identical elements a. Writing strings in order, for example $a^n b^m$, with $a, b \in \mathcal{A}$ and $n, m \in \mathbb{N}$, denotes the string that is a concatenation of the individual strings.

We write $\mu(\omega_i^j)$ for the measure of the cylinder set

$$[\omega_i^j] = \{\tilde{\omega} : \tilde{\omega}_i = \omega_i, \ldots, \tilde{\omega}_j = \omega_j\}.$$

These sets are of special importance, as they generate both the topologies and the σ-algebras of the spaces above. The shift operator $S : X_+ \to X_+$ is defined as $S(\omega_0\omega_1\cdots) = (\omega_1\omega_2\cdots)$. Similarly, we define a shift operator on X and a right shift, S_-, on X_-; note that the shift operator on X is invertible. In the present paper, we study g-measures and Gibbs measures that are translation-invariant: $\mu(S^{-1}A) = \mu(A)$ for all measurable sets A in the relevant σ-algebra. The set of translation-invariant measures will be denoted by $\mathcal{M}_S(X_+)$, $\mathcal{M}_S(X)$ or $\mathcal{M}_{S_-}(X_-)$.

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The natural extension, as a dynamical system, uniquely maps a translation-invariant measure on X_+ (or X_-) to a translation-invariant measure on X. Let, for a bidirectional sequence, $\boldsymbol{\omega} \in X$, the projection $\pi : X \to X_+$ be defined by $\pi(\boldsymbol{\omega}) = \omega_0^\infty$. The corresponding projection for the measures on these spaces, $\pi^* : \mathcal{M}_S(X) \to \mathcal{M}_S(X_+)$, is given by $\mu_+(A) = (\pi^*\mu)(A) = \mu(\pi^{-1}(A))$, for $\mu \in \mathcal{M}_S(X)$ and $A \in \mathcal{B}_+$ a Borel measurable subset of X_+. Similarly a projection $\pi_- : X \to X_-$, given by $\pi_-(\boldsymbol{\omega}) = \omega_{-\infty}^0$, relates X_- to X. In this way we can identify the translation-invariant measures in $\mathcal{M}_S(X_+)$, $\mathcal{M}_S(X)$ and $\mathcal{M}_{S_-}(X_-)$ with each other.

2.2. *The class of g-measures.* Let $G(X_+)$ be the set of all positive continuous functions $g : X_+ \to (0, 1)$ which are *normalized*,

$$\sum_{a \in \mathcal{A}} g(a\boldsymbol{\omega}) = 1 \quad \text{for all } \boldsymbol{\omega} \in X_+ = \mathcal{A}^{\mathbb{Z}_+}.$$

Definition 2.1. A translation-invariant measure μ_+ on X_+ is called a g-measure for $g \in G(X_+)$ if

$$\mu_+(\omega_0 \mid \omega_1^\infty) = g(\omega_0^\infty)$$

for μ_+-almost everywhere (a.e.) $\boldsymbol{\omega} = (\omega_0, \omega_1, \ldots) \in X_+$.

Equivalently, one can say that μ_+ is a g-measure if, for any continuous function $f : X_+ \to \mathbb{R}$, one has

$$\int f(\boldsymbol{\omega})\mu_+(d\boldsymbol{\omega}) = \int \sum_{a \in \mathcal{A}} f(a\boldsymbol{\omega})g(a\boldsymbol{\omega})\mu_+(d\boldsymbol{\omega}).$$

g-measures on X_- are defined analogously. Finally, if μ_+ is a g-measure on X_+, we will also call the natural extension μ on X a g-measure. Using the projections defined above we can obtain the reverse of a g-measure μ_+ by extending it to X, resulting in a measure $\mu \in M_S(X)$, and then projecting onto X_- to obtain its reverse $\mu_- \in M_S(X_-)$. We call the g-measure reversible if μ_- is a g-measure. As not every g-measure is reversible, when we call $\mu \in \mathcal{M}_S(X)$ a g-measure, we have to keep in mind with respect to which shift. The following result characterizes g-measures by uniform convergence of conditional probabilities [**23**].

THEOREM 2.1. (Palmer, Parry, and Walters [**23**]) *A fully supported translation-invariant probability measure μ_+ on X_+ is a g-measure if and only if the sequence of functions $[g_n]_{n \in \mathbb{N}}$, with*

$$g_n(\boldsymbol{\omega}) := \mu_+(\omega_0 \mid \omega_1^n), \quad n \geq 1, \boldsymbol{\omega} \in X_+,$$

converges uniformly in $\boldsymbol{\omega}$, as $n \to \infty$, to $g(\boldsymbol{\omega})$, for some $g \in G(X_+)$.

Note that the natural extension, μ, of μ_+ satisfies the same convergence of one-sided conditional probabilities if and only if μ_+ satisfies the conditions of the above theorem.

2.3. *The class of DLR Gibbs measures.* The classical definition of Gibbs measures for lattice systems $\mathcal{A}^{\mathbb{Z}^d}$, $d \geq 1$, involves the notions of interactions, Hamiltonians and specifications. We will use a novel equivalent definition, for translation-invariant measures

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in one dimension, that stresses the similarity with g-measures. Denote by $\mathcal{G}(X)$ the class of continuous functions $\gamma : X \to (0, 1)$ that are normalized:

$$\sum_{a\in\mathcal{A}} \gamma(\ldots, \omega_{-2}, \omega_{-1}, a, \omega_1, \omega_2, \ldots) = 1,$$

for all $\boldsymbol{\omega} = (\ldots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \ldots) \in X$.

Definition 2.2. A translation-invariant measure μ is called Gibbs for $\gamma \in \mathcal{G}(X)$ if

$$\mu(\omega_0 \mid \omega_{-\infty}^{-1}, \omega_1^{\infty}) = \gamma(\ldots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \ldots) = \gamma(\boldsymbol{\omega})$$

for μ-almost all $\boldsymbol{\omega} \in X$. Equivalently, for all continuous functions $f : X \to \mathbb{R}$ one has

$$\int f(\boldsymbol{\omega})\mu(d\boldsymbol{\omega}) = \int \sum_{a\in\mathcal{A}} f(\boldsymbol{\omega}^{(a)})\gamma(\boldsymbol{\omega}^{(a)})\mu(d\boldsymbol{\omega}),$$

where $\boldsymbol{\omega}^{(a)} = (\omega_n^{(a)})$, for $\boldsymbol{\omega} \in X$ and $a \in \mathcal{A}$, is defined as

$$\omega_n^{(a)} = \begin{cases} a, & n = 0, \\ \omega_n, & n \neq 0. \end{cases}$$

Theorem 2.1, for g-measures, has a counterpart for Gibbs measures in the following form.

THEOREM 2.2. (Folklore) *A fully supported translation-invariant probability measure μ on X is Gibbs if and only if the double-indexed sequence of functions $[g_{n,m}]_{n,m\in\mathbb{N}}$, with*

$$g_{m,n}(\boldsymbol{\omega}) := \mu(\omega_0 \mid \omega_1^n, \omega_{-m}^{-1}), \quad n, m \geq 1,$$

converges, as $n, m \to \infty$, to a function $\gamma \in \mathcal{G}(X)$, uniformly in $\boldsymbol{\omega}$.

Definition 2.2 seems to be new, we show the equivalence between the classical definition of Gibbs states and the one above in §3.1. Theorem 2.2 is a folklore result, for the convenience of readers we provide a proof.

Proof of Theorem 2.2. Let us start by showing that the convergence of finite range conditional probabilities is sufficient for Gibbsianity. Since the sequence converges uniformly the limit will be continuous. Hence the measure will be consistent with a continuous uniformly non-null specification on single sites. This means that the measure satisfies the conditions of Definition 2.2 and therefore it is a Gibbs measure. We postpone the proof that the new definition is indeed equivalent to the classical definition to Theorem 3.1.

In the other direction let μ be a translation-invariant Gibbs measure and let $\{\gamma_V : V \Subset \mathbb{Z}\}$ be the corresponding continuous specification. Assume $N > 0$ and $n, m > N$. Then

$$\mu(\omega_0 \mid \omega_{-m}^{-1}\omega_1^n) = \frac{1}{\sum_{\sigma_0} (\mu(\omega_{-m}^{-1}\sigma_0\omega_1^n)/\mu(\omega_{-m}^{-1}\omega_0\omega_1^n))}. \tag{1}$$

If the quantity

$$\frac{\mu(\omega_{-m}^{-1}\sigma_0\omega_1^n)}{\mu(\omega_{-m}^{-1}\omega_0\omega_1^n)} = \frac{\int_X \gamma_{[-m,n]}(\omega_{-m}^{-1}\sigma_0\omega_1^n \mid \xi_{[-m,n]^c})\mu(d\xi)}{\int_X \gamma_{[-m,n]}(\omega_{-m}^{-1}\omega_0\omega_1^n \mid \xi_{[-m,n]^c})\mu(d\xi)}, \tag{2}$$

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is uniformly convergent, then it is bounded away from 0 and ∞ and therefore the denominator in (1) is uniformly convergent, bounded away from 1 and ∞. It then follows that the conditional probability (1) does converge uniformly. To show uniform convergence of (2) we will use the consistency relation for specifications, this is discussed later in §3.1. For $V \Subset \mathbb{L}$ and $V' \subset V$ the specification satisfies

$$\gamma_V(\sigma_V \mid \omega_{V^c}) = \sum_{\eta_{V'}} \gamma_{V'}(\sigma_{V'} \mid \sigma_{V\setminus V'}\omega_{V^c})\gamma_V(\eta_{V'}\sigma_{V\setminus V'} \mid \omega_{V^c}).$$

Applying this relation for $V = \{-m, -m+1, \ldots, n-1, n\}$ and $V' = \{0\}$ results in

$$\frac{\mu(\omega_{-m}^{-1}\omega_0\omega_1^n)}{\mu(\omega_{-m}^{-1}\sigma\omega_1^n)} = \frac{\int_X \gamma(\omega_0 \mid \omega_{-m}^{-1}\omega_1^n\xi_{[-m,n]^c}) \sum_{\eta_0\in\mathcal{A}} \gamma(\omega_{-m}^{-1}\eta_0\omega_1^n \mid \xi_{[-m,n]^c})\mu(d\xi)}{\int_X \gamma(\sigma_0 \mid \omega_{-m}^{-1}\omega_1^n\xi_{[-m,n]^c}) \sum_{\eta_0\in\mathcal{A}} \gamma(\omega_{-m}^{-1}\eta_0\omega_1^n \mid \xi_{[-m,n]^c})\mu(d\xi)}.$$

As all factors under the integrals are positive, one can estimate this quantity from above by

$$\begin{aligned}\frac{\mu(\omega_{-m}^{-1}\omega_0\omega_1^n)}{\mu(\omega_{-m}^{-1}\sigma_0\omega_1^n)} &\le \frac{\int_X \sup_{\lambda\in X} \gamma(\omega_0 \mid \omega_{-m}^{-1}\omega_1^n\lambda_{[-m,n]^c})\sum_{\eta_0\in\mathcal{A}}\gamma(\omega_{-m}^{-1}\eta_0\omega_1^n \mid \xi_{[-m,n]^c})\mu(d\xi)}{\int_X \inf_{\lambda\in X} \gamma(\sigma_0 \mid \omega_{-m}^{-1}\omega_1^n\lambda_{[-m,n]^c})\sum_{\eta_0\in\mathcal{A}}\gamma(\omega_{-m}^{-1}\eta_0\omega_1^n \mid \xi_{[-m,n]^c})\mu(d\xi)} \\ &= \frac{\sup_{\lambda\in X} \gamma(\omega_0 \mid \omega_{-m}^{-1}\omega_1^n\lambda_{[-m,n]^c})}{\inf_{\lambda\in X} \gamma(\sigma_0 \mid \omega_{-m}^{-1}\omega_1^n\lambda_{[-m,n]^c})} \\ &\le \frac{\sup_{\lambda\in X} \gamma(\omega_0 \mid \omega_{-N}^{-1}\omega_1^N\lambda_{[-N,N]^c})}{\inf_{\lambda\in X} \gamma(\sigma_0 \mid \omega_{-N}^{-1}\omega_1^N\lambda_{[-N,N]^c})},\end{aligned}$$

for any $N \le \min\{m, n\}$. In the same way we get a lower bound:

$$\frac{\mu(\omega_{-m}^{-1}\omega_0\omega_1^n)}{\mu(\omega_{-m}^{-1}\sigma_0\omega_1^n)} \ge \frac{\inf_{\lambda\in X} \gamma(\omega_0 \mid \omega_{-N}^{-1}\omega_1^N\lambda_{[-N,N]^c})}{\sup_{\lambda\in X} \gamma(\sigma_0 \mid \omega_{-N}^{-1}\omega_1^N\lambda_{[-N,N]^c})}.$$

By quasilocality of γ both the upper and lower bound converge uniformly to the same value, as $N \to \infty$, and therefore $\mu(\omega_0 \mid \omega_{-m}^{-1}\omega_1^n)$ converges uniformly. □

2.4. *The class of Bowen–Gibbs measures.* Sometimes measures with a slightly different definition, by Bowen [**4**], are also referred to as Gibbs measures. Bowen's definition is not equivalent to the original definition of Gibbs states due to Dobrushin, Lanford, and Ruelle. This non-equivalence was deliberate: as a definition, Bowen chose a particular result valid for one-dimensional Gibbs measures, giving a convenient expression for the probability of any cylindrical event.

Definition 2.3. A translation-invariant measure μ on X_+ or X is Bowen–Gibbs for a continuous potential ϕ, if there exist constants $c > 1$ and $P \in \mathbb{R}$ such that, for all $\boldsymbol{\omega}$ and every $n \in \mathbb{N}$,

$$\frac{1}{c} \le \frac{\mu([\omega_0^{n-1}])}{\exp(\sum_{j=0}^{n-1} \phi(S^j\boldsymbol{\omega}) - nP)} \le c.$$

The constant $P = P(\phi)$ is called the *topological pressure* of ϕ, and can be defined independently [**28**]. We propose to use the name 'Bowen–Gibbs measure', rather than

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'Gibbs measure' to avoid confusion. The naming problem also extends to the class of measures μ satisfying

$$\frac{1}{c_n} \leq \frac{\mu([\omega_0^{n-1}])}{\exp(\sum_{j=0}^{n-1} \phi(S^j \boldsymbol{\omega}) - nP)} \leq c_n, \tag{3}$$

where the c_n grow at most sub-exponentially in n, i.e. $(1/n) \log c_n \to 0$. These measures were called weak Gibbs in [**32**]. However, there exists an independent notion of weak Gibbs states in statistical mechanics [**10**, **11**].

2.5. *Equilibrium states.* Finally, let us recall the notion of equilibrium states. The three classes of invariant measures defined above turn out to be equilibrium states for appropriate potentials.

Definition 2.4. A translation-invariant measure μ on X_+ or X is called an *equilibrium state* for a continuous function (potential) ϕ, defined on X_+ or X, respectively, if

$$h(\mu, S) + \int \phi(\boldsymbol{\omega})\mu(d\boldsymbol{\omega}) = \sup_{\nu} \left[h(\nu, S) + \int \phi(\boldsymbol{\omega})\nu(d\boldsymbol{\omega}) \right], \tag{4}$$

where the supremum is taken over $\mathcal{M}_S(X_+)$ or $\mathcal{M}_S(X)$, and $h(\mu, S)$ is the Kolmogorov–Sinai entropy.

By the variational principle [**7**, **16**], the right-hand side in (4) coincides with the topological pressure $P(\phi)$ of ϕ. It is easy to check that any g-measure is also an equilibrium state for $\phi = \log g$ [**22**], and every Bowen–Gibbs measure for the potential ϕ is also an equilibrium state for the same potential. In the case of Gibbs measures the corresponding function ϕ can be expressed in terms of an underlying interaction.

2.6. *The relation between Gibbs and g-measures.* In the present paper we investigate the relation between the classes mentioned above. This problem has been addressed before. For example, for *sufficiently smooth* potentials, meaning potentials that have a sufficiently fast decay, such as Hölder continuous potentials or those that have summable variation, the corresponding g-measures are Gibbs, and *vice versa*.

The renewed interest in this problem was sparked by a recent example constructed by Gallo, Fernández, and Maillard [**14**] of a g-measure μ_+^{GFM} on X_+ such that its extension μ^{GFM} to X is not Gibbs. This example was found in the class of so-called variable length Markov models. Even earlier, Walters [**30**] produced an example of a g-measure μ_+^W on X_+ such that its reversal μ_-^W is not a g-measure. We will show that this example is also an example of a g-measure that is not Gibbs. We will also show that the Gallo–Fernandez–Maillard measure μ_+^{GFM} is in fact more regular than Walters' example: its reversal μ_-^{GFM} is a g-measure. Unknown to the authors of [**14**], their g-function belongs to the so-called R-class, introduced earlier by Walters [**31**]. We will use this class to discuss the measure μ_+^{GFM} and generate other relevant examples. Our main result gives the necessary and sufficient condition for a g-measure to be Gibbs. The problem of finding necessary and sufficient conditions for Gibbs measures to be g-measures remains open and will be discussed briefly in §6.

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3. *Further properties of Gibbs measures*

In the previous section we gave a definition of Gibbs measures adapted for a comparison with g-measures. Now we will discuss how it compares to the standard way of defining a Gibbs measure.

3.1. *Gibbs measures.*

One option for defining Gibbs measures is to start with the notion of an interaction.

Definition 3.1. An interaction is a collection of functions, $\{\Phi_V\}$ on X, indexed by finite subsets $V \Subset \mathbb{Z}$ ($\Subset$ indicates that the subset is finite), such that

$$\Phi_V(\boldsymbol{\omega}) = \Phi_V(\boldsymbol{\omega}|_V),$$

i.e. $\Phi_V(\boldsymbol{\omega})$ depends only on the values of $\boldsymbol{\omega}$ in V. An interaction $\Phi = \{\Phi_V\}_{V \Subset \mathbb{Z}}$ is called *uniformly absolutely convergent* (UAC) if

$$\|\Phi\| := \sup_{n \in \mathbb{Z}} \sum_{n \in V \Subset \mathbb{Z}} \sup_{\boldsymbol{\omega} \in X} |\Phi_V(\boldsymbol{\omega})| = \sup_{n \in \mathbb{Z}} \sum_{n \in V \Subset \mathbb{Z}} \|\Phi_V\|_\infty < \infty.$$

If $\Phi = \{\Phi_V\}$ is a UAC interaction, for a finite set (volume) $\Lambda \Subset \mathbb{Z}$, the corresponding Hamiltonian is defined as

$$H_\Lambda(\boldsymbol{\omega}) = \sum_{V \cap \Lambda \neq \varnothing} \Phi_V(\boldsymbol{\omega}).$$

Finally, a Gibbsian specification density γ^Ψ is a collection of probability kernels $\gamma_V^\Psi : X_V \times X_{V^c} \to (0, 1)$ indexed by $V \Subset \mathbb{Z}$, determined by

$$\gamma_V^\Phi(\boldsymbol{\sigma}_V \mid \boldsymbol{\omega}_{V^c}) = \frac{1}{Z^\Psi(\boldsymbol{\omega}_{V^c})} \exp(-H_V(\boldsymbol{\sigma}_V \boldsymbol{\omega}_{V^c})),$$

where $\boldsymbol{\sigma}_V \boldsymbol{\omega}_{V^c}$ is an element in X, equal to $\boldsymbol{\sigma}$ on V, and to $\boldsymbol{\omega}$ on V^c. The normalizing constant $Z^\Psi(\boldsymbol{\omega}_{V^c}) = \sum_{\boldsymbol{\sigma}_V \in \mathcal{A}^V} \exp(-H_V(\boldsymbol{\sigma}_V \boldsymbol{\omega}_{V^c}))$ is known as a partition function. We will refer to the specification density as a specification since the notions are equivalent in the context of this paper.

Definition 3.2. A probability measure μ on X is Gibbs for an interaction Φ (denoted by $\mu \in \mathcal{G}(\Phi)$) if it is *consistent* with the corresponding specification γ^Φ ($\mu \in \mathcal{G}(\gamma^\Phi)$), meaning that for every $V \Subset \mathbb{Z}$,

$$\mu(\boldsymbol{\omega}_V \mid \boldsymbol{\omega}_{V^c}) = \gamma_V^\Phi(\boldsymbol{\omega}_V \mid \boldsymbol{\omega}_{V^c}) \quad \mu\text{-almost surely.}$$

Equivalently, μ is Gibbs if for any continuous function f on X and every $V \Subset \mathbb{Z}$,

$$\int f(\boldsymbol{\omega})\mu(d\boldsymbol{\omega}) = \int \sum_{\boldsymbol{\sigma}_V \in \mathcal{A}^V} f(\boldsymbol{\sigma}_V \boldsymbol{\omega}_{V^c}) \gamma_V^\Phi(\boldsymbol{\sigma}_V \mid \boldsymbol{\omega}_{V^c}) \mu(d\boldsymbol{\omega}).$$

The principal result due to Dobrushin, Lanford, and Ruelle, is that for every UAC interaction Φ, there exists at least one Gibbs measure, i.e. $\mathcal{G}(\Phi) = \mathcal{G}(\gamma^\Phi) \neq \varnothing$. The specification $\gamma^\Phi = \{\gamma_V^\Phi\}$ has the following important properties.

- *Uniform non-nullness*: for every $V \Subset \mathbb{Z}$ there exist positive constants a_V, b_V such that

  $$a_V \leq \gamma_V^\Phi(\boldsymbol{\sigma}_V \mid \boldsymbol{\omega}_{V^c}) \leq b_V$$

  for all $\boldsymbol{\sigma}_V \in \mathcal{A}^V$ and every $\boldsymbol{\omega} \in X$.

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- *Quasilocality*: for every $V \Subset \mathbb{Z}$,

$$v_{V,n} := \sup_{\sigma_V \in \mathcal{A}^V} \sup_{\omega \in X} \sup_{\eta,\chi \in X} |\gamma_V^{\Phi}(\sigma_V | \omega_{[-n,n]\setminus V} \eta_{[-n,n]^c \setminus V}) - \gamma_V^{\Phi}(\sigma_V | \omega_{[-n,n]\setminus V} \xi_{[-n,n]^c \setminus V})| \to 0,$$

  as $n \to \infty$.

A third important property is that the conditional probabilities are *consistent* between different sets $V \Subset \mathbb{Z}$, for all external configurations.

Based on these properties a specification can be defined without mentioning an interaction. Define a specification $\gamma = \{\gamma_V : V \Subset \mathbb{Z}\}$ as a family of probability kernels on $(X, \mathcal{B}(X))$ where the kernels $\gamma_V : \mathcal{B}(X) \times X \to (0, 1)$ are such that for all $V \Subset \mathbb{Z}$ one has:

- for every $C \in \mathcal{B}(X)$, $\gamma_V(C \mid \cdot)$ is measurable with respect to $\mathcal{B}_{V^c} = \sigma\{[\omega_W] : W \Subset V^c\}$;
- for every $C \in \mathcal{B}_{V^c}(X)$, $\gamma_V(C \mid \omega) = \mathbb{1}_C(\omega)$;
- for every $W \subset V$, one has $\gamma_V \gamma_W = \gamma_V$, where the product $\gamma_W \gamma_V$ is given by

$$\gamma_V \gamma_W(C \mid \omega) = \int_X \gamma_W(C \mid \tilde{\omega}) \gamma_V(d\tilde{\omega} \mid \omega),$$

  for any $C \in \mathcal{B}(X)$.

A fundamental result of Kozlov and Sullivan [**19**, **26**] states that a specification $\gamma = \{\gamma_V\}$ that is uniformly non-null and quasilocal is a Gibbsian specification for some UAC interaction Φ, i.e. $\gamma = \gamma^{\Phi}$. As g-measures are translation-invariant, we will only consider those Gibbs measures that are translation-invariant.

Definition 3.3. A specification is translation-invariant if, for any $a \in \mathbb{Z}$ and $V \Subset \mathbb{Z}$,

$$\gamma_V((S^a\omega)_V | (S^a\omega)_{V^c}) = \gamma_{V+a}(\omega_{V+a} | \omega_{(V+a)^c}).$$

For a translation-invariant quasilocal specification there exists a translation-invariant (Gibbs) measure consistent with this specification [**24**]. If a continuous potential is constructed as $\varphi(\omega) = \sum_{V \ni 0} (1/|V|)\Phi_V(\omega_0 \mid \omega_{\{0\}^c})$ from a UAC interaction Φ and μ is a translation-invariant measure, then μ is an equilibrium state for φ if and only if it is a Gibbs measure for Φ.

Specifications are defined for all finite subsets, for Definition 2.2 to make sense however one needs to reconstruct a full specification from singletons.

THEOREM 3.1. *Let μ be a translation-invariant measure on X, then μ is a Gibbs measure if and only if it satisfies the conditions of Definition 2.2, for some $\gamma \in \mathcal{G}(X)$.*

Proof. If the translation-invariant measure μ is Gibbs then it is consistent with a specification, and therefore, for any finite $V \Subset \mathbb{Z}$,

$$\mu(\omega_V \mid \omega_{V^c}) = \gamma_V(\omega_V \mid \omega_{V^c}),$$

for μ-a.e. ω. Thus, in particular, this holds for $V = \{0\}$.

In the opposite direction, suppose μ is a fully supported translation-invariant measure satisfying

$$\mu(\omega_0 \mid \omega_{-\infty}^{-1}, \omega_1^{\infty}) = \gamma(\ldots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \ldots) = \gamma(\omega),$$

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for μ-a.e. $\boldsymbol{\omega} \in X$ and some $\gamma \in \mathcal{G}(X)$. From [**12**, Theorem A.4], which also applies to non-invariant measures, it follows that a specification can be uniquely constructed from a family of non-null single-site kernels $\gamma_{\{i\}}$ if they satisfy the following consistency conditions: the expression

$$\frac{\gamma_{\{i\}}(\eta_i \mid \eta_j \omega_{\{i,j\}^c})}{\sum_{\alpha_i} ((\gamma_{\{i\}}(\alpha_i \mid \eta_j \omega_{\{i,j\}^c}))/(\gamma_{\{j\}}(\eta_j \mid \alpha_i \omega_{\{i,j\}^c})))}$$

must be invariant under the exchange $i \leftrightarrow j$. The reason for this consistency condition is that the expression above is for μ-a.e. $\boldsymbol{\omega} \in X$ equal to

$$\mu(\eta_i \mid \eta_j \omega_{\{i,j\}^c}) \mu(\eta_j \mid \omega_{\{i,j\}^c}) = \mu(\eta_i \eta_j \mid \omega_{\{i,j\}^c}),$$

which is symmetrical under the $i \leftrightarrow j$ exchange. The construction in the proof of Theorem A.4 of [**12**] shows that the continuity of the single-site kernels extend to all finite-volume kernels. Alternatively the theorem can be proven using the conditions from [**6**]. □

4. *Main results*

4.1. *Main theorems.* Known conditions for g-measures to be Gibbs relate to rapid decay of variations of the corresponding g-functions. Likewise our main result is a regularity condition on the function g that is necessary and sufficient for the corresponding g-measure to be Gibbs.

THEOREM 4.1. *Let μ be a g-measure on $X_+ = \mathcal{A}^{\mathbb{Z}_+}$. Viewed as a measure on $X = \mathcal{A}^{\mathbb{Z}}$, μ is Gibbs if and only if the sequence of functions $[\tilde{f}_n^{\sigma_0,\eta_0}]_{n\in\mathbb{N}}$, given by*

$$\tilde{f}_n^{\sigma_0,\eta_0}(\boldsymbol{\omega}) = \prod_{i=-n}^{-1} \frac{g(\omega_i^{-1}\sigma_0\omega_1^\infty)}{g(\omega_i^{-1}\eta_0\omega_1^\infty)}, \tag{5}$$

converges, for all σ_0, $\eta_0 \in \mathcal{A}$, uniformly in $\boldsymbol{\omega} \in X$, as $n \to \infty$.

Proof. Let us start by showing that condition (5) is sufficient for μ to be Gibbs. We can express two-sided conditional probabilities $\mu(\omega_0 \mid \omega_{-n}^{-1}, \omega_1^m)$ as follows:

$$\mu(\omega_0 \mid \omega_{-n}^{-1}, \omega_1^m) = \frac{\mu(\omega_{-n}^{-1}\omega_0\omega_1^m)}{\sum_{\sigma_0\in\mathcal{A}} \mu(\omega_{-n}^{-1}\sigma_0\omega_1^m)} = \frac{1}{\sum_{\sigma_0\in\mathcal{A}} (\mu(\omega_{-n}^{-1}\sigma_0\omega_1^m)/\mu(\omega_{-n}^{-1}\omega_0\omega_1^m))}. \tag{6}$$

Let us introduce the following functions on X: for fixed σ_0, $\eta_0 \in \mathcal{A}$ put

$$f_{n,m}^{\sigma_0,\eta_0}(\boldsymbol{\omega}) = \frac{\mu(\omega_{-n}^{-1}\sigma_0\omega_1^m)}{\mu(\omega_{-n}^{-1}\eta_0\omega_1^m)},$$

$$f_n^{\sigma_0,\eta_0}(\boldsymbol{\omega}) = \tilde{f}_n^{\sigma_0,\eta_0}(\boldsymbol{\omega}) \times \frac{g(\sigma_0\omega_1^\infty)}{g(\eta_0\omega_1^\infty)} = \prod_{i=-n}^{-1} \frac{g(\omega_i^{-1}\sigma_0\omega_1^\infty)}{g(\omega_i^{-1}\eta_0\omega_1^\infty)} \times \frac{g(\sigma_0\omega_1^\infty)}{g(\eta_0\omega_1^\infty)}.$$

Since μ is a g-measure, by Theorem 2.1, for every $i \in \mathbb{Z}$, one has

$$\mu(\omega_i \mid \omega_{i+1}^m) \rightrightarrows g(\omega_i^\infty),$$

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as $m \to \infty$, uniformly in $\boldsymbol{\omega} \in X$. Therefore, for each $n \in \mathbb{N}$,

$$f_{n,m}^{\sigma_0,\eta_0}(\boldsymbol{\omega}) = \frac{\mu(\omega_{-n}^{-1}\sigma_0\omega_1^m)}{\mu(\omega_{-n}^{-1}\eta_0\omega_1^m)} = \prod_{i=-n}^{-1} \frac{\mu(\omega_i \mid \omega_{i+1}^{-1}\sigma_0\omega_1^m)}{\mu(\omega_i \mid \omega_{i+1}^{-1}\eta_0\omega_1^m)} \times \frac{\mu(\sigma_0 \mid \omega_1^m)}{\mu(\eta_0 \mid \omega_1^m)}$$
$$\rightrightarrows \prod_{i=-n}^{-1} \frac{g(\omega_i^{-1}\sigma_0\omega_1^\infty)}{g(\omega_i^{-1}\eta_0\omega_1^\infty)} \times \frac{g(\sigma_0\omega_1^\infty)}{g(\eta_0\omega_1^\infty)} = f_n^{\sigma_0,\eta_0}(\boldsymbol{\omega}),$$

as $m \to \infty$ uniformly in $\boldsymbol{\omega}$. Since g is uniformly bounded away from 0, the functions $f_n^{\sigma_0,\eta_0}$, for fixed n, are continuous and bounded away from 0,

$$\inf_{\boldsymbol{\omega}\in X} f_n^{\sigma_0,\eta_0}(\boldsymbol{\omega}) \geq \left(\frac{\inf_{\boldsymbol{\omega}} g(\boldsymbol{\omega})}{\sup_{\boldsymbol{\omega}} g(\boldsymbol{\omega})}\right)^{n+1} > 0.$$

Uniform convergence of (5) immediately implies that

$$f_n^{\eta_0,\sigma_0}(\boldsymbol{\omega}) = \tilde{f}_n^{\sigma_0,\eta_0}(\boldsymbol{\omega}) \times \frac{g(\sigma_0\omega_1^\infty)}{g(\eta_0\omega_1^\infty)}$$

converges, uniformly in $\boldsymbol{\omega}$, as $n \to \infty$. Thus the limiting function,

$$f^{\sigma_0,\eta_0}(\boldsymbol{\omega}) = \prod_{i=-\infty}^{-1} \frac{g(\omega_i^{-1}\sigma_0\omega_1^\infty)}{g(\omega_i^{-1}\eta_0\omega_1^\infty)} \times \frac{g(\sigma_0\omega_1^\infty)}{g(\eta_0\omega_1^\infty)},$$

is continuous and is bounded away from 0. Indeed, since for every n, one has $f_n^{\sigma_0,\eta_0}(\boldsymbol{\omega}) f_n^{\eta_0,\sigma_0}(\boldsymbol{\omega}) = 1$, we also have $f^{\sigma_0,\eta_0}(\boldsymbol{\omega}) f^{\eta_0,\sigma_0}(\boldsymbol{\omega}) = 1$; f^{σ_0,η_0} is bounded from above as a continuous function on the compact X, it is bounded below by $1/\|f^{\sigma_0,\eta_0}\|_\infty$, for all σ_0, η_0. Finally, equation (6) implies that

$$\lim_{n\to\infty} \lim_{m\to\infty} \mu(\omega_0 \mid \omega_{-n}^{-1}, \omega_1^m) = \frac{1}{\sum_{\sigma_0\in\mathcal{A}} f^{\sigma_0,\omega_0}(\boldsymbol{\omega})} =: \gamma(\boldsymbol{\omega})$$

for all $\boldsymbol{\omega} \in X$. Therefore μ is consistent with a continuous uniformly non-null specification; hence it is a Gibbs measure.

Conversely assume that μ is a g-measure such that $\tilde{f}_n^{\sigma,\eta}$ does not converge uniformly, furthermore assume μ is a Gibbs measure. Then $f_{n,m}^{\eta,\sigma}$, for any n, converges uniformly to $f_n^{\sigma,\eta}$ as $m \to \infty$. It follows that if $f_n^{\sigma_0,\eta_0}$ does not converge uniformly then $f_{n,m}^{\sigma,\eta}$ does not converge uniformly either. However, for a Gibbs measure μ,

$$f_{n,m}^{\sigma_0,\eta_0}(\omega) = \frac{\mu(\omega_{-n}^{-1}\sigma_0\omega_1^m)}{\mu(\omega_{-n}^{-1}\eta_0\omega_1^m)} = \frac{\mu(\sigma_0 \mid \omega_{-m}^{-1}, \omega_1^n)}{\mu(\eta_0 \mid \omega_{-m}^{-1}, \omega_1^n)}$$

must converge uniformly. This follows from uniform non-nullness and Theorem 2.2 combined with uniform continuity of fractions on the relevant part of the domain. This leads to a contradiction, therefore a g-measure is a Gibbs measure if and only if the conditions of Theorem 4.1 are satisfied. □

Theorem 4.1 shows how the regularity of g-functions determines the continuity of two-sided conditional probabilities. A reversibility condition for g-measures should have a similar form.

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THEOREM 4.2. *A g-measure μ is reversible if and only if the sequence*

$$\hat{f}_n(\boldsymbol{\omega}) \equiv \left[\prod_{i=-n}^{-1} \frac{\mu(\omega_i \mid \omega_{i+1}^0)}{\mu(\omega_i \mid \omega_{i+1}^{-1})} \right]_{n\in\mathbb{N}} \tag{7}$$

converges uniformly in $\boldsymbol{\omega} \in X$, bounded away from 0, as $n \to \infty$.

Proof. The proof is almost immediate as $\hat{f}_n(\boldsymbol{\omega}) = \mu(\omega_{-n} \cdots \omega_{-1} \mid \omega_0)/\mu(\omega_{-n} \cdots \omega_{-1})$, hence, if $[\hat{f}_n]_{n\in\mathbb{N}}$ converges in $C(X)$ as $n \to \infty$, then $\mu(\omega_0)\hat{f}_n(\boldsymbol{\omega}) = \mu(\omega_{-n} \cdots \omega_{-1}\omega_0)/\mu(\omega_{-n} \cdots \omega_{-1})$ converges uniformly. Thus by Theorem 2.1 μ_-, the reverse of μ_+, is a g-measure. The corresponding g-function, $g_-(\boldsymbol{\omega})$, is given by $g_-(\boldsymbol{\omega}) = \lim_{n\to\infty} \mu(\omega_0)\hat{f}_n(\boldsymbol{\omega})$. □

Convergence of sequence (7) is often more difficult to show then the convergence of (5). The reason is that it is not easily reduced to a property of the function g or the potential, the quantities one usually knows or controls.

4.2. *Good future and uniqueness.* In this section we will relate Theorem 4.1 to a known sufficient condition, called *good future* [**12**], introduced by Fernández and Maillard. This condition applies to processes that are not necessarily translation-invariant and in the case of g-measures it reduces to the following: for $f \in C(X_+)$ let

$$\partial_k(f) \equiv \sup_{\omega\in X_+, \sigma_k, \eta_k \in \mathcal{A}} |f(\omega_0^{k-1}\sigma_k\omega_{k+1}^\infty) - f(\omega_0^{k-1}\eta_k\omega_{k+1}^\infty)|, \tag{8}$$

then the g-function $g \in G(X_+)$ has good future if the following summability condition is satisfied:

$$\sum_{k=1}^{\infty} \partial_k(g) < \infty. \tag{9}$$

Furthermore, we define $\mathrm{GF}(X_+)$ to be the set of g-functions satisfying this condition.

Remark 4.1. Summable variation of $g \in G(X_+)$ is equivalent to summable variation of $\log(g)$. Likewise, $g \in \mathrm{GF}(X_+)$ is equivalent to

$$\sum_{k=1}^{\infty} \partial_k(\log(g)) < \infty.$$

THEOREM 4.3. (Fernández and Maillard [**12**]) *If μ is a g-measure and $g \in G(X_+)$, if $g \in \mathrm{GF}(X_+)$, then μ is Gibbs.*

Proof. As μ is a g-measure there exists $c > 0$ such that for all ω one has $c < g(\boldsymbol{\omega}) < 1 - c$. Thus, for any $\sigma_k \in \mathcal{A}$,

$$\frac{g(\omega) - \partial_k(g)}{g(\omega)} \le \frac{g(\omega_0^{k-1}\sigma_k\omega_{k+1}^\infty)}{g(\omega)} \le \frac{g(\omega) + \partial_k(g)}{g(\omega)}$$

and therefore

$$\max\left\{0, 1 - \frac{\partial_k(g)}{1-c}\right\} \le \frac{g(\omega_0^{k-1}\sigma_k\omega_{k+1}^\infty)}{g(\omega)} \le 1 + \frac{\partial_k(g)}{c}.$$

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The function g is bounded away from 0 and 1 and continuous so we can choose an $N > 0$ such that $1 - (\partial_k(g)/c) > 0$, for $k \geq N$. Now notice that, due to the convergence of $\sum_{k=1}^{\infty} \partial_k(g)$, the following two products converge, for N as above, to a non-zero value:

$$\prod_{k=N}^{\infty}\left(1 - \frac{\partial_k(g)}{1-c}\right), \quad \prod_{k=N}^{\infty}\left(1 + \frac{\partial_k(g)}{c}\right).$$

Moreover, for sufficiently large $N > 0$,

$$\prod_{k=N+1}^{\infty}\left(1 - \frac{\partial_k(g)}{1-c}\right) \leq \prod_{i=-\infty}^{-N-1} \frac{g(\omega_i \omega_{i+1}^{-1} \sigma_0 \omega_1^{\infty})}{g(\omega_i^{\infty})} \leq \prod_{k=N+1}^{\infty}\left(1 + \frac{\partial_k(g)}{c}\right).$$

It follows that for any $\epsilon > 0$ there exists an N such that the upper and lower bounds above are ϵ-close to 1. This implies that the sequence

$$\prod_{i=-N}^{-1} \frac{g(\omega_i \omega_{i+1}^{-1} \sigma_0 \omega_1^{\infty})}{g(\omega_i \omega_{i+1}^{\infty})}$$

is uniformly Cauchy and thus, uniformly convergent. Therefore, by Theorem 4.1, a g-measure for which (9) converges uniformly is Gibbs. □

An interesting consequence of the above result is that the smoothness required for Gibbsianity does not imply uniqueness. Hulse [**17**] has shown that for any $\lambda > 1$ there exists a function $g \in G(X_+)$, with multiple g-measures, such that

$$\sum_{k=1}^{\infty} \partial_k(g) < \lambda < \infty.$$

An important distinction between the good future condition and Theorem 4.1 is that, in equation (5), the supremum is taken after the product, meaning single factors in the product in equation (5) are not always of the order of $\partial_k(g)$. Furthermore, different factors in the product might cancel, therefore g-measures with slowly decaying $\partial_k(g)$ could still be Gibbs states. We will give an example of a measure having such behavior in §5.4.

4.3. *Special classes of g-measures.* There are several extensively studied classes of g-measures. We will discuss some of them from the point of view of being Gibbs. Hölder continuity was already known as a sufficient condition for being Gibbs [**25**]. Functions $g \in G(X_+)$ with $\sum_n \mathsf{var}_n(g) < \infty$ are said to have summable variation, the corresponding g-measures are known to be Gibbs [**9**, **21**]. The set of functions with summable variation contains the Hölder continuous functions as a proper subset. In turn these are a subset of a set of functions by Walters [**27**]. Let

$$S_n\phi \equiv \sum_{i=0}^{n-1} \phi \circ S^i,$$

the set of functions $\mathrm{Wal}(X_+, S)$ is defined as

$$\mathrm{Wal}(X_+, S) = \left\{\phi \in C(X_+) : \sup_{n \geq 1} \mathsf{var}_{n+p}(S_n\phi) \to 0 \text{ as } p \to \infty\right\}.$$

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Remark 4.2. In order to compare Walters' class with Theorem 4.1 we point out that the uniform convergence in the theorem is equivalent to

$$\sup_{n\geq 0} \partial_{n+p}(S_n \log g) \to 0 \quad \text{as } p \to \infty.$$

Given a potential $\phi \in \mathrm{Wal}(X_+, S)$, there exists a unique equilibrium state for ϕ that is also a g-measure [**27**]. Hölder continuity or summable variation of a potential $\phi \in C(X_+)$ implies $\phi \in \mathrm{Wal}(X_+, S)$.

THEOREM 4.4. (Walters [**30**]) *If μ is (the natural extension of) a g-measure with $\log(g) \in \mathrm{Wal}(X_+, S)$ then it is also a g-measure in the reverse direction, with $\log(g_-) \in \mathrm{Wal}(X_-, S_-)$, and a Gibbs measure.*

Proof. The reversibility with $\log(g_-) \in \mathrm{Wal}(X_-, S_-)$ has been shown by Walters [**30**].

If $\log(g) \in \mathrm{Wal}(X_+, S)$ then the corresponding condition can be written as

$$\sup_{n\geq 1} \sup_{\omega_0^{n+p-1}=\eta_0^{n+p-1}} \left(\sum_{i=0}^{n-1} \log(g(S^i\omega)) - \sum_{i=0}^{n-1} \log(g(S^i\eta)) \right) \to 0 \quad \text{as } p \to \infty,$$

which is equivalent to

$$\sup_{n\geq 1} \sup_{\omega_0^{n+p-1}=\eta_0^{n+p-1}} \log\left(\prod_{i=0}^{n-1} \frac{g(S^i\omega)}{g(S^i\eta)} \right) \to 0 \quad \text{as } p \to \infty.$$

Writing down the previous expression for the condition imposed on $\log(g)$ in a more explicit form one gets

$$\sup_{n\geq 1} \sup_{\omega,\eta\in X_+} \log\left(\prod_{i=0}^{n-1} \frac{g(\omega_i\omega_{i+1}^{n-1}\omega_n^{n+p-1}\omega_{n+p}^{\infty})}{g(\omega_i\omega_{i+1}^{n-1}\omega_n^{n+p-1}\eta_{n+p}^{\infty})} \right) \to 0 \quad \text{as } p \to \infty.$$

We can, for $\omega \in X$, let g act on ω_i^{∞}, with $i \in \mathbb{Z}$. For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $|\log(1+x)| < \epsilon$ then $|x| < \epsilon$, and therefore

$$\sup_{n\geq 1} \sup_{\omega,\eta\in X} \left| \left(\prod_{i=-n-p}^{-p-1} \frac{g(\omega_i\omega_{i+1}^{-p-1}\omega_{-p}^{-1}\omega_0^{\infty})}{g(\omega_i\omega_{i+1}^{-p-1}\omega_{-p}^{-1}\eta_0^{\infty})} \right) - 1 \right| \to 0 \quad \text{as } p \to \infty.$$

Here the entire tail can be different in the fraction. We only need to allow them to differ at a single site. So, for all $\epsilon > 0$ there exists an $N > 0$ such that if $p > 0$ then

$$\sup_{n\geq N} \sup_{\omega\in X,\sigma\in\mathcal{A}} \left| \left(\prod_{i=-n-p}^{-n-1} \frac{g(\omega_i\omega_{i+1}^{-1}\omega_0^{\infty})}{g(\omega_i\omega_{i+1}^{-1}\sigma\omega_1^{\infty})} \right) - 1 \right| < \epsilon.$$

As the finite products are bounded away from 0 and ∞ the tail being arbitrarily close to 1 implies uniform convergence and boundedness, so that we can use Theorem 4.1. Therefore the natural extension of a g-measure with $\log(g) \in \mathrm{Wal}(X_+, S)$ is Gibbs. □

An even larger set of well-behaved potentials is given by

$$\mathrm{Bow}(X_+, S) = \left\{ \phi \in C(X_+) : \sup_{n\geq 1} \mathsf{var}_n(S_n\phi) < \infty \right\}.$$

For any potential $\phi \in \mathrm{Bow}(X_+, S)$ a unique equilibrium state exists and this is a Bowen–Gibbs measure. However, there is no guarantee that an equilibrium state for a potential $\phi \in \mathrm{Bow}(X_+, S)$ is a g-measure.

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4.4. *Reversibility of g-measures.* As was shown by Walters [**30**] and discussed above, if μ is a g-measure with $\log(g) \in \mathrm{Wal}(X_+, S)$, then its reverse μ_- is a g-measure and has $\log(g_-) \in \mathrm{Wal}(X_-, S_-)$. Walters proved that if $\log(g) \in \mathrm{Bow}(X_+, S)$ and if the measure is also a reverse g-measure then the reverse g-function, g_-, satisfies $\log(g_-) \in \mathrm{Bow}(\mathrm{X}_-, \mathrm{S}_-)$. However it remained an open question whether μ_- is a g-measure if the potential of μ_+ satisfies $\log(g) \in \mathrm{Bow}(X_+, S)$. Borrowing elements from both non-Gibbsian examples [**14**, **30**] we can show that this is not always the case.

PROPOSITION 4.1. *There exists a g-measure μ_+ with $\log(g) \in \mathrm{Bow}(X_+, S)$ for which the reverse measure μ_- is not a g-measure.*

Proof. We will show this using an example. First we define a g-function, then show that $\log(g) \in \mathrm{Bow}(X_+, S)$ and finally show that the reverse is not a g-measure.

Step 1. Take $\{\nu_k\}_{k\geq 0}$ a sequence of reals satisfying:

(i) $\nu_k \to 0$, as $n \to \infty$;
(ii) there exists a $K > 0$ with $|\sum_{k=0}^{N} \nu_k| < K$ for all $N > 0$;
(iii) $\sum_{k=0}^{\infty} \nu_k$ does not converge;
(iv) $k \to \nu_k$ is injective.

The fourth requirement is not needed for the validity of the example, but it simplifies the combinatorial aspects of the proof. It is easy to see that such a sequence exists.

Let $\xi \in (1, 2^{1/2K})$, $\mathcal{A} = \{0, 1\}$, $n \geq 1$. Define the function g as follows:

$$\begin{cases} g(00^n 10\boldsymbol{\eta}) = \frac{1}{2}\xi^{-\nu_n}, \\ g(10^n 10\boldsymbol{\eta}) = 1 - \frac{1}{2}\xi^{-\nu_n}, \\ g(00^n 11\boldsymbol{\eta}) = \frac{1}{2}\xi^{\nu_n}, \\ g(10^n 11\boldsymbol{\eta}) = 1 - \frac{1}{2}\xi^{\nu_n}, \end{cases} \tag{10}$$

for all $\boldsymbol{\eta} \in \mathcal{A}^{\infty}$ and let $g(\boldsymbol{\omega}) = \frac{1}{2}$ otherwise. Then $0 < g < 1$ and $\sum_{\sigma\in\mathcal{A}} g(\sigma\omega_1^{\infty}) = 1$, for any $\boldsymbol{\omega} \in X_+$. Furthermore

$$\begin{aligned} \mathrm{var}_m(g) &= |g(\underbrace{00^n1}_{m}\,0\boldsymbol{\eta}) - g(\underbrace{00^n1}_{m}\,1\boldsymbol{\sigma})| = |\tfrac{1}{2}\xi^{\nu_{m-2}} - \tfrac{1}{2}\xi^{-\nu_{m-2}}| \\ &= \sinh(\log(\xi)\nu_{m-2}) \to 0 \quad \text{as } m \to \infty, \end{aligned}$$

since $\nu_k \to 0$ as $k \to \infty$. Thus g is a continuous function.

Step 2. We will now show that $\log(g) \in \mathrm{Bow}(X_+, S)$. It is sufficient to show that $\mathrm{var}_n(S_n \log g)$ is uniformly bounded for all $n \geq 1$. Note that

$$\mathrm{var}_n(S_n \log g) = \sup_{\boldsymbol{\omega},\boldsymbol{\eta},\boldsymbol{\sigma}\in X_+} \log\left(\prod_{i=0}^{n-1} \frac{g(\omega_i^{n-1}\eta_n^{\infty})}{g(\omega_i^{n-1}\sigma_n^{\infty})}\right). \tag{11}$$

Suppose $n \geq 1$ and let $\boldsymbol{\omega}$, $\boldsymbol{\eta}$, $\boldsymbol{\sigma} \in X$ be the points at which the supremum in (11) is attained. Suppose $0 \leq i < n - 2$ is such that

$$\frac{g(\omega_i^{n-1}\eta_n^{\infty})}{g(\omega_i^{n-1}\sigma_n^{\infty})} \neq 1.$$

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Then, by the definition of g, one necessarily has

$$\omega_i\omega_{i+1} = 00 \quad \text{or} \quad \omega_i\omega_{i+1} = 10.$$

Put

$$I_1^{(n)} = \left\{1 \le i < n-1 : \omega_i\omega_{i+1} = 00 \text{ and } \frac{g(\omega_i^{n-1}\eta_n^\infty)}{g(\omega_i^{n-1}\sigma_n^\infty)} \ne 1\right\},$$

$$I_2^{(n)} = \left\{1 \le i < n-1 : \omega_i\omega_{i+1} = 10 \text{ and } \frac{g(\omega_i^{n-1}\eta_n^\infty)}{g(\omega_i^{n-1}\sigma_n^\infty)} \ne 1\right\},$$

then

$$\prod_{i=0}^{n-1} \frac{g(\omega_i^{n-1}\eta_n^\infty)}{g(\omega_i^{n-1}\sigma_n^\infty)} = \prod_{i\in I_1^{(n)}\cup I_2^{(n)}\cup\{n-1\}} \frac{g(\omega_i^{n-1}\eta_n^\infty)}{g(\omega_i^{n-1}\sigma_n^\infty)}. \tag{12}$$

We proceed by evaluating the contribution of factors for indices in $I_1^{(n)}$. Note that since $\omega_i^{n-1}\boldsymbol{\eta}$ and $\omega_i^{n-1}\boldsymbol{\sigma}$ start with two 0's: $\omega_i\omega_{i+1} = 00$, one has

$$\frac{g(\omega_i^{n-1}\eta_n^\infty)}{g(\omega_i^{n-1}\sigma_n^\infty)} = \frac{(1/2)\xi^{\pm\nu_m}}{(1/2)\xi^{\pm\nu_k}} = \xi^{\pm\nu_m\mp\nu_k},$$

where $m, k \in \mathbb{N}\cup\{+\infty\}$ depend on the first occurrence of 10 or 11 in the corresponding sequences; with a minor abuse of notation we let $m = +\infty$ or $k = +\infty$ if $\omega_i^{n-1}\boldsymbol{\eta} = 0^\infty$ or $\omega_i^{n-1}\boldsymbol{\sigma} = 0^\infty$, respectively. Note that the first 1 in $\omega_i^{n-1}\boldsymbol{\eta}$ and $\omega_i^{n-1}\boldsymbol{\sigma}$ cannot occur before position $n-1$. Indeed, if 1 appears earlier, i.e. $\omega_j = 1$ for some $j \le n-1$, then necessarily

$$\frac{g(\omega_i^{n-1}\eta_n^\infty)}{g(\omega_i^{n-1}\sigma_n^\infty)} = 1.$$

Finally, since we concluded that the first 1 in $\omega_i^{n-1}\boldsymbol{\eta}$ and $\omega_i^{n-1}\boldsymbol{\sigma}$ occurs in position n or later, one has

$$\omega_i, \omega_{i+1}, \ldots, \omega_{n-2} = 0, 0, \ldots, 0.$$

Hence, if $I_1^{(n)} \ne \varnothing$ and $i_* = \min I_1^{(n)}$, then

$$\{i_*, i_*+1, \ldots, n-3\} \subseteq I_1^{(n)} \subseteq \{i_*, i_*+1, \ldots, n-2\}, \tag{13}$$

where we used the injectivity of the sequence. A similar argument applies to $I_2^{(n)}$: if $i \in I_2^{(n)}$, i.e. $\omega_i\omega_{i+1} = 10$, and

$$\frac{g(\omega_i^{n-1}\eta_n^\infty)}{g(\omega_i^{n-1}\sigma_n^\infty)} \ne 1,$$

then 1 does not appear in $\omega_{i+1}, \ldots, \omega_{n-2}$, and hence if $I_2^{(n)} \ne \varnothing$, $I_2^{(n)}$ is a singleton, say $I_2^{(n)} = \{i\}$ and $\omega_{i+1}, \ldots, \omega_{n-2} = 0, \ldots, 0$.

We are now able to derive uniform bounds for (12). Firstly, since g is a continuous positive function, and thus uniformly bounded away from 0 and 1, by taking into account that $|I_2^{(n)}| \le 1$, one has

$$\prod_{i=0}^{n-1} \frac{g(\omega_i^{n-1}\eta_n^\infty)}{g(\omega_i^{n-1}\sigma_n^\infty)} = \prod_{i\in I_1^{(n)}\cup I_2^{(n)}\cup\{n-1\}} \frac{g(\omega_i^{n-1}\eta_n^\infty)}{g(\omega_i^{n-1}\sigma_n^\infty)} \le \left(\frac{\sup g}{\inf g}\right)^2 \prod_{i\in I_1^{(n)}} \frac{g(\omega_i^{n-1}\eta_n^\infty)}{g(\omega_i^{n-1}\sigma_n^\infty)}. \tag{14}$$

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If $I_1^{(n)}$ is empty this completes the proof that $\log g$ is in the Bowen class. Alternatively if $I_1^{(n)}$ is not empty, taking (13) into account, one has

$$\prod_{i\in I_1^{(n)}} \frac{g(\omega_i^{n-1}\eta_n^\infty)}{g(\omega_i^{n-1}\sigma_n^\infty)} \le \left(\frac{\sup g}{\inf g}\right) \prod_{i=i_*}^{n-3} \frac{g(\omega_i^{n-1}\eta_n^\infty)}{g(\omega_i^{n-1}\sigma_n^\infty)}.$$

Consider $(\omega_{i_*} \dots \omega_{n-2}\omega_{n-1}\eta_n\eta_{n+1}\dots) = (0, \dots, 0, \omega_{n-1}\eta_n\eta_{n+1}\dots)$. Let k be the position of the first 1 in this string. Note $k \ge n-1$, we let $k = +\infty$ if 1 does not appear. Depending on the next symbol: 0 or 1, one has

$$\prod_{i=i_*}^{n-3} g(\omega_i^{n-1}\eta_n^\infty) = \frac{1}{2^{n-i_*-2}} \xi^{-\sum_{i=i_*}^{n-3} \nu_{k-i-1}} \quad \text{or}$$

$$\prod_{i=i_*}^{n-3} g(\omega_i^{n-1}\eta_n^\infty) = \frac{1}{2^{n-i_*-2}} \xi^{+\sum_{i=i_*}^{n-3} \nu_{k-i-1}},$$

respectively. One has a similar expression for $(\omega_{i_*} \cdots \omega_{n-2}\omega_{n-1}\sigma_n\sigma_{n+1} \cdots)$. Since $|\sum_{j=m_1}^{m_2} \nu_j| = |\sum_{j=0}^{m_2} \nu_j - \sum_{j=0}^{m_1} \nu_j| \le 2K$ for all $m_1, m_2 \in \mathbb{N}$, we conclude that $\mathrm{var}_n(S_n \log g)$ is uniformly bounded.

Step 3. In the last part of the proof we show that the reverse of the unique g-measure μ_+ for the function g, given by (10), is not a g-measure. Let μ be the natural extension to X of μ_+. By Theorem 2.1 it suffices to present an element $\boldsymbol{\omega} \in X_-$ such that the sequence

$$a_n = \mu(\omega_0 \mid \omega_{-1}, \dots, \omega_{-n}), \quad n \in \mathbb{N},$$

does not converge as $n \to \infty$. Let

$$\boldsymbol{\omega} = (\dots, \omega_{-n}, \dots, \omega_{-2}, \omega_{-1}, \omega_0) = (\dots, 0, \dots, 0, 1, 0) \in X_-.$$

We use $0_{-\infty}^{-2}1_1 0_0$ as a shorthand notation for $\boldsymbol{\omega}$. Since μ is a fully supported g-measure, for $n \ge 2$, one has

$$\begin{aligned} a_n = \mu(0_0 \mid 1_{-1}0_{-n}^{-2}) &= \frac{\mu(0_{-n}^{-2}1_{-1}0_0)}{\mu(0_{-n}^{-2}1_{-1}0_0) + \mu(0_{-n}^{-2}1_{-1}1_0)} \\ &= \frac{1}{1 + (\mu(0_{-n}^{-2}1_{-1}1_0))/(\mu(0_{-n}^{-2}1_{-1}0_0))}. \end{aligned}$$

Furthermore, note that

$$\begin{aligned} b_n = \frac{\mu(0_{-n}^{-2}1_{-1}1_0)}{\mu(0_{-n}^{-2}1_{-1}0_0)} &= \prod_{j=-n}^{-3} \frac{\mu(0_j \mid 0_{j+1}^{-2}1_{-1}1_0)}{\mu(0_j \mid 0_{j+1}^{-2}1_{-1}0_0)} \times \frac{\mu(0_{-2}1_{-1}1_0)}{\mu(0_{-2}1_{-1}0_0)} \\ &= \left(\prod_{j=-n}^{-3} \xi^{2\nu_{-j-2}}\right) \frac{\mu(0_{-2}1_{-1}1_0)}{\mu(0_{-2}1_{-1}0_0)} = C\xi^{2\sum_{j=1}^{n-2} \nu_j}, \end{aligned}$$

for some $C > 0$. However, since the partial sums $\sum_{j=1}^{n} \nu_j$ are bounded and do not converge, the sequences $\{b_n\}$ and $\{a_n\}$ do not converge as $n \to \infty$. Thus we constructed an example μ_+ of a g-measure with $\log(g) \in \mathrm{Bow}(X_+, S)$, such that its reverse is not a g-measure. □

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5. *Examples and an overview*

As a large number of conditions are relevant to this paper, this section aims at providing some insight in how these conditions compare. First we will discuss a class of measures introduced by Walters [**31**], as it provides a convenient framework for discussing the example in [**14**], of a non-Gibbsian g-measure. Furthermore we will use this class to generate more examples. The first examples we will discuss are the non-Gibbsian g-measures that can be found in the literature. Then we will give two examples to clarify the distinction between Walters' condition and Good Future. A fifth example shows how the decay of variation of μ_+ can differ from the decay of variation of its reverse, μ_-, even if both are g-measures. Finally we will give a table showing how the classes differ, supported by examples.

5.1. *Walters' natural space $R(X_+)$.* In order to investigate the set of g-measures that are Gibbsian the space of functions, $R(X_+)$, introduced by Walters [**31**] is very useful. For example, the non-Gibbsian g-function from [**14**] is in this class. Consider the alphabet $\mathcal{A} = \{0, 1\}$. The set $R(X_+) \cap G(X_+)$ is parameterized by two sequences, conditions for being in $\mathrm{Wal}(X_+, S)$ and $\mathrm{Bow}(X_+, S)$ depend in an elegant way on these sequences. Walters showed that cylinder sets of the corresponding g-measures can be calculated explicitly in terms of these sequences and that the measures are unique.

To be precise, let $(\gamma_p)_2^\infty$ and $(\delta_p)_2^\infty$ be two sequences taking values in $(c, 1-c)$, with $c \in (0, \frac{1}{2})$, converging to γ and δ respectively. Define, for $p \geq 2$ and any $\boldsymbol{\eta} \in \mathcal{A}^\infty$,

$$g(0^p 1\boldsymbol{\eta}) = \gamma_p, \quad g(10^{p-1}1\boldsymbol{\eta}) = 1 - \gamma_q,$$
$$g(1^p 0\boldsymbol{\eta}) = \delta_p, \quad g(01^{p-1}0\boldsymbol{\eta}) = 1 - \delta_p.$$

For this to be consistent with $g \in G(X_+)$ the function must be continuous, we must then require that, for $q \geq 1$,

$$g(0^\infty) = \gamma, \quad g(1^\infty) = \delta,$$
$$g(10^\infty) = 1 - \gamma, \quad g(01^\infty) = 1 - \delta.$$

This is what defines an element of $G(X_+) \cap R(X_+)$. For such a function Walters proved the following.

THEOREM 5.1. (Walters [**31**]) *Let $g \in G(X_+) \cap R(X_+)$ be given in terms of $(\gamma_p)_2^\infty$ and $(\delta_p)_2^\infty$ as above. Then the following statements hold:*

(1) $\log(g) \in \mathrm{Bow}(X_+, S)$ *if and only if there exists $A > 1$ with*

$$A^{-1} \leq \gamma_2 \cdots \gamma_{n+1}/\gamma^n \leq A$$

and

$$A^{-1} \leq \delta_2 \cdots \delta_{n+1}/\delta^n \leq A$$

for all $n \geq 1$.

(2) $\log(g) \in \mathrm{Wal}(X_+, S)$ *if and only if $\sum_{n=2}^\infty \log(\gamma_n/\gamma)$ and $\sum_{n=2}^\infty \log(\delta_n/\delta)$ are both convergent.*

All functions $g \in G(X_+) \cap R(X_+)$ define a unique g-measure that is reversible, even those for which $\log(g) \notin \mathrm{Bow}(X_+, S)$.

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THEOREM 5.2. (Walters [**31**]) *For $g \in G(X_+) \cap R(X_+)$ there exists a unique g-measure μ. Furthermore the measure of a cylinder set is equal to the measure of the reverse cylinder,*

$$\mu(\omega_0\omega_1 \cdots \omega_{n-1}) = \mu(\omega_{n-1} \cdots \omega_1\omega_0)$$

for all $\omega_0, \omega_1, \ldots, \omega_{n-1} \in \{0, 1\}$, where $n \geq 1$.

Due to the ease of construction of elements in $R(X_+)$ and good control of their properties this class constitutes a very nice source of examples.

5.2. *A well-behaved g-measure that is not Gibbs.* Let us start with the example of a non-Gibbsian g-measure constructed in [**14**]. The measure is defined as follows. Let $p_\infty \in (0, 1)$ and $\xi \in (1, (1 - p_\infty)^{-2})$. For $k \geq 0$ put

$$\nu_k = \frac{(-1)^{r_k}}{r_k} \quad \text{with } r_k = \inf\left\{ i \geq 1 : \sum_{j=1}^{i} j \geq k + 1 \right\}.$$

The first few terms of this sequence are

$$-1, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{1}{5}, -\tfrac{1}{5}, -\tfrac{1}{5}, -\tfrac{1}{5}, -\tfrac{1}{5}.$$

The function g is in the class $R(X_+) \cap G(X_+)$ and can thus be defined in terms of two sequences:

$$\gamma_p = (1 - p_\infty)\xi^{\nu_{p-1}},$$
$$\delta_p = 1 - (1 - p_\infty)\xi^{-1}.$$

In [**14**] the following was shown:

$$\mu^{\mathrm{GFM}}(0_0 \mid 1_{-m}0^{-1}_{-m+1}, 0^{n-1}_1 1_n) = (1 + c_{n,m}\xi^{\sum_{k=0}^{m-1} \nu_k - \nu_{k+n}})^{-1},$$

where $c_{n,m}$ is a positive converging sequence as $m, n \to \infty$, bounded away from 0. It follows that the limiting behavior is determined by a factor, $\xi^{\sum_{k=0}^{m-1} \nu_k - \nu_{k+n}}$. For any $M > 0$ one can choose $m, n > M$ such that this factor equals ξ^{-1}; likewise $m, n > M$ can be chosen such that the factor becomes 1. This oscillatory behavior implies that the sequence does not converge, resulting in an essential discontinuity of the two-sided conditional probability at $\boldsymbol{\omega} = 0^{+\infty}_{-\infty}$.

Alternatively, it is easy to see that

$$\prod_{i=1}^{\infty} \frac{g(0^i 10^\infty)}{g(0^\infty)} = \prod_{i=1}^{\infty} \xi^{\nu_i}$$

does not converge and therefore we can use Theorem 4.1 to show that the measure is not Gibbs. Note that this example still satisfies the uniform non-nullness required for Gibbs measures. Additionally, $\sum_{n=2}^{\infty} \log(\gamma_n/\gamma)$ does not converge due to oscillating partial sums, while $\gamma_2 \cdots \gamma_{n+1}/\gamma^n$ and $\delta_2 \cdots \delta_{n+1}/\delta^n$ are bounded away from 0 and ∞, this implies, by Theorem 5.1, that $\log(g) \in \mathrm{Bow}(X_+, S) \backslash \mathrm{Wal}(X_+, S)$, showing that μ^{GFM} is a Bowen–Gibbs measure as well.

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5.3. *A more severely non-Gibbsian g-measure.* Another example of a g-measure that is not Gibbsian, was, for different purposes, constructed by Walters [**30**]. This example was constructed to show the existence of a g-measure μ_+^W on X_+ such that its reversal on X_- is not a g-measure. It turns out that Walters' construction also provides a non-Gibbsian measure. While the example in §5.2 subtly breaks continuity, but still has the uniform non-nullness property, in the example by Walters both conditions are violated.

Let $(p_k)_{k=0}^\infty$, with $p_k \in [0, 1)$ for all $k \in \mathbb{Z}_+$, $p_k \to 0$ as $k \to \infty$ and $\sum_{k=0}^\infty (p_k/(1+p_k)) = \infty$. For $k, l \geq 0$ let

$$g(000^k1^l101\boldsymbol{\eta}_{k+l+4}^\infty) = \tfrac{1}{2}(1-p_{k+l}) \quad \text{and} \quad g(100^k1^l101\boldsymbol{\eta}_{k+l+4}^\infty) = \tfrac{1}{2}(1+p_{k+l}),$$
$$g(000^k1^l100\boldsymbol{\eta}_{k+l+4}^\infty) = \tfrac{1}{2}(1+p_{k+l}) \quad \text{and} \quad g(100^k1^l100\boldsymbol{\eta}_{k+l+4}^\infty) = \tfrac{1}{2}(1-p_{k+l}),$$

for any $\boldsymbol{\eta} \in X_+$. For all other $\boldsymbol{\omega} \in X_+$ put $g(\boldsymbol{\omega}) = \frac{1}{2}$. Clearly, g is a continuous normalized function on X_+, for any such g at least one g-measure exists, let μ_+^W be such a measure and let μ^W be the natural extension of μ_+^W. Then we have the following recurrence relation: for $m \geq 2$,

$$\frac{\mu^W(1^m00)}{\mu^W(1^m0)} = \frac{\mu^W(1 \mid 1^{m-1}00)}{\mu^W(1 \mid 1^{m-1}0)} \frac{\mu^W(1^{m-1}00)}{\mu^W(1^{m-1}0)} = \frac{\mu^W(1^{m-1}00)}{\mu^W(1^{m-1}0)},$$

where the last equality holds as $\mu^W(1 \mid 1^{m-1}00)$ and $\mu^W(1 \mid 1^{m-1}0)$ are both equal to $\frac{1}{2}$. This implies that $\mu^W(1^m00)/\mu^W(1^m0)$ is constant in m and due to μ^W being a g-measure this value is not 0 or 1:

$$\begin{aligned}\frac{\mu^W(0^n1^m00)}{\mu^W(0^n1^m0)} &= \frac{\mu^W(0^n1^m00)}{\mu^W(0^n1^m00)+\mu^W(0^n1^m01)} = \frac{1}{1+\mu^W(0^n1^m01)/\mu^W(0^n1^m00)} \\ &= \frac{1}{1+(\prod_{i=1}^{n-1}(\mu^W(0 \mid 0^i1^m01)/\mu^W(0 \mid 0^i1^m00)))\mu^W(01^m01)/\mu^W(01^m00)} \\ &= \frac{1}{1+(\prod_{i=1}^{n-1}((1-p_{i+m-2})/(1+p_{i+m-2})))(\mu^W(01^m01)/\mu^W(01^m00))}.\end{aligned}$$

For any $m \geq 1$ the product $\prod_{i=1}^{n-1}((1-p_{i+m-2})/(1+p_{i+m-2}))$ tends, by the choice of $(p_k)_{k=0}^\infty$, to 0 as $n \to \infty$. Then it follows that

$$\lim_{n\to\infty} \frac{\mu^W(0^n1^m00)}{\mu^W(0^n1^m0)} = 1,$$

for any $m \geq 1$. It follows that there exists an $\epsilon > 0$ such that for any $m \geq 1$ there exists an $n > 0$ such that $|\mu^W(1^m00)/\mu^W(1^m0) - \mu^W(0^n1^m00)/\mu^W(0^n1^m0)| > \epsilon$. It follows that $\mu_-^W(0_0 \mid \cdot)$ is discontinuous at $0_{-1}1_{-\infty}^{-1}$. This shows that the reverse of μ_+^W is not a g-measure. We now show that non-Gibbsianness follows from similar arguments. The conditional probability $\mu^W(0_0 \mid 1_{-n}^{-2}0_{-1}0_1^m)$ is, for a given $n \geq 2$, bounded away from 0 and 1, for all $m \geq 1$, as μ^W is a g-measure. Now for any $m \geq 1$ we have that $\mu^W(0_0 \mid 1_{-n}^{-2}0_{-1}0_1^m)$ is constant in $n \geq 2$, hence the conditional probability is bounded away from 0 and 1. On the other hand, $\mu^W(0_0 \mid 0_{-n-p}^{-n-1}1_{-n}^{-2}0_{-1}0_1^m)$ tends, for any $n \geq 2$ and $m \geq 1$, to 1, as $p \to \infty$. Therefore a consistent specification would necessarily be discontinuous in $1_{-\infty}^{-2}0_{-1}^\infty$, furthermore uniform non-nullness is violated. It follows that μ^W is not a Gibbs measure.

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5.4. *A g-measure for a potential in Walters' class without good future.* Neither Walters' condition nor good future implies the other. In this example we will construct a g-function with oscillating dependencies on far away spins. The requirements for $\log(g) \in \mathrm{Wal}(X_+, S)$ are such that a strong influence from individual spins far away can be allowed, by having individual contributions cancel. Using this we can let the decay of the dependencies fall off rather slowly and yet have $\log(g) \in \mathrm{Wal}(X_+, S)$. Let, for $p \geq 2$,

$$\gamma_p = \frac{1}{2} + \frac{(-1)^p}{2p}, \quad \gamma = \frac{1}{2},$$
$$\delta_p = \frac{1}{2}, \quad \delta = \frac{1}{2}.$$

Now let $g \in G(X_+, S) \cap R(X_+)$ be the function defined by these sequences, as explained in §5.1. In this case $\sum_{n=2}^{\infty} \log(\gamma_n/\gamma)$ and $\sum_{n=2}^{\infty} \log(\delta_n/\delta)$ converge, and hence $\log(g) \in \mathrm{Wal}(X_+, S)$. However

$$\partial_k(g) = \sup_{l>k}\{|\gamma_k - \gamma_l|\} = |\gamma_k - \gamma_{k+1}| = \frac{2k+1}{2k(k+1)},$$

and hence

$$\sum_{k=2}^{\infty} \partial_k(g) = \infty.$$

Thus we conclude that there exists a $g \in G(X_+)$ such that $\log(g) \in \mathrm{Wal}(X_+, S)\backslash \mathrm{GF}(X_+)$.

5.5. *A g-measure for a potential with good future outside Walters' class.* Conversely, potentials satisfying Walters' condition have to behave well even if the entire tail is changed, whereas good future depends on single symbol changes. We can construct an example satisfying the Good Future condition, while failing Walters' condition by exploiting this property. Define, on a two-letter alphabet $\mathcal{A} = \{0, 1\}$, a g-function $g : X_+ \to (0, 1)$ as

$$g(0\omega_1\omega_2\cdots) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{i=1}^{\infty}\frac{\omega_i}{i^2},$$
$$g(1\omega_1\omega_2\cdots) = \frac{1}{2} - \frac{2}{\pi^2}\sum_{i=1}^{\infty}\frac{\omega_i}{i^2}.$$

As $\sum_{j=1}^{\infty}(1/j^2) = \pi^2/6$ this function is an element of $G(X_+)$. Also $g \in \mathrm{GF}(X_+)$ because $\delta_k = 2/\pi^2 k^2$, in fact it even satisfies Dobrushin's uniqueness criterion [**13**], which is stronger than good future. Thus the function defines a unique g-measure, which is also a Gibbs measure. However,

$$\sup_{n\geq 1}\sup_{\eta_0^{n-1}=\omega_0^{n-1}}\left\{\sum_{i=0}^{n-1}\log\left(\frac{g(S^i\omega)}{g(S^i\eta)}\right)\right\} \geq \sup_{n\geq 1}\left\{\log\left(\prod_{i=0}^{n-1}\frac{g(0_i^{n-1}1_n^{\infty})}{g(0_i^{\infty})}\right)\right\}$$
$$= \sup_{n\geq 1}\left\{\log\left(\prod_{i=0}^{n-1}\left(1+\frac{4}{\pi^2}\sum_{j=n-i}^{\infty}\frac{1}{j^2}\right)\right)\right\} = \sup_{n\geq 1}\left\{\log\left(\prod_{i=1}^{n}\left(1+\frac{4}{\pi^2}\sum_{j=i}^{\infty}\frac{1}{j^2}\right)\right)\right\}$$
$$\geq \sup_{n\geq 1}\left\{\log\left(\prod_{i=1}^{n}\left(1+\frac{4}{\pi^2 i}\right)\right)\right\} = +\infty.$$

From this divergence it follows that $\log(g) \in \mathrm{GF}(X_+)\backslash \mathrm{Bow}(X_+, S)$.

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5.6. *A reversible g-measure with slower decay of variation in the reverse direction.* We have seen that $\log(g) \in \mathrm{Wal}(X_+, S)$ is a sufficient condition for the corresponding g-measure to be reversible. Moreover the reverse measure μ_- is a g-measure which has $\log(g_-) \in \mathrm{Wal}(X_-)$. Similarly, if $\log(g) \in \mathrm{Bow}(X_+, S)$ and if the measure is also a reverse g-measure, then the reverse function satisfies $\log(g_-) \in \mathrm{Bow}(\mathrm{X}_-)$ [**30**]. Suggesting that, to some extent, the regularity of the function g and its reverse counterpart g_- are comparable. However, we would like to point out that there exist g-measures for which the decay of variation of g_- is slower that the decay of variation of g. We demonstrate this by the following adaptation of Walters' example [**30**].

Define the g-function by a sequence $(p_k)_{\geq 0}$, taking values in (0, 1), converging to $\frac{1}{2}$, as follows:

$$\begin{aligned} g(00^{k-2}101\boldsymbol{\eta}) &= p_k, \quad g(10^{k-2}101\boldsymbol{\eta}) = 1 - p_k, \\ g(00^{k-2}100\boldsymbol{\eta}) &= 1 - p_k, \quad g(10^{k-2}100\boldsymbol{\eta}) = p_k, \end{aligned}$$

for $k \geq 3$ and any $\boldsymbol{\eta} \in X_+$. Let $g(\boldsymbol{\omega}) = \frac{1}{2}$, for all other $\boldsymbol{\omega} \in X_+$. Now let $p_k = \frac{1}{2}\xi^{1/k^2}$, with $\xi \in (0, 1)$, then $\mathrm{var}_n(g) = \mathcal{O}(1/n^2)$, therefore $\log(g) \in \mathrm{Wal}(X_+, S)$. It follows that the reverse, μ_-, is a g-measure with $\log(g_-) \in \mathrm{Wal}(X_-, S_-)$. Let $a_k = \mu(0_0 \mid 0_{-1}1_{-2}0_{-k+1}^{-3})$, then a_k is bounded away from 0 and 1 by some constant $c > 0$. Then

$$\begin{aligned} \prod_{k=m}^{\infty} \frac{\mu(0_{-k} \mid 0_{-k+1}^{-3}1_{-2}0_{-1}1_0)}{\mu(0_{-k} \mid 0_{-k+1}^{-3}1_{-2}0_{-1})} &= \prod_{k=m}^{\infty} \frac{p_k}{a_k(1-p_k) + (1-a_k)p_k} \\ &= \prod_{k=m}^{\infty} \frac{1}{1 + a_k(1/p_k - 2)} \leq 1 + \frac{2c\log(\xi)}{m} + \mathcal{O}\left(\frac{1}{m^2}\right), \end{aligned}$$

for $m > 3$. Note that $\log(\xi)$ is negative, therefore the above estimate establishes a bound away from 1. As we already established that μ_- is a g-measure, Theorem 2.1 applies. Furthermore

$$\mu(\omega_0 \mid \omega_1^n) = \sum_{\sigma_{n+1} \in \mathcal{A}} \mu(\omega_0 \mid \omega_1^n \sigma_{n+1}) \mu(\sigma_{n+1} \mid \omega_1^n) \leq \sup_{\sigma_{n+1}} \mu(\omega_0 \mid \omega_1^n \sigma_{n+1}).$$

This can be used to determine the following lower bound to the variation of g_-, the reverse g-function:

$$\begin{aligned} \mathrm{var}_n(g_-) &\geq \sup_{\omega \in X} |\mu(\omega_0 | \omega_{-n+1}^{-1}) - g_-(\omega_{-\infty}^0)| \\ &\geq \left| \mu(1_0 \mid 0_{-1}1_{-2}0_{-n+1}^{-3}) \left(1 - \prod_{k=n}^{\infty} \frac{\mu(0_{-k} \mid 0_{-k+1}^{-3}1_{-2}0_{-1}1_0)}{\mu(0_{-k} \mid 0_{-k+1}^{-3}1_{-2}0_{-1})}\right)\right| \\ &\geq \frac{2c^2|\log(\xi)|}{n} + \mathcal{O}\left(\frac{1}{n^2}\right), \end{aligned}$$

for sufficiently large n. Hence $\mathrm{var}_n(g) = \mathcal{O}(1/n^2)$ while the variation of g_- can be estimated from below by a term proportional to $1/n$.

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5.7. *Overview of the relevant classes.* In this section we will give an overview of a relation between various classes of measures discussed in this paper. Some classes of g-measures are not included in the overview. Let us only mention the following growing chain of classes of measures:

- Markov measures;
- g-measures/Bowen–Gibbs measures with Hölder continuous potentials;
- g-measures/Bowen–Gibbs measures with summable variation potentials;
- g-measures/Bowen–Gibbs measures with $\log g \in \mathrm{Wal}(X_+, S)$.

Let us now turn to summarizing the properties of various classes.

Wal If $\log(g) \in \mathrm{Wal}(X_+, S)$, then there exists a unique g-measure $\mu = \mu_g$, which is also Gibbs and Bowen–Gibbs; its reversal μ_- is also a g-measure for some g_- with $\log g_- \in \mathrm{Wal}(X_-, S_-)$. However μ_g does not necessarily satisfy the good future condition, cf. §5.4.

Bow If $\log(g) \in \mathrm{Bow}(X_+, S)$, then there exists a unique g-measure $\mu = \mu_g$, which is also Bowen–Gibbs, but not necessarily Gibbs. Moreover, μ does not necessarily have Good Future, nor does its reversal μ_- have to be a g-measure, cf. §§5.5, 4.4 respectively.

GF If $g \in G(X_+)$ satisfies the good future condition: $\sum_k \partial_k(g) < \infty$, then any g-measure is necessarily Gibbs. Note that there exist g-functions with good future that are consistent with several g-measures; and hence, $\log g \notin \mathrm{Bow}(X_+, S)$. It is not known whether the reverse of a g-measure with $g \in \mathrm{GF}(X_+)$ must be a g-measure.

Gibbs If μ is a Gibbs measure and a g-measure at the same time then it is not necessarily Bowen–Gibbs, nor does it have to have good future, this is highlighted in the examples in §§5.5 and 5.4. It is not known whether the reverse μ_- is a g-measure.

Table 1 in this section refers to examples of measures having properties discussed above. One example in the overview we have not mentioned before. Namely a measure for a g-function in the R class, that does not belong to Bowen's class. Such a measure can easily be constructed. Take for example the sequences $\gamma_p = \frac{1}{2}(\frac{9}{4})^{1/p}$, $\delta_p = \frac{1}{2}$. These sequences take values in the interval (0, 1) and converge to $\frac{1}{2}$ and therefore define a g-function in the space $R(X_+)$, as described in §5.1. Furthermore $\prod_{p=2}^{\infty}(\frac{9}{4})^{1/p} = (\frac{9}{4})^{\sum_{p=2}^{\infty}(1/p)} = \infty$, hence $\log(g) \notin \mathrm{Bow}(X_+, S)$.

The set of g-measures with $\log(g) \in \mathrm{Bow}(X_+, S)\backslash\mathrm{Wal}(X_+, S)$ is particularly interesting in the context of this paper. This set of measures contains examples of both reversible and non-reversible g-measures and an example of a g-measure that is not Gibbs. It would be interesting to know if there can be a g-measure with $\log(g) \in \mathrm{Bow}(X_+, S)\backslash\mathrm{Wal}(X_+, S)$ that is a Gibbs measure, we do not have a result on this.

6. *When Gibbs measures are g-measures*

Suppose μ is a translation-invariant Gibbs measure on $X = \mathcal{A}^{\mathbb{Z}}$, corresponding to a uniformly absolutely convergent interaction $\Phi = \{\Phi(\Lambda, \cdot), \Lambda \Subset \mathbb{Z}\}$. A natural question is to identify conditions under which μ (more precisely, a restriction μ_+ of μ to $X_+ = \mathcal{A}^{\mathbb{Z}_+}$)

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TABLE 1. The symbol in row A and column B represents an example which belongs to A, but not to B. The categories, in order, are g-measures with $\log(g) \in \mathrm{Wal}(X_+, S)$, $\log(g) \in \mathrm{Bow}(X_+, S)$, $g \in \mathrm{GF}(X_+)$, μ is a DLR Gibbs measure, the reverse of μ_+ is a g-measure, g has a unique corresponding g-measure and $g \in R(X_+) \cap G(X_+)$. We use the following symbols: 'X.X' refers to a section in this paper; '$\varnothing$' means no example exists; '?' means no example is known; and 'RnB' is an R class example with $\log(g) \notin \mathrm{Bow}(X_+, S)$, as discussed below.

	Wal	Bow	GF	Gibbs	reversible	unique	R ∩ G
Wal		$\varnothing$	5.4	$\varnothing$	$\varnothing$	$\varnothing$	5.6
Bow	5.2		5.4	5.2	4.4	$\varnothing$	5.6
GF	5.5	5.5		$\varnothing$	?	4.2	5.5
Gibbs	5.5	5.5	5.4		?	4.2	5.5
reversible	5.2	RnB	5.4	5.2		?	5.6
unique	5.2	RnB	5.4	5.2	4.4		5.6
R ∩ G	5.2	RnB	5.2	5.2	$\varnothing$	$\varnothing$	

is a g-measure. For completeness we will discuss two known types of solutions to this problem.

The first approach to deciding whether a given Gibbs measure is a g-measure is based on the variational description of Gibbs states. Let μ be a translation-invariant Gibbs measure for a UAC interaction Φ. It is well-known [**24**, Ch. 3], [**15**, Ch. 15], that the functions

$$f_\Phi(\boldsymbol{\omega}) = \sum_{0 \in V \Subset \mathbb{Z}} \frac{1}{|V|} \Phi_V(\omega_\Lambda),$$
$$\tilde{f}_\Phi(\boldsymbol{\omega}) = \sum_{0 \in V \Subset \mathbb{Z}_+} \Phi_V(\omega_\Lambda),$$

are continuous on X. Note that $\tilde{f}_\Phi$ depends only on $\{\omega_k,\ k \geq 0\}$, and hence, can be viewed as a function on $X_+ = \mathcal{A}^{\mathbb{Z}_+}$. Moreover, μ is an equilibrium state for f_Φ and $\tilde{f}_\Phi$. If μ is a g-measure, then μ is an equilibrium state for $\log g \in C(X_+, \mathbb{R})$, for some $g \in G(X_+)$. Thus the question boils down to the problem of deciding whether for a given $f_\Phi \in C(X, \mathbb{R})$ or $\tilde{f}_\Phi \in C(X_+, \mathbb{R})$ there exists a positive continuous normalized function $g \in C(X_+, \mathbb{R})$ such that $\log g$ has the same set of equilibrium states. One typically distinguishes two types of equivalence conditions on continuous functions resulting in identical sets of equilibrium states:

- two continuous functions $f_1,\ f_2$ are called *cohomologous* if there exists a continuous function h and a constant c such that
$$f_1 - f_2 = h \circ \sigma - h + c,$$
where σ is the left shift. Equivalently, one says that $f_1 - f_2$ is a *coboundary*;
- by analogy with the standard notion in statistical mechanics, we say that two continuous functions $f_1,\ f_2$ are *physically equivalent* if
$$\int (f_1 - f_2)\, d\nu = \text{const}$$
for all translation-invariant probability measures ν.

Clearly, if $f_1,\ f_2$ are cohomologous, then they are physically equivalent. The opposite is not true in general, cf. [**3**].

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Walters [**29**] identified the necessary and sufficient conditions for a continuous function f_1 on the space of two-sided sequences X to be cohomologous to a continuous function f_2 which depends only on the values of $\{\omega_k : k \geq 0\}$, i.e. f_2 can be viewed as a function on X_+.

THEOREM 6.1. (Walters [**29**]) *Denote by π the natural projection from $X = \mathcal{A}^{\mathbb{Z}}$ to $X_+ = \mathcal{A}^{\mathbb{Z}_+}$. Then $f_1 \in C(X, \mathbb{R})$ satisfies*

$$f_1(\boldsymbol{\omega}) = f_2 \circ \pi(\boldsymbol{\omega}) + h \circ \sigma(\boldsymbol{\omega}) - h(\boldsymbol{\omega}) + c$$

for some $f_2 \in C(X_+, \mathbb{R})$, $h \in C(X, \mathbb{R})$, and $c \in \mathbb{R}$ if and only if f_1 belongs to the class of functions satisfying the extended Walters condition, $W_\pi(X, S)$: namely, if for every $\epsilon > 0$ there exists $N > 0$ such that whenever $\boldsymbol{\omega}, \tilde{\boldsymbol{\omega}}$ are such that $\pi(\boldsymbol{\omega}) = \pi(\tilde{\boldsymbol{\omega}})$ and $\boldsymbol{\omega}_{-N}^{N} = \tilde{\boldsymbol{\omega}}_{-N}^{N}$ one has

$$\left| \sum_{k=0}^{n-1} [f_1(\sigma^k \boldsymbol{\omega}) - f_1(\sigma^k \tilde{\boldsymbol{\omega}})] \right| < \epsilon.$$

for all $n \geq 1$.

This still leaves the question whether the one-sided potential can be chosen such that it is normalized. In the literature several results are known that will guarantee this. Sinai showed this for Hölder continuous potentials on X [**25**]. This result was later improved to potentials on X with summable variation [**5**] and then to potentials in the two-sided counterpart of Walters' class, $\mathrm{Wal}(X, S)$ [**2**]. Such a potential has been shown to be cohomologous to a one-sided potential in $\mathrm{Wal}(X_+, S)$, implying that the corresponding equilibrium state is unique and a g-measure. At present, it is unknown whether potentials in Bowen's class, $\mathrm{Bow}(X_+, S)$, will always result in a continuous g-function.

The second approach is aimed at establishing continuity of versions of conditional probabilities. Suppose μ is a translation-invariant Gibbs measure on $X = \mathcal{A}^{\mathbb{Z}}$. Denote by γ the corresponding (translation-invariant) Gibbs specification. Denote by μ_+ the restriction of μ to $X_+ = \mathcal{A}^{\mathbb{Z}_+}$. The objective is to identify sufficient conditions on the Gibbsian specification γ (equivalently, on the potential Φ) such that μ_+ is a g-measure for some positive normalized continuous function $g : X_+ \to (0, 1)$. Note that μ_+ is a g-measure if and only if for every continuous $f : X_+ \to \mathbb{R}$ the conditional expectation

$$\mathbb{E}_{\mu_+}(f \mid \mathcal{F}_1^{\infty}),$$

defined μ_+-a.e., admits a continuous version. It is sufficient to check the above condition for $f(\omega) = \mathbb{1}_{[a_0]}(\omega)$, $a_0 \in \mathcal{A}$. The natural approach to the construction of a one-sided conditional expectation from the known two-sided conditional expectations (given by specification γ) is as follows. Consider again some $f \in C(X_+)$ and conditional expectations

$$\mathbb{E}_{\mu}(f \mid \mathcal{F}_1^{\infty} \cap \mathcal{F}_{-\infty}^{-n-1}), \quad n \geq 1.$$

If $f(\omega) = \mathbf{I}_{[a_0]}(\omega)$, $a_0 \in \mathcal{A}$, and $\omega \in \mathcal{A}^{\mathbb{Z}_+}$ and $\xi \in \mathcal{A}^{\mathbb{Z}_{<0}}$, then

$$\mathbb{E}_{\mu}(f \mid \mathcal{F}_1^{\infty} \cap \mathcal{F}_{-\infty}^{-n-1})(\xi_{-\infty}^{-1}, \omega_0^{+\infty}) = \gamma_{[-n,0]}([a_0] \mid \omega_1^{\infty}, \xi_{-\infty}^{-n-1}).$$

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Note that $\gamma_{[-n,0]}(\cdot \mid \omega_1^\infty, \xi_{-\infty}^{-n-1})$ is indeed $\mathcal{F}_1^\infty \cap F_{-\infty}^{-n-1}$-measurable and depends continuously on ω_1^∞,$\xi_{-\infty}^{-n-1}$. Therefore, if we assume that $\gamma_{[-n,0]}(\cdot \mid \omega_1^\infty, \xi_{-\infty}^{-n-1})$ converges as $n \to \infty$, and the limit is *independent* of ξ, then the limit

$$\lim_{n\to\infty} \gamma_{[-n,0]}([a_0] \mid \omega_1^\infty, \xi_{-\infty}^{-n-1})$$

is a natural candidate for the g-function of μ_+. The hereditary uniqueness condition (HUC) [**12**] is a sufficient condition for $g(\boldsymbol{\omega}) = \lim_{n\to\infty} \gamma_{[-n,0]}([a_0] \mid \omega_1^\infty, \xi_{-\infty}^{-n-1})$ to exist, with g a continuous function that is a version of $\mathbb{E}_{\mu_+}(f \mid \mathcal{F}_1^\infty)$. The HUC itself is a mostly technical condition related to the uniqueness of measures consistent with alterations of the specification γ. More interesting is that HUC is implied by two known uniqueness conditions for Gibbs measures. The first condition is Dobrushin's uniqueness condition [**8**, **20**]. The second condition is Georgii's boundary uniformity [**15**].

Definition 6.1. A specification $\{\gamma_V\}$ satisfies boundary-uniformity if there exists a constant $K > 0$ so that for every cylinder set $A \in \mathcal{B}$ there exists a finite interval $W = [i, \ldots, j] \subset \mathbb{Z}$ such that $\gamma_W(A \mid \xi) \geq K\gamma_W(A \mid \eta)$ for all $\xi, \eta \in X$.

If μ satisfies the boundary uniformity condition then the resulting measure μ_+ will satisfy the one-sided boundary uniformity condition, meaning that, for the corresponding g-function g, there exists a $K > 0$ such that

$$\prod_{i=0}^{n-1} \frac{g(S^i\omega_0^n\eta_n^\infty)}{g(S^i\omega_0^n\xi_n^\infty)} < K,$$

for all $\boldsymbol{\omega}, \boldsymbol{\eta}, \boldsymbol{\xi} \in X$. This means the resulting g-measure will have $\log(g) \in \mathrm{Bow}(X_+, S)$.

7. *Concluding remarks*

We discussed a condition and properties of g-measures that are Gibbs. Either by directly applying Theorem 4.1 or one of the sufficient conditions. The question how to determine for a given Gibbs measure whether it is a g-measure is largely open and we merely gave an overview of methods in the literature. Recently a first example of a Gibbs measure that is not a g-measure has been given in [**1**]. A particularly interesting question would be if a Gibbs measure with a potential for which $f_\Phi(\omega) = \sum_{X \ni \min 0} (1/|X|)\Phi_X(\omega) \in \mathrm{Bow}(X_+, S) \backslash \mathrm{Wal}(X_+, S)$ can be a g-measure, as this precisely forms a boundary where the continuity of g via the Ruelle operator theorem becomes uncertain.

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