Since $f \neq 0$, there is a point $x$ such that $f(x) = 1$. Thus $X$ has a non-empty subset $B$ with the following property: $B$ consists of linearly independent vectors, and $f(x) = 1$ for all $x \in B$. By Zorn's lemma, we may assume that $B$ is maximal with respect to this property. With this assumption, let $x_0 \in B$ and let $x$ be any point of $X$. Then $x$ is in the subspace generated by $B$: for, if it were not we could enlarge $B$ by adjoining $x + x_0$ if $f(x) = 0$ or $[f(x)]^{-1}x$ if $f(x) \neq 0$. Thus $B$ is a basis of $X$.

Now let

$$\|x\| = \sup_{1 \leqslant i \leqslant n} |\alpha_i|,$$

where $x = \alpha_1 x_1 + \ldots + \alpha_n x_n$ and $x_1, \ldots, x_n$ are distinct elements of $B$. This defines a norm for $X$. By hypothesis, there is a positive number $k$ such that $|f(x)| \leqslant k\|x\|$ for all $x \in X$. Taking

$$x = x_1 + \ldots + x_n,$$

with $x_1, \ldots, x_n$ in $B$, we find that $n \leqslant k$. Thus the number of elements in $B$ cannot exceed $k$, so that $X$ is finite-dimensional.

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# A NOTE ON THE RELATION BETWEEN SCATTERING PHASE AND BOUND STATES

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*ABSTRACT.* The form of Levinson's relation is obtained for the case where the free Hamiltonian contains discrete eigenstates. The relevance of this to the ambiguous solutions of Low equations and dispersion relations is discussed.

1. Levinson (5) has shown that the scattering phase $\delta(k)$ for $S$-states in a spherically symmetric potential $V$ satisfying a certain regularity condition is related to $n$, the number of bound $S$-states, by the equation

$$\delta(0) - \delta(\infty) = n\pi. \tag{1}$$

Jauch (4) has extended the result to elastic scattering by an arbitrary potential satisfying a given condition on the behaviour of the scattering wave functions at infinite energy. His argument uses the orthogonality and completeness relations for the eigenstates of the total Hamiltonian. Both authors consider the case where the total Hamiltonian, $H = H_0 + V$, and the free Hamiltonian, $H_0$, possess an identical continuous spectrum with eigenkets $|k^+\rangle$ and $|k\rangle$, respectively. In addition $H$ possesses certain discrete bound states $|\alpha\rangle$. In this note we shall also consider the possibility that $H_0$ possesses discrete eigenstates $|\beta\rangle$.

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The wave operator, $\Omega$, is defined by

$$\Omega = \sum_k |k^+\rangle\langle k|, \tag{2}$$

the summation sign symbolizing an integration over the continuous spectrum. $\Omega$ satisfies

$$\Omega^*\Omega = 1 - \sum_\beta |\beta\rangle\langle\beta|, \tag{3}$$

$$\Omega\Omega^* = 1 - \sum_\alpha |\alpha\rangle\langle\alpha|, \tag{4}$$

the sum in (3) being over the discrete eigenstates of $H_0$, and in (4) over the discrete eigenstates of $H$. Therefore

$$\mathrm{Tr}\,[\Omega^*, \Omega] = n - m, \tag{5}$$

where $n$ is the number of bound states of $H$ and $m$ the number of bound states of $H_0$. It follows immediately from the arguments given by Jauch (4) that

$$\delta(0) - \delta(\infty) = (n-m)\,\pi. \tag{6}$$

2. The derivation of the generalized Levinson relation (6) has been quite trivial. Its interest lies in the connexion of a relation of this type with the manifold of solutions of Low equations and dispersion relations. Haag (3) has pointed out that the solutions of a simple Low equation satisfy a relation (6), where $m$ is the number of poles in the generalized $R$-function generating the solution. Recently Fairlie and the author (1) have given a direct interpretation of the significance of the generalized $R$-function in terms of bound states of the free Hamiltonian. The relation (6) can then be considered as another aspect of this connexion.

It also provides a link between the ambiguities of solutions of Low equations and those of dispersion relations. Gasiorowicz and Ruderman (2) have recently exhibited different solutions of the same dispersion relation for scattering by a given potential. They satisfy relations of type (6). Presumably $m$ is again to be interpreted as relating to the 'hidden structure' of $m$ bound states of the free Hamiltonian.

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