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Weak Gibbs measures for certain non-hyperbolic systems

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Abstract. We study a weak Gibbs property of equilibrium states for potentials of weak bounded variation and for maps admitting indifferent periodic points. We further establish statistical properties of the weak Gibbs measures and bounds of their pointwise dimension. We apply our results to higher-dimensional maps (which are not necessarily conformal) with indifferent periodic points and show that their absolutely continuous finite invariant measures are weak Gibbs measures.

1. *Introduction*

In this paper, we shall develop a generalized thermodynamic formalism for maps T defined on a subset of a compact metric space which exhibit certain non-hyperbolic phenomena. We direct our attention to potentials ϕ which may fail both summable variations and bounded distortion but satisfy the *weak bounded variation* (see the definition in §2). In [**20**], we have already established for such systems a generalized variational principle for the pressure $P(\phi)$ and a characterization of $P(\phi)$ in terms of the radius of convergence of Artin–Mazur–Ruelle zeta functions. In particular, we obtained equilibrium states μ for ϕ which are absolutely continuous with respect to natural reference measures ν called *conformal measures.* The method used for solving these problems was constructing derived maps T^* which are uniformly expanding and guarantee a weak Hölder type property of the potential ϕ^* associated to T^*. In non-hyperbolic situations, the Gibbs property in the sense of Bowen typically fails for both ν and μ even if the equilibrium states μ^* for ϕ^* satisfy it. However, we could still establish a version of Gibbs property which is called *weak Gibbs* (see the definition in §2) for the conformal measure ν. On the other hand, the problem of the weak Gibbs property of the equilibrium state μ still remains open. The main purpose of this paper is to solve it (Theorem 3.2) and to establish statistical properties of the weak Gibbs measure μ (Theorems 6.1 and 6.2). The key to the proof of the weak Gibbs property is to clarify the order of divergence of the invariant density $d\mu/d\nu$ near indifferent periodic points (Theorem 3.1). In particular, if ν-measures of the sets of

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points at which the stopping time with respect to an expanding region is greater than n decays polynomial fast, then $d\mu/d\nu$ is unbounded and it may cause interesting phenomena from a statistical point of view [**4, 5, 9, 21, 22**].

The paper is organized as follows. In §3, first we give a criterion of finiteness of μ in terms of weak bounded variation (WBV) property. Next we give an exact estimate of the density of μ with respect to the weak Gibbs measure ν and then establish the weak Gibbs property of μ. In §4, we show several piecewise C^1-smooth maps with indifferent periodic points for which $-\log|\det DT|$ are potentials of WBV and we can verify all conditions in our theorems. In §5, we study the pointwise dimension of μ and the Haussdoff dimension of the set of forward generic points. In particular, we apply these results to non-conformal parabolic repellers arising from Brun's map which admits indifferent fixed points (cf. [**6**]). In §6, we shall establish the convergence of the iterated Perron–Frobenius operators with respect to the weak Gibbs measure μ in the L^∞-norm (Theorem 6.1) and we shall give an upper bound on the convergence in the L^1-norm (Theorem 6.2). Both results can be applied to examples in §4. In the appendix, we study a relation between the Gibbs property for the derived systems (T^*, μ^*) and the weak Gibbs property for the original system (T, μ).

2. *Preliminaries*

Let $(X, \mathcal{F}, \nu)$ be a non-atomic probability space. Suppose that $T : X \to X$ is a non-singular transformation with respect to $\nu(\nu T^{-1} \sim \nu)$ and that there exists a countable measurable partition $Q = \{X_a\}_{a\in I}$ of X, generating $\mathcal{F}$ satisfying the local invertibility, i.e. for every $X_a \in Q$, $\nu(X_a) > 0$ and $T|_{X_a} : X_a \to TX_a$ is an invertible map satisfying $d(\nu T)|_{X_a}/d\nu|_{X_a} > 0$ (ν-a.e). Such a system (T, X, Q) is called a *piecewise invertible system* [**14–22**]. Typically, such a non-singular measure exists in the following setting for a map T defined on a subset X of a compact metric space $(\bar{X}, d)$. Let $\mathcal{F}$ be the σ-algebra of Borel subsets of X. We suppose that there exists a disjoint generating partition $Q = \{X_a\}_{a\in I}$ of X with $X_a \in \mathcal{F}$ $(\forall a \in I)$ such that $X_0 = \bigcup_{a\in I} \operatorname{int} X_a$ is an open dense subset of $\bar{X}$. Let $T : X \to \bar{X}$ be a non-invertible map which is continuous on X_0 with $X_0 \subset TX_0$ and $T|_{X_a} : X_a \to TX_a$ is a homeomorphism $(\forall a \in I)$. We denote $X_{a_1} \cap T^{-1}X_{a_2} \cap \cdots \cap T^{-(n-1)}X_{a_n}$ by $X_{a_1\dots a_n}$ if its interior is not empty. We impose the next condition from which we can obtain nice countable symbolic dynamics similar to sofic shifts [**16, 18**].

Finite range structure (FRS). $\mathcal{U} = \{T^n(\operatorname{int} X_{a_1\dots a_n}) : \forall X_{a_1\dots a_n}, \forall n > 0\}$ consists of finitely many open subsets of $\bar{X}$.

Let us denote the local inverse $(T|_{X_a})^{-1}$ by ψ_a and $(T^n|_{X_{a_1\dots a_n}})^{-1}$ by $\psi_{a_1\dots a_n}$.

Definition. A Borel probability measure ν is called a *weak Gibbs measure* for a function ϕ with a constant P if there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n\to\infty}(1/n)\log K_n = 0$ such that ν-a.e. x,

$$K_n^{-1} \le \frac{\nu(X_{a_1\dots a_n}(x))}{\exp\bigl(\sum_{i=0}^{n-1}\phi T^i(x) + nP\bigr)} \le K_n,$$

where $X_{a_1\dots a_n}(x)$ denotes the cylinder containing x.

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Definition. We say that a function ϕ is of WBV if there exists a sequence $\{C_n\}$ satisfying $\lim_{n\to\infty}(1/n)\log C_n = 0$ and $\forall X_{a_1\dots a_n} \in \bigvee_{i=0}^{n-1} T^{-i}Q, \forall n \geq 1$,

$$\frac{\sup_{x\in X_{a_1\dots a_n}} \exp\big(\sum_{i=0}^{n-1}\phi(T^i x)\big)}{\inf_{x\in X_{a_1\dots a_n}} \exp\big(\sum_{i=0}^{n-1}\phi(T^i x)\big)} \leq C_n.$$

If a function ϕ is a piecewise Hölder potential of WBV and ψ_a is Lipschitz continuous with the Lipschitz constant L_a satisfying $\sup_{a\in I} L_a < \infty$, then under $\mathcal{L}_\phi 1(x) = \sum_{a\in I}\exp(\phi\circ\psi_a(x)) \leq K < \infty$ $(\forall x \in X)$ and $\mathcal{U} = \{\operatorname{int} X\}$ there exist $p > 0$ and a Borel probability measure ν such that ν is a weak Gibbs measure for ϕ with $-\log p$ satisfying $\mathcal{L}_\phi^*\nu = p\nu$, where $\mathcal{L}_\phi^*$ is the dual of $\mathcal{L}_\phi$ (see [**13, 20**]). Then $d(\nu T|_{X_a})/d(\nu|_{X_a}) = \exp(\log p - \phi) > 0$ holds ν-a.e. for each $a \in I$ and so (T, X, Q) is a piecewise invertible system with FRS. We can describe $\log p$ as the reciprocal of the radius of convergence of *Artin–Muzer–Ruelle* zeta function (see [**20**, Theorem 2.1]) and even if uniformly expanding property fails, the potential ϕ of WBV still admits an equilibrium state μ equivalent to ν and $\log p$ coincides with the measure theoretical pressure $P(\phi)$. Such results were established in [**20**] by constructing a derived system T^* which is called a *jump transformation* [**11**]. Even if we weaken $\mathcal{U} = \{\operatorname{int} X\}$ by the FRS condition, we still have a weak Gibbs measure for ϕ under additional conditions [**3**]. We shall recall the construction of the jump transformation in [**20**]. Assume that $L_a \leq 1$ $(\forall a \in I)$. For $L < 1$, define $\mathcal{B}_1 = \{X_b : L_b \leq L\}$, $\mathcal{D}_1 = \{X_d : L_d > L\}$, and inductively define

$$\mathcal{B}_n = \{X_{b_1\dots b_n} : X_{b_1\dots b_{n-1}} \in \mathcal{D}_{n-1}, L_{b_n} \leq L\},$$
$$\mathcal{D}_n = \{X_{d_1\dots d_n} : X_{d_1\dots d_{n-1}} \in \mathcal{D}_{n-1}, L_{d_n} > L\}.$$

We put $B_n = \bigcup_{X_{b_1\dots b_n}\in\mathcal{B}_n} X_{b_1\dots b_n}$ and $D_n = \bigcup_{X_{d_1\dots d_n}\in\mathcal{D}_n} X_{d_1\dots d_n}$. Put $D_0 = X$. Define $I^* = \bigcup_{n=1}^\infty\{(a_1\dots a_n) : X_{a_1\dots a_n} \in \mathcal{B}_n\}$. Then $Q^* = \{X_\alpha\}_{\alpha\in I^*}$ is a countable partition of $\bigcup_{i=1}^\infty B_i$. Now we define $T^* : \bigcup_{i=1}^\infty B_i \to X$ by $T^*x = T^i x$ for $x \in B_i$. Put $X^* = X\backslash\big(\bigcup_{m=0}^\infty T^{*-m}\big(\bigcap_{n\geq 0} D_n\big)\big)$. Then (T^*, X^*, Q^*) is a piecewise invertible system which is uniformly expanding. Let $\mathcal{R}$ be the set of all cylinders $X_{a_1\dots a_n}$ of which the last symbol a_n satisfies $X_{a_n} \in \mathcal{B}_1$. Then $\mathcal{R}$ satisfies the *strong playback property*; $X_{a_1\dots a_n b_1\dots b_m} \in \mathcal{R}$ whenever $X_{b_1\dots b_m} \in \mathcal{R}$. Let $R(x)$ be the stopping time with respect to $\mathcal{R}$, i.e. $R(x) = \inf\{n \in \mathbb{N} : X_{a_1\dots a_n}(x) \in \mathcal{R}\}$. Then a regularity condition on $\phi^* = \sum_{i=0}^{R-1}\phi T^i$ ([**20**, (C8)]) allows one to have the following key property for establishing the existence of absolutely continuous T^*-invariant finite measure μ^* with respect to ν whose density is bounded away from infinity.

Measure theoretical bounded distortion. There exists $C \geq 1$ such that

$$\sup_n \sup_{X_{\alpha_1\dots\alpha_n}\in\bigvee_{i=0}^{n-1} T^{*-i}Q^*} \sup_{x,y\in T^{*n}X_{\alpha_1\dots\alpha_n}} \frac{\psi'^*_{\alpha_1\dots\alpha_n}(x)}{\psi'^*_{\alpha_1\dots\alpha_n}(y)} \leq C$$

where $\psi^*_{\alpha_1\dots\alpha_n} = (T^{*n}|_{X_{\alpha_1\dots\alpha_n}})^{-1}$ and

$$\psi'^*_{\alpha_1\dots\alpha_n}(x) = \frac{d(\nu\psi^*_{\alpha_1\dots\alpha_n})}{d\nu}(x)$$
$$= \exp\left(\sum_{i=0}^{n-1}\phi^*(T^{*i}\psi^*_{\alpha_1\dots\alpha_n}x) - \sum_{i=0}^{n-1}R(T^{*i}\psi^*_{\alpha_1\dots\alpha_n}x)\log p\right).$$

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If the next condition is satisfied, then $d\mu^*/d\nu$ is bounded away from zero and μ^* is exact.

Transitivity. $\forall U \in \mathcal{U}$, $\exists X_{a_1\dots a_s} \subset U$ such that $T^s X_{a_1\dots a_s} = X$ and $X_{a_s} \in \mathcal{R}$.

The T^*-invariant measure μ^* is a Gibbs measure (in the sense of Bowen) and a unique equilibrium state for $\phi^* - R\log p$. If $\lim_{n\to\infty} \nu(D_n) = 0$, then $\nu(X^*) = \nu(X)$ and the following formula gives a T-invariant σ-finite measure $\mu \sim \nu$, which is exact [**16**]:

$$\mu(E) = \sum_{n=0}^{\infty} \mu^*(D_n \cap T^{-n}E) \quad (\forall E \in \mathcal{F}). \tag{F1}$$

In particular, if $\sum_{n=0}^{\infty} \nu(D_n) < \infty$ then μ is finite. The density of μ with respect to ν is written as follows [**11, 15–17**]:

$$\frac{d\mu}{d\nu}(x) = \sum_{n=0}^{\infty} \sum_{X_{d_1\dots d_n} \in \mathcal{D}_n} \psi'_{d_1\dots d_n}(x) \frac{d\mu^*}{d\nu}(\psi_{d_1\dots d_n}x) 1_{T^n X_{d_1\dots d_n}}(x), \tag{F2}$$

where $\psi'_{d_1\dots d_n} = d(\nu\psi_{d_1\dots d_n})/d\nu$ $\big(= \exp\big(\sum_{i=0}^{n-1} \phi T^i - n\log p\big)\psi_{d_1\dots d_n}\big)$. Furthermore, a generalized thermodynamic formalism for ϕ was established in [**20**] via one for ϕ^*. We shall see in §3 that singularities of $d\mu/d\nu$ happen when $\nu(D_n)$ is polynomial decay. We should mention that the jump transformaton is not uniquely determined so that other constructions would be possible as we shall see it in the appendix (cf. [**20**, Remark B]).

3. *Weak Gibbs property of equilibrium states for potentials of WBV*

In this section, we assume all facts in the previous section. That is, let ϕ be a potential of WBV satisfying $\mathcal{L}_\phi 1(x) \le K < \infty$ $(\forall x \in X)$ and we suppose the existence of a Borel probability measure ν and $p > 0$ satisfying $\mathcal{L}_\phi^* \nu = p\nu$ so that (T, X, Q) is a piecewise invertible system with respect to ν. Then the measure ν satisfies the weak Gibbs property for ϕ [**20**, Proposition 2.2]. The situation typically happens for piecewise C^1-smooth maps admitting indifferent periodic points ($T^q x_0 = x_0$, $|\det DT^q(x_0)| = 1$), more precisely, let ν be the normalized Lebesgue measure of $X \subset \mathbb{R}^N$ and $\phi = -\log|d(\nu T)/d\nu|$ $(= -\log|\det DT|)$. Then $\mathcal{L}_\phi{}^*\nu = \nu$ and we can see for all examples in §4 that ϕ is a potential of weak bounded variation and ν is a weak Gibbs measure for ϕ. Let us define the partition function $Z_n(\phi) = \sum_{T^n x = x} \exp\big(\sum_{i=0}^{n-1} \phi(T^i x)\big)$. Then $\lim_{n\to\infty} 1/n \log Z_n(\phi) = \log p$ (see [**20**, Theorem 2.1] and [**18**, Theorem 5.1]). Furthermore, suppose that $\exists\mathcal{R}$ a subset of the set of all admissible cylinders satisfying the strong playback property and we can construct a jump transformation T^* with respect to $\mathcal{R}$ as in §2 which is (piecewise) uniformly expanding (see the appendix). If equi-Hölder continuity of $\{\phi^*\psi_\alpha^*\}_{\alpha\in I^*}$ holds for the derived map T^* satisfying the transitivity in §2, then there exists an exact finite invariant measure μ^* equivalent to ν which is Gibbs for $\phi^* - R\log p$ (see [**20**]) and $d\mu^*/d\nu$ is a piecewise Hölder continuous function [**17, 20, 21**]. For example, the next condition implies equi-Hölder continuity of $\{\phi^*\psi_\alpha^*\}_{\alpha\in I^*}$:

$$\forall X_{a_1\dots a_n} \in \mathcal{B}_n,\ \exists 0 < L_\phi(a_1\dots a_n) < \infty \tag{C0}$$

such that

$$|\phi(\psi_{a_1\dots a_n}x) - \phi(\psi_{a_1\dots a_n}y)| \le L_\phi(a_1\dots a_n)d(x, y)^\theta$$

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for $x, y \in T^n X_{a_1 \dots a_n}$ and

$$L_\phi(\infty) = \sup_{n>0} \sup_{X_{a_1 \dots a_n} \in \mathcal{B}_n} \sum_{i=1}^{n} L_\phi(a_i \dots a_n) < \infty.$$

If $\lim_{n\to\infty} \nu(D_n) = 0$, then the formula (F1) gives a T-invariant exact measure $\mu \sim \nu$ and in particular, when $\sum_{n=0}^{\infty} \nu(D_n) < \infty$ (which implies $\mu(X) < \infty$) sufficient conditions for μ being an equilibrium state for ϕ and for $\log p = P(\phi)$ (where $P(\phi)$ denotes the measure theoretical pressure) were given in [**20**]. Define

$$Z_n^*(\phi) = \sum_{\substack{T^n x = x, \\ x \in X_{d_1 \dots d_n} \in \mathcal{D}_n}} \exp\left(\sum_{i=0}^{n-1} \phi(T^i x) \right).$$

Let $\mathcal{U}_\mathcal{D} = \{\operatorname{int} T^n X_{a_1 \dots a_n} : \forall X_{a_1 \dots a_n} \in \mathcal{D}_n, \forall n \geq 1\}$. The next lemma gives a criterion of finiteness of μ in terms of the WBV constants.

LEMMA 3.1. *Suppose that* $\mathcal{U}_\mathcal{D} = \{\operatorname{int} X\}$. *If* $\sum_{n=1}^{\infty} p^{-n} C_n Z_n^*(\phi) < \infty$, *then* $\mu(X) < \infty$.

Proof of Lemma 3.1. It suffices to show that $\sum_{n=0}^{\infty} \nu(D_n) < \infty$ which implies $\mu(X) < \infty$. The weak Gibbs property of ν gives that

$$p^{-n} C_n^{-1} \leq \frac{\nu(X_{a_1 \dots a_n}(x))}{\exp\left(\sum_{i=0}^{n-1} \phi T^i(x) \right)} \leq p^{-n} C_n.$$

Then

$$\begin{aligned} \nu(D_n) &= \sum_{X_{d(n)} \in \mathcal{D}_n} \nu(X_{d(n)}) \\ &\geq p^{-n} C_n^{-1} \sum_{\substack{T^n x = x, \\ x \in X_{d(n)} \in \mathcal{D}_n}} \exp\left(\sum_{i=0}^{n-1} \phi T^i(x) \right) = p^{-n} C_n^{-1} Z_n^*(\phi), \end{aligned}$$

where $d(n)$ denote a sequence $d_1 \dots d_n$ of length n. Similarly we have $\nu(D_n) < p^{-n} C_n Z_n^*(\phi)$ by using $\mathcal{U}_\mathcal{D} = \{\operatorname{int} X\}$. □

As we have mentioned in §2, under the transitivity condition the T-invariant measure μ obtained by (F1) is equivalent to a weak Gibbs measure ν for ϕ (with constant $-\log p$). Now we ask the following.

Question 1. Is the equilibrium state μ weak Gibbs for ϕ with $(-\log p)$?

In order to answer to this question, first we shall study properties of $h(x) = (d\mu/d\nu)(x)$. We recall that there exists $1 \leq G < \infty$ such that $G^{-1} \leq (d\mu^*/d\nu)(x) \leq G$.

LEMMA 3.2. *Suppose that the following conditions are satisfied:*

$$\sum_{n=0}^{\infty} \nu(D_n) < \infty. \tag{C1}$$

$$\mathcal{U}_\mathcal{D} = \{\operatorname{int} X\}. \tag{C2}$$

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$\exists C' > 1$ *such that* $\forall X_{d(n)}$, $\forall n \geq 1$ *and* $\forall k \geq 1$,

$$\sup_{x,y \in T^n X_{d(n)} \cap B_k} \frac{\psi'_{d(n)}(x)}{\psi'_{d(n)}(y)} \leq C'. \tag{C3}$$

Then $\forall x \in B_k$, $\forall k \geq 1$, $h(x) \leq GC' \sum_{n=0}^{k-1} \nu(D_n)/\nu(B_k)$.

We remark that the conditions (C1)–(C3) can be easily verified for all piecewise C^1-smooth examples in §4 of which $\bigcap_{n\geq 0} D_n$ consist of indifferent periodic points. Typically for such non-hyperbolic maps the invariant densities h with respect to the normalized Lebesgue measures ν are unbounded at the indifferent periodic points (see [**15–17**]). We generalize this fact for potentials of WBV.

THEOREM 3.1. (Order of divergence of the invariant density $h = d\mu/d\nu$ near $\bigcap_{n\geq 0} D_n$)

(1) *Suppose that (C1) holds and*

$$B_{n+1} = D_n \cap T^{-n} B_1. \tag{C4}$$

Then $\mu(B_{n+1}) = \mu^*(D_n)$ *and* $\mu(D_n) = \int_{D_n} (R(x) - n)\, d\mu^*$. *We assume further that*

$$\exists 1 \leq H < \infty \quad \text{such that} \quad \forall x, y \in B_k, \forall k \geq 1, h(x)/h(y) \leq H. \tag{C5}$$

If $\nu(D_n) = n^{-\alpha}$ $(\exists \alpha > 1)$, *then* $\exists 1 \leq R < \infty$ *such that* $R^{-1} n \leq h|_{B_n} \leq Rn$.

(2) *Suppose that (C1) holds and*

$$\begin{aligned} &\exists t_0 > 0 \text{ such that } X_{a_1 \dots a_n} \in \mathcal{D}_n^c \text{ with } X_{a_2 \dots a_n} \in \mathcal{D}_{n-1} \\ &\text{implies } X_{a_1 \dots a_n} \subseteq \bigcup_{i=1}^{t_0} B_i \ (n > t_0). \end{aligned} \tag{C6}$$

Then $\mu(B_{n+1}) \geq \mu^*(D_n)$. *Assume further that (C2) and (C3) are satisfied. If* $\nu(D_n) = n^{-\alpha} (\alpha > 1)$ *then* $\exists 1 \leq R < \infty$ *such that* $R^{-1} n \leq h|_{B_n} \leq Rn^{1+\alpha}$.

The next lemma gives a sufficient condition for (C5) which can be easily verified for all examples in §4.

LEMMA 3.3. *If (C2) and (C3) hold, then (C5) is satisfied.*

Remark A. As we will see in the proof of Theorem 3.1, the assumption $\nu(D_n) = n^{-\alpha}$ $(\alpha > 1)$ can be replaced by the next condition on $\nu(B_n)$,

$$\exists 0 < r_1 < r_2 < \infty \quad \text{such that} \quad r_1 n^{-(\alpha+1)} \leq \nu(B_n) \leq r_2 n^{-(\alpha+1)},$$

which allows one to show that

$$\exists 0 < r'_1 < r'_2 < \infty \quad \text{such that} \quad r'_1 n^{-\alpha} \leq \nu(D_n) \leq r'_2 n^{-\alpha}$$

and hence to establish the same results as those in (1) and (2).

The following formula will be useful to estimate exact rates of the return sequences $a_n(T)$ in case of $\mu(X) = \infty$ (see [**1, 16**] for details).

COROLLARY 3.1. (Order of return sequences)

$$a_n(T) \sim \frac{n}{\sum_{k=1}^{n} \nu(D_{k-1})}.$$

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Remark B. (C4) is valid for maps with one indifferent fixed point. On the other hand, we will see that a two-dimensional map with infinitely many indifferent fixed points (example 4) also satisfy (C4). (C6) can be verified for a inhomogeneous diophantine transformation which fails (C4) (see [**14, 15**]).

Proof of Lemma 3.2. Let $x \in B_k$. Then the formula (F1) and (C2) and (C3) allow us to have the following inequalities:

$$\begin{aligned} h(x) &= \sum_{n=0}^{\infty} \sum_{X_{d(n)} \in \mathcal{D}_n} \psi'_{d(n)}(x) 1_{T^n X_{d(n)}}(x) (d\mu^*/d\nu)(\psi_{d(n)} x) \\ &\le GC' \Bigg(\sum_{n=0}^{\infty} \sum_{X_{d(n)} \in \mathcal{D}_n} \inf_{x' \in T^n X_{d(n)} \cap B_k} \psi'_{d(n)}(x') \Bigg) \\ &\le GC' \Bigg(\sum_{n=0}^{\infty} \sum_{X_{d(n)} \in \mathcal{D}_n} \frac{\int_{T^n X_{d(n)} \cap B_k} \psi'_{d(n)}(x')\, d\nu(x')}{\nu(T^n X_{d(n)} \cap B_k)} \Bigg) \\ &\le GC' \nu(B_k)^{-1} \sum_{n=0}^{\infty} \sum_{X_{d(n)} \in \mathcal{D}_n} \nu(X_{d(n)} \cap T^{-n} B_k). \end{aligned}$$

Since $D_n \cap T^{-n} B_k \subset \bigcup_{i=1}^{k} B_{n+i}$ (because of the strong playback property of $\mathcal{R}$), we have that

$$h(x) \le GC' \nu(B_k)^{-1} \sum_{n=0}^{\infty} \sum_{i=1}^{k} \nu(B_{n+i}).$$

Then the equality $\sum_{i=1}^{k} \nu(B_{n+i}) = \nu(D_n) - \nu(D_{n+k})$ gives the desired result. □

Proof of Lemma 3.3. It suffices to see the following equality for $x, y \in B_k$,

$$\begin{aligned} &|h(x) - h(y)| \\ &= \left| \sum_{n=0}^{\infty} \sum_{\substack{X_{d(n)} \in \mathcal{D}_n, \\ x, y \in T^n X_{d(n)}}} \psi'_{d(n)}(x) \frac{d\mu^*}{d\nu}(\psi_{d(n)} x) \left(1 - \frac{\psi'_{d(n)}(y)\, d\mu^*/d\nu\,(\psi_{d(n)} y)}{\psi'_{d(n)}(x)\, d\mu^*/d\nu\,(\psi_{d(n)} x)} \right) \right|. \end{aligned}$$

Then (C3) and boundness of $d\mu^*/d\nu$ give (C5). □

Proof of Theorem 3.1. Let (C4) be satisfied. Then the first and the second assertions of (1) follow from (F1), the formula of invariant measure μ. For the last assertion of (1), we see that

$$\begin{aligned} \mu(B_{n+1})/\nu(B_{n+1}) &= \mu^*(D_n)/\nu(B_{n+1}) \le G\nu(D_n)/\nu(B_{n+1}) \\ &= G(\nu(D_{n+1}) + \nu(B_{n+1}))/\nu(B_{n+1}) = G(\nu(D_{n+1})/\nu(B_{n+1}) + 1) \end{aligned}$$

and similarly we see that

$$\mu(B_{n+1})/\nu(B_{n+1}) \ge G^{-1}(\nu(D_{n+1})/\nu(B_{n+1}) + 1).$$

Since $\mu(B_{n+1})/\nu(B_{n+1}) = \int_{B_{n+1}} h(y)\, d\nu(y) / \int_{B_{n+1}} d\nu(y)$ and $\exists 0 < H_1 < H_2 < \infty$ such that $H_1(n+1) \le \nu(D_{n+1})/\nu(B_{n+1}) \le H_2(n+1)$, the mean value theorem in integration allows us to establish the required fact.

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Next we assume that (C6) is satisfied. The first assertion of (2) was established in [**19**, Theorems 4.1 and 4.2]. Using this fact gives the lower bounds of $h|_{B_n}$ similarly to (1). It follows from Lemma 3.2 that the desired upper bound of $h|_{B_n}$ is obtained. □

Now we come to establish the weak Gibbs property of μ.

THEOREM 3.2. (Weak Gibbs property of μ) *Let ν be the weak Gibbs measure for ϕ with constant $-\log p$ and let (C1) be satisfied. We assume either ((C4) and (C5)) or ((C2), (C3) and (C6)) in Theorem 3.1 and the following condition.*

(Polynomial decay of cylinder measure near $\cap_{n\geq 0} D_n$) there exist $0 < r_1 < r_2 < \infty$ and $\exists \alpha > 1$ such that

$$r_1 n^{-(\alpha+1)} \leq \nu(B_n) \leq r_2 n^{-(\alpha+1)}. \tag{C7}$$

Suppose that $\exists 0 < H < \infty$ and $\exists 0 < \eta < \infty$ such that

$$\sup_{X_{d_1\dots d_n}\in\mathcal{D}_n} \frac{\mu(X_{d_1\dots d_n})}{\nu(X_{d_1\dots d_n})} \leq H n^{\eta}. \tag{C8}$$

Then μ is a weak Gibbs measure for ϕ with constant $-\log p$.

Remark C. As we have observed in Remark A, the assumption (C7) allows us to establish the following fact : $\exists 0 < r_1' < \infty$ such that $r_1' n^{-\alpha} \leq \nu(D_n)$. Hence the next condition is sufficient for establishing (C8), which can be easily verified for all examples in §4: $\exists 0 < G_1 < \infty$ such that $\forall X_{d_1\dots d_n} \in \mathcal{D}_n$, $\nu(D_n) \leq G_1 \nu(X_{d_1\dots d_n})$. In fact, we can take $H = G_1\mu(X)/r_1'$ and $\eta = \alpha$.

Remark D. Though (C4) fails for Example 3 (on the other hand (C6) is valid), an exact formula of the invariant density h with respect to the normalized Lebesgue measure (see [**15**]) and a direct calculation allow one to have $R^{-1}k \leq h|_{B_k} \leq Rk$.

Proof of Theorem 3.2. It follows from the formula (F1) of μ that $\mu(E) \geq \mu^*(E) \geq G^{-1}\nu(E)$ $(\forall E \in \mathcal{F})$. Then we see that

$$\frac{\mu(X_{a_1\dots a_n}(x))}{\exp\bigl(\sum_{i=0}^{n-1}\phi(T^i(x)) - n\log p\bigr)} \geq G^{-1}\frac{\nu(X_{a_1\dots a_n}(x))}{\exp\bigl(\sum_{i=0}^{n-1}\phi(T^i(x)) - n\log p\bigr)} \geq (GK_n)^{-1}.$$

Next we note that

$$\frac{\mu(X_{a_1\dots a_n}(x))}{\exp\bigl(\sum_{i=0}^{n-1}\phi(T^i(x)) - n\log p\bigr)} = \frac{\nu(X_{a_1\dots a_n}(x))}{\exp\bigl(\sum_{i=0}^{n-1}\phi(T^i(x)) - n\log p\bigr)} \times \frac{\mu(X_{a_1\dots a_n}(x))}{\nu(X_{a_1\dots a_n}(x))}$$

and

$$\frac{\mu(X_{a_1\dots a_n}(x))}{\nu(X_{a_1\dots a_n}(x))} \leq \max\left\{ \sup_{X_{a_1\dots a_n}\subseteq\bigcup_{i=1}^n B_i} \frac{\mu(X_{a_1\dots a_n})}{\nu(X_{a_1\dots a_n})}, \sup_{X_{a_1\dots a_n}\in\mathcal{D}_n} \frac{\mu(X_{a_1\dots a_n})}{\nu(X_{a_1\dots a_n})} \right\}.$$

Since Theorem 3.1 and Remark A give either

$$\sup_{x\in\bigcup_{i=1}^n B_i} h(x) = \max_{1\leq i\leq n} \sup_{x\in B_i} h(x) \leq O(n)$$

or

$$\sup_{x\in\bigcup_{i=1}^n B_i} h(x) \leq O(n^{1+\alpha}),$$

the weak Gibbs property of μ follows from (C8). □

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4. *Examples and applications*

In this section, we show several piecewise C^1-smooth maps of which $-\log|\det DT|$ are potentials of WBV and absolutely continuous finite invariant measures μ (with respect to Lebesgue measures) are weak Gibbs in our sense. We refer to the readers [**17, 18, 20**] for a characterization of μ as an equilibrium state for $-\log|\det DT|$.

First we remark that if ϕ is a piecewise Hölder potential with exponent θ and the Hölder constant is L_ϕ, then

$$C_n = \exp\left(\sum_{i=1}^{n} V_i(\phi)\right) \le \exp\left(L_\phi \sum_{i=1}^{n} \sigma(i)^\theta\right),$$

where

$$\sigma(i) = \sup_{X_{a_1\dots a_i} \in \bigvee_{j=0}^{i-1} T^{-j}Q} \operatorname{diam} X_{a_1\dots a_i}$$

and

$$V_i(\phi) = \sup_{X_{a_1\dots a_i} \in \bigvee_{j=0}^{i-1} T^{-j}Q} \sup_{x,y \in X_{a_1\dots a_i}} |\phi(x) - \phi(y)|.$$

Hence the generator condition that we imposed on $Q(\sigma(n) \to 0$ as $n \to \infty)$ implies the WBV property.

LEMMA 4.1. *Let ϕ be a piecewise Hölder potential with exponent θ and $\sigma(i) = i^{-q}$ $(\exists q > 0)$:*

(1) *if $q\theta = 1$, then $C_n = O(n^\gamma)$ $(\exists \gamma > 0)$;*

(2) *if $q\theta \ne 1$, then $C_n = O(n^{1-q\theta}/(1-q\theta))$.*

Proof. We see that $\sum_{i=1}^{n} \sigma(i)^\theta = \sum_{i=1}^{n} i^{-\theta q} = O(\int_1^n x^{-q\theta}\,dx)$ and then the above remark gives the desired fact. □

In the case when $q\theta \ne 1$, if $q\theta > 1$, then $C_n = \exp\big(\sum_{i=1}^{n} V_i(\phi)\big) \le C < \infty$ which implies summability of variations $V_n(\phi)$. This property is satisfied for hyperbolic systems and for Hölder potentials. On the other hand, $q\theta = 1$ typically happens for maps admitting indifferent periodic points and for $\phi = -\log|\det DT|$. In fact, the following examples satisfy $\sum_{k=1}^{n} V_k(\phi) = O(\log n)$ and so $\sum_{k=1}^{\infty} V_k(\phi) = \infty$, but $V_n(\phi) \to 0$ as $n \to \infty$.

Example 1 (One parameter family of maps admitting an indifferent fixed point). Let $X = [0, 1]$ and for β with $0 < \beta$ define

$$T_\beta(x) = \begin{cases} \dfrac{x}{(1-x^\beta)^{1/\beta}}, & \text{on } X_0 = [0, (1/2)^{1/\beta}) \\ \dfrac{x}{(1/2)^{1/\beta} - 1} + \dfrac{1}{1-(1/2)^{1/\beta}}, & \text{on } X_1 = [(1/2)^{1/\beta}, 1]. \end{cases}$$

Then $(T_\beta, X, Q = \{X_i : i = 0, 1\})$ is a piecewise invertible system with $\mathcal{U} = \{\operatorname{int} X\}$. The map T_β has an indifferent fixed point 0. We remark that T_β with $\beta = 1$ is just the interval map which was studied by Lasota and Yorke. We also can think that the Manneville–Pomeau equation, $f_\beta = x + x^{1+\beta}$ (mod 1), of which asymptotic behaviour near the indifferent fixed point is essentially similar to T_β. Both $-\log|T'_\beta|$ and $-\log f'_\beta$

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are piecewise Hölder continuous with exponent β and $q = 1/\beta$. Thus $q\theta = 1$ and in fact $C_n = O(n^{1+\beta})$ (see [**14, 15**]). Define

$$D_n = X_{\underbrace{0\ldots0}_{n}} \quad \text{and} \quad B_n = D_{n-1} \setminus D_n = \{X_{\underbrace{0\ldots01}_{n}}\}.$$

The jump transformation $T^* : \bigcup_{i=1}^{\infty} B_n \to X$ given by $T^*x = T^i x$ for $x \in B_i$ satisfies the bounded distortion property for T^*. T^* admits an exact finite invariant measure μ^* which is Gibbs in the sense of Bowen. The formula (F1) gives an exact σ-finite invariant measure μ equivalent to the Lebesgue measure ν of which density satisfies $h(x) \sim O(x^{-\beta})(x \to 0)$ (cf. [**11, 12**]). Let $\beta < 1$. Then (C1) is satisfied so that μ is finite. (C4) is immediately verified. (C3) was proved, for example in [**17, 19**]. Since $\nu(D_n) = (n+1)^{-1/\beta}$, (C7) and (C8) are easily verified by using the formula (F1) as follows:

$$\sup_{X_{d_1\ldots d_n} \in \mathcal{D}_n} \frac{\mu(X_{d_1\ldots d_n})}{\nu(X_{d_1\ldots d_n})} \le G \frac{\sum_{m=n}^{\infty} \nu(D_m)}{\nu(D_n)} = O(n+1).$$

Hence (1) in Theorem 3.1 and Theorem 3.2 are valid. On the other hand, the asymptotic formula of h allows one to verify conditions in Theorems 3.1 and 3.2, directly.

*Example 2 (Brun's map [**11, 15, 20**]).* Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_2 \le x_1 \le 1\}$, and let for $i = 0, 1, 2$

$$X_i = \{(x_1, x_2) \in X : x_i + x_1 \ge 1 \ge x_{i+1} + x_1\},$$

where we put $x_0 = 1$ and $x_3 = 0$. T is defined by

$$T(x_1, x_2) = \begin{cases} \left(\dfrac{x_1}{1-x_1}, \dfrac{x_2}{1-x_1}\right) & \text{on } X_0, \\ \left(\dfrac{1}{x_1} - 1, \dfrac{x_2}{x_1}\right) & \text{on } X_1, \\ \left(\dfrac{x_2}{x_1}, \dfrac{1}{x_1} - 1\right) & \text{on } X_2. \end{cases}$$

Since $TX_i = X (i = 0, 1, 2)$, $(T, X, Q = \{X_i\}_{i=0}^2)$ is a piecewise invertible system with $\mathcal{U} = \{\text{int } X\}$. T is a continuous, piecewise C^2-map and $\phi = -\log|\det DT|$ is piecewise Lipschitz continuous, i.e. $\theta = 1$. T admits an indifferent fixed point $(0, 0)$ and we can easily see that $q = 1$. It follows from Lemma 4.1 that C_n is at most polynomial. Define

$$\mathcal{D}_n = \{X_{\underbrace{0\ldots0}_{n}}\}, \quad \mathcal{B}_n = \{X_{\underbrace{0\ldots0i}_{n}}\} \quad (i = 1, 2).$$

Then we can show that $\exists 0 < \theta^* < 1, \exists 0 < \delta < \infty$ and $\exists n_0 > 0$ such that for $\forall (x_1, x_2), (x_1', x_2') \in X$

$$\|\psi^*_{\alpha_1\ldots\alpha_n}(x_1, x_2) - \psi^*_{\alpha_1\ldots\alpha_n}(x_1', x_2')\| \le \delta\theta^{*n}\|(x_1, x_2) - (x_1', x_2')\|$$

for $\forall(\alpha_1 \ldots \alpha_n) \in I^{*n}$ and $\forall n \ge n_0$. A direct calculation allows us to verify equi-Lipschitz continuity of $\{\phi^*\psi_{\alpha^*}\}_{\alpha^* \in I^*}$ so that T^* satisfies the bounded distortion property and admits an exact finite invariant measure μ^* equivalent to the Lebesgue measure ν (see [**15, 20**]).

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Since $\nu(D_n) = 1/2n^2$ (C7) is immediately verified and the formula (F1) gives an exact finite invariant measure μ of which density is unbounded at the indifferent fixed point $(0, 0)$. (C3) is easily verified by using the following fact; $\forall X_{a_1 \dots a_n}, \exists a, b \in \mathbb{Z}^+, \exists c \in \mathbb{N}$ such that $|\det D\psi_{a_1 \dots a_n}(x_1, x_2)| = (ax_1 + bx_2 + c)^{-3}$. Hence (C5) is valid. On the other hand the exact formula of the invariant density : $h(x_1, x_2) = (2x_1(1 + x_2))^{-1}$ allows one to verify (C5) directly. (C8) is also immediately verified by using Remark C. Thus we have established all conditions in Theorems 3.1 and 3.2.

The next example shows that $\phi = -\log|T'|$ is not piecewise Hölder but still C_n is polynomial, in fact $V_n(\phi) \to 0$ and $\sum_{k=1}^{n} V_k(\phi) = O(\log n)$.

Example 3 (Inhomogeneous Diophantine algorithm). Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_2 \le 1, -x_2 \le x_1 \le -x_2 + 1\}$. The following map T defined on X is related to a Diophantine approximation problem in inhomogeneous linear class. We refer to the reader for further details of number theoretical meaning [**11, 21**] (cf. [**14–17**]):

$$T(x_1, x_2) = (1/x_1 - [(1 - x_2)/x_1] + [-(x_2/x_1)], -[-(x_2/x_1)] - (x_2/x_1)).$$

Let

$$a(x_1, x_2) = \left[\frac{(1 - x_2)}{x_1}\right] - \left[\frac{-x_2}{x_1}\right] \quad \text{and} \quad b(x_1, x_2) = -\left[-\frac{x_2}{x_1}\right].$$

We define

$$I = \{(a, b) \in \mathbb{Z}^2 : a > b > 0 \text{ or } a < b < 0\}$$

and define for each $(a, b) \in I$,

$$X_{(a,b)} = \{(x_1, x_2) \in X : a(x_1, x_2) = a, b(x_1, x_2) = b\}.$$

Then $Q = \{X_{(a,b)}\}_{(a,b)\in I}$ is a countable partition of X and $TX_{(a,b)} = X$ $(\forall (a, b) \in I)$. Therefore (T, X, Q) is a piecewise invertible system with $\mathcal{U} = \{\text{int } X\}$. Both $(1, 0)$ and $(-1, 1)$ are indifferent periodic points with period 2. For $\phi = -\log|\det DT|$, we can verify $V_n(\phi) \le \log(1 + \sigma(n - 1))$ and $\sigma(n) = O(n^{-1})$. $C_n = O(n^3)$ were established in [**14**] (cf. [**15, 17, 19, 21, 22**]). Let $\mathcal{D}_n$ be the set of cylinders of rank n containing indifferent periodic points and $D_n = \bigcup_{X_{a(n)}\in\mathcal{D}_n} X_{a(n)}$. Denote $B_n = D_{n-1}\backslash D_n$. Then the jump transformation $T^* : \bigcup_{i=1}^{\infty} B_i \to X$ defined by $T^*x = T^i x$ for $x \in B_i$ satisfies the bounded distortion property so that T^* admits an exact finite invariant measure μ^* equivalent to the Lebesgue measure ν. In [**21**] exponential decay of diameter of cylinders with respect to T^* was established. Since $\nu(D_n) = 1/[(n + 1)(2n + 1)]$, (C7) is immediately verified and the formula (F1) gives an exact finite invariant measure μ equivalent to the Lebesgue measure ν (see [**14, 16**]). (C1) and (C2) are immediately verified. Denote $x_0 = (1, 0), Tx_0 = (-1, 1)$ and $D_n = \bigcup_{i=0}^{1} D_n^{(i)}$, where $D_n^{(i)}(\subset D_n)$ contains $T^i x_0$. Then we can write $B_n = \bigcup_{i=0}^{1} B_n^{(i)}$, where $B_n^{(i)} = D_{n-1}^{(i)}\backslash D_n^{(i)}$ and replacing (C3) by

$$\exists C' > 1 \text{ such that } \forall X_{d(n)}, \forall n \ge 1 \text{ and } \forall k \ge 1,$$
$$\sup_{x,y\in T^n X_{d(n)}\cap B_k^{(i)}} \frac{\psi'_{d(n)}(x)}{\psi'_{d(n)}(y)} \le C' \quad (i = 0, 1) \tag{C3)*}$$

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still allows us to establish Lemma 3.2. (C3)* is readily verified by observing that

$$\forall (x_1, x_2), (x_1', x_2') \in B_k^{(i)}, \quad \frac{2k-3}{2k-1} \le |x_1|, |x_1'| \le \frac{k}{k+1}$$

and

$$\left| 1 - \frac{\psi'_{a_1 \dots a_n}(x_1, x_2)}{\psi'_{a_1 \dots a_n}(x_1', x_2')} \right| \le 21 \frac{|x_1 - x_1'|}{1 - |x_1|}$$

(cf. [**9, 17, 19**]). On the other hand, the formula of the invariant density: $h(x_1, x_2) = 1/((2 \log 2)(1 - x_1^2))$ allows one to verify (C5) directly (see [**17**, p. 991]).

Since $\sharp \mathcal{D}_n = 2$ $(\forall n \ge 1)$ and $\nu(D_n) = 2\nu(D_n^{(i)})$ $(i = 0, 1)$, (C8) is easily verified. Thus all conditions in Theorems 3.1 and 3.2 are verified.

Example 4 (A two-dimensional map related to uniform distribution of a cylinder flow). Let $X = \{(x, y) : 0 < x \le 1, 0 \le y \le 1\}$ and T is defined by

$$T(x, y) = \left(-\frac{1}{x} - \left[-\frac{1}{x} \right]_{(-)}, \frac{y}{x} - \left[\frac{y}{x} \right] \right),$$

where $[x]_{(-)} = \max\{n \in \mathbb{Z} : n < x\}$. The first component of T is known as a map related to negative continued fraction which admits a σ-finite infinite absolutely continuous invariant measures with respect to the Lebesgue measure ([**16, 21**]). Let $a(x) = -[-1/x]_{(-)}$, and $b(x, y) = [y/x]$. The index set I is given by

$$I = \{(a, b) \in \mathbb{N} \times (\mathbb{N} \cup \{0\}) : a \ge 2, a > b\}.$$

The partition $Q = \{X_{(a,b)}\}_{(a,b) \in I}$ is defined as follows:

$$(x, y) \in X_{(a,b)} \text{ iff } a(x) = a \text{ and } b(x, y) = b.$$

It is easy to see that $\mathcal{U}$ consists of $U_0 = X$ and $U_1 = \{(x, y) \in X : x + y \le 1\}$ and that Q is not Markov, but satisfies the FRS condition. T admits a σ-finite infinite ergodic invariant measure equivalent to ν the normalized Lebesgue measure and it was shown that $\mathcal{D}_n$ is the set of cylinders touching $\{1\} \times [0, 1]$ which consists of indifferent fixed points, i.e.

$$\mathcal{D}_n = \{X_{((2,b_1),(2,b_2),\dots(2,b_n))} : b_1 = b_2 = \dots = b_n = 0 \text{ or } b_i = 1\ (\exists i) \text{and } b_j = 0\ (\forall j \ne i)\}.$$

Although $\phi = -\log |\det DT|$ is not a piecewise Hölder potential, a direct computation gives $V_n(\phi) \to 0 (n \to \infty)$ so that ϕ is a potential of WBV. By Corollary 3.1, we have $a_n(T) \sim n/\log(n + 1)$ as $n \to \infty$ [**21**].

5. *Pointwise dimension of equilibrium states for potentials of weak bounded variation*

Let (T, X, Q) be a piecewise invertible system with FRS satisfying $\sigma(n) \to 0$ $(n \to \infty)$ and let $X_J = \bigcap_{i=0}^{\infty} T^{-i}(\cup_{a \in J} \mathrm{cl} X_a)$ where J is a subset of I. Then $TX_J \subseteq X_J$ and X_J is a closed subset of X if J is a finite set. The system $(T|_{X_J}, X_J, Q_J)$, where $Q_J = \{X_a \cap X_J\}_{a \in J}$, gives a subsystem of (T, X, Q) which is also a piecewise invertible

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system. Without loss of generality, (T, X, Q) denotes such a subsystem. In this section, for a surjective piecewise invertible system (T, X, Q) we shall consider the local dimension of the weak Gibbs measure μ for ϕ a potential of weak bounded variation. We recall that in the category of conformal repellers for smooth expanding maps T with positive Lyapunov exponents $\log \gamma$ the following equalities hold for x a forward generic point of μ,

$$h_\mu(T, Q, x) = -\lim_{n\to\infty} 1/n \log \mu(X_{a_1\ldots a_n}(x)) = (\log \gamma) d_\mu(x),$$

where $d_\mu(x)$ is the pointwise dimension and $h_\mu(T, Q, x)$ is the local entropy of μ (see [**7, 8**]). In particular, if Q is a generating partition and μ is an ergodic invariant measure, then the Shannon–McMillan–Breiman (SMB) theorem allows us to have for μ-a.e. x

$$h_\mu(T) = h_\mu(T, Q, x) = (\log \gamma)\, d_\mu(x).$$

We establish an analogy of this fact for non-conformal maps with indifferent periodic points under the existence of the derived system (T^*, μ^*) mentioned in §2, i.e. $T^* : X^* \to X^*$ admits an exact finite invariant measure $\mu^* \sim \nu$ which is Gibbs and $\nu(X^*) = \nu(X) = 1$. μ is a T-invariant measure equivalent to ν defined by (F1). We assume $\mu(X) = 1$. Let $\underline{d}_\mu(x)$ and $\bar{d}_\mu(x)$ denote the lower and upper pointwise dimensions of μ at x, respectively. The next theorem gives estimates of $\underline{d}_\mu(x)$ and $\bar{d}_\mu(x)$.

THEOREM 5.1. *Let (C4) be satisfied. Suppose that there exist $\gamma^* > 1$ and $0 < \bar{\Gamma} < \infty$ such that*

$$\sigma^*(n) = \sup_{Y \in \bigvee_{i=0}^{n-1} T^{*-i} Q^*} \operatorname{diam} Y \le \bar{\Gamma} \gamma^{*-n} \ (\forall n \ge 1). \tag{C9}$$

Then for μ-a.e. $x \in X^*$ we have the following upper bound of $\bar{d}_\mu(x)$.

$$\bar{d}_\mu(x) \le \frac{h_{\mu_0^*}(T^*)}{\log \gamma^*},$$

where $\mu_0^* = \mu^*/\mu^*(X^*)$.

Let $X_G^* (\subset X^*)$ be the set of all points for which the Birkhoff ergodic theorem holds for the stopping time R with respect to μ_0^* and the SMB theorem holds with respect to μ. For example, the set of all forward generic points with respect to both μ and μ_0^* satisfies such a property. Theorem 5.1 allows us to have an estimate of the Haussdorff dimension of X_G^* (see [**7**, Theorem 7.2]).

COROLLARY 5.1.

$$\dim_H X_G^* \le \frac{h_{\mu_0^*}(T^*)}{\log \gamma^*}.$$

Remark E. (C9) can be easily verified for all examples in §4 [**20, 21**].

THEOREM 5.2. *Let (C4) be satisfied. If T is a piecewise C^1 map and satisfies*

$$\exists \underline{\Gamma} > 0 \quad \textit{and} \quad \exists \gamma_* > 1$$

$$\textit{such that} \quad \inf_{X_{\alpha_1\ldots\alpha_n} \in \bigvee_{i=0}^{n-1} T^{*-i} Q^*} \ \inf_{x \in X_{\alpha_1\ldots\alpha_n}} \|DT^{*n}|_{X_{\alpha_1\ldots\alpha_n}}(x)\|^{-1} \ge \underline{\Gamma} {\gamma_*}^{-n}, \tag{C10}$$

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then for μ-a.e. $x \in X^*$

$$\underline{d}_\mu(x) \geq \frac{h_{\mu_0^*}(T^*)}{\log \gamma_*}.$$

In particular, if T is a piecewise C^1-conformal map, then the bounded distortion property for $|\det DT^*|$ allows us to have the following (cf. [7]).

COROLLARY 5.2. *For μ-a.e. $x \in X^*$,*

$$\underline{d}_\mu(x) = \bar{d}_\mu(x) = d_\mu(x) = \frac{h_\mu(T)}{\int_X \log \|DT(x)\| \, d\mu(x)}$$

and if X_G^* is the set of all points for which the Birkhoff ergodic theorem holds for R and for $-\log \|DT^*\|$ with respect to ${\mu^*}_0$ and the SMB theorem holds with respect to μ then

$$\dim_H {X_G}^* = \frac{h_\mu(T)}{\int_X \log \|DT(x)\| \, d\mu(x)}.$$

We remark that the set $X^* = \bigcap_{m=0}^\infty T^{*-m}(\bigcup_{i=1}^\infty B_i)$ is invariant not only under T^*, but also under T, so that $T X_G^* \subseteq X_G^*$.

PROPOSITION 5.1. *If (C4) is satisfied, then $T X^* = X^*$ and $X^* = \{x \in X : T^n x \in X^* \ (\forall n \geq 1)\}$.*

Proof of Proposition 5.1. The first assertion is obvious because of (C4). Let $x \in X^*$ and there exist $n_0 > 0$ such that $T^{n_0} x \in \bigcup_{i=0}^\infty T^{*-i}(\bigcap_{n\geq 0} D_n)$. Then $\exists m_0$ such that $T^{*m_0} T^{n_0} x \in \bigcap_{n\geq 0} D_n$. Since $\bigcap_{n\geq 0} D_n$ is positively invariant, x can visit cylinders which belong to $\mathcal{R}$ only finitely many times. We have a contradiction immediately. On the other hand, if $x \in X$ satisfies $T^n x \in X^*$ for all $n \geq 1$ and $x \notin X^*$, then there exists $m_0 > 0$ such that $T^{*m_0} x \in \bigcap_{n\geq\infty} D_n$ which also gives a contradiction. □

Proof of Theorem 5.1. We first note that the generator condition $\sigma(n) \to 0$ as $n \to \infty$ allows one to have

$$\underline{d}_\mu(x) = \liminf \frac{\log \mu(X_{a_1 \ldots a_n}(x))}{\log \operatorname{diam} X_{a_1 \ldots a_n}(x)}, \quad \bar{d}_\mu(x) = \limsup \frac{\log \mu(X_{a_1 \ldots a_n}(x))}{\log \operatorname{diam} X_{a_1 \ldots a_n}(x)}.$$

Suppose that $x \in X^*$. Define $N_n(x) = \max\left\{l \geq 0 : \sum_{i=0}^l R(T^{*i}x) \leq n\right\}$. Then

$$\sum_{i=0}^{N_n+1} R(T^{*i}x) > n \geq \sum_{i=0}^{N_n} R(T^{*i}x).$$

It follows from the pointwise ergodic theorem for (T^*, μ_0^*) that

$$\lim_{n\to\infty} n/N_n = \int_{X^*} R(x) \, d\mu^*(x)/\mu^*(X^*) = \mu(X)/\mu^*(X^*) = \mu^*(X^*)^{-1}.$$

Let $M_n(x) = \sum_{i=0}^{N_n(x)} R(T^{*i}x)$. Then $M_n(x) < n$ and so $\operatorname{diam} X_{\alpha_1 \ldots \alpha_{N_n}}(x) \geq \operatorname{diam} X_{a_1 \ldots a_n}(x)$, where $\sum_{i=1}^{N_n(x)+1} |\alpha_i| = M_n$. Combining these observations allows us to have

$$\frac{\log \mu(X_{a_1 \ldots a_n}(x))}{\log \operatorname{diam} X_{a_1 \ldots a_n}(x)} \leq -\frac{\log \mu(X_{a_1 \ldots a_n}(x))}{n} \frac{n}{N_n} \left(\frac{N_n}{-\log(\Gamma(\gamma^{*-N_n}))} \right)$$

and by the SMB theorem we have the upper bound of $\bar{d}_\mu(x)$ for μ-a.e. $x \in X^*$. □

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Proof of Theorem 5.2. First we note that $\operatorname{diam} X_{\alpha_1\dots\alpha_n} \geq \text{const}\,\gamma_*^{-n}$. In fact, we have that

$$\|\psi^*_{\alpha_1\dots\alpha_n}x - \psi^*_{\alpha_1\dots\alpha_n}y\| \geq \|x-y\| \inf_{z\in X_{\alpha_1\dots\alpha_n}} (\|DT^{*n}|_{X_{\alpha_1\dots\alpha_n}}(z)\|^{-1})$$

which implies the desired inequlity. Next define $L_n(x) = \min\{l \geq 0 : \sum_{i=0}^{l} R(T^{*i}x) \geq n\}$. Then using L_n instead of N_n allows one to have the desired lower bound similarly to the proof of Theorem 5.1. □

Example 5 (Non-conformal parabolic repellers arising from Brun's map.) Let $T : X \to X$ be the Brun's map (see Example 2) and let $S = T^3$. Then $(S, X, \bigvee_{i=0}^{2} T^{-i}Q)$ is a piecewise invertible system satisfying $\mathcal{U} = \{\operatorname{int} X\}$. For each $i \in \{0,1,2\}$ define $J_i \subset I^3$ by $J_i = \{\mathbf{a} = (a_1,a_2,a_3) : a_3 \neq i\}$ and $X_{J_i} = \bigcap_{k=0}^{\infty} S^{-k}\big(\bigcup_{\mathbf{a}\in J_i} \operatorname{cl}(X_{\mathbf{a}})\big)(\subset X)$. Then X_{J_i} is a closed subset of X satisfying $SX_{J_i} = X_{J_i}$ for each $i \in \{0,1,2\}$. In fact, by the definition of X_{J_i}, $SX_{J_i} \subset X_{J_i}$ is obvious. On the other hand, for $x \in X_{J_i}$, $\psi_{\mathbf{a}}x \in X_{J_i}$ for $\mathbf{a} \in J_i$ and so $x = S\psi_{\mathbf{a}}x \in SX_{J_i}$. Note that J_1 and J_2 contain the parabolic indexes (i.e. the index for which corresponding cylinders touched the indifferent fixed point $(0,0)$) which is not true of J_0. See Figures 1–3. Put $Q_{J_i} = X_{J_i} \cap \bigvee_{m=0}^{2} T^{-m}Q$ $(i \in \{0,1,2\})$. Since $(S|_{X_{J_i}}, X_{J_i}, Q_{J_i})$ is a subsystem of (T, X, Q), all results in the previous sections established for (T, X, Q) are also true for $(S|_{X_{J_i}}, X_{J_i}, Q_{J_i})$. In particular, we can apply Theorem 5.1 and Corollary 5.1 to $(S|_{X_{J_i}}, X_{J_i}, Q_{X_{J_i}})$ $(i = 0,1,2)$ (see §4).

6. *Statistical properties of weak Gibbs measures*

In this section, we assume all facts in §2, i.e. $\mathcal{L}^*_{\phi}\nu = p\nu$ and (T, X, Q) is a piecewise invertible system with respect to ν satisfying the FRS condition and the transitivity. We recall that $\phi - \log p = -\log d(\nu T)/d\nu$ and $\psi'_{a_1\dots a_n} = d(\nu\psi_{a_1\dots a_n})/d\nu$. Let $\mathcal{V}$ be the disjoint partition generated by $\mathcal{U}$.

THEOREM 6.1. (Uniform convergence of $\mathcal{L}^n_{\phi-\log p}f$ to $(\int f\,d\nu)h$) *We suppose that (C0) is satisfied and $\{\psi^*_\alpha\}_{\alpha\in I^*}$ is equi-Lipschitz continuous with Lipschitz constant $L_\alpha \leq L < 1$. Assume further that $\exists\{E_k\}_{k>0}$ with $0 < E_k < \infty$ such that*

$$\left|1 - \frac{\psi'_{a_1\dots a_n}(x)}{\psi'_{a_1\dots a_n}(y)}\right| \leq E_k d(x,y)^{\theta},$$

$$\forall x, y \in D_k^c \cap T^n X_{a(n)},\ \forall X_{a(n)} \in \bigvee_{i=0}^{n-1} T^{-i}Q \text{ and } \forall n > 0. \tag{C11}$$

$\exists\{F_k\}_{k>0}$ with $0 < F_k < \infty$ such that

$$d(\psi_{a_1\dots a_n}(x), \psi_{a_1\dots a_n}(y)) \leq F_k d(x,y),$$

$$\forall x, y \in D_k^c \cap T^n X_{a(n)},\ \forall X_{a(n)} \in \bigvee_{i=0}^{n-1} T^{-i}Q \text{ and } \forall n > 0. \tag{C12}$$

If $\sum_{n\geq 0}\nu(D_n) < \infty$ and h^{-1} is continuous on $\bar{X}$ then for all continuous functions f on $\bar{X}$, $\mathcal{L}^n_{\phi-\log p}f \Rightarrow (\int f\,d\nu)h$ uniformly on compact subsets of $X\setminus(\partial\mathcal{V} \cup \bigcap_{n\geq 0} D_n)$.

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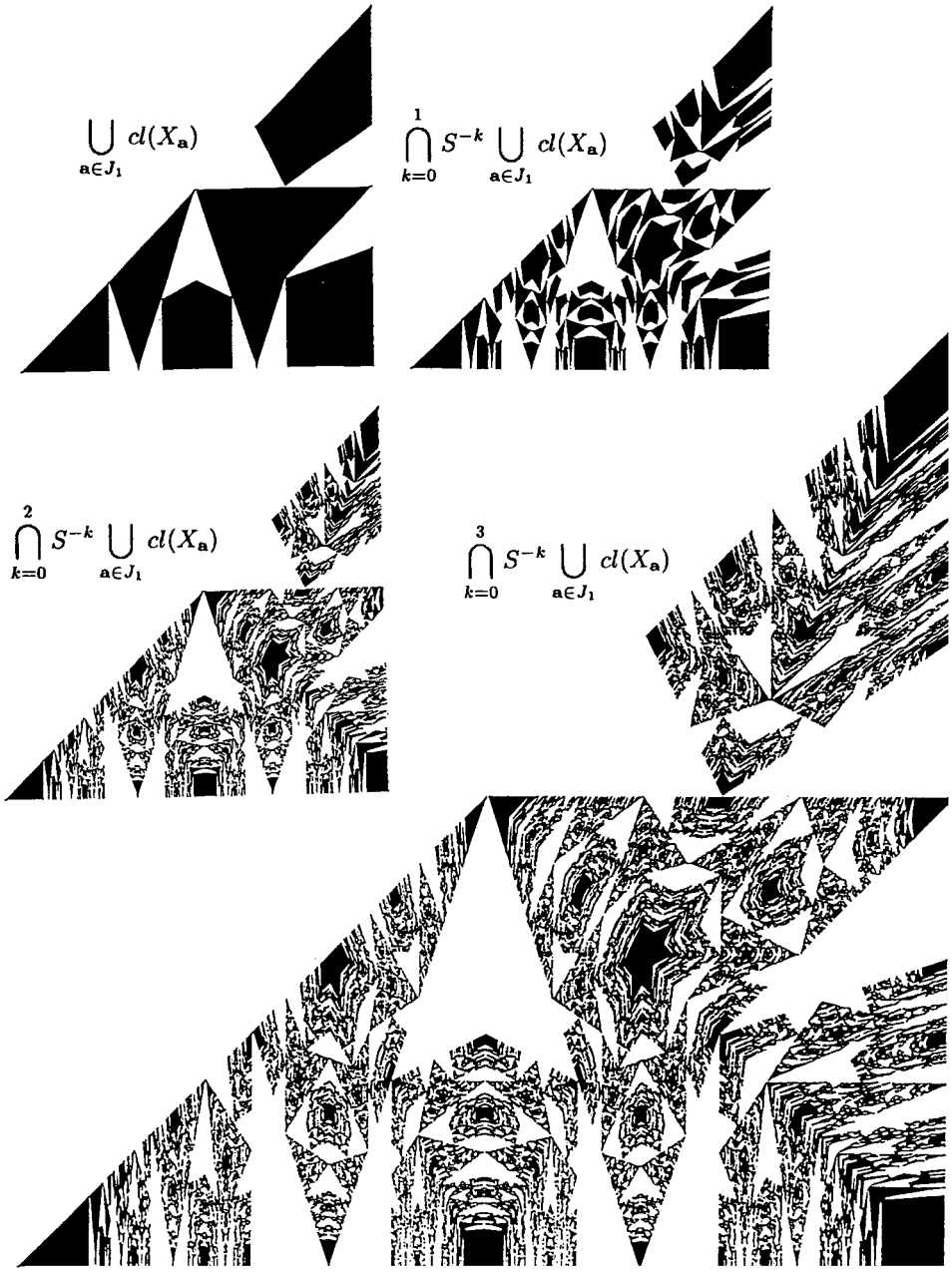


FIGURE 1.

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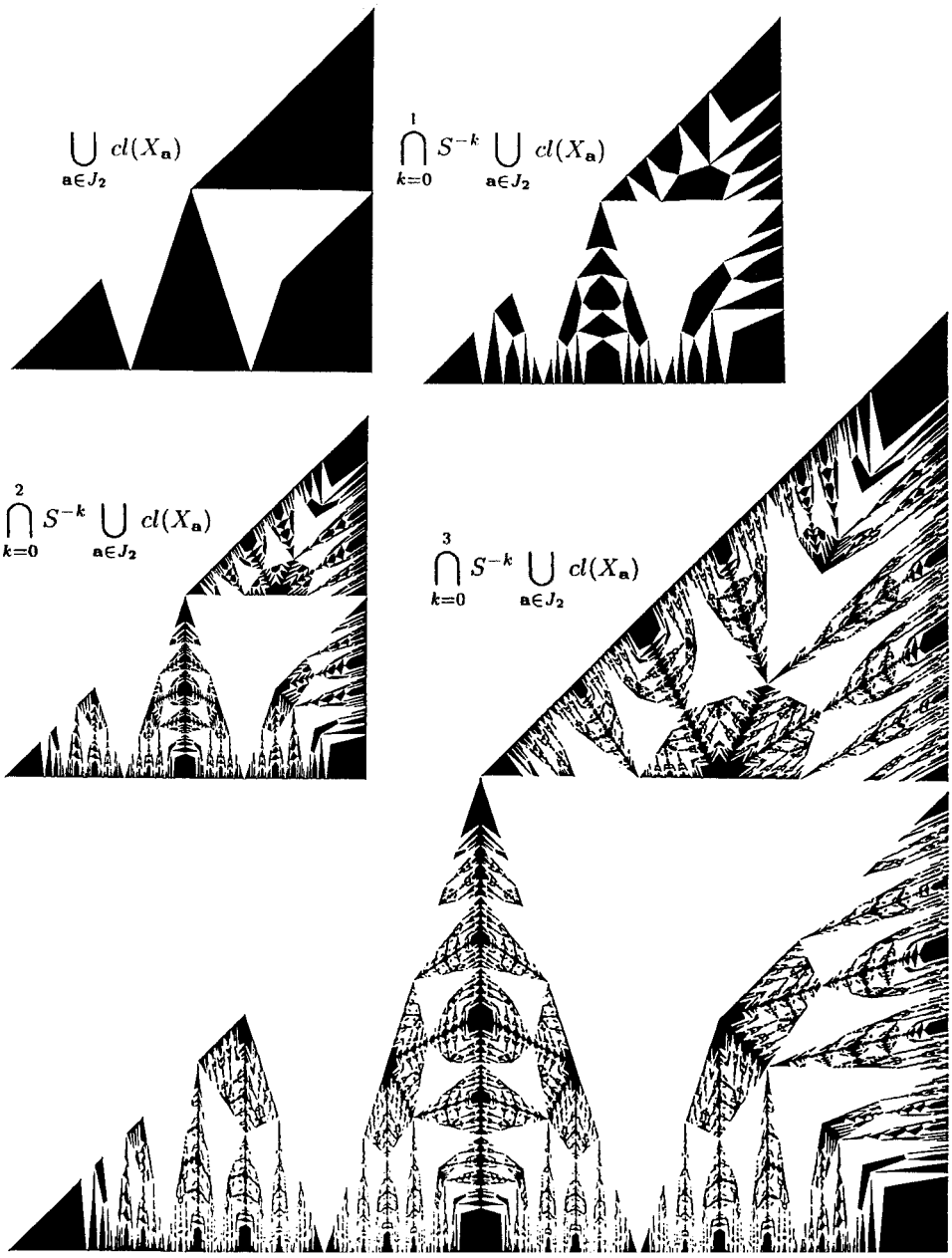


FIGURE 2.

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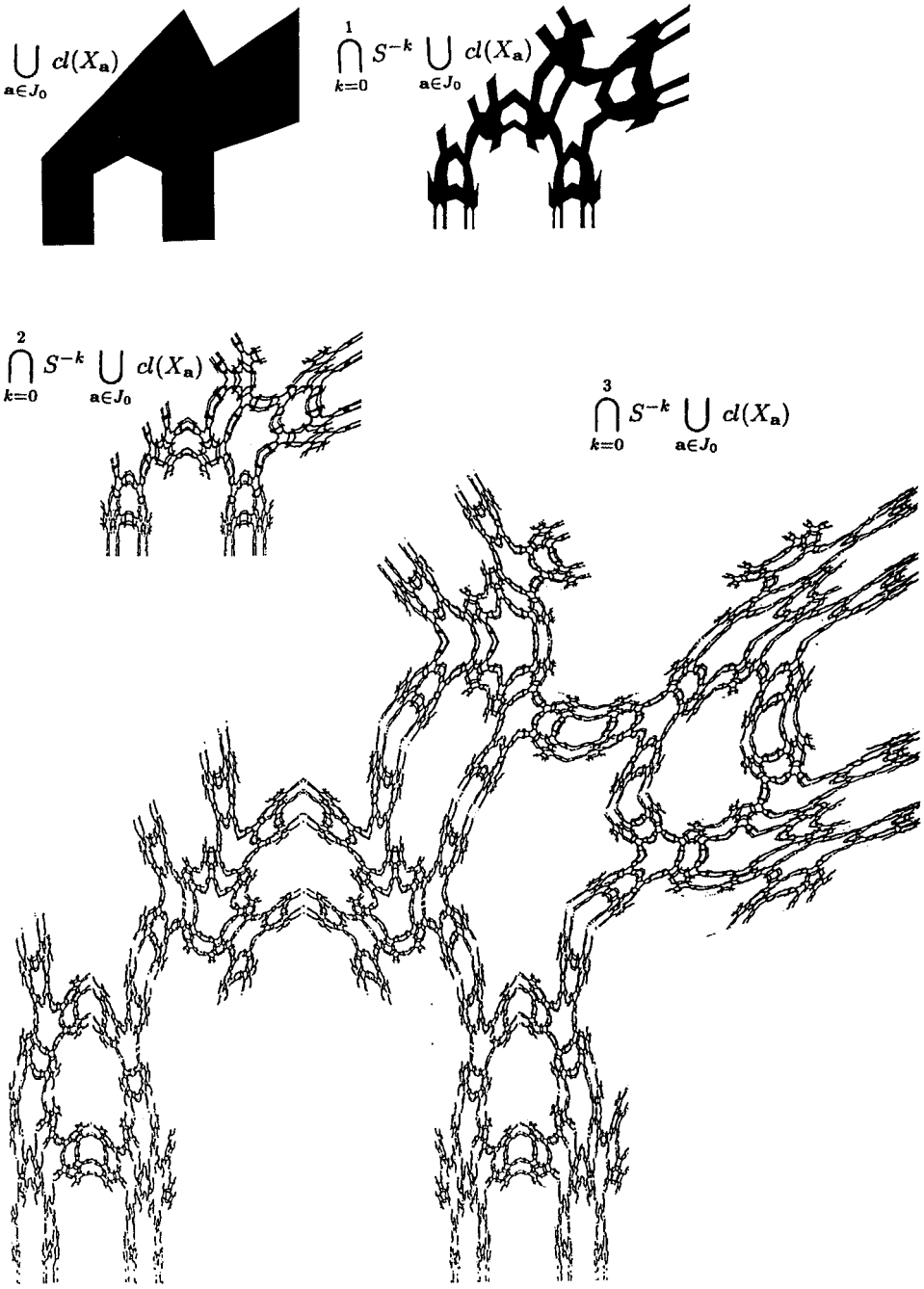


FIGURE 3.

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LEMMA 6.1. *Let us define* $g = (h/hT)(d(\nu T)/d\nu)^{-1}$ *and let* f *be a continuous function on* $\bar{X}$. *Then* $\{\mathcal{L}^n_{\log g} f\}_{n\geq 1}$ *is equicontinuous and uniformly bounded on compact subsets of* $X\setminus(\partial\mathcal{V} \cup \bigcap_{n\geq 0} D_n)$.

Proof of Lemma 6.1 First we note that $\{\log \psi^{*\prime}_{\alpha_1\dots\alpha_n} : (\alpha_1\dots\alpha_n) \in I^{*n}\}$ is equi-Hölder continuous so that $\{\mathcal{L}^n_{\phi^*-R\log p}1\}$ (with respect to T^*) is equi-continuous and uniformly bounded on each $V \in \mathcal{V}$. Then a standard argument allows us to obtain a finite T^*-invariant measure $\mu^* \sim \nu$ of which density $d\mu^*/d\nu$ is Hölder continuous on each $V \in \mathcal{V}$. Next we note that $\forall x, y \in V \subset T^m X_{a(m)}$, $\exists V' \in \mathcal{V}$ such that $\psi_{a(m)}x, \psi_{a(m)}y \in V' \cap X_{a(m)}$ and $\forall X_{d(n)}$ with $T^n X_{d(n)} \supset V' \cap X_{a(m)}$

$$\frac{\psi'_{d(n)}(\psi_{a(m)}y)}{\psi'_{d(n)}(\psi_{a(m)}x)} = \frac{\psi'_{d(n)a(m)}(y)}{\psi'_{d(n)a(m)}(x)}\frac{\psi'_{d(n)}(x)}{\psi'_{d(n)}(y)}.$$

Then the formula (F2) gives

$$|h(\psi_{a(m)}x) - h(\psi_{a(m)}y)| \leq \sum_{n=0}^{\infty} \sum_{\substack{X_{d(n)}\in\mathcal{D}_n,\\ T^n X_{d(n)}\supset X_{a(m)}\cap V'}} \psi'_{d(n)}(\psi_{a(m)}x)\frac{d\mu^*}{d\nu}(\psi_{d(n)a(m)}x) \times \left|1 - \frac{\psi'_{d(n)}(\psi_{a(m)}y)}{\psi'_{d(n)}(\psi_{a(m)}x)}\frac{d\mu^*/d\nu(\psi_{d(n)a(m)}y)}{d\mu^*/d\nu(\psi_{d(n)a(m)}x)}\right|.$$

It follows from (C11) and (C12) and piecewise Hölder continuity of $d\mu^*/d\nu$ that $\forall x, y \in D_k^c \cap V$, $\forall X_{a(m)}$ with $T^m X_{a(m)} \supset V$,

$$\left|1 - \frac{h(\psi_{a(m)}x)}{h(\psi_{a(m)}y)}\right| \leq O(E_k + F_k)d(x, y)^\theta,$$

which enables us to establish equi-continuity and uniformly boundness of $\{\mathcal{L}^n_{\log g} f\}_{n\geq 1}$ on compact subsets of $X\setminus(\partial\mathcal{V} \cup \bigcap_{n\geq 0} D_n)$. □

Proof of Theorem 6.1. In Lemma 6.1 we established [**17**, Lemma 3.3] so that we can apply the same argument as the one in the proof of [**17**, Theorem 3.2]. □

Remark F. We remark that (C0) can be replaced by equi-Hölder continuity of $\{\log \psi^{*\prime}_\alpha : \alpha \in I^*\}$. In [**17**], the uniform convergence theorem was established under (C11) and (C12) and more additional conditions which guarantee equi-Lipschitz continuity of $\{\log \psi^{*\prime}_{\alpha_1\dots\alpha_n} : (\alpha_1\dots\alpha_n) \in I^{*n}\}$. Our proof of Theorem 6.1 simplified some technical arguments in [**17**].

The validity of (C11) and (C12) is verified in [**17**] for all examples in §4 except Example 4 in the case when ν is the normalized Lebesgue measure of X. Next we study the speed of L^1-convergence of $\{\mathcal{L}^n_{-\log(d(\mu T)/d\mu)}\}_{n\geq 1}$. Let $-\log(d(\nu T)/d\nu)$ $(= \phi - \log p)$ be a potential of WBV. Define for $n > m \geq 1$ and for $(x, y) \in X \times X$

$$k_\nu^{(n,m)}(x, y) = \sum_{a_1\dots a_m} \mathcal{L}^{n-m}_{-\log(d(\nu T)/d\nu)}\left(\frac{\psi'_{a_1\dots a_m}}{\nu(X_{a_1\dots a_m})}\right)(x)1_{X_{a_1\dots a_m}}(y).$$

Then we see that

$$\mathcal{L}^n_{-\log(d(\nu T)/d\nu)} E_\nu(f \mid \vee_{i=0}^{m-1} T^{-i}Q)(x) = \int_X k_\nu^{(n,m)}(x, y) f(y)\, d\nu(y).$$

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Definition. We say that x_0 is an *indifferent periodic point with period q with respect to ν* if

$$\frac{d(\nu T^q|_{X_{a_1\dots a_q}(x_0)})}{d\nu|_{X_{a_1\dots a_q}(x_0)}}(x_0)\left(= \exp\left[q\log p - \sum_{i=0}^{q-1}\phi T^i(x_0)\right]\right) = 1.$$

LEMMA 6.2. *Suppose that $\cap_{n\geq 0} D_n$ consists of an indifferent periodic orbit with respect to ν and $\mathcal{U} = \{\operatorname{int} X\}$. Assume further that: $\exists 0 < \pi_1, \pi_2 < 1$ such that for all $X_{a_1\dots a_m} \in \mathcal{D}_m$ and $\forall m \geq 1$,*

$$\frac{\inf_{y\in D_1}\psi'_{a_1\dots a_m}(y)}{\sup_{y\in B_1}\psi'_{a_1\dots a_m}(y)} \geq \pi_1 \quad \textit{and} \quad \inf_{n>m}\frac{\nu(X_{a_1\dots a_m}\cap T^{-n}B_1)}{\nu(X_{a_1\dots a_m}\cap T^{-n}D_1)} \geq \pi_2. \tag{C13}$$

Then there exists $0 < \gamma_1 < 1$ such that $\forall n > m+1 > 1$

$$\mathcal{L}^{n-m}_{-\log(d(\nu T)/d\nu)}\left(\frac{\psi'_{a_1\dots a_m}}{\nu(X_{a_1\dots a_m})}\right)(x) \geq \gamma_1 > 0 \quad (\forall x\in X,\ \forall X_{a_1\dots a_m} \in \vee_{i=0}^{m-1}T^{-i}Q)$$

and $k_\nu^{(n,m)}(x,y) \geq \gamma_1$ $(\forall x, y \in X)$.

Proof. Suppose that $a(m) = b_1\dots b_l d_1\dots d_{m-l}$, where $X_{b_1\dots b_l} \in \mathcal{R}$ and $X_{d_1\dots d_{m-l}} \in \mathcal{D}_{m-l}$. Then it follows from the measure theoretical bounded distortion (see §2) that

$$\mathcal{L}^{n-m}_{-\log(d(\nu T)/d\nu)}\left(\frac{\psi'_{a_1\dots a_m}}{\nu(X_{a_1\dots a_m})}\right)(x) \geq C^{-1}\mathcal{L}^{n-m}_{-\log(d(\nu T)/d\nu)}\left(\frac{\psi'_{d_1\dots d_{m-l}}}{\nu(X_{d_1\dots d_{m-l}})}\right)(x).$$

We assume that $X_{a_1\dots a_m} \in \mathcal{D}_m$. Then the WBV property and (C13) allow us to establish

$$\mathcal{L}^{n-m}_{-\log(d(\nu T)/d\nu)}\left(\frac{\psi'_{a_1\dots a_m}}{\nu(X_{a_1\dots a_m})}\right)(x) \geq C^{-1}C_1^{-1}\pi_1\pi_2\nu(D_1) \quad (\forall x\in X).$$

Hence $\gamma_1 := C^{-2}C_1^{-1}\pi_1\pi_2\nu(D_1)$ gives the desired bound. □

Remark G. Let $\bigcap_{n\geq 0} D_n$ consist of a single indifferent periodic orbit and let each point of the orbit be contained in a unique cylinder. If we assume (C3) and (C7), then by Theorem 3.1 we can show the property

$$\frac{\inf_{X_{a_1\dots a_m}\in\mathcal{D}_m}\inf_{x\in X_{a_1\dots a_m}}h(x)}{\sup_{X_{a_1\dots a_m}\in\mathcal{D}_m}\mu(X_{a_1\dots a_m})/\nu(X_{a_1\dots a_m})} \geq H' > 0$$

which plays an important role with respect to the uniform lower estimate in Lemma 6.2, for establishing Theorem 6.2 by using the method in [**9**] (cf. [**4, 5**]). This property can be readily verified for Examples 1–3.

THEOREM 6.2. (Rates of L^1-convergence of $\mathcal{L}^n_{-\log(d(\mu T)/d\mu)}f$ to $\int f d\mu$) *We assume all conditions in Lemma 6.2, (C3) and (C7). Suppose that*

$$\Delta_1(m) = \sup_{(b_1\dots b_l),l\geq 1}\ \sup_{X_{a_1\dots a_m}\subset D_m^c}\ \sup_{x,y\in X_{a_1\dots a_m}\cap T^l X_{b_1\dots b_l}}\left|1-\frac{\psi'_{b_1\dots b_l}(x)}{\psi'_{b_1\dots b_l}(y)}\right| \to 0 \quad (m\to\infty),$$

$$\Delta_2(m) = \sup_{X_{a_1\dots a_m}\subset D_m^c}\ \sup_{x,y\in X_{a_1\dots a_m}}\left|1-\frac{h(x)}{h(y)}\right| \to 0 \quad (m\to\infty).$$

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Then for any $f \in L^\infty(\mu)$ *such that* $\exists 0 < L_f < \infty$ *satisfying*

$$\sup_{X_{a_1\dots a_m} \subset D_m^c} \sup_{x,y \in X_{a_1\dots a_m}} |f(x) - f(y)| \le L_f \sigma(m) \quad (\forall m > 0)$$

and $\forall 0 < \epsilon < 1$,

$$\left\| \mathcal{L}^n_{-\log(d(\mu T)/d\mu)} f - \int f \, d\mu \right\|_1 \le O(\max\{\mu(D_{[n^\epsilon]}), \sigma([n^\epsilon]), \Delta_1([n^\epsilon]), \Delta_2([n^\epsilon])\}).$$

(Cf. [**9**, Theorem 1], [**22**].)

Remark H. Under Hölder continuity of $d\mu^*/d\nu$ (with exponent θ), $\Delta_2(m)$ can be bounded from above by $O(\Delta_1(m)) + O(\sigma(m)^\theta)$.

Appendix. Jump transformations

In previous sections, we have established approachs to study non-hyperbolic phenomena caused by indifferent periodic points by using jump transformations. In this section, we shall develop these approachs for piecewise invertible systems with FRS in a more general setting. Let a Borel probability measure ν and a positive number $p > 0$ satisfy $\mathcal{L}^*_\phi \nu = p\nu$. $\mathcal{A}$ denotes the set of all admissible cylinders $\bigcup_{n=1}^\infty \{X_{a_1\dots a_n} \in \bigvee_{i=0}^{n-1} T^{-i} Q : \nu(X_{a_1\dots a_n}) > 0\}$. Suppose that $\exists \mathcal{R} \subset \mathcal{A}$ satisfying the following conditions.

(a) $X_{a(n)b(m)} \in \mathcal{R}$ whenever $X_{b(m)} \in \mathcal{R}$ (*the strong playback property*).

(b) Define $\mathcal{D}_n = \{X_{a_1\dots a_n} : X_{a_1\dots a_i} \notin \mathcal{R} \ (i = 1 \dots n)\}$ and $\mathcal{B}_n = \{X_{a_1\dots a_n} : X_{a_1\dots a_{n-1}} \in \mathcal{D}_{n-1}, X_{a_1\dots a_n} \in \mathcal{R}\}$ inductively. We define T^* and denote the local inverses ψ^*_α, $\psi^*_{\alpha_1\dots\alpha_n}$ as in §2 and define $\sigma^*(n)$ as in §5. Then there exist $0 < \Gamma < \infty$, $\gamma^* > 1$ and $n_0 > 0$ such that

$$d(\psi^*_{\alpha_1\dots\alpha_n} x, \psi^*_{\alpha_1\dots\alpha_n} y) \le \Gamma \gamma^{*-n} d(x, y) \quad (\forall n \ge n_0)$$

and $\sum_{i=1}^{n_0} \sigma^*(i) < \infty$.

We impose the next condition on the potential ϕ which implies equi-Hölder continuity of $\{\phi^* \psi^*_\alpha\}_{\alpha \in I^*}$.

(c) $\exists 0 < L^* < \infty$, $\exists \theta \in \mathbb{R}$ such that $\forall X_\alpha \in Q^*$,

$$|\phi^*(x) - \phi^*(y)| \le L^* d(T^* x, T^* y)^\theta \quad (\forall x, y \in X_\alpha \in Q^*).$$

Then it follows from equi-Hölder continuity of $\{\log \psi^{*\prime}_{\alpha_1\dots\alpha_n} : (\alpha_1 \dots \alpha_n) \in I^{*n}\}$ that the measure theoretical bounded distortion is valid for ϕ^* associated to T^*. In particular, if $n_0 = 0$ then $\{\mathcal{L}^n_{\phi^* - R\log p}\}$ (with respect to T^*) is piecewise equi-continuous with respect to $\mathcal{V}$ and uniformly bounded. Then we have a T^*-invariant measure μ^* of which density is bounded and piecewise Hölder continuous with respect to $\mathcal{V}$. Furthermore, if each $U \in \mathcal{U}$ contains a cylinder $\operatorname{int} X_{a_1\dots a_s} \subset U$ such that $T^s(\operatorname{int} X_{a_1\dots a_s}) = \operatorname{int} X$ and $X_{a_s} \in \mathcal{R}$, then the density is bounded away from zero and μ^* is a Gibbs measure for $\phi^* - R \log p$ which is exact. If $\lim_{n\to\infty} \nu(D_n) = 0$, then $\nu(X) = \nu(X^*)$. We say that ϕ is of WBV with respect to ν if ϕ satisfies the inequality in the definition of WBV in terms of ess sup (ess inf) with respect to ν.

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THEOREM A.1 *Suppose that $\mathcal{L}^*_\phi \nu = p\nu$ $(p > 0)$. Assume that we can construct a jump transformation T^* under (a)–(c) for which $\phi^* - R\log p$ satisfies the measure theoretical bounded distortion and $\sum_{n=0}^{\infty} \nu(D_n) < \infty$. If $\frac{1}{n}\sum_{i=0}^{n-1} R \cdot T^{*i}$ converges uniformly on a full measure set and $\inf_{x\in D_1} \phi > -\infty$, then ϕ is a potential of WBV with respect to ν and ν is a weak Gibbs measure for ϕ with $-\log p$.*

LEMMA A.1 *Suppose that $\mathcal{L}^*_{\phi-\log p}\nu = \nu$. Then ϕ is a potential of WBV with respect to ν iff ν is a weak Gibbs measure for ϕ with $-\log p$.*

LEMMA A.2 *Suppose that $\mathcal{L}^*_{\phi^*-R\log p}\nu = \nu$ (with respect to T^*). Then $\phi^* - R\log p$ satisfies the measure theoretical bounded distortion iff ν is a Gibbs measure for $\phi^* - R\log p$ with $P(\phi^* - R\log p) = 0$.*

COROLLARY A.1 *Suppose that all conditions in Theorem A.1 are satisfied. Then ν is Gibbs for $\phi^* - R\log p$ with respect to T^* and weak Gibbs for ϕ with $\log p$ with respect to T.*

Remark I. Examples (1–4) satisfy $\inf_{x\in D_1} \phi > -\infty$ for $\phi = -\log|\det DT|$.

Proof of Lemma A.1 Suppose that ϕ is a potential of WBV. Then the weak Gibbs property of ν was established in [**20**]. Conversely, suppose that ν is a weak Gibbs measure for ϕ with $-\log p$. Then $\exists\{K_n\}_{n\geq 1}$ satisfying $\lim_{n\to\infty} n^{-1}\log K_n = 0$ such that for each $n > 0$ and for all $X_{a_1\dots a_n}$

$$K_n^{-1} \leq \frac{\nu(X_{a_1\dots a_n})}{\exp\left(\sum_{i=0}^{n-1}\phi T^i(x) - n\log p\right)} \leq K_n \quad (\forall x \in X_{a_1\dots a_n})$$

and so $\forall x, y \in X_{a_1\dots a_n}$,

$$K_n^{-1}\exp\left(\sum_{i=0}^{n-1}\phi T^i(x) - n\log p\right) \leq \nu(X_{a_1\dots a_n}) \leq \exp\left(\sum_{i=0}^{n-1}\phi T^i(y) - n\log p\right)K_n.$$

Hence the WBV property is satisfied. □

Proof Lemma A.2 It immediately follows from the proof of Lemma A.1. □

Proof of Theorem A.1. Let $N_n(x) = \max\{l \geq 0 : \sum_{i=0}^{l} R(T^{*i}x) \leq n\}$. Then for $x \in X_{a_1\dots a_n}$

$$\begin{aligned}
&\frac{\exp\left(\sum_{i=0}^{n-1}\phi T^i(x) - n\log p\right)}{\exp\left(\sum_{i=0}^{N_n(x)}\phi^* T^{*i}(x) - (\log p)\sum_{i=0}^{N_n(x)} R(T^{*i}x)\right)} \\
&\quad = \exp\left(\sum_{i=\sum_{j=0}^{N_n(x)} R(T^{*j}x)}^{n-1} \phi T^i(x) - \left(n - \sum_{i=0}^{N_n(x)} R(T^{*i}x)\right)\log p\right) \\
&\quad \leq \exp\left(\left(n - \sum_{j=0}^{N_n(x)} R(T^{*j}x)\right)\left(\sup_X \phi - \log p\right)\right)
\end{aligned}$$

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and

$$\frac{\exp\bigl(\sum_{i=0}^{n-1}\phi T^i(x)-n\log p\bigr)}{\exp\bigl(\sum_{i=0}^{N_n(x)}\phi^* T^{*i}(x)-(\log p)\sum_{i=0}^{N_n(x)}R(T^{*i}x)\bigr)} \geq \exp\left(\left(n-\sum_{j=0}^{N_n(x)}R(T^{*j}x)\right)\left(\inf_{D_1}\phi-\log p\right)\right).$$

Since we have a T^*-invariant ergodic finite measure μ^*, μ^*-a.e. x

$$\lim_{n\to\infty}\frac{\sum_{j=0}^{N_n(x)-1}R(T^{*j}x)}{N_n(x)}=\frac{\int_{X^*}R(x)\,d\mu^*(x)}{\mu^*(X^*)}=\mu^*(X^*)^{-1}$$

and $\lim_{n\to\infty}N_n(x)/n=\mu^*(X^*)$. Additionally it follows from equi-Hölder continuity of $\{\log\psi^{*\prime}_{\alpha_1\dots\alpha_n}:(\alpha_1\dots\alpha_n)\in I^{*n}\}$ that [**21**, Theorems 2.2 and 2.3] allow us to have a sequence Γ_n satisfying $\Gamma_n\to 0$ as $n\to\infty$, which can be used to deduce the rate of uniform convergence a.e. in the Birkhoff ergodic theorem [**9**, p. 3]. Finally putting $\bar{C}_n=\exp(nO(\Gamma_n)|\sup\phi-\log p|)$ gives

$$\frac{\exp\bigl(\sum_{i=0}^{n-1}\phi T^i(x)-n\log p\bigr)}{\exp\bigl(\sum_{i=0}^{N_n(x)}\phi^* T^{*i}(x)-(\log p)\sum_{i=0}^{N_n(x)}R(T^{*i}x)\bigr)}\leq\bar{C}_n$$

and $\lim_{n\to\infty}1/n\log\bar{C}_n=0$. Similarly putting $\underline{C_n}=\exp(nO(\Gamma_n)|\inf_{D_1}\phi-\log p|)$ gives

$$\frac{\exp\bigl(\sum_{i=0}^{n-1}\phi T^i(x)-n\log p\bigr)}{\exp\bigl(\sum_{i=0}^{N_n(x)}\phi^* T^{*i}(x)-(\log p)\sum_{i=0}^{N_n(x)}R(T^{*i}x)\bigr)}\geq\underline{C_n}$$

and $\lim_{n\to\infty}1/n\log\underline{C_n}=0$. Then the measure theoretical bounded distortion for $\phi^*-R\log p$ allows one to have the WBV property of $\phi-\log p$. Then applying Lemma A.1 gives the desired result. □

Even if $\operatorname{diam}X<\infty$ fails, we can still construct a jump transformation T^* which satisfies (a)–(c). For example, let us consider a countable aperiodic Markov shift T and let ϕ be a continuous potential with $\sum_{n=2}^{\infty}V_n(\phi)<\infty$ which admits a Borel probability measure ν and $p>0$ satisfying $\mathcal{L}^*_\phi\nu=p\nu$ (cf. [**10**]). Suppose that there exists a cylinder X_b with $\operatorname{diam}TX_b<\infty$ such that $\exists L_b(\phi)<\infty$ and $\theta\in\mathbb{R}$ satisfying

$$|\phi(\psi_b x)-\phi(\psi_b y)|\leq L_b(\phi)d(x,y)^\theta\quad(\forall x,y\in TX_b)$$

and $I_b=\{a\in I:X_a\subset TX_b\}$ is a finite set. Under the above situation, we have no evidence of the measure theoretical bounded distortion with respect to ν. For such a T, we shall construct a jump transformation satisfying (a)–(c) as follows. Let $\mathcal{R}$ be the set of all cylinders with last symbol b so that (a) is valid. Then $\mathcal{B}_1=\{X_b\}$, $\mathcal{D}_1=\{X_d:d\neq b\}$ and $\mathcal{D}_n=\{X_{d_1\dots d_n}:d_i\neq b\ (i=1\dots n)\}$, $\mathcal{B}_n=\{X_{d_1\dots d_{n-1}b}:X_{d_1\dots d_{n-1}}\in\mathcal{D}_{n-1}\}$. T^* and ϕ^* can be defined as in the §2 and since for $X_{a_1\dots a_{n-1}b}\in\bigvee_{i=0}^{n-1}T^iQ$,

$$\left|\sum_{i=0}^{n-1}(\phi T^i(\psi_{a_1\dots a_{n-1}b}x)-\phi T^i(\psi_{a_1\dots a_{n-1}b}y))\right|\leq L_b(\phi)\operatorname{diam}TX_b+\sum_{n=2}^{\infty}V_n(\phi),$$

(c) can be readily verified.

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