THE REVIEW OF SYMBOLIC LOGIC
Volume 3, Number 3, September 2010

PARACONSISTENT LOGICS INCLUDED IN LEWIS' S4

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Abstract. As is known, a logic S is *paraconsistent* if the rule ECQ (E contradictione quodlibet) is not a rule of S. Not less well known is the fact that Lewis' modal logics are not paraconsistent. Actually, Lewis vindicates the validity of ECQ in a famous proof currently known as the "Lewis' proof" or "Lewis' argument." This proof essentially leans on the Disjunctive Syllogism as a *rule of inference*. The aim of this paper is to define a series of paraconsistent logics included in S4 where the Disjunctive Syllogism is valid only as a *rule of proof*.

§1. Introduction. In logics where the following rules hold

$$\mathrm{E}\wedge .\ A \wedge B \vdash A, B$$

$$\mathrm{I}\wedge .\ A, B \vdash A \wedge B$$

the rule ECQ (E contradictione quodlibet)

$$\mathrm{ECQ}.\ A, \neg A \vdash B$$

can equivalently be formulated in the form

$$\mathrm{ECQ}.\ A \wedge \neg A \vdash B.$$

As E$\wedge$ and I$\wedge$ hold in all logics defined in this paper, by ECQ we shall refer to this second version of the rule.

As it is well known (cf. Priest & Tanaka, 2004), a logic S is *paraconsistent* if the rule ECQ is not a rule of S. Not less well known is the fact that Lewis' logics are not paraconsistent. But this fact is, indeed, no accident. Actually, Lewis holds that B is deducible from $A \wedge \neg A$ (equivalently, $(A \wedge \neg A) \rightsquigarrow B$ — $\rightsquigarrow$ stands for strict implication), according to the following argument (Lewis & Langford, 1932, p. 250) currently known as "Lewis' argument" (cf. Anderson & Belnap, 1975, Section 16.1):

1. $A \wedge \neg A$	Hyp.
2. A	E$\wedge$, 1
3. $\neg A$	E$\wedge$, 1
4. $A \vee B$	I$\vee$, 2
5. B	Disjunctive Syllogism, 3, 4

Received: July 25, 2009

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doi:10.1017/S1755020310000109

https://doi.org/10.1017/S1755020310000109 Published online by Cambridge University Press

The aim of this paper is not to discuss this challenged argument again (cf. e.g., Anderson & Belnap, 1975 and Anderson *et al.*, 1992) but to define a series of paraconsistent logics included in Lewis' S4. In this sense, let us remark that in "Lewis argument" the Disjunctive Syllogism (d.s) is used as a *rule of inference*, that is, in the form

$$\text{From } \neg A \text{ and } A \vee B \text{ to infer } B.$$

In other words, it is used as a rule that may be applied to whichever premises. However, if d.s is considered in the form

$$\text{If } \neg A \text{ and } A \vee B \text{ are provable, then } B \text{ is provable}$$

then it is a *rule of proof*: it can only be applied to theorems. And if d.s is a rule of proof, then the proof of ECQ by Lewis does not, of course, hold.

Now, in all logics in this paper, d.s is understood as a rule of proof. So, theories built upon these logics are not necessarily closed by d.s, and consequently, in case of inconsistency, triviality does not follow automatically. Nevertheless, consistent, prime theories are of course always closed by d.s.

But let us return to "Lewis' argument". As Lewis strongly insists on, this demonstration is a paradigm of the general principle that can be expressed as follows:

(i) "Any proposition one chooses may be deduced from the denial of a tautological or necessary truth." (Lewis & Langford, 1932, pp. 250–251).

According to Lewis, this "states a fact about deducibility" (Lewis & Langford, 1932, p. 251). This fact can equivalently be rendered as follows:

(ii) "Tautologies in general are derivable from any premise we please" (Lewis & Langford, 1932, p. 251).

Now, in order to validate d.s, we need the "principle of excluded middle"

$$\text{pem. } A \vee \neg A$$

which, in the context of a De Morgan negation, is equivalent to the "principle of noncontradiction"

$$\text{pnc. } \neg(A \wedge \neg A).$$

Consequently, principles (i) and (ii) have to be invalidated, as it is in fact the case with the logics to be defined.

These principles can generally be expressed as follows:

$$(\text{i}') \text{ If } A \text{ is provable, then } \neg A \rightarrow B \text{ is provable}$$

and

$$(\text{ii}') \text{ If } A \text{ is provable, then } B \rightarrow A \text{ is provable.}$$

Theses (i′) and (ii′) are the rules EFQ (E falso quodlibet) and K, respectively, and, as just remarked, they are not derivable in any of the logics defined in this paper. Moreover, as we shall see, EFQ and K are not even *admissible* rules.

Nevertheless, special cases of these rules would be derivable, to wit,

$$\text{ECQ}'. \text{ If } A \rightarrow B \text{ is provable, then } \neg(A \rightarrow B) \rightarrow C \text{ is provable}$$

and

$$(\text{K}'). \text{ If } A \rightarrow B \text{ is provable, then } C \rightarrow (A \rightarrow B) \text{ is provable.}$$

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Consequently, Lewis' general principles are restricted as follows:

(i″) Any proposition one chooses may be deduced from the denial of a tautological entailment.

(ii″) Tautological entailments are in general derivable from any premise we please.

Of course, by "entailment" we are not here referring to Anderson and Belnap's conception of the term but to any provable conditional.

Therefore, the essential difference between the modal logics defined in this paper and Lewis' modal logics is that pem (equivalently, pnc) is not a necessary truth, in the sense that pem does not necessarily hold in any possible world, as it is just the case with relevant logics. Or put another way, this essential difference lies in the fact that only provable conditionals are necessary truths in the logics to be defined.

It has been noted above that in theories built upon the logics to be here presented, inconsistency does not lead necessarily to triviality. Notice, however, that theories containing the negation of an entailment —a provable conditional— being trivial, would be useless, which might not be unreasonable: who needs a theory denying a conditional theorem? Anyway, this point shall not be defended here. Let us only remark that the price to pay is a high one: provable conditionals are entailed by any proposition whichever.

The series of logics referred to above shall begin with Routley and Meyer's basic logic B (cf. Routley *et al*., 1982a) supplemented with the characteristic S4 axiom

$$B \to (A \to A)$$

and the rules *reductio*

$$\text{If } A \to \neg A \text{ is provable, then } \neg A \text{ is provable}$$

and d.s. This logic is labelled $\mathrm{B}_{\mathrm{K2r}\gamma}$ (for this label, cf. Section 2). Then, $\mathrm{B}_{\mathrm{K2r}\gamma}$ is extended with some well-known theses up to a sublogic of S4 that, from an intuitive point of view, can variously (and equivalently) be described as, for example (cf. Proposition 16 in Section 8):

1. S4 without the reductio rules, as e.g.,

$$\text{If } A \to B \text{ is provable, then } (A \to \neg B) \to \neg A \text{ is provable}$$

or

$$\text{If } A \to B \text{ is provable, then } (\neg A \to B) \to B \text{ is provable.}$$

2. S4 without rule K, or, equivalently, without the rule of assertion

$$\text{If } A \text{ is provable, then } (A \to B) \to B \text{ is provable.}$$

3. S4 without the axiom ECQ.

All logics in this paper contain classical logic in the sense that all tautologies in $\wedge$, $\vee$ and $\neg$ are provable. Moreover, as d.s is a primitive rule of each one of them, modus ponens for $\supset$ is available.

The structure of the paper is as follows. In Section 2, the logic $\mathrm{B}_{\mathrm{K2r}\gamma}$ is defined and some useful theorems of this logic are remarked. In Section 3, a Routley–Meyer semantics for $\mathrm{B}_{\mathrm{K2r}\gamma}$ is provided and soundness is proved. In Sections 4–6, we show that $\mathrm{B}_{\mathrm{K2r}\gamma}$ is complete with respect to this semantics. In Section 7 some extensions of $\mathrm{B}_{\mathrm{K2r}\gamma}$ are considered, and soundness and completeness of each one of these extensions are proved,

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In Section 8, we study the relations between logics introduced in Section 7, with some detail. In Section 9, it is shown that all the logics in this paper are paraconsistent. Finally, an appendix providing some sets of matrices is included. These sets are useful for checking some facts claimed throughout the paper. Knowledge of the Routley–Meyer semantics for relevant logics is presupposed.

§2. The logic $\mathbf{B_{K2r\gamma}}$. As is known, Routley and Meyer's basic positive logic B_+ can be axiomatized as follows (cf. Routley & Meyer, 1972 or Routley *et al.*, 1982b):

Axioms:

A1. $A \to A$

A2. $(A \wedge B) \to A \ / \ (A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $A \to (A \vee B) \ / \ B \to (A \vee B)$

A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

Rules of inference:

Modus Ponens (MP). $(\vdash A \to B \ \ \& \ \vdash A) \Rightarrow \vdash B$

Adjunction (Adj). $(\vdash A \ \ \& \ \vdash B) \Rightarrow \vdash A \wedge B$

Suffixing (Suf). $\vdash A \to B \Rightarrow \vdash (B \to C) \to (A \to C)$

Prefixing (Pref). $\vdash B \to C \Rightarrow \vdash (A \to B) \to (A \to C)$

Next, Routley and Meyer's basic logic B (cf. Routley *et al.,* 1982b) is the result of adding to B_+ the axioms

A7. $A \to \neg\neg A$

A8. $\neg\neg A \to A$

and the rule

Contraposition (Con). $\vdash A \to B \Rightarrow \vdash \neg B \to \neg A$

REMARK 1. *We note that Meyer and Routley's original B also contained the principle of excluded middle $A \vee \neg A$ together with the truth constant t and the binary connective fusion $\circ$ (cf. Meyer & Routley, 1972).*

The following are some theorems and rules of B (a proof for each one of them is sketched to their right. The biconditional ($\leftrightarrow$) is defined in the usual way):

T1. $\vdash A \to \neg B \Rightarrow \vdash B \to \neg A$ — Con, A7

T2. $\vdash \neg A \to B \Rightarrow \vdash \neg B \to A$ — Con, A8

T3. $\vdash \neg A \to \neg B \Rightarrow \vdash B \to A$ — A7, T2

T4. $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$ — Con, T2

T5. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ — Con, T1

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By adding the following axiom

$$\text{A9. } B \to (A \to A)$$

to B, we obtain an axiomatization of the logic $\mathrm{B_{K2}}$ where, in addition to T1-T5, we have:

$$\text{T6. } \vdash A \to B \Rightarrow \vdash C \to (A \to B) \qquad \text{A9}$$

$$\text{T7. } (A \to B) \to [A \to (A \wedge B)] \qquad \text{T6}$$

$$\text{T8. } \vdash A \to B \Rightarrow \vdash \neg(A \to B) \to C \qquad \text{T2, T6}$$

REMARK 2. *In Robles & Méndez, 2008 the logic B_{K+} is defined. It is the result of adding the rule K to B_+. The label B_{K2} is then intended to distinguish the present logic (with a restricted form of rule K) from B_{K+} and its extensions. On the other hand, notice that A1 is not independent in B_{K2}.*

Next, the logic $\mathrm{B_{K2r}}$ is the result of adding the rule reductio

$$\text{r. } \vdash A \to \neg A \Rightarrow \vdash \neg A$$

to $\mathrm{B_{K2}}$. In addition to T1–T7, the following are provable in $\mathrm{B_{K2r}}$:

$$\text{T9. } (\vdash A \to B \ \ \& \ \vdash A \to \neg B) \Rightarrow \vdash \neg A \qquad \text{Con, r}$$

$$\text{T10. } \neg(A \wedge \neg A) \qquad \text{T9}$$

$$\text{T11. } (\vdash \neg A \to B \ \ \& \ \vdash A \to B) \Rightarrow \vdash B \qquad \text{Con, T9, A8}$$

$$\text{T12. } A \vee \neg A \qquad \text{T4, T10, A8}$$

Concerning T9–T12, we note the following:

PROPOSITION 1. *Given the logic B, r, T9, T10, T11 and T12 are equivalent.*

Proof.

1. *r and T9 are equivalent:* r is immediate by A1 and T9.
2. *r and T11 are equivalent:* r is immediate by A1 and T11.
3. *r and T10 are equivalent:* r is derivable by A1, Con and T10.
4. *T10 and T12 are equivalent:* $\neg(A \wedge \neg A) \leftrightarrow (\neg A \vee A)$ is provable by A7, A8 and T4. Then, this equivalence follows from the previous one. □

Finally, $\mathrm{B_{K2r\gamma}}$ is axiomatized when adding the rule γ (the label γ is Ackermann's (cf. Ackermann, 1956)

$$\gamma\text{. } (\vdash A \ \ \& \ \vdash \neg A \vee B) \Rightarrow \vdash B$$

to $\mathrm{B_{K2r}}$. We note that, in addition to T1–T12, the following variant of γ is provable:

$$\gamma'\text{. } (\vdash A \ \ \& \ \vdash \neg(A \wedge B)) \Rightarrow \vdash \neg B \qquad \gamma\text{, T4}$$

In the next section, a semantics for $\mathrm{B_{K2r\gamma}}$ is provided.

§3. Semantics for $\mathbf{B_{K2r\gamma}}$.

First $\mathrm{B_{K2r\gamma}}$-models and $\mathrm{B_{K2r\gamma}}$-validity are defined.

DEFINITION 1. *A $B_{K2r\gamma}$-model is a structure $\langle K, O, R, *, \vDash\rangle$ where O is a nonempty subset of K (the set of points in the model), R is a ternary relation on K, and $*$ a*

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unary operation on K subject to the following definitions and postulates for all a, b, c, d ∈ K:

$$d1.\ a \leq b =_{df} (\exists x \in O)Rxab$$

$$d1'.\ a = b =_{df} a \leq b\ \ \&\ \ b \leq a$$

$$P1.\ a \leq a$$

$$P2.\ (a \leq b\ \ \&\ \ Rbcd) \Rightarrow Racd$$

$$P3.\ Rabc \Rightarrow b \leq c$$

$$P4.\ a = a**$$

$$P5.\ a \leq b \Rightarrow b* \leq a*$$

$$P6.\ a \in O \Rightarrow a = a*$$

Finally, ⊨ is a relation from K to the formulas of the propositional language such that the following conditions are satisfied for all propositional variables p, wff A, B and a ∈ K

$$(i).\ (a \leq b\ \ \&\ \ a \vDash p) \Rightarrow b \vDash p$$

$$(ii).\ a \vDash A \wedge B \text{ iff } a \vDash A \text{ and } a \vDash B$$

$$(iii).\ a \vDash A \vee B \text{ iff } a \vDash A \text{ or } a \vDash B$$

$$(iv).\ a \vDash A \rightarrow B \text{ iff for all } b, c \in K\ (Rabc\ \&\ b \vDash A) \Rightarrow c \vDash B$$

$$(v).\ a \vDash \neg A \text{ iff } a* \nvDash A.$$

We note that a B-model is defined similarly as a $B_{K2r\gamma}$-model except that P3 and P6 are dropped.

Validity is defined as follows:

DEFINITION 2. *A formula A is $B_{K2r\gamma}$-valid ($\vDash_{B_{K2r\gamma}} A$) iff $a \vDash A$ for all $a \in O$ in all $B_{K2r\gamma}$-models.*

Then, it is proved:

THEOREM 1. *(Soundness of $B_{K2r\gamma}$) If $\vdash_{B_{K2r\gamma}} A$, then $\vDash_{B_{K2r\gamma}} A$.*

Proof. It is left to the reader (cf., e.g., Routley *et al.*, 1982b). Anyway, A9 is valid by P3; and that r and γ preserve validity is proved by P6. □

This section is ended with the following proposition that will be used in the following section.

PROPOSITION 2. *Let A be a theorem of $B_{K2r\gamma}$. Then, ¬A is false in every $a \in O$ in all models.*

Proof. Suppose $\vdash_{B_{K2r\gamma}} A$, and, for reductio, $a \vDash \neg A$ for $a \in O$ in some model. By clause (*v*), $a* \nvDash A$; by Theorem 1, $a \vDash A$, whence $a* \vDash A$ by P6. Therefore $a \nvDash \neg A$ for every $a \in O$ in all models. □

§4. Completeness of $B_{K2r\gamma}$ I. The canonical model. Weak consistency. We begin by recalling some definitions.

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DEFINITION 3.

1. *A $B_{K2r\gamma}$-theory is a set of formulas closed under adjunction and provable $B_{K2r\gamma}$-entailment. That is, a is a $B_{K2r\gamma}$-theory if whenever $A, B \in a$, then $A \wedge B \in a$; and if whenever $A \rightarrow B$ is a theorem of $B_{K2r\gamma}$ and $A \in a$, then $B \in a$.*
2. *A theory a is prime if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$.*
3. *A theory is regular iff all theorems of $B_{K2r\gamma}$ belong to it.*
4. *A theory is null iff no wff belongs to it.*
5. *A theory a is w-inconsistent (inconsistent in a weak sense) iff $\neg A \in a$, A being a theorem of $B_{K2r\gamma}$. Then, a is w-consistent (consistent in a weak sense) iff a is not w-inconsistent.*
6. *A theory a is a-inconsistent (inconsistent in an absolute sense) iff a is trivial, i.e., iff every wff belongs to a. Then, a theory is a-consistent (consistent in an absolute sense) iff it is not a-inconsistent.*
7. *A theory a is sc-inconsistent (inconsistent according to the standard concept) iff for some wff A, $A \wedge \neg A \in a$. Then, a is sc-consistent (consistent according to the standard concept) iff a is not sc-inconsistent.*

REMARK 3. *The concept of w-inconsistency is introduced in (Robles & Méndez, 2008). But it is, of course, no novelty. For example, in defining the canonical model for E, the authors of (Routley et al., 1982a) remark "T is consistent provided T does not contain the negation of some theorem of E." (Routley et al., 1982a, p. 412)*

Next, the canonical model is defined.

DEFINITION 4. *Let K^T be the set of all $B_{K2r\gamma}$-theories and R^T be defined on K^T as follows: for all $a, b, c \in K^T$ and wff A, B, $R^T abc$ iff $(A \rightarrow B \in a \ \& \ A \in b) \Rightarrow B \in c$. Now, let K^C be the set of all nonnull, prime, a-consistent theories, and O^C the set of all regular, prime, w-consistent theories. On the other hand, let R^C be the restriction of R^T to K^C and $*^C$ be defined on K^C as follows: for any $a \in K^C$, $a*^C = \{A \mid \neg A \notin a\}$. Finally, $\vDash^C$ is defined as follows: for any $a \in K^C$, $a \vDash^C A$ iff $A \in a$. Then, the $B_{K2r\gamma}$-canonical model is the structure $\langle K^C, O^C, R^C, *^C, \vDash^C \rangle$.*

REMARK 4. *What distinguishes the $B_{K2r\gamma}$-canonical model from those for standard relevant logics is only the following important fact: in the latter members in K^C need not be regular, non-null or consistent in any sense of this term. Moreover, members in O^C are not necessarily consistent in any sense of the term.*

We now prove a lemma on nonnull theories and a couple of easy propositions on a-consistency, on the one hand, and on the relationship between w-consistency and the standard concept of consistency (sc-consistency) as absence of any contradiction, on the other hand

LEMMA 1. *Let a be a nonnull element in K^T. Then, a contains every implicative theorem of $B_{K2r\gamma}$.*

Proof. Let $A \rightarrow B$ be a theorem and $C \in a$. Then, $A \rightarrow B \in a$ by T6. □

REMARK 5. *Notice, however, that a nonnull theory need not contain all theorems.*

PROPOSITION 3. *Let $a \in K^T$. Then, a is a-inconsistent iff a contains the negation of an implicative theorem of $B_{K2r\gamma}$.*

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Proof. From left to right, Proposition 3 is obvious. In the inverse direction, it is immediate by T8. □

Regarding the relationship between w-consistency and sc-consistency in the present context, we note the following:

PROPOSITION 4.

1. *Let $a \in K^T$. Then, if a is w-consistent, then a is sc-consistent.*
2. *Let a be a regular member in K^T. Then, if a is sc-consistent, a is w-consistent.*

Proof.

1. Suppose $B \wedge \neg B \in a$ for some wff B. By A7, $\neg\neg(B \wedge \neg B) \in a$. Then, a is w-inconsistent by T10.
2. Suppose $\neg B \in a$, B being a theorem. As a is regular, $B \in a$. So, $B \wedge \neg B \in a$, and then, a is sc-inconsistent. □

So, in the case of $\mathrm{B_{K2r\gamma}}$-theories, only if regularity is present w.-consistency and sc-consistency are equivalent. In this sense, notice that, as pointed out above (Remark 5), members in K^C are not necessarily regular even though they contain all implicative theorems. On the other hand, if any member in K^T is w-consistent or sc-consistent, it is, of course, a-consistent. The converse, however, does not hold.

Next, we prove that the logic $\mathrm{B_{K2r\gamma}}$ is w-consistent, and the fundamental lemma on w-consistent theories: each w-consistent theory can be extended to a regular, w-consistent one.

PROPOSITION 5. *The logic $B_{K2r\gamma}$ is w-consistent.*

Proof. $\mathrm{B_{K2r\gamma}}$ is of course a $\mathrm{B_{K2r\gamma}}$-theory. Now, let A be a theorem. By Proposition 2, $\neg A$ is not valid. So, $\neg A \notin \mathrm{B_{K2r\gamma}}$ by the soundness theorem (Theorem 1). □

LEMMA 2. *Let a be a w-consistent element in K^T. Then, there is a regular, w-consistent x in K^C such that $a \subseteq x$.*

Proof. If a is null, the proof of Lemma 2 is trivial, the set of regular w-consistent theories being nonempty (cf. Proposition 5). So, let us assume that a is a nonnull, w-consistent member in K^T. Then, consider the set $x = \{B \mid \exists A, C[\vdash_{\mathrm{B_{K2r\gamma}}} A, C \in a \ \& \ \vdash_{\mathrm{B_{K2r\gamma}}} (A \wedge C) \rightarrow B]\}$. It is easy to prove that x is a theory. Moreover, it is proved:

1. $a \subseteq x$: by A2.
2. x is regular: by A2.
3. x is w-consistent: suppose $\neg A \in x$, A being a theorem. By definition of x, $\vdash (C \wedge B) \rightarrow \neg A$ for some $B \in a$ and theorem C. By T1, $\vdash A \rightarrow \neg(C \wedge B)$. So, $\vdash \neg(C \wedge B)$ by MP. Therefore $\vdash \neg B$ by γ'. But $\neg\neg B \in a$ ($B \in a$, A7). Then, a is w-inconsistent, which is impossible. □

Concerning the proof just given, notice that if consistency is understood in the standard sense (i.e., absence of contradiction), Lemma 2 is not provable in the present context: suppose that a contains no contradictions, define x as above, and suppose that x contains a contradiction for reductio. Similarly, as in the proof of Lemma 2, we can show that B is in a, $\neg B$ being a theorem. But a is not necessarily regular. Therefore, we cannot conclude $B \wedge \neg B \in a$, and consequently, the supposition that x contains a contradiction is not absurd.

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Lemma 2 is used in an essential way in the following section. The proofs of Lemma 7, Lemma 10 and Lemma 11 (the Fundamental Lemma) actually depend on it. And as was just remarked, in order to prove this lemma, consistency cannot be understood as the absence of any contradiction, but as the absence of the negation of any theorem. In this sense, we say that $\mathrm{B}_{\mathrm{K2r}\gamma}$ (and in fact all logics defined in this paper) is *adequate to w-consistency*, but it is *not adequate to consistency as the absence of any contradiction (sc-consistency).* On the other hand, this is the reason why members in O^C are required to be w-consistent instead of sc-consistent.

REMARK 6. *Notice the essential role of γ' (γ) in the proof of Lemma 2.*

Next, we prove a lemma on the extension of w-consistent theories to prime w-consistent ones.

LEMMA 3. *Let a be a w-consistent element in K^T. Then, there is a prime, w-consistent x in K^C such that $a \subseteq x$.*

Proof. Define from a a maximal w-consistent theory x such that $a \subseteq x$. If x is not prime, then there are wff A, B such that $A \vee B \in x$, $A \notin x$, $B \notin x$. Define the set $[x, A] = \{C \mid \exists D[D \in x \ \ \& \ \vdash_{\mathrm{B}_{\mathrm{K2r}\gamma}} (A \wedge D) \rightarrow C]\}$. Define $[x, B]$ similarly. It is easy to prove that $[x, A]$ and $[x, B]$ are theories strictly including x. By the maximality of x, they are w-inconsistent. That is, $\neg C \in [x, A]$, $\neg D \in [x, B]$ for some theorems C, D. By definitions of $[x, A]$ and $[x, B]$, we have $\vdash (A \wedge E) \rightarrow \neg C$, $\vdash (B \wedge E') \rightarrow \neg D$ for some E, $E' \in x$, whence by basic theorems of B_+, $\neg C \vee \neg D \in x$, and, by T4, $\neg(C \wedge D) \in x$. But, by Adj, $\vdash C \wedge D$. Therefore, if x is not prime, it is w-inconsistent, which is impossible. □

To end this section, we prove a lemma on a-consistency and the relation R^T. But, first, we have:

LEMMA 4. *Let a, b be nonnull elements in K^T. The set $x = \{B \mid \exists A[A \in b \ \ \& \ \ A \rightarrow B \in a]\}$ is a nonnull theory such that $R^T abx$.*

Proof. It is easy to prove that x is a theory. Moreover, x is nonnull: let $A \in b$; by A1 and Lemma 1, $A \rightarrow A \in a$. So, $A \in x$. Finally, $R^T abx$ is immediate by definition of R^T. (cf. Definition 4.) □

Suppose now that a and b in Lemma 4 are w-consistent or a-consistent. It is clear that the theory x in this lemma cannot be generally be proved w-consistent or a-consistent. However, it is proved:

LEMMA 5. *Let a, b be nonnull elements in K^T and c an a-consistent member in K^T such that $R^T abc$. Then, a and b are a-consistent.*

Proof.

(1) Suppose a is a-inconsistent. Now, let $C \in b$ and $A \rightarrow B$ be a theorem. Clearly, $C \rightarrow \neg(A \rightarrow B) \in a$. So, $\neg(A \rightarrow B) \in c$, contradicting the a-consistency of c.
(2) Suppose b is a-inconsistent. Then, $\neg(A \rightarrow B) \in b$, $A \rightarrow B$ being a theorem. As a is nonnull, $\neg(A \rightarrow B) \rightarrow \neg(A \rightarrow A) \in a$ by Lemma 1, T6 and Con. So, $\neg(A \rightarrow A) \in c$, which is impossible. □

Let us briefly comment on the proof of this lemma. Suppose that c in Lemma 5 is w-consistent (respectively, sc-consistent), in order to prove that a and b are w-consistent

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(respectively sc-consistent). It is easy to see that the following rules are required in the first case:

$$(\vdash A \ \& \ \vdash B) \Rightarrow \vdash \neg A \rightarrow (B \rightarrow C)$$

$$(\vdash A \ \& \ \vdash B) \Rightarrow \vdash \neg A \rightarrow \neg B$$

whereas in the second case the following restriction of axiom ECQ

$$(A \wedge \neg A) \rightarrow (B \rightarrow C)$$

or in fact the weaker formula

$$(A \wedge \neg A) \rightarrow [B \rightarrow (A \wedge \neg A)]$$

is needed. But these rules and axioms are not provable in any of the logics in this paper (cf. Matrix set VIII in the Appendix). Therefore, members in K^C cannot be required to be w-consistent and, of course, neither can they be required to be sc-consistent.

On the other hand, notice that Lemma 5 is essential in the completeness proof: it is used in Lemma 9, and, consequently, in Proposition 8, which is used in its turn to prove the soundness and completeness of the extensions of $\mathrm{B}_{\mathrm{K2r}\gamma}$ considered in Section 7.

§5. Completeness of $\mathbf{B_{K2r\gamma}}$ II. $*^C$-theories. In this section, some facts about theories in K^C and O^C and their $*^C$ images are proved.

We first prove a trivial lemma:

LEMMA 6. *The following hold for any wff A and a, $b \in K^C$:*

1. $\neg A \in a*^C$ *iff* $A \notin a$
2. $a *^C *^C = a$
3. $a*^C \subseteq b*^C \Leftrightarrow b \subseteq a$

Proof. 1 follows by A7 and A8; 2 and 3 follow by 1 and definitions. □

Next, it is proved that $*^C$ is an operation on K^C (and on O^C). We also prove a proposition on the relations between regularity and w-consistency in the context of K^C.

PROPOSITION 6. $*^C$ *is an operation on* K^C.

Proof. Let $a \in K^C$. By Con, T4 and T5 $a*^C$ is a prime theory (cf. Routley *et al.*, 1982b). Next, we prove that $a*^C$ is non-null and a-consistent. (1) Suppose $a*^C$ is null and let $A \rightarrow B$ be a theorem. Then, $A \rightarrow B \notin a*^C$. So, $\neg(A \rightarrow B) \in a$, contradicting the a-consistency of a (cf. Proposition 3). (2) Suppose $a*^C$ is a-inconsistent. Then, $\neg(A \rightarrow B) \in a*^C$, $A \rightarrow B$ being a theorem. So, $A \rightarrow B \notin a$ (Lemma 6(1)), which is impossible by Lemma 1. □

PROPOSITION 7.

1. *If a is a w-consistent element in* K^C, *then* $a*^C$ *is regular. Moreover,* $a \subseteq a*^C$.
2. *If a is a regular element in* K^C, *then* $a*^C$ *is w-consistent. Moreover,* $a* \subseteq a$.
3. $*^C$ *is an operation on* O^C.
4. $a \in O^C \Rightarrow a*^C = a$.

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Proof.

1. (a) Suppose a is a w-consistent member in K^C, and let B be a theorem. If $B \notin a*^C$, then $\neg B \in a$, contradicting the w-consistency of a. (b) Suppose $B \in a$, $B \notin a*^C$ for some wff B. Then, $B \wedge \neg B \in a$, contradicting the w-consistency of a (cf. Proposition 4(1)).
2. (a) Suppose a is a regular member in K^C, and let B be a theorem. If $\neg B \in a*^C$, then $B \notin a$ (Lemma 6(1)), contradicting the regularity of a. (b) Suppose $B \in a*^C$, $B \notin a$ for some wff B. Then, $B \wedge \neg B \in a*^C$, contradicting the w-consistency of $a*^C$ (cf. Proposition 4(1)).

Now, 3 and 4 are immediate by 1 and 2. □

Finally, an essential lemma on the relationship between prime w-consistent theories and their $*^C$-images is proved:

LEMMA 7. *Let a be a w-consistent element in K^T. Then, there is some $x \in O^C$ such that $x \subseteq a*^C$.*

Proof. Assume the hypothesis of the lemma. By Lemma 2, there is some regular w-consistent theory y such that $a \subseteq y$. By Lemma 3, there is a prime, w-consistent (regular) theory z (i.e., some z in O^C) such that $y \subseteq z$. So, $a \subseteq z$. By Lemma 6(3), $z*^C \subseteq a*^C$. Moreover, by Proposition 7(3), $z*^C \in O^C$. Therefore, $z*^C$ is the required x. □

We make a note of the following corollary of Lemma 7.

COROLLARY 1. *Let a be a regular element in K^C. Then, there is some $x \in O^C$ such that $x \subseteq a$.*

Proof. Assume the hypothesis of the lemma. By proposition 7(2), $a*^C$ is a w-consistent element in K^C. By Lemma 7, $x \subseteq a*^C*^C$ for some $x \in O^C$. So, by Lemma 6(2), $x \subseteq a$. □

§6. Completeness of $\mathbf{B_{K2r\gamma}}$. The fundamental lemma. The canonical model is in fact a model. First, we prove some auxiliary lemmas that will be useful in this and subsequent sections.

LEMMA 8. *Let a be a nonnull member in K^T and B a wff such that $B \notin a$. Then, there is some x in K^C such that $a \subseteq x$ and $B \notin x$.*

Proof. Build up a prime nonnull theory x such that $a \subseteq x$ and $B \notin x$ (cf. Routley *et al.*, 1982b). This theory is a-consistent. Therefore, $x \in K^C$. □

LEMMA 9. *Let a, b be nonnull elements in K^T and $c \in K^C$ such that $R^T abc$. Then, there are x, $y \in K^C$ such that $a \subseteq x$, $b \subseteq y$, $R^T xbc$ and $R^T ayc$.*

Proof. Build up prime nonnull theories x and y such that $R^T xbc$ and $R^T ayc$ (cf. Routley *et al.*, 1982b). By Lemma 5, x and y are, in addition, a-consistent. □

Next, in order to demonstrate that the canonical model is in fact a model, we prove the following essential lemmas.

LEMMA 10. *For any a, $b \in K^C$, $a \leq^C b$ iff $a \subseteq b$.*

Proof. From left to right, it is immediate. So, suppose $a \subseteq b$. Clearly, $R^T \mathrm{B}_{\mathrm{K2r}\gamma} aa$ (cf. Definition 4). Then, by Lemma 9, there is some (regular) y in K^C such that $\mathrm{B}_{\mathrm{K2r}\gamma} \subseteq y$ and

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$R^C yaa$. By Proposition 7(2), $y*^C$ is a w-consistent member in K^C such that $y*^C \subseteq y$. Therefore, by Lemma 7, there is some x in O^C such that $x \subseteq y *^C *^C$, that is, $x \subseteq y$ by Lemma 6(2). Then, obviously, $R^C xaa$, and $R^C xab$, by hypothesis. Finally, $a \leq^C b$ by Definition 4. □

LEMMA 11. *(The fundamental Lemma) Let a be a regular member in K^T and B be a wff such that $B \notin a$. Then, there is some x in O^C such that $B \notin x$.*

Proof. By Lemma 8, there is some (regular) element y in K^C such that $a \subseteq y$ and $B \notin y$. If y is w-consistent, then it is the required x. If y is not w-consistent, then $y*^C$ is w-consistent anyway because y is regular (Proposition 7(2)). Moreover, $\neg B \in y*^C$ (Lemma 6(1)). Then, by Lemma 2, there is a regular, w-consistent z in K^T such that $y*^C \subseteq z$. By Lemma 3, there is some prime (regular) w-consistent member u in K^T (i.e., some $u \in O^C$) such that $z \subseteq u$. As $\neg B \in u$ ($\neg B \in y*^C$), $B \notin u$: otherwise, u would be w-inconsistent (cf. Proposition 4(1)). Consequently, if y is not w-consistent, u is the required x. □

From Lemma 11, we obtain the following corollary:

COROLLARY 2. *Let A be a wff such that $\nvdash_{B_{K2r\gamma}} A$. Then, there is some x in O^C such that $A \notin x$.*

We can now prove:

LEMMA 12. *The $B_{K2r\gamma}$ canonical model (cf. Definition 4) is indeed a $B_{K2r\gamma}$-model.*

Proof. We have to prove:

1. The set O^C is not empty.
2. Clauses (*i*)-(*v*) are satisfied by the canonical model.
3. Postulates P1-P6 hold in the canonical model.

Case (1) is immediate by Proposition 5 and Lemma 11.

Case (2): clause (*i*) is immediate by Lemma 10. Clauses (*ii*), (*iii*), (*v*), and clause (*iv*) from left to right are proved as in the semantics for B (cf., e.g., Routley *et al.*, 1982b). So, let us prove clause (*iv*) from right to left. Suppose for wff A, B and $a \in K^C$, $A \rightarrow B \notin a$. We prove that there are $x, y \in K^C$ such that $R^C axy$, $A \in x$ and $B \notin y$. The sets $z = \{C \mid \vdash_{B_{K2r\gamma}} A \rightarrow C\}$, $u = \{C \mid \exists D[D \rightarrow C \in a \ \& \ D \in z]\}$ are theories such that $R^T azu$. Now, z is a-consistent: if it is not, then $B \in z$, that is, $\vdash A \rightarrow B$, and then, $A \rightarrow B \in a$ (Lemma 1), contradicting the hypothesis. Moreover, $A \in z$ (by A1). So, u is nonnull (Lemma 4). On the other hand, $B \notin u$ (if $B \in u$, then $A \rightarrow B \in a$, contradicting the hypothesis). Therefore, u is a-consistent. Consequently, we have nonnull, a-consistent theories z, u such that $R^T azu$, $A \in z$ and $B \notin u$. Now, by Lemma 8, there is some $y \in K^C$ such that $u \subseteq y$ and $B \notin y$. Obviously, $R^C azy$. Next, by Lemma 9, there is some $x \in K^C$ such that $z \subseteq x$ and $R^C axy$. Clearly, $A \in x$. Therefore, we have prime nonnull a-consistent theories x, y such that $A \in x$, $B \notin y$ and $R^C axy$, as it was required.

Case (3): it is easy to prove with the assistance of Lemma 10: P1 and P2 are trivial and P3 is easy by Lemma 1; Then, P4 and P5 follow by Lemma 6(2) and Lemma 6(3), respectively. Finally P6 is provable with Proposition 7(4). □

Then, the completeness of $B_{K2r\gamma}$ is immediate by Corollary 2 and Lemma 12:

THEOREM 2. *(Completeness of $B_{K2r\gamma}$) If $\vDash_{B_{K2r\gamma}} A$, then $\vdash_{B_{K2r\gamma}} A$.*

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§7. Extensions of $\mathbf{B_{K2r\gamma}}$. Consider the following axioms:

A10. $(B \to C) \to [(A \to B) \to (A \to C)]$
A11. $(A \to B) \to [(B \to C) \to (A \to C)]$
A12. $[A \to (A \to B)] \to (A \to B)$
A13. $[(A \to A) \to B] \to B$
A14. $(A \to B) \to [C \to (A \to B)]$
A15. $(A \to B) \to (\neg B \to \neg A)$

Axioms A10–A13 are, respectively, the axioms of *prefixing* (p), *suffixing* (s), *contraction* (c) and *specialized assertion* (a); A14 is a restricted (S4) version of axiom K (k); A15 is (one of the forms of) the axiom of weak contraposition (con).

Consider now the following definition and semantical postulates:

d2. $R^2abcd =_{\text{df}} (\exists x \in K)(Rabx \ \& \ Rxcd)$
PA10. $R^2abcd \Rightarrow (\exists x \in K)(Rbcx \ \& \ Raxd)$
PA11. $R^2abcd \Rightarrow (\exists x \in K)(Racx \ \& \ Rbxd)$
PA12. $Raaa$
PA13. $(\exists x \in K)Raxa$
PA14. $R^2abcd \Rightarrow Racd$
PA15. $Rabc \Rightarrow Rac * b*$

PROPOSITION 8. *Given the logic $B_{K2r\gamma}$ and $B_{K2r\gamma}$-semantics, PA10–PA15 are the corresponding postulates (c.p) to A10–A15, respectively.*

That is to say, given $\mathrm{B_{K2r\gamma}}$ semantics, A10–A15 are proved valid with PA10–PA15, respectively; and, given the logic $\mathrm{B_{K2r\gamma}}$, PA10–PA15 are proved canonically valid with A10–A15, respectively.

Proof. The proof of Proposition 8 can be found in (or can easily be derived from Routley *et al.*. 1982b), with the exception perhaps of the cases regarding the correspondence between A12 and PA12 and A13 and PA13. So, let us prove both cases.

1. *PA12 is the c.p to A12:*

   (a) For reductio, suppose that for wff A, B and $a \in K$ in some $\mathrm{B_{K2r\gamma}}$-model where PA12 holds, $a \vDash A \to (A \to B)$, $a \nvDash A \to B$. Then, $b \vDash A$, $c \nvDash B$ for $b, c \in K$ such that $Rabc$. By P3, $b \leq c$. So, $c \vDash A \to B$ ($Rabc$, $a \vDash A \to (A \to B)$, $b \vDash A$) . Then, $c \vDash C$ by PA12 ($c \vDash A \to B$, $Rccc$, $c \vDash A$), a contradiction.

   (b) That $R^C aaa$ holds is immediate by using the *modus ponens* axiom $[(A \to B) \wedge A] \to B$, which is derivable from the contraction axiom, given Routley and Meyer's basic positive logic $\mathrm{B_+}$ (cf. Proposition 11 in Section 8. Regarding $\mathrm{B_+}$, cf. Section 2).

3. *PA13 is the c.p to A13:*

   (a) We prove that A13 is valid in any $\mathrm{B_{K2r\gamma}}$-model in which PA13 holds. First, note that for any $x \in K$ in any model like this $x \vDash A \to A$. For suppose $x \nvDash A \to A$ for some $x \in K$. Then $y \vDash A$, $z \nvDash A$ for $y, z \in K$ such that $Rxyz$. By P3, $y \leq z$. So, $z \vDash A$, a contradiction. Now, suppose that for some wff A,

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B and $a \in K$ in some $B_{K2r\gamma}$-model where PA13 holds, $a \vDash (A \to A) \to B$. By PA13, $Raxa$ for some $x \in K$. As $x \vDash A \to A$, $a \vDash B$, as it was required.

(b) We prove: let $a \in K^C$, then there is some x in K^C such that $R^C axa$. Let $B_{K2r\gamma\,A13}$ be the logic axiomatized by adding A13 to $B_{K2r\gamma}$. In addition, it is then supposed that the rules of MP, Adj, r and γ are now applicable to theorems of $B_{K2r\gamma\,A13}$. Consider now the set $y = \{B \mid \exists A[\vdash_{B_{K2r\gamma\,A13}} (A \to A) \to B]\}$. We prove that y is a nonnull theory such that $R^T aya$.

i. *y is closed by $B_{K2r\gamma\,A13}$-entailment*: suppose $\vdash_{B_{K2r\gamma\,A13}} A \to B$ and $A \in y$. Then, $\vdash_{B_{K2r\gamma\,A13}} (C \to C) \to A$ for some wff C. By Suf, $\vdash_{B_{K2r\gamma\,A13}} (C \to C) \to B$. So, $B \in y$.

ii. *y is closed by adjunction*: suppose $A \in y$, $B \in y$. Then, $\vdash_{B_{K2r\gamma\,A13}} (C \to C) \to A$, $\vdash_{B_{K2r\gamma\,A13}} (D \to D) \to B$ for wff C, D. By A9, $\vdash_{B_{K2r\gamma\,A13}} (C \to C) \to B$. So, $\vdash_{B_{K2r\gamma\,A13}} (C \to C) \to (A \wedge B)$, that is, $A \wedge B \in y$.

iii. *y is nonnull*: by A9, y contains all formulas of the form $A \to A$.

iv. $R^T aya$: suppose $A \to B \in a$, $A \in y$ for wff A, B. Then, $\vdash_{B_{K2r\gamma\,A13}} (C \to C) \to A$ for some wff C. By Suf, $(C \to C) \to B \in a$ whence by A13, $B \in a$, as it was to be proved.

Next, by Lemma 9, there is a prime, a-consistent theory x such that $y \subseteq x$ and $R^C axa$. □

REMARK 7. *Notice that PA13 is not the c.p to A13 in relevant logics. In fact, the c.p to A13 is a bit more complicated in those logics (cf., e.g., Anderson et al., 1992, pp. 171–172). On the other hand, note that in relevant logics, PA12 is the c.p to the modus ponens axiom $[(A \to B) \wedge A] \to B$, the c.p to A12 being, in general,*

$$PA12'.\ Rabc \Rightarrow R^2abbc.$$

Now, let $SB_{K2r\gamma}$ be any extension of $B_{K2r\gamma}$ with any selection of A10–A15. (As in the case of $B_{K2r\gamma}$, it is understood that MP, Adj, r and γ are now applicable to the theorems of $SB_{K2r\gamma}$.) And let $SB_{K2r\gamma}$-models be defined, similarly, as $B_{K2r\gamma}$-models except for the addition of the c.p to the new axioms added. Furthermore, define $SB_{K2r\gamma}$-validity in a similar way to which $B_{K2r\gamma}$-validity was defined. Finally, provide a definition of the $SB_{K2r\gamma}$ canonical model similar to that of the $B_{K2r\gamma}$ canonical model. It is clear that from Proposition 8 and Theorems 1 and 2, we have:

THEOREM 3. *(Soundness and completeness of $SB_{K2r\gamma}$) $SB_{K2r\gamma}$ is sound and complete in respect of $SB_{K2r\gamma}$-models. That is, $\vDash_{SB_{K2r\gamma}} A$, iff $\vdash_{SB_{K2r\gamma}} A$.*

§8. A closer look at the $SB_{K2r\gamma}$ logics. Let us take a closer look at the logics defined in the preceding section. Firstly, we prove some syntactical facts. Let $B_{K2r\gamma\,con}$ be the result of adding A15 to $B_{K2r\gamma}$, then the following propositions are provable.

PROPOSITION 9. *Given $B_{K2r\gamma\,con}$, A10 (p) and A11 (s) are equivalent.*

Proof. We prove that A11 is derivable from $B_{K2r\gamma\,con}$ and A10 (the other part of the proof is similar).

$$1.\ (\neg B \to \neg A) \to [(\neg C \to \neg B) \to (\neg C \to \neg A)] \qquad \text{A10}$$

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By A15,

$$2.\ (A \to B) \to [(\neg C \to \neg B) \to (\neg C \to \neg A)]$$

By A15, 2 and Suf.

$$3.\ (A \to B) \to [(B \to C) \to (\neg C \to \neg A)]$$

Now, by A7, A8 and A15

$$4.\ (\neg C \to \neg A) \to (A \to C)$$

whence by 3 and Pref.

$$5.\ (A \to B) \to [(B \to C) \to (A \to C)].$$

□

PROPOSITION 10. *Given B_+, A14 (k) is derivable from A10 (s) (or A11 (p)).*

Proof. We prove that A14 is derivable from B_+ and A10 (the proof from A11 is similar).

$$1.\ (A \to B) \to [(A \to A) \to (A \to B)] \qquad \text{A10}$$

By A9, Suf and 1

$$2.\ (A \to B) \to [C \to (A \to B)].$$

□

PROPOSITION 11. *Given B_+ plus A12 (c), the following are derivable:*

1. $[A \to (B \to C)] \to [(A \wedge B) \to C]$.
2. $[(A \to B) \wedge A)] \to B$
3. $[[(A \to B) \wedge (B \to C)] \wedge A] \to C$

Proof.

1. *Proof of case 1*: by A12 and Suf.

$$1.\ [A \to (B \to C)] \to [(A \wedge B) \to (B \to C)]$$

and

$$2.\ (B \to C) \to [(A \wedge B) \to C]$$

By Pref. and 2,

$$3.\ [(A \wedge B) \to (B \to C)] \to [(A \wedge B) \to [(A \wedge B) \to C]]$$

whence by A12,

$$4.\ [(A \wedge B) \to (B \to C)] \to [(A \wedge B) \to C].$$

Finally, by 1 and 4,

$$5.\ [A \to (B \to C)] \to [(A \wedge B) \to C].$$

2. *Proof of case 2*: immediate by A1 and case 1.
3. *Proof of case 3*: by case 2,

$$1.\ [(A \to B) \wedge A] \to B$$

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and

$$2.\ [(B \to C) \wedge B] \to C.$$

By B_+,

$$3.\ [[(A \to B) \wedge (B \to C)] \wedge A] \to (B \to C).$$

By B_+ and 1,

$$4.\ [[(A \to B) \wedge (B \to C)] \wedge A] \to B.$$

By A3, 3 and 4,

$$5.\ [[(A \to B) \wedge (B \to C)] \wedge A] \to [(B \to C) \wedge B].$$

Finally, by 2 and 5,

$$6.\ [[(A \to B) \wedge (B \to C)] \wedge A] \to C.$$

□

PROPOSITION 12. *Given B_+ plus A12 (c) and A14 (k), A13 (a) is derivable.*

Proof. First, notice that A9 is immediate by A1 and A14. Then, by Proposition 11 (2),

$$1.\ [[(A \to A) \to B] \wedge (A \to A)] \to B.$$

Now, we have,

$$2.\ [(A \to A) \to B] \to [(A \to A) \to B] \qquad \text{A1}$$
$$3.\ [(A \to A) \to B] \to (A \to A) \qquad \text{A9}$$

By A3, 2, and 3,

$$4.\ [(A \to A) \to B] \to [[(A \to A) \to B] \wedge (A \to A)]$$

whence by 1

$$5.\ [(A \to A) \to B] \to B$$

From the propositions just proved, it follows: □

COROLLARY 3. *Given $B_{K2r\gamma\, con}$ plus A10 (p) and A12 (c) (or A11 (s) and A12 (c)), A13 (a) and A14 (k) are derivable.*

Proof. Propositions 9, 10, and 12. □

Next, we have,

PROPOSITION 13. *Given B_+ plus A14 (k), the following are derivable:*

(1). $[(A \to B) \wedge (C \to D)] \to [E \to [[(A \to B) \wedge (C \to D)] \wedge E]]$

(2). $\vdash [[(A \to B) \wedge (C \to D)] \wedge E] \to F \Rightarrow \vdash [(A \to B) \wedge (C \to D)] \to (E \to F)$.

Proof. Proof of (1): first, we note that A9 and T7 (cf. Section 2) are immediate: by A1 and A14. That is,

$$1.\ B \to (A \to A)$$

and

$$2.\ (A \to B) \to [A \to (A \wedge B)].$$

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Now,

$$3.\ (A \to B) \to [E \to (A \to B)] \qquad \text{A14}$$
$$4.\ (C \to D) \to [E \to (C \to D)] \qquad \text{A14}$$

By 2 and 3 and commutativity of conjunction,

$$5.\ [E \to (A \to B)] \to [E \to [(A \to B) \wedge E]]$$

and

$$6.\ [E \to (C \to D)] \to [E \to [(C \to D) \wedge E]]$$

By 3 and 5,

$$7.\ (A \to B) \to [E \to [(A \to B) \wedge E]]$$

By 4 and 6,

$$8.\ (C \to D) \to [E \to [(C \to D) \wedge E]]$$

By B_+, 7 and 8,

$$9.\ [(A \to B) \wedge (C \to D)] \to [E \to [(A \to B) \wedge E]]$$

and

$$10.\ [(A \to B) \wedge (C \to D)] \to [E \to [(C \to D) \wedge E]]$$

By A3, 9 and 10,

$$11.\ [(A \to B) \wedge (C \to D)] \to \{[E \to [(A \to B) \wedge E]] \wedge [E \to [(C \to D) \wedge E]]\}$$

Now, by A3

$$12.\ \{[E \to [(A \to B) \wedge E]] \wedge [E \to [(C \to D) \wedge E]]\} \to \{E \to [[(A \to B) \wedge E] \wedge [(C \to D) \wedge E]]\}$$

whence by 11,

$$13.\ [(A \to B) \wedge (C \to D)] \to \{E \to [[(A \to B) \wedge E] \wedge [(C \to D) \wedge E]]\}$$

and by B_+,

$$14.\ [(A \to B) \wedge (B \to C)] \to [E \to [[(A \to B) \wedge (C \to D)] \wedge E]]$$

Proof of (2):

1. $\vdash [[(A \to B) \wedge (C \to D)] \wedge E] \to F$ — Hyp.
2. $\vdash \{E \to [[(A \to B) \wedge (C \to D)] \wedge E]\} \to (E \to F)$ — Pref., 1
3. $\vdash \{[(A \to B) \wedge (C \to D)] \to [E \to [[(A \to B) \wedge (C \to D)] \wedge E]]\} \to [[(A \to B) \wedge (C \to D)] \to (E \to F)]$ — Pref., 2
4. $[(A \to B) \wedge (C \to D)] \to (E \to F)$ — MP (1), 3

□

PROPOSITION 14. *Given B_+ plus A12 (c) and A14 (k), A11 (s) is derivable.*

Proof. Immediate by (cf. Proposition 11(3))

$$[[(A \to B) \wedge (B \to C)] \wedge A] \to C$$

and the rule (2) in Proposition 13 just proved. □

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In Méndez (1988), it is proved that A1, A11, A12, and A14 together with MP axiomatize the implicative fragment of Lewis' S4, Therefore, B_+ plus A12 and A14 contains this fragment, whence it is easy to see that B_+ plus A12 and A14 is a logic deductively equivalent to the positive fragment of Lewis' S4, $S4_+$ (cf. Hacking, 1963). Now, let us use the label S to refer to $B_{K2r\gamma\,con}$. Furthermore, let S_a be the result of adding A13 (a) to $B_{K2r\gamma\,con}$; S_{ap}, the result of adding A13 (a) and A11 (p) to $B_{K2r\gamma\,con}$, etc. From propositions 9–14 and by using MaGIC, the matrix generator developed by J. Slaney (see Slaney 1995; cf. Appendix), it follows that there are exactly seven different logics definable by extending $B_{K2r\gamma\,con}$ with A10–A14. In the following diagram (the arrow (→) stands for set inclusion), the deductive relations these logics maintain to each other are summarized. That no other relations are obtained is proved by MaGIC (cf. Appendix).

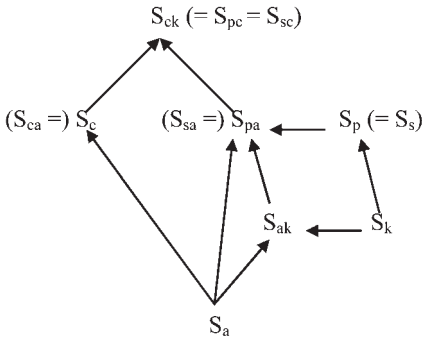


Fig. 1. Diagram

Now, S_{ck}, that is, $B_{K2r\gamma\,con}$ plus A12 and A14 is a strong logic contained in Lewis' S4. Let us refer to this logic by $S4_{K2r\gamma}$. It has been pointed out above that it contains the positive fragment of S4. In addition, it has the De Morgan laws (T4, T5), the principle of noncontradiction (T10), the principle of excluded middle (T12), and all forms of the contraposition axioms (they follow from T15 with A7 and A8). However, it lacks the axiom ECQ and all forms of the axioms of reductio, although it has the rules of weak reductio (cf. T9, T11). In fact, we have the following proposition whose proof is left to the reader.

PROPOSITION 15.

1. *$S4_{K2r\gamma}$ plus any of the theses and rules that follow (all of them provable in S4) is a logic deductively equivalent to S4 (cf. Hacking, 1963):*

(a). $(A \wedge \neg A) \rightarrow B$
(b). $(A \rightarrow \neg A) \rightarrow \neg A$
(c). $(\neg A \rightarrow A) \rightarrow A$
(d). $\vdash A \rightarrow B \Rightarrow \vdash (A \rightarrow \neg B) \rightarrow \neg A$
(e). $\vdash \neg A \rightarrow B \Rightarrow \vdash (\neg A \rightarrow \neg B) \rightarrow A$
(f). $\vdash A \rightarrow B \Rightarrow \vdash (\neg A \rightarrow B) \rightarrow B$
(g). $(A \rightarrow B) \rightarrow \neg(A \wedge \neg B)$
(h). $(A \rightarrow \neg B) \rightarrow \neg(A \wedge B)$
(i). $[B \rightarrow (A \wedge \neg A)] \rightarrow \neg B$
(j). $[(A \rightarrow B) \wedge (A \rightarrow \neg B)] \rightarrow \neg A$.

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2. *$S4_{K2r\gamma}$ supplemented with either rule K*

$$(k).\ \vdash A \Rightarrow \vdash B \rightarrow A$$

or else, the rule of assertion

$$(l).\ \vdash A \Rightarrow \vdash (A \rightarrow B) \rightarrow B$$

is a logic deductively equivalent to S4.

3. *None of (a)–(l) are provable in $S4_{K2r\gamma}$ (cf. Matrix set VIII in the Appendix).*

Moreover, we note that rule K and the rule of assertion are not even admissible: if rule K is admissible, then the ECQ axiom is immediate by using T10. If the rule of assertion is admissible, then rule K is admissible as well, as it can be shown by using A9.

In the following section, it is proved that $S4_{K2r\gamma}$ (and so, all logics included in it) is paraconsistent.

§9. The logic $S4_{K2r\gamma}$ is paraconsistent. It is shown that any logic included in or equivalent to $B_{K2r\gamma}$ plus A10–A15 is paraconsistent.

As was recalled in Section 1, we have to prove that the ECQ rule

$$\text{ECQ. } A \wedge \neg A \vdash B$$

is not a rule of $B_{K2r\gamma}$ plus A10–A15. In order to do this, a relation of semantical consequence in each $SB_{K2r\gamma}$ logic (cf. Theorem 3) is defined as follows:

DEFINITION 5. *Let Γ be a set of wff and A a wff. Then, $\Gamma \vDash A$ ($\Gamma \vDash A$ is $SB_{K2r\gamma}$-valid) iff for all $a \in K$ in all $SB_{K2r\gamma}$-models, if $a \vDash \Gamma$, then $a \vDash A$ ($a \vDash \Gamma$ iff $a \vDash B$ for every $B \in \Gamma$).*

As in standard relevant logics, it is proved.

LEMMA 13. $(a \leq b \ \ \& \ \ a \vDash A) \Rightarrow b \vDash A$

Proof. Induction on the length of A (cf., e.g., Routley *et al*., 1982b). □

LEMMA 14. $\vDash_{SB_{K2r\gamma}} A \rightarrow B$ *iff for all* $a \in K$ *in all models,* $a \vDash A \Rightarrow a \vDash B$

Proof. (cf. e.g., Routley *et al*., 1982b). By using Lemma 13, P1 and d1 (cf. Section 3). □

Then, we have:

PROPOSITION 16. *Each $SB_{K2r\gamma}$ is paraconsistent.*

Proof. We prove that rule ECQ is not valid. As is shown in Proposition 15, the axiom ECQ

$$(A \wedge \neg A) \rightarrow B$$

is not provable in $S4_{K2r\gamma}$. Therefore, it is not provable in any of the $SB_{K2r\gamma}$ logics. By the completeness theorem (Theorem 3), $(A \wedge \neg A) \rightarrow B$ is not $SB_{K2r\gamma}$-valid. That is, there are wff A, B such that $\nvDash_{SB_{K2r\gamma}} (A \wedge \neg A) \rightarrow B$. By Lemma 14, there is some $a \in K$ in some model such that $a \vDash A \wedge \neg A$, $a \nvDash B$. Therefore, $A \wedge \neg A \nvDash B$ by Definition 5. Actually, it is easy to prove that rule ECQ fails in the $SB_{K2r\gamma}$-canonical model. Consider the following set of formulas where p_i is the i-th propositional variable $x = \{B \mid \vdash_{SB_{K2r\gamma}} A \ \& \ \vdash_{SB_{K2r\gamma}} [A \wedge (p_i \wedge \neg p_i)] \rightarrow B\}$. It is easy to prove that x is a $SB_{K2r\gamma}$-theory (it is closed

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under Adjunction and $SB_{K2r\gamma}$ -entailment). Moreover, x is regular (it contains all theorems of $SB_{K2r\gamma}$) but it is inconsistent. Nevertheless, x is a-consistent because the rule

$$\vdash A \Rightarrow \vdash [A \wedge (p_i \wedge \neg p_i)] \rightarrow B$$

is not provable in $SB_{K2r\gamma}$ (cf. Matrix set VIII in Appendix). So, there is some wff C such that for some theorem E, $\nvdash_{SB_{K2r\gamma}} [E \wedge (p_i \wedge \neg p_i)] \rightarrow C$. By definition of x, $C \notin x$. Therefore, though inconsistent, x is a-consistent. Now, by Lemma 8, there is a prime, regular, a-consistent theory y such that $x \subseteq y$, $(p_i \wedge \neg p_i) \in y$ and $C \notin y$. As the $SB_{K2r\gamma}$-canonical model is a $SB_{K2r\gamma}$-model, $y \vDash^C p_i \wedge \neg p_i$, $y \nvDash^C C$. By Definition 5, $p_i \wedge \neg p_i \nvDash C$, whence $A \wedge \neg A \nvDash B$. □

Next, let us investigate to which proof-theoretical consequence relation the semantical relation introduced in Definition 5 corresponds.

Consider the following rule: for each $SB_{K2r\gamma}$ logic:

$SB_{K2r\gamma}$ *-entailment*: Given $\vdash_{SB_{K2r\gamma}} A \rightarrow B$, from A to infer B.

Now, let us define:

DEFINITION 6. *Let* Γ *be a set of wff and* A *be a wff. Then,* $\Gamma \vdash_{SB_{K2r\gamma}} A$ *iff there is a finite sequence of wff* $B_1, \ldots B_m$ *such that* B_m *is* A *and for each* i $(1 \leq i \leq m)$*, one of the following is the case: (a)* $B_i \in \Gamma$ *(b)* B_i *is obtained from preceding wffs in the sequence by Adjunction (c)* B_i *is by* $SB_{K2r\gamma}$ *-entailment.*

On the other hand, the set of consequences of a set of wff Γ (CONΓ) is classically defined as follows:

DEFINITION 7. $CON\Gamma = \{A \mid \Gamma \vdash_{SB_{K2r\gamma}} A\}$

Then, we immediately have:

PROPOSITION 17. *For any set of wff* Γ*,* $CON\Gamma$ *is a* $SB_{K2r\gamma}$ *-theory.*

And, unsurprisingly enough, we obtain:

THEOREM 4. *For any set of wff* Γ *and wff* A*,* $\Gamma \vdash_{SB_{K2r\gamma}} A$ *iff* $\Gamma \vDash_{SB_{K2r\gamma}} A$.

Proof.

1. If $\Gamma \vdash_{SB_{K2r\gamma}} A$, then $\Gamma \vDash A$. Induction on the length of the proof $\Gamma \vdash_{SB_{K2r\gamma}} A$. If $A \in \Gamma$ or A is by Adjunction, the proof is trivial. Suppose then, that A is by $SB_{K2r\gamma}$-entailment. Then, $\Gamma \vdash_{SB_{K2r\gamma}} C$, $C \rightarrow A$ being a theorem of $SB_{K2r\gamma}$. Now, let $a \vDash \Gamma$ for $a \in K$ in an arbitrary model. By hypothesis, $\Gamma \vDash C$. So, $a \vDash C$. On the other hand, $\vDash_{SB_{K2r\gamma}} C \rightarrow A$ by Theorem 3. Then, by Lemma 14, $a \vDash A$, as was to be proved.
2. If $\Gamma \vDash A$ then $\Gamma \vdash_{SB_{K2r\gamma}} A$. If Γ is empty, then the proof is immediate by Theorem 3. So, suppose that Γ is a nonempty set and $\Gamma \nvdash_{SB_{K2r\gamma}} A$. Clearly, $A \notin CON\Gamma$. Then, by Lemma 8, there is some x in K^C such that $CON\Gamma \subseteq x$ and $A \notin x$. As $\Gamma \subseteq CON\Gamma$, $\Gamma \subseteq x$. Now, the $SB_{K2r\gamma}$-canonical model is an $SB_{K2r\gamma}$-model. So, $x \vDash^C \Gamma$ (i.e., for every $B \in \Gamma$, $x \vDash^C B$), but $x \nvDash^C A$. Therefore, $\Gamma \nvDash^C A$, as was to be proved. □

Let us now return to the relationship between w-consistency and the standard concept of consistency, sc-consistency, briefly discussed in Section 4. It is clear that the concept of

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consistency underlying the notion of paraconsistency is sc-consistency, that is, absence of any contradiction. Nevertheless, in Section 4, it has been argued that the logics defined in this paper are adequate to w-consistency, not to sc-consistency (cf. Robles & Méndez, 2009). Consequently, paraconsistency would have to be defined in the case of $\mathrm{SB}_{\mathrm{K2r}\gamma}$ logics as follows:

DEFINITION 8. *A logic S is w-paraconsistent (paraconsistent according to w-consistency) iff the rule*

$$\vdash A \Rightarrow \neg A \vdash B$$

is not provable in S.

Paraconsistency and w-paraconsistency are by no means equivalent. For instance, consider Łukasiewicz's logics SŁ. These logics can be considered as paraconsistent logics. Actually, they *could* belong to one of the four groups in which paraconsistent logics are classified by Priest & Tanaka (2004). But they are not w-paraconsistent, because the EFQ ('E falso quodlibet') axioms

$$\neg A \rightarrow (A \rightarrow B)$$

and

$$A \rightarrow (\neg A \rightarrow B)$$

are provable in SŁ. Therefore, and incidentally, they are not paraconsistent in respect of w-consistency, the concept of consistency they are adequate to (cf. Robles & Méndez, 2009).

REMARK 8. *As pointed out by a referee of RSL, Łukasiewicz's logics SŁ are not paraconsistent in a standard sense because ECQ vacuously holds in SŁ. However, these logics are paraconsistent if a consequence relation similar to the one defined in this paper is introduced, that is, for any set of wff* $\{B_1, ..., B_m, A\}$, $\{B_1, ..., B_m\} \models A$ *iff* $\models_{SŁ} \{B_1 \wedge ... \wedge B_m\} \rightarrow A$. *Notice, in this sense, that prime, regular, w-inconsistent theories built upon SŁ are not necessarily trivial, as shown by the failure in Ł3 (the strongest of Łukasiewizc's logics) of the rule* $\vdash A \Rightarrow \vdash [A \wedge (p_i \wedge \neg p_i)] \rightarrow B$. *Just set* $v(A) = 2$, $v(B) = 0$, $v(p_i) = 1$ *(cf. Proposition 16 and Robles & Méndez, 2009).*

However, consider a logic S in which

$$\text{A7. } A \rightarrow \neg\neg A$$

and

$$\text{T10. } \neg(A \wedge \neg A)$$

are theorems. Clearly, if S is paraconsistent, S is w-paraconsistent. Consequently, we have the following corollary of Proposition 16:

COROLLARY 4. *Each* $SB_{K2r\gamma}$ *logic is w-paraconsistent.*

Thus, the logics defined in this paper are paraconsistent in respect of the concept of consistency they are adequate to. Nevertheless, an important fact has to be remarked. As we have seen, sc-inconsistent (so, w-inconsistent —cf. Proposition 4) theories are not necessarily trivial. So, the logics defined in this paper are adequate for dealing with inconsistent (the term being understood either in the standard or in the weak sense) situations. But there is a notable exception: reasonably enough, we think, if a theory built upon B_{K2} or any

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logic including it contains the negation of a provable entailment, this theory collapses into a trivial one (cf. T8).

A last remark. It has been essential that $SB_{K2r\gamma}$-theories are not in general closed by γ in order to invalidate ECQ. Nevertheless, any w-consistent, prime $SB_{K2r\gamma}$-theory is, of course, closed by γ. And γ is not derivable from B_{K2r} plus A10-A15 (cf. Matrix set IX in Appendix). The question of its admissibility in $SB_{K2r\gamma}$ logics is left open.

§10. Appendix. The following sets of matrices have been found by MaGIC (see Slaney 1995). Designated values are starred. All of them satisfy the axioms and rules of $SB_{K2r\gamma\,con}$ (see Section 8)

Matrix set I (MSI): S_p does not include S_c, S_a.

$\rightarrow$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	1	3	3	3	2
*2	1	1	3	3	1
*3	1	1	1	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

Satisfies: A10, A11 and A14.

Falsifies: A12 ($A = 1$, $B = 0$).

Matrix set II (MSII): S_c does not include S_p, S_s, S_k.

$\rightarrow$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	3	3	2
*2	0	1	3	3	1
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

Satisfies: A12 and A13.

Falsifies: A10 ($A = B = 2$, $C = 1$); A11 ($A = 2$, $B = C = 1$); A14 ($A = 2$, $B = 1$, $C = 3$).

Matrix set III (MSIII): S_a does not include S_p, S_s, S_c, S_k.

$\rightarrow$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	1	3	3	3	2
*2	0	1	3	3	1
*3	0	0	1	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

Satisfies: A13.

Falsifies: A10 ($A = B = 1$, $C = 0$); A11 ($A = 1$, $B = C = 0$); A12 ($A = 1$, $B = 0$); A14 ($A = 1$, $B = 0$, $C = 3$).

Matrix set IVa (MSIVa): S_k does not include S_p, S_s.

$\rightarrow$	0	1	2	3	4	5	$\neg$
0	5	5	5	5	5	5	5
1	3	5	5	5	5	5	4
2	3	3	5	3	5	5	3
3	0	3	3	5	5	5	2
*4	0	3	3	3	5	5	1
*5	0	0	0	3	3	5	0

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	1	2	2
3	0	1	1	3	3	3
*4	0	1	2	3	4	4
*5	0	1	2	3	4	5

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$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	3	4	5
2	2	2	2	4	4	5
3	3	3	4	3	4	5
*4	4	4	4	4	4	5
*5	5	5	5	5	5	5

Satisfies: A14.
Falsifies: A10 and A11 ($A = 3$, $B = 1$, $C = 0$)

Matrix set IVb (MSIVb): S_k does not include S_a, S_c.

$\rightarrow$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	1	3	3	3	2
*2	1	1	3	3	1
*3	1	1	1	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

Satisfies: A14.
Falsifies: A12 ($A = 1$, $B = 0$); A13 ($A = B = 0$).
Matrix set V (MSV): S_{pa} does not include S_c.

$\rightarrow$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	1	3	3	3	2
*2	1	1	3	3	1
*3	0	1	1	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

Satisfies: A10, A11, A13 and A14.
Falsifies: A12 ($A = 1$, $B = 0$).

Matrix set VI (MSVI): S_c does not include S_p, S_s, S_k.

$\rightarrow$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	3	3	2
*2	0	1	3	3	1
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

Satisfies: A12 and A13.
Falsifies: A10 ($A = B = 2$, $C = 1$); A11 ($A = 2$, $B = C = 1$); A14 ($A = 2$, $B = 1$, $C = 3$).

Matrix set VII (MSVII): S_{ak} does not include S_p, S_s, S_c.

$\rightarrow$	0	1	2	3	4	5	$\neg$
0	5	5	5	5	5	5	5
1	3	5	5	5	5	5	4
2	3	3	5	3	5	5	3
3	0	3	3	5	5	5	2
*4	0	3	3	3	5	5	1
*5	0	0	0	3	3	5	0

$\wedge$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	1	2	2
3	0	1	1	3	3	3
*4	0	1	2	3	4	4
*5	0	1	2	3	4	5

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$\vee$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	3	4	5
2	2	2	2	4	4	5
3	3	3	4	3	4	5
*4	4	4	4	4	4	5
*5	5	5	5	5	5	5

Satisfies: A13 and A14.
Falsifies: A10 and A11 ($A = 3$, $B = 1$, $C = 0$); A12 ($A = 1$, $B = 0$).

Matrix set VIII (MSVIII): satisfies the axioms and rules of $S4_{K2r\gamma}$.

$\rightarrow$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	3	3	2
*2	0	0	3	3	1
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
*2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
*2	2	2	2	3
*3	3	3	3	3

It falsifies the following axioms and rules:

1. $(A \wedge \neg A) \rightarrow B$ $(A = 1, B = 0)$
2. $(A \rightarrow \neg A) \rightarrow \neg A$ $(A = 1)$
3. $(\neg A \rightarrow A) \rightarrow A$ $(A = 2)$
4. $\vdash A \rightarrow B \Rightarrow \vdash (A \rightarrow \neg B) \rightarrow \neg A$ $(A = B = 1)$
5. $\vdash \neg A \rightarrow B \Rightarrow \vdash (\neg A \rightarrow \neg B) \rightarrow A$ $(A = 2, B = 1)$
6. $\vdash A \rightarrow B \Rightarrow \vdash (\neg A \rightarrow B) \rightarrow B$ $(A = 1, B = 2)$
7. $(A \rightarrow B) \rightarrow \neg(A \wedge \neg B)$ $(A = B = 1)$
8. $(A \rightarrow \neg B) \rightarrow \neg(A \wedge B)$ $(A = B = 1)$
9. $[B \rightarrow (A \wedge \neg A)] \rightarrow \neg B$ $(A = B = 1)$
10. $[(A \rightarrow B) \wedge (A \rightarrow \neg B)] \rightarrow \neg A$ $(A = B = 1)$
11. $\vdash A \Rightarrow \vdash B \rightarrow A$ $(A = 2, B = 3)$
12. $\vdash A \Rightarrow \vdash (A \rightarrow B) \rightarrow B$ $(A = B = 2)$
13. $\vdash A \Rightarrow \vdash [A \wedge (p_i \wedge \neg p_i)] \rightarrow C$ $(A = 3, p_i = 1, B = 0)$
14. $(A \wedge \neg A) \rightarrow (B \rightarrow C)$ $(A = 1, B = 2, C = 0)$
15. $(A \wedge \neg A) \rightarrow [B \rightarrow (A \wedge \neg A)]$ $(A = 1, B = 2)$
16. $(\vdash A \ \& \vdash B) \Rightarrow \vdash \neg A \rightarrow (B \rightarrow C)$ $(A = B = 2, C = 0)$
17. $(\vdash A \ \& \vdash B) \Rightarrow \vdash \neg A \rightarrow \neg B$ $(A = 2, B = 3)$

Matrix set IX (MSIX): satisfies the axioms and rules of B_{K2r} plus A10–A15.

$\rightarrow$	0	1	2	$\neg$
0	2	2	2	2
*1	0	2	2	1
*2	0	0	2	0

$\wedge$	0	1	2
0	0	0	0
*1	0	1	1
*2	0	1	2

$\vee$	0	1	2
0	0	1	2
*1	1	1	2
*2	2	2	2

It falsifies γ $(A = 1, B = 0)$.

https://doi.org/10.1017/S1755020310000109 Published online by Cambridge University Press

§11. Acknowledgments. Work supported by researchs project FFI 2008-05859/FISO and FFI2008-01205/FISO, financed by the Spanish Ministry of Science and Innovation.

G. Robles is supported by Juan de la Cierva Program of the Spanish Ministry of Science and Innovation.

We thank the referees of the RSL for their suggestions and comments on a previous version of our paper.

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https://doi.org/10.1017/S1755020310000109 Published online by Cambridge University Press