Math. Proc. Camb. Phil. Soc. (1985), **97**, 473
Printed in Great Britain

Abelian semigroups whose Stone-Čech compactifications have left ideal decompositions

By D. J. PARSONS
University of Sheffield†

(*Received* 27 *July* 1984; *revised* 1 *November* 1984)

1. *Introduction*

Stone-Čech compactifications of semigroups have aroused a good deal of interest recently. Several authors, for example Milnes [6], Marcri [5] and Baker and Butcher [1], have concentrated on problems of the existence of continuous extensions to βS of the operation in a topological semigroup S. For a discrete semigroup a separately continuous extension always exists, and others such as Pym and Vasudeva [8] have studied the compactifications of particular classes of semigroups. Further interest has centred on the algebraic structure of these compactifications; see for example Hindman [3].

Young [9] and others have related the properties of βS to those of the semigroup algebra $l^1(S)$ and the measure algebra of a topological semigroup in order to study their Arens regularity. This approach was used in our previous paper [7] to find Abelian groups for which the centre of the second dual algebra of the semigroup algebra was equal to the semigroup algebra itself.

In doing this, it was found to be useful to consider discrete semigroups S for which βS contained a pair of disjoint closed left ideals whose union was $\beta S \backslash S$, since these gave rise to corresponding ideals in $l^1(S)''$. Semigroups for which this was the case were said to have property B. (Property A was used by Baker and Butcher for semigroups with a separately continuous extension to βS). It was shown that an Abelian group has property B if and only if either it is a countable torsion group, or it is the direct product of the infinite cyclic group and a finite group. It is interesting to note that for Abelian groups property B is equivalent to the possession of more than one end, or to having first cohomological dimension greater than zero (see Cohen [2]).

In the present paper the result for Abelian groups is extended to commutative semigroups in Sections 4 and 5 by means of a quotient construction developed in Section 3.

2. *Notation and definitions*

The positive integers will be denoted by $\mathbb{N}$ and the non-negative integers by $\mathbb{Z}^+$.

All semigroups considered in this paper will be commutative and written additively.

Let S be a semigroup and let E be a subset of S. Then $\langle E \rangle$ denotes the subsemigroup generated by E. If $E = \{e_1, \ldots, e_n\}$, then $\langle E \rangle$ may be written $\langle e_1, \ldots, e_n \rangle$. The semigroup S is said to be *periodic* if $\langle e \rangle$ is finite for every $e \in S$. If $e \in S$ and $n \in \mathbb{N}$ then

$$ne = e + e + \ldots + e \ (n \text{ terms}) \quad \text{and} \quad \mathbb{N}e = \{ne : n \in \mathbb{N}\} = \langle e \rangle.$$

This notation for $\langle e \rangle$ will be used only when S is non-periodic.

† Present address: Crop Engineering Division, The National Institute of Agricultural Engineering, Wrest Park, Silsoe, Bedford, MK45 4HS.

https://doi.org/10.1017/S0305004100063040 Published online by Cambridge University Press

If $A \subset S$ and $s \in S$ then $A+s$ denotes $\{t+s: t \in A\}$ and $A-s$ denotes $\{t \in S: t+s \in A\}$. The semigroup S will be said to be *almost cancellative* if $A-s$ is finite for every $s \in S$ and every finite subset A of S.

A subset U of S will be said to be *almost invariant* if $(U+s)\backslash U$ is finite for all $s \in S$.

An almost cancellative semigroup S will be said to have *property B* if S is the union of two disjoint, infinite, almost invariant sets.

The following result is proved in [7], where the details of the construction of a semigroup operation in the Stone-Čech compactification βS which extends the operation in S may be found.

THEOREM 2.1. *Let S be an infinite discrete semigroup. Then $\beta S \backslash S$ is the union of two disjoint closed left ideals of βS if and only if S is almost cancellative and has property B.*

3. *Quotients of semigroups*

In this section a construction is described for commutative semigroups which generalizes the construction of the quotient of a group by a subgroup. The method seems to be novel; however, its application is effectively restricted to almost cancellative semigroups, where one can be sure that the 'cosets' of an infinite subsemigroup will be infinite. The proofs of all the results are straightforward.

The definition of congruence and the proposition which follows it may be found in Sections 1.4 and 1.5 of [4].

Definition. An equivalence relation $\sim$ on a semigroup S is said to be a *congruence* if $s_1 \sim t_1$ and $s_2 \sim t_2$ together imply $s_1+s_2 \sim t_1+t_2$ for $s_1, s_2, t_1, t_2 \in S$.

PROPOSITION 3·1. *Let $\sim$ be a congruence on a semigroup S. The set $S/\sim$ of equivalence classes is a semigroup with the operation $\bar{s}+\bar{t} = \overline{s+t}$ where $\bar{s}$ denotes the equivalence class containing s.*

PROPOSITION 3·2. *Let T be a subsemigroup of S. The binary relation $\sim$ on S defined by $s_1 \sim s_2$ if $(T+s_1) \cap (T+s_2) \neq \varnothing$ is a congruence.*

Definitions. Let T be a subsemigroup of a semigroup S. Then S/T will denote the quotient semigroup $S/\sim$, where $\sim$ is the congruence constructed above. The equivalence class containing s will be denoted by $\bar{s}$ and the equivalence classes will be referred to as *cosets*.

The properties of S/T which will be required later will now be developed.

PROPOSITION 3·3. *Let T be a subsemigroup of a semigroup S and define S/T as above. Then*

(*a*) *$T+s \subset \bar{s}$ and $\bar{s} = (T+s)-T$ for all $s \in S$;*

(*b*) *$\bar{t} = T-T$ for all $t \in T$;*

(*c*) *if S is almost cancellative and T is infinite then every coset of T is infinite and, in particular, if T is countably infinite then every coset of T is countably infinite.*

PROPOSITION 3·4. *Let T' be a subsemigroup of a semigroup S. Let $T = T'-T'$. Then T is a semigroup and $T = T-T$.*

https://doi.org/10.1017/S0305004100063040 Published online by Cambridge University Press

4. *Periodic semigroups and uncountable semigroups*

The cases of periodic semigroups and uncountable semigroups are dealt with first, as the proofs are very similar to those given in section 7 of [7] for the corresponding groups.

THEOREM 4·1. *Let S be a countably infinite, almost cancellative, periodic semigroup. Then S has property B.*

Proof. Let $\{e_1, e_2, \ldots\}$ be a set of generators of S. For $E \subset S$ let $[E]$ denote the subsemigroup $\langle E\rangle - \langle E\rangle$ of S. If $E = \{s_1, s_2, \ldots, s_n\}$, write $[s_1, s_2, \ldots, s_n]$ for $[E]$. Since S is almost cancellative, $[E]$ is finite for any finite subset E of S.

Let $S_0 = [e_1]$ and for $n \in \mathbb{N}$ let $s_n = [e_1, \ldots, e_{n+1}] \backslash [e_1, \ldots, e_n]$. Suppose that $s \in S_m$ and $t \in S_n$, where $n > m$. Then $s+t \in [e_1, \ldots, e_{n+1}]$ and $s \in [e_1, \ldots, e_n]$, but $t \notin [e_1, \ldots, e_n]$. Hence $s+t \notin [e_1, \ldots, e_n]$, since $[e_1, \ldots, e_n] - [e_1, \ldots, e_n] = [e_1, \ldots, e_n]$ by (3·4). Thus $s+t \in S_n$.

Now each S_n is finite and $S = \bigcup_{n=0}^{\infty} S_n$, therefore $S_n \neq \varnothing$ for infinitely many n. Let $\{n_i : i \in \mathbb{N}\} = \{n \in \mathbb{Z}^+ : S_n \neq \varnothing\}$ indexed so that $n_i < n_j$ whenever $i < j$. Then

$$S = \bigcup_{i=1}^{\infty} S_{n_i} \quad \text{and} \quad S_{n_i} \neq \varnothing$$

for each i.

Let $U_- = \bigcup_{i \in D} S_{n_i}$ and $U_+ = \bigcup_{i \in V} S_{n_i}$, where D is the set of odd natural numbers and V is the set of even natural numbers. Then U_- and U_+ are disjoint infinite sets whose union is S. It remains to show that U_- and U_+ are almost invariant.

Let $s \in S$. Then $s \in S_{n_j}$ for some j. By above, $(U_- + s) \cap U_+ \subset S_{n_j}$ and $(U_+ + s) \cap U_- \subset S_{n_j}$. Hence U_- and U_+ are almost invariant and S has property B. ∎

THEOREM 4·2. *Let S be an uncountable almost cancellative semigroup. Then S does not have property B.*

Proof. Let T be a countably infinite subsemigroup of S. Since S is almost cancellative, $T - T$ is countably infinite. By (3·4), $T - T$ is a semigroup. So, without loss of generality, assume that $T = T - T$.

Let $R = S/T$ and let $\{s_r : r \in R\}$ be a set of representatives of the cosets of T. By (3·3c), R is uncountable.

Suppose that $S = U_- \cup U_+$ where U_- and U_+ are disjoint almost invariant sets. Define a mapping f of S into $\{0, 1\}$ by $f(s) = 0$ if $s \in U_-$ and $f(s) = 1$ if $s \in U_+$.

For each $t \in T$ let $V_t = \{r \in R : f(s+t) \neq f(s)$ for some $s \in \bar{s}_r\}$. Since U_- and U_+ are almost invariant, V_t is finite. Let $V_T = \bigcup_{t \in T} V_t$. Then V_T is either finite or countably infinite. If $r \in R \backslash V_T$ then $f(s+t) = f(s)$ for all $t \in T$ and $s \in r$ so, since $r = (T + s_r) - s_r$, either $r \subset U_-$ or $r \subset U_+$.

Now suppose that there exists $q \in R$ such that $q \subset U_-$. We show that, for all $r \in R \backslash V_T$, $r \subset U_-$ and deduce that U_+ is finite.

Let $r \in R \backslash V_T$. Then either $T + s_r \subset U_-$ or $T + s_r \subset U_+$. Now

$$(T + s_r) + s_q = (T + s_q) + s_r \subset U_- + s_r,$$

so $(T + s_r) \cap U_+$ is finite and hence empty. Thus $U_+ \subset \bigcup_{r \in V_T} r$ which is either empty or countably infinite.

https://doi.org/10.1017/S0305004100063040 Published online by Cambridge University Press

Therefore $U_+ - U_+$ is at most countably infinite and hence $S\backslash(U_+ - U_+)$ is non-empty. Let $u \in S\backslash(U_+ - U_+)$. Then $U_+ + u \subset U_-$, so U_+ is finite.

Similarly, if there exists $q \in R$ such that $q \subset U_+$, then U_- is finite.

Hence S does not have property B. ∎

5. *Countable non-periodic semigroups*

We now turn to the remaining case, that of countably infinite, non-periodic, almost cancellative semigroups; though in fact it is not necessary to assume countability. We begin with an example of one such semigroup which does not have property B.

PROPOSITION 5.1. *The semigroup* $(\mathbb{N}, +)$ *does not have property* B.

Proof. Suppose that $\mathbb{N} = U_- \cup U_+$, where $U_- \cap U_+ = \varnothing$ and that $(U_- + 1) \cap U_+$ is finite. Then there exists $N \in \mathbb{N}$ such that for all $n > N$, if $n \in U_-$ then $n+1 \in U_-$. Thus either $\{n \in \mathbb{N} | n > N\} \subset U_+$ or for some $M > N$ $\{n \in \mathbb{N} | n > M\} \subset U_-$. In the former case U_- is finite and in the latter U_+ is finite. Hence $\mathbb{N}$ does not have property B. ∎

For the remainder of this section let S be an infinite, non-periodic, almost cancellative semigroup. Let e be an element of S such that $\mathbb{N}e$ is infinite. If $s \in S$ then $s + ne$ where $n = 0$ will be taken to mean s, so that $\mathbb{Z}^+e + s$ denotes $\{s\} \cup (\mathbb{N}e + s)$. Since S is almost cancellative, $\mathbb{Z}^+e + s$ is infinite for each $s \in S$.

The quotient $S/\mathbb{N}e$ may be formed as described in Section 3, and $\mathbb{Z}^+e + s \subset \bar{s}$ for each $s \in S$. Let $A = S/\mathbb{N}e$.

Definitions. Define a relation $\leqslant$ on S by $s \leqslant t$ if there exist $m, n \in \mathbb{Z}^+$ with $m \leqslant n$ such that $s + ne = t + me$. Then each coset of $\mathbb{N}e$ is pre-ordered by $\leqslant$. Furthermore, if s and t belong to the same coset then either $s \leqslant t$ or $t \leqslant s$.

Now let $L_S = \{\alpha \in A : \alpha$ has no minimal element$\}$. Choose a set $\{t_\alpha : \alpha \in A\}$ of representatives of the cosets as follows: if $\alpha \in A\backslash L_S$ then let t_α be a minimal element of α; otherwise let t_α be any element of α. We now construct a map $R: S \to \mathbb{Z}$ whose restriction to any coset is order-preserving. Although not generally a homomorphism, R has some of the properties of one.

PROPOSITION 5.2. *There is a mapping* $R: S \to \mathbb{Z}$ *such that*:

(*a*) $R(t_\alpha) = 0$ *for each* $\alpha \in A$;

(*b*) $R(s + ke) = R(s) + k$ *for each* $s \in S$ *and* $k \in \mathbb{N}$;

(*c*) *if* $s \in \alpha$ *and* $t \in \beta$ *where* $\alpha, \beta \in A$ *then*

$$R(s + t) = R(t_\alpha + t_\beta) + R(s) + R(t);$$

(*d*) *if* $\alpha \in L_S$ *then* $R(\alpha) = \mathbb{Z}$, *otherwise* $R(\alpha) = \mathbb{Z}^+$.

Proof. Let $\alpha \in A$ and let $s \in \alpha$. By definition, $(\mathbb{Z}^+e + s) \cap (\mathbb{Z}^+e + t_\alpha) \neq \varnothing$, so there is a minimal $n \in \mathbb{Z}^+$ such that $s + ne \in \mathbb{Z}^+e + t_\alpha$. The elements $t_\alpha, t_\alpha + e, t_\alpha + 2e, \ldots$ must be distinct, since otherwise $\mathbb{Z}^+e + t_\alpha$ would be finite, contradicting the almost-cancellativity of S, so there exists a unique $m \in \mathbb{Z}^+$ such that $s + ne = t_\alpha + me$. Let $R(s) = m - n$. Clearly $R(t_\alpha) = 0$; and if $s + n'e = t_\alpha + m'e$, where $m', n' \in \mathbb{Z}^+$ then $m' - n' = R(s)$.

If $s + ne = t_\alpha + me$ then $s + ke + ne = t_\alpha + (m + k)e$, so

$$R(s + ke) = (m + k) - n = R(s) + k,$$

proving (*b*).

https://doi.org/10.1017/S0305004100063040 Published online by Cambridge University Press

Now let $s \in \alpha$ and $t \in \beta$ where $\alpha, \beta \in A$. Then there are $m, n, m', n' \in \mathbb{Z}^+$ such that $s + ne = t_\alpha + me$ and $t + n'e = t_\beta + m'e$. Let $\gamma = \alpha + \beta$. Then there are $p, q \in \mathbb{Z}^+$ such that $t_\alpha + t_\beta + qe = t_\gamma + pe$. So

$$s + t + (n + n' + q)e = t_\gamma + (m + m' + p)e$$

and hence

$$R(s+t) = R(s) + R(t) + R(t_\alpha + t_\beta).$$

Now, by (*a*) and (*b*), $\mathbb{Z}^+ \subset R(\alpha)$ for all $\alpha \in A$. If $\alpha \in A \setminus L_S$ then $s + ne = t_\alpha + me$ implies that $m \geqslant n$; hence $R(\alpha) = \mathbb{Z}^+$. On the other hand, if $\alpha \in L_S$ then for each $k \in \mathbb{N}$ there exist $s \in \alpha$ and $n \in \mathbb{Z}^+$ such that $s + (n+k)e = t_\alpha + ne$; hence $-k \in R(s)$ for all $k \in \mathbb{N}$. ∎

COROLLARY 5·3. *The set L_S is an ideal of A.*

Notation. From now on define R as above and for each $n \in \mathbb{Z}$ let $R_n = R^{-1}(n)$.

PROPOSITION 5·4. (*a*) *$R_n \neq \varnothing$ for all $n > 0$.*

(*b*) *Either $R_n = \varnothing$ for all $n < 0$ or $R_n \neq \varnothing$ for all $n < 0$.*

(*c*) *If $\alpha \cap R_n$ is infinite for some $\alpha \in A$ and $n \in \mathbb{Z}$, then $\alpha \cap R_m$ is infinite for all $m > n$.*

(*d*) *If R_n is infinite for some $n \geqslant 0$, then R_m is infinite for all $m > n$.*

(*e*) *If R_n is infinite for some $n < 0$, then R_m is infinite for all $m \in \mathbb{Z}$.*

Proof. (*a*) and (*b*) follow immediately from (5·2).

(*c*) Suppose that $\alpha \cap R_n$ is infinite. Then, by (5·2*b*), $(\alpha \cap R_n) + ke \subset \alpha \cap R_{n+k}$ for all $k \in \mathbb{N}$. Hence $\alpha \cap R_{n+k}$ is infinite for all $k \in \mathbb{N}$, since S is almost cancellative.

(*d*) also follows from (5.2*b*).

(*e*) Suppose that R_n is infinite for some $n < 0$. There are two possible cases: either L_S is infinite, in which case the result follows from (5·2*d*); or L_S is finite and $\alpha \cap R_n$ is infinite for some $\alpha \in A$. Suppose that the latter holds and let $k = R(t_\alpha + t_\alpha)$. Given $p \in \mathbb{Z}^+$ let $u \in R_{-k-p} \cap \alpha$. Then $R(s+u) = R(s) + R(u) + R(t_\alpha + t_\alpha) = n - p$ for all $s \in R_n \cap \alpha$. Hence, since S is almost cancellative, R_{n-p} is infinite for all $p > 0$. ∎

Notation. For $s \in S$ let $P(s) = \{t \in S : s \in \mathbb{Z}^+ e + t\}$ (the set of 'predecessors' of s). For $\alpha \in A$ let $Q(t_\alpha) = P(t_\alpha)$ and for $n \in \mathbb{N}$ let $Q(t_\alpha + ne) = P(t_\alpha + ne) \setminus P(t_\alpha + (n-1)e)$.

PROPOSITION 5·5. (*a*) *$P(s) \cap R_n$ is finite for all $s \in S$ and $n \in \mathbb{Z}$.*

(*b*) *For all $n \in \mathbb{Z}$, for all $s, t \in R_n$, if $P(s) \cap P(t) \neq \varnothing$ then $s = t$.*

(*c*) *$P(s) + t \subset P(s+t)$ for all $s, t \in S$ and if $t \in P(s)$ then $P(t) \subset P(s)$.*

(*d*) *For $\alpha \in A$ and $n, m \in \mathbb{Z}^+$, if $n \neq m$ then $Q(t_\alpha + ne) \cap Q(t_\alpha + me) = \varnothing$.*

(*e*) *If $\alpha \in A$, then $\alpha = \bigcup_{n \in \mathbb{Z}^+} Q(t_\alpha + ne)$.*

Proof. (*a*) Let $s \in S$ and let $m = R(s)$. Then $P(s) \cap R_m = \{s\}$. If $n > m$ then

$$P(s) \cap R_n = \varnothing.$$

If $n < m$ then $(P(s) \cap R_n) + (m-n)e = \{s\}$ so, since S is almost cancellative, $P(s) \cap R_n$ is finite.

(*b*) Let $s, t \in S$ with $R(s) = R(t) = n$. Suppose that $u \in P(s) \cap P(t)$ and $R(u) = m$. Then $s = u + (n-m)e = t$.

(*c*) Let $s, t \in S$ and let $u \in P(s)$. Then $s \in \mathbb{Z}^+ e + u$, so $s + t \in \mathbb{Z}^+ e + (u+t)$; that is, $u + t \in P(s+t)$.

Now let $s \in S$, $t \in P(s)$ and $u \in P(t)$. Then $s \in \mathbb{Z}^+ e + t$ and $t \in \mathbb{Z}^+ e + u$, so $s \in \mathbb{Z}^+ e + u$.

(*d*) is immediate from the definition.

https://doi.org/10.1017/S0305004100063040 Published online by Cambridge University Press

(*e*) Let $\alpha \in A$. By definition, $s \in \alpha$ if and only if $s \in P(t_\alpha + ne)$ for some $n \in \mathbb{Z}^+$. Hence $\alpha \subset \bigcup_{n\in\mathbb{Z}^+} P(t_\alpha + ne) = \bigcup_{n\in\mathbb{Z}^+} Q(t_\alpha + ne)$. ∎

THEOREM 5·6. *The non-periodic, almost cancellative semigroup S discussed in this section has property B if and only if R_{-1} is finite but non-empty.*

Proof. Suppose that $S = U_- \cup U_+$, where U_- and U_+ are disjoint almost invariant sets. For each $\alpha \in A$ let $U_-^\alpha = U_- \cap \alpha$ and $U_+^\alpha = U_+ \cap \alpha$.

Since $\mathbb{N}$ does not have property B, either $U_- \cap \mathbb{N}e$ is finite or $U_+ \cap \mathbb{N}e$ is finite. Without loss of generality assume that $U_- \cap \mathbb{N}e$ is finite. Then, since U_+ is almost invariant, $U_- \cap (\mathbb{Z}^+e + s)$ is finite for each $s \in S$, and in particular for each $\alpha \in A$ there exists $N_\alpha \in \mathbb{Z}^+$ such that $t_\alpha + ne \in U_+$ for all $n \geqslant N_\alpha$.

Let $B = \{\alpha \in A : U_-^\alpha \neq \varnothing\}$. If $\alpha \in B$ and $s \in U_-^\alpha$ then there exists $n \in \mathbb{Z}^+$ such that $s + ne \in U_-$ and $s + (n+1)e \in U_+$. Hence, since U_- is almost invariant, B is finite.

Now take $\beta \in B$ and suppose that $U_- \cap Q(t_\beta + ne) \neq \varnothing$ for some $n > N_\beta$. Then, since $t_\beta + ne \in U_+$, there exists $s \in U_- \cap Q(t_\beta + ne)$ such that $a + e \in U_+$. Hence, since the sets $Q(t_\beta + ne)$ for different n are disjoint, there are only finitely many $n > N_\beta$ such that $U_- \cap Q(t_\beta + ne)$ is non-empty. Thus $R(U_-)$ has an upper bound M. Let $V = R_{M+1}$. Then $U_- \subset \bigcup_{s\in V} P(s)$ and these $P(s)$ are disjoint by (5·5*b*).

Let $s \in V$ such that $U_- \cap P(s) \neq \varnothing$. Then, since $s \in U_+$, there exists $t \in U_- \cap P(s)$ such that $t + e \in U_+$. Hence there are only finitely many $s \in V$ such that $U_- \cap P(s) \neq \varnothing$. Call these $s_1, s_2, \ldots, s_q$. By (5·5*a*), $P(s_i) \cap R_m$ is finite for $i = 1, 2, \ldots, q$ and all $m \in \mathbb{Z}$. Hence $U_- \cap R_m$ is finite for all $m \in \mathbb{Z}$.

It is now clear that if R_{-1} is empty then U_- is finite and S does not have property B.

Now suppose that R_{-1} is infinite so that, by (5·4*e*), R_n is infinite for all $n \in \mathbb{Z}$. There are two possible cases: either (*a*) L_s is infinite; or (*b*) L_s is finite and $\alpha \cap R_{-1}$ is infinite for some $\alpha \in L_s$.

Consider case (*a*) and take $\beta \in B$. For $\alpha \in L_s$ let $n(\alpha) = -R(t_\alpha + t_\beta)$. Then $R_{n(\alpha)} \cap \alpha \neq \varnothing$, so take $v_\alpha \in R_{n(\alpha)} \cap \alpha$. Then $R(s + v_\alpha) = R(s)$ for all $s \in \beta$. Let $V_\beta = \{v_\alpha : \alpha \in L_s\}$, which is infinite.

Now consider case (*b*). By the argument used in the proof of (5.4*e*), there exists $\gamma \in A$ such that $\gamma \cap R_n$ is infinite for all $n \in \mathbb{Z}$. Take $\beta \in B$ and let $n = -R(t_\beta + t_\gamma)$. Then $R(s + v) = R(s)$ for all $s \in \beta$ and $v \in R_n \cap \gamma$. Let V_β be the infinite set $R_n \cap \gamma$.

Now take both cases together and let V_β be as defined above for each $\beta \in B$. Recall that $U_- \subset P(s_1) \cup \ldots \cup P(s_q)$ where $s_1, \ldots, s_q \in R_{M+1}$.

Take $i \in \{1, \ldots, q\}$ and let $\beta = \bar{s}_i$. Then $\beta \in B$, so $V_\beta + s_i$ is an infinite subset of R_{M+1}. Hence there exists $v \in V_\beta$ such that $\{v + s_i\} \cap \{s_1, \ldots, s_q\} = \varnothing$. By (5·4*c*),

$$P(s_i) + v \subset P(s_i + v) \subset U_+,$$

so $U_- \cap P(s_i)$ is finite.

Hence U_- is finite and S does not have property B.

It has now been shown that if R_{-1} is empty or infinite then S does not have property B. It remains to prove the converse.

Suppose that R_{-1} is finite but non-empty. Let $U_- = \bigcup_{n\in\mathbb{N}} R_{-n}$ and $U_+ = \bigcup_{n\in\mathbb{Z}^+} R_n$. Then U_- and U_+ are disjoint infinite sets whose union is S.

Let $s \in S$. If $(U_+^\alpha + s) \cap U_- \neq \varnothing$ for some $\alpha \in A$ then $t + s \in R_{-1}$ for some $t \in U_+^\alpha$. Hence, since S is almost cancellative, the set of such α is finite. Also, if $(U_+^\alpha + s) \cap U_- \neq \varnothing$ then

$$(U_\alpha^+ + s) \cap U_- \subset R_{-n} \cup R_{-n+1} \cup \ldots \cup R_{-1},$$

https://doi.org/10.1017/S0305004100063040 Published online by Cambridge University Press

where $-n = R(t_\alpha + s)$, so $(U_+^\alpha + s) \cap U_-$ is finite. Hence $(U_+ + s) \cap U_-$ is finite. Thus U_+ is almost invariant.

It is straightforward to check that U_- is almost invariant and hence that S has property B. |

Remark. The construction performed above began with the choice of an element e of S; however property B is clearly independent of this choice. In fact, for a given semigroup S the set R_{-1} will be either empty for all e such that $\langle e \rangle$ is infinite, or finite for all such e, or infinite for all such e.

REFERENCES

[1] J. W. Baker and R. J. Butcher. The Stone–Čech compactification of a topological semigroup. *Math. Proc. Cambridge Philos. Soc.* **80** (1976), 102–107.

[2] D. E. Cohen. *Groups of Cohomological Dimension One.* Lecture Notes in Math. vol. 245 (Springer-Verlag, 1972).

[3] N. Hindman. Minimal ideals and cancellation in $\beta\mathbb{N}$. *Semigroup Forum* **25** (1982), 291–310.

[4] J. M. Howie. *An Introduction to Semigroup Theory* (Academic Press, 1976).

[5] N. Macri. The continuity of Arens product on the Stone–Čech compactification of semigroups. *Trans. Amer. Math. Soc.* **191** (1974), 185–193.

[6] P. Milnes. Compactifications of semitopological semigroups. *J. Austral. Math. Soc.* **15** (1973), 488–503.

[7] D. J. Parsons. The centre of the second dual of a commutative semigroup algebra. *Math. Proc. Cambridge Philos. Soc.* **95** (1984), 71–92.

[8] J. S. Pym and H. L. Vasudeva. Semigroup structure in compactifications of ordered semigroups. *Czech. Math. J.* **27** (**102**) (1977), 528–544.

[9] N. J. Young. The irregularity of multiplication in group algebras. *Quart. J. Math. Oxford Ser.* (2), **24** (1973), 59–62.

https://doi.org/10.1017/S0305004100063040 Published online by Cambridge University Press