Ergod. Th. & Dynam. Sys. (2001), **21**, 1411–1427
Printed in the United Kingdom © *2001 Cambridge University Press*

Chaos for functions of discrete and continuous weighted shift operators

R. DELAUBENFELS† and H. EMAMIRAD‡
† *Scientia Research Institute, 22000 Jordan Run Road, Guysville, Ohio 45735, USA*
(e-mail: scientia@frognet.net)
‡ *Université de Poitiers, Laboratoire de Modélisation Mécanique et de Mathématiques Appliquées, SP2MI, BP 30179, F 86962 Futuroscope Cedex, France*
(e-mail: emamirad@l3ma.univ-poitiers.fr)

(*Received* 5 *January* 2000 *and accepted in revised form* 20 *September* 2000)

Abstract. For L equal to the unilateral or bilateral shift on a weighted sequence space, we characterize, in terms of f and the weight function, those f holomorphic on the spectrum of L for which $f(L)$ is a chaotic operator. For B equal to d/dx, the generator of left translation, on weighted L^p spaces on $[0, \infty)$ or $\mathbb{R}$, we similarly characterize those polynomials Q for which the differential operator $Q(B)$ generates a chaotic semigroup.

1. *Introduction and preliminaries*

Throughout this paper, all operators are linear, on a Banach space X. The operator $T \in B(X)$, the space of bounded operators from X to itself, is said to be *hypercyclic* if there exists $x \in X$ whose orbit $\{T^n x \mid n - 1 \in \mathbb{N}\}$ is dense in X. Note that this implies the existence of a dense set of vectors with dense orbits. A *periodic point* for T is a vector x such that $T^n x = x$ for some $n \in \mathbb{N}$. Finally, T is said to be *chaotic* if T is hypercyclic and there exists a dense set of periodic points for T.

Note that, since T is linear, being hypercyclic is equivalent to the more familiar condition of being topologically transitive: for any pair of non-empty open sets U and V, there exists a natural number n so that $T^n(U) \cap V$ is non-empty ([**5**, Theorem 1.2]; for a continuous analogue, see [**3**, Theorem 2.2]). The definition of chaotic given here also implies a sensitive dependence on initial conditions; in [**5**, Proposition 6.1], it is shown that then $\{(T^n x - T^n y) \mid n - 1 \in \mathbb{N}\}$ is dense in X, for all x, y in a dense set; see also [**1**]. Thus the definition of chaos given here is equivalent, for T linear, to that given in [**4**, Definition 8.5, p. 50].

Similarly, the strongly continuous semigroup of operators $\{T(t)\}_{t \geq 0}$ is *hypercyclic* if there exists $x \in X$ such that the orbit $\{T(t)x \mid t \geq 0\}$ is dense in X, and is *chaotic* if, in addition, there is a dense set of *periodic points*: x such that $T(t)x = x$ for some $t > 0$.

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

Let $X := \ell^p(\mathbb{N}, \vec{\beta})$ $(1 \leq p < \infty)$, the weighted sequence space with the weight vector $\vec{\beta}$, equipped with the norm:

$$\|\vec{x}\|^p := \sum_{k=1}^{\infty} |x(k)|^p \beta(k).$$

We choose the weight vector $\vec{\beta}$ in such a way that the left shift L on X defined by

$$(L\vec{x})(k) := x(k+1) \quad (k \in \mathbb{N}).$$

becomes a bounded operator on X. For f holomorphic on a neighborhood of the spectrum of L, let $f(L)$ be defined by the usual Riesz functional calculus. Let D be the open unit disc in the complex plane and R the radius of convergence of the power series

$$z \mapsto \sum_{k=1}^{\infty} z^k \beta(k).$$

Then we show (Theorem 2.3) that, for f non-constant, when $\sum_{k=1}^{\infty} R^k \beta(k) = \infty$ and $R > 0$, then $f(L)$ is a chaotic operator if and only if $f(R^{1/p} D)$ intersects the unit circle ∂D if and only if $f(L)$ has a non-trivial periodic point. To characterize $g(L)$ generating a chaotic semigroup (Theorem 2.13), replace ∂D with $i\mathbb{R}$; this is immediately seen by choosing $f(z) := e^{tg(z)}$. For f or g an affine function, even simpler characterizations are given. When g is a polynomial or f is a polynomial other than a complex multiple of $z \mapsto z^n$, then the first equivalence holds without the hypothesis of divergence on the boundary of the disc of convergence (Lemma 2.7 and Theorems 2.8 and 2.14).

§3 considers the left shift on the bilateral weighted sequence space $\ell^p(\mathbb{Z}, \vec{\beta})$; the results are the same as in §2, with discs replaced by annuli (since a Laurent series replaces a power series).

In §§4 and 5 we characterize differential operators $Q(d/dx)$, Q a polynomial, that generate chaotic semigroups on weighted L^p $(1 \leq p < \infty)$ spaces. On $L^p([0, \infty), \rho)$, when we have divergence on the boundary of the half-plane of convergence of the Laplace transform of the weight function, a strongly continuous semigroup generated by $Q(B)$ is chaotic if and only if the image, under Q, of an appropriate half-plane intersects the imaginary axis if and only if the semigroup has a non-trivial periodic point (Theorem 4.6). On $L^p(\mathbb{R}, \rho)$, the result is the same, with the half-plane replaced by a vertical strip (Theorem 5.1). For first-order differential operators we have simpler characterizations (Corollaries 4.7 and 5.2).

In the following we show that many results from [**3, 5, 12–14**] are immediate special cases of the results in this paper.

We will use the following very useful result.

LEMMA 1.1. [**5**, Corollary 1.5] *Suppose $A \in B(X)$, Y_1, Y_2 are dense subsets of X and $Z : Y_1 \to Y_1$ such that:*

(1) $AZy = y, \forall y \in Y_1$;

(2) $\lim_{n\to\infty} Z^n y = 0, \forall y \in Y_1$; *and*

(3) $\lim_{n\to\infty} A^n w = 0, \forall w \in Y_2$.

Then A is hypercyclic.

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

In many of the classes of operators considered in this paper, an operator T (not a multiple of the identity) is chaotic if and only if it has a non-trivial periodic point. A similar result is obtained in [**10**] for the unilateral backward shift on Köthe echelon spaces and is generalized by Grosse-Erdmann [**8**] for an arbitrary Fréchet-sequence space in which canonical unit vectors e_n ($e_n(z) = z^n$) form an unconditional basis. Given all that chaos means, this is quite surprising. This result also has immense practical value, because finding a (single non-trivial) periodic point seems so much easier than demonstrating all the properties of chaos directly.

What is also surprising in our results is the abundance of chaos, in such a standard, linear setting. For example, we show (Corollary 5.6) that, if there exists real μ so that $\int_{-\infty}^{\infty} e^{\mu s}\rho(s)\,ds$ is finite, then d^2/dx^2, on $L^p(\mathbb{R}, \rho)$, generates a strongly continuous chaotic semigroup. Thus we have numerous chaotic diffusion semigroups, under the minimal, clearly necessary hypothesis that d/dx has an eigenvalue.

We also show (Corollary 2.16 and Remarks 3.3(iv)) that, if L has an eigenvalue, then for *any* non-constant f holomorphic on $\sigma(L)$, there exists a positive constant c such that $cf(L)$ is chaotic. Similarly, if d/dx has an eigenvalue, then for any strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ generated by a differential operator $Q(d/dx)$ (Q a polynomial) there exists a real constant c such that $\{e^{ct}T(t)\}_{t\geq 0}$ is a chaotic semigroup (Corollaries 4.10 and 5.5).

Thus, in considering the algebra of operators

$$\{f(L) \mid f \text{ is holomorphic in a neighborhood of the spectrum of } L\},$$

it is natural to wonder how typical the behavior we have seen is. Specifically, for what algebras of operators $\mathcal{A}$ is the following true?

CONJECTURE. *Suppose $\mathcal{A}$ is a commutative algebra of operators that contains a chaotic operator. Let $T \in \mathcal{A}$ and T is not a multiple of the identity. Then there exists a constant $c > 0$ such that cT is chaotic.*

Given the results of Theorems 2.3, 3.2, 4.4 and 5.1, it is also natural to ask the following.

Open question. For what algebras of operators $\mathcal{A}$ is it the case that $T \in \mathcal{A}$ is chaotic if and only if T has a non-trivial periodic point?

Throughout this paper, $\sigma(B)$ is the spectrum of the operator B and $P\sigma(B)$ is the point spectrum of B.

2. *Discrete weighted one-side shifts*

Throughout this section, we will denote by $\ell^p(\mathbb{N}, \vec{\beta})$ ($1 \leq p < \infty$) the Banach space X of complex-valued sequences $\vec{x} := \{x(k)\}_{k=1}^{\infty}$ such that

$$\|\vec{x}\|^p := \sum_{k=1}^{\infty} |x(k)|^p \beta(k)$$

is finite. Here the sequence $\vec{\beta}$ satisfies

$$\sup_{k\geq 1} \frac{\beta(k)}{\beta(k+1)} < \infty. \tag{2.0}$$

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

We will denote by $\ell^p(\mathbb{N})$, $\ell^p(\mathbb{N}, \vec{\beta})$ with $\beta(k) := 1$.

Denote by L the one-side left shift on $\ell^p(\mathbb{N}, \vec{\beta})$

$$(L\vec{x})(k) := x(k+1) \quad (k \in \mathbb{N}).$$

Note that (2.0) is equivalent to L being bounded; the supremum in (2.0) is $\|L\|^p$.

Let D denote the *open* unit disc in the complex plane, ∂D the unit circle, and let f be holomorphic in rD, for some $r > \|L\|$, where $f(L)$ is defined by the usual Riesz functional calculus

$$f(L) := \frac{1}{2\pi i} \int_{|z|=r} f(z)(z-L)^{-1}\, dz.$$

We also denote by

$$\vec{h}_\mu := (\mu, \mu^2, \mu^3, \dots)$$

the eigenvector for L corresponding to the eigenvalue μ.

Let

$$R := \underline{\lim}_{k\to\infty} (\beta(k))^{-1/k}$$

be the radius of convergence of $z \mapsto \sum_{k=1}^\infty \beta(k) z^k$.

Note that $\vec{h}_\mu \in \ell^p(\mathbb{N}, \vec{\beta})$ if and only if $\sum_{k=1}^\infty \beta(k)|\mu|^k < \infty$. Thus

$$P\sigma(L) = \begin{cases} R^{1/p} D, & \text{if } \sum_{k=1}^\infty \beta(k) R^k = \infty \\ R^{1/p} \overline{D}, & \text{if } \sum_{k=1}^\infty \beta(k) R^k < \infty \end{cases}$$

LEMMA 2.1. *Suppose that the open set $\Omega \subseteq R^{1/p} D$ has an accumulation point. Then*

$$\mathcal{A}_\mu := \text{span}\{\vec{h}_\mu \mid \mu \in \Omega\}$$

is dense in $\ell^p(\mathbb{N}, \vec{\beta})$.

Proof. Suppose $\vec{\phi} \in \mathcal{A}_\mu^\perp$. Then

$$g(\mu) := \langle \vec{\phi}, \vec{h}_\mu \rangle := \sum_{k=1}^\infty \phi_k \mu^k \quad (\mu \in R^{1/p} D)$$

is an analytic function in $R^{1/p} D$. Since $g(\mu) = 0$ for all $\mu \in \Omega$, which is a set with an accumulation point, $g := 0$, so that $\vec{\phi} = \vec{0}$, as desired. □

LEMMA 2.2. *Suppose f is holomorphic and non-constant in a neighborhood of $\|L\|\overline{D}$ and $R > 0$.*

(1) *If $f(R^{1/p} D) \cap \partial D$ is non-empty, then $f(L)$ is chaotic.*
(2) *If $f(L)$ is chaotic, then $f(P\sigma(L)) \cap \partial D$ is infinite.*
(3) *If $f(L)$ generates a chaotic semigroup, then $f(P\sigma(L)) \cap i\mathbb{R}$ is infinite.*

Proof. (1) Let

$$\Omega_1 := \{\mu \in R^{1/p} D \mid |f(\mu)| > 1\}, \quad \Omega_2 := \{\mu \in R^{1/p} D \mid |f(\mu)| < 1\},$$
$$\Omega_3 := \{\mu \in R^{1/p} D \mid f(\mu) \in \exp(2\pi i \mathbb{Q})\}.$$

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

Suppose $z_0 \in f(R^{1/p}D) \cap \partial D$. Since f is holomorphic, $f(R^{1/p}D)$ is an open set. It follows that an open ball containing z_0 is contained in $f(R^{1/p}D)$. Thus Ω_1 and Ω_2 are open, non-empty subsets of $R^{1/p}D$, and $\{\mu \in R^{1/p}D \mid |f(\mu)| = 1\}$ includes f^{-1} of an interval on the unit circle. This implies that, for $j = 1, 2, 3$, Ω_j has an accumulation point.

For $j = 1, 2, 3$, let

$$Y_j := \text{span}\{\vec{h}_\mu \mid \mu \in \Omega_j\}.$$

By Lemma 2.1, each of Y_1, Y_2 and Y_3 is dense in X.

Let $A := f(L)$, and define $Z := (f(L))^{-1}$ on Y_1; that is,

$$Z\left(\sum_{k=1}^{N} \alpha_k \vec{h}_{\mu_k}\right) := \sum_{k=1}^{N} \alpha_k (f(\mu_k))^{-1} \vec{h}_{\mu_k},$$

for $\mu_k \in \Omega_1, \alpha_k \in \mathbb{C}$.

It is clear that we have the hypotheses of Lemma 1.1, so that the hypercyclicity of A follows.

To see that Y_3 is contained in the set of all periodic points of $f(L)$, suppose, for $\alpha_k \in \mathbb{C}, n_k, m_k \in \mathbb{Z}$, $f(\mu_k) = \exp(2\pi i n_k/m_k)$,

$$y := \sum_{k=1}^{N} \alpha_k \vec{h}_{\mu_k};$$

then, if $M := \prod_{k=1}^{N} m_k$,

$$[f(L)]^M y = y,$$

as desired.

(2) Let $\mathcal{P}$ be the set of all periods for $f(L)$; that is, $n \in \mathcal{P}$ if and only if there exists non-trivial $x \in X$ such that n is the first positive integer such that $[f(L)]^n x = x$.

We claim that $\mathcal{P}$ must be infinite. To see this, suppose for the sake of contradiction that $\mathcal{P} = \{n_j\}_{j=1}^{N}$. Let $M \equiv \prod_{j=1}^{N} n_j$; then $[f(L)]^M x = x$ for all periodic points x; since the set of all periodic points is dense, this holds for all $x \in X$. This contradicts the hypercyclicity of $f(L)$, proving the claim.

Fix $n \in \mathcal{P}$ and let x be as in the definition of $\mathcal{P}$. Then

$$[(z \mapsto (f(z))^n)(L)]x = x,$$

thus $1 \in P\sigma([(z \mapsto (f(z))^n)(L)])$. By point-spectral mapping theorems [**9**, Theorem 5.12.2], there exists $\mu \in P\sigma(L)$ such that $[f(\mu)]^n = 1$, so that

$$f(\mu) = \exp\left(\frac{2\pi i k}{n}\right)$$

for some integer k, relatively prime with respect to n, between 0 and n.

Since $\exp(2\pi i k/n) \in f(P\sigma(L)) \cap \partial D$, for any $n \in \mathcal{P}$, and $\mathcal{P}$ is infinite, this proves (2).

(3) Let $\mathcal{P}$ be the set of all periodic points of $\{e^{tf(L)}\}_{t \geq 0}$; that is, $t \in \mathcal{P}$ if and only if there exists non-trivial x such that t is the first positive number such that $e^{tf(L)}x = x$.

As in the proof of (2), if $\mathcal{P}$ were finite, then there would exist $T > 0$ such that

$$e^{Tf(L)}x = x$$

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

for all $x \in X$. This would imply that all orbits of $\{e^{tf(L)}\}_{t\geq 0}$ are compact, hence the hypercyclicity of $\{e^{tf(L)}\}_{t\geq 0}$ would be violated. Thus $\mathcal{P}$ is infinite.

As in the proof of (2), $\{2\pi i/t \mid t \in \mathcal{P}\}$ is contained in $f(P\sigma(L)) \cap i\mathbb{R}$. This concludes the proof. □

A surprising result is that the existence of a single non-trivial periodic point can generate in many cases the chaoticity of the operator. This phenomenon is observed in a particular case by Grosse-Erdmann [**7, 8**] and Martinez-Gimenez and Peris [**10**]. For some results we will impose on the weight vector $\vec{\beta}$, the following hypothesis:

$$R > 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \beta(k) R^k \text{ diverges.} \tag{H1}$$

THEOREM 2.3. *Suppose f is holomorphic and non-constant in a neighborhood of $\|L\|\overline{D}$. Under assumption (H1) the following are equivalent:*

(a) *$f(L)$ is chaotic;*

(b) *$f(R^{1/p}D) \cap \partial D$ is non-empty;*

(c) *$f(L)$ has a non-trivial periodic point.*

Proof. (a) $\Longrightarrow$ (c) is clear from the definition of chaos. (c) $\Longrightarrow$ (b). By the same proof as the second half of the proof of Lemma 2.2(2), $f(P\sigma(L)) \cap \partial D$ is non-empty. Since (H1) implies $P\sigma(L) = R^{1/p}D$, (b) follows. (b) $\Longrightarrow$ (a) is Lemma 2.2(1). □

Remark 2.4. Although (b) $\Longrightarrow$(a) $\Longrightarrow$ (c) of Theorem 2.3 always holds, none of the equivalences are true if we suppress (H1). The following two counterexamples are due to Alfredo Peris (personal communication). Take $\beta(k) \equiv 1/k^2$ $(k \in \mathbb{N})$. If we suppress the assumption (H1), by [**10**, Theorem 2.4], L is chaotic, so that (a) of Theorem 2.3 holds, with $f(z) \equiv z$, but $f(R^{1/p}D) = f(D) = D$, hence (b) of Theorem 2.3 fails.

Now take $f(z) \equiv \frac{1}{2}(1+z)$, so that $f(L) = \frac{1}{2}(I+L)$. Since $L\vec{h}_1 = \vec{h}_1$ and $f(1) = 1$, if we suppress the assumption (H1), $\vec{h}_1$ is a non-trivial periodic point for $f(L)$. Thus $f(L)$ satisfies (c) of Theorem 2.3. But since $P\sigma(L) = \overline{D}$ and $f(\overline{D}) \cap \partial D = \{1\}$, Lemma 2.2(2) implies that $f(L)$ is not chaotic; that is, $f(L)$ does not satisfy (a) of Theorem 2.3.

Now we will present many special cases with more readily verifiable equivalences. We will begin by seeing how Theorem 2.3 looks on the unweighted sequence space $\ell^p(\mathbb{N})$.

COROLLARY 2.5. *The following are equivalent, for f holomorphic and non-constant in a neighborhood of $\overline{D}$:*

(a) *$f(L)$ on $\ell^p(\mathbb{N})$ is chaotic;*

(b) *$f(\overline{D}) \cap \partial D$ is non-empty;*

(c) *$f(L)$ on $\ell^p(\mathbb{N})$ has a non-trivial periodic point.*

Remark 2.6. In [**13**] it is asserted that $f(L)$ on $\ell^1(\mathbb{N})$ is chaotic whenever $\sigma(f(L)) \cap \partial D$ is non-empty. This is not quite correct, as can be seen by choosing $f(z) := z$: $f(L) = L$ is not chaotic on $\ell^1(\mathbb{N})$, but $\sigma(f(L)) = \sigma(L) = \overline{D}$.

For many functions f, (a) and (b) of Theorem 2.3 remain equivalent, without the restriction on $\vec{\beta}$.

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

LEMMA 2.7. *Suppose f is holomorphic and non-constant in a neighborhood of $\|L\|\overline{D}$, $R > 0$ and $f(R^{1/p}\partial D) \cap \partial D$ is finite. Then the following are equivalent:*
(a) *$f(L)$ is chaotic;*
(b) *$f(R^{1/p}D) \cap \partial D$ is non-empty.*

Proof. (a) $\Longrightarrow$ (b). By Lemma 2.2(2), $f(R^{1/p}\overline{D}) \cap \partial D$ is infinite. Since $f(R^{1/p}\partial D) \cap \partial D$ is finite, (b) follows. (b) $\Longrightarrow$(a) is Lemma 2.2(1). □

THEOREM 2.8. *Suppose Q is a polynomial other than a complex multiple of $z \mapsto z^n$, for some non-negative integer n and $R > 0$. Then the following are equivalent:*
(a) *$Q(L)$ is chaotic;*
(b) *$Q(R^{1/p}D) \cap \partial D$ is non-empty.*

Proof. By Lemma 2.7, it is sufficient to show that, if $Q(R^{1/p}\partial D) \cap \partial D$ is infinite, then $Q(z) = \alpha z^n$, for some $n \in \mathbb{N}, \alpha \in \mathbb{C}$.

To this end, suppose Q is an nth degree polynomial,

$$Q(z) \equiv \sum_{k=0}^{n} \alpha_k z^k.$$

If $Q(R^{1/p}e^{i\theta}) \in Q(R^{1/p}\partial D) \cap \partial D$, then

$$1 = [Q(R^{1/p}e^{i\theta})]\overline{[Q(R^{1/p}e^{i\theta})]} = \sum_{k,j=0}^{n} R^{(k+j)/p}\alpha_k\overline{\alpha_j}e^{i(k-j)\theta},$$

so that

$$0 = \left[\sum_{k=0}^{n} |\alpha_k|^2 R^{2k/p} - 1\right] + \sum_{k,j=0,k\neq j}^{n} R^{(k+j)/p}\alpha_k\overline{\alpha_j}e^{i(k-j)\theta}. \tag{2.9}$$

If $Q(R^{1/p}\partial D) \cap \partial D$ is infinite, then (2.9) holds for infinitely many points $R^{1/p}e^{i\theta} \in R^{1/p}\partial D$. Since a non-constant trigonometric polynomial has only finitely many zeroes and $R > 0$, this implies that $\alpha_k\overline{\alpha_j} = 0$ whenever $k \neq j$, $0 \leq k, j \leq n$. This is saying that $\alpha_k = 0$ for $0 \leq k < n$, as desired. □

Remark 2.10. Theorem 2.8 and Lemma 2.7, combined with [**10**, Theorem 2.4], imply that, for Q a polynomial, $Q(L)$ is chaotic if and only if $Q(P\sigma(L)) \cap \partial D$ is infinite.

Affine functions of L may be of particular interest (see [**14**]).

COROLLARY 2.11. *Suppose a and b are non-zero complex numbers. Then the following are equivalent:*
(a) *$a + bL$ is chaotic;*
(b) *$|b|R^{1/p} > |1 - |a||$.*

Proof. By Theorem 2.8, we must show that (b) holds if and only if $(|a| + |b|R^{1/p}D) \cap \partial D$ is non-trivial. By rotation, we may assume both a and b are non-negative real numbers. Then $(a + |b|R^{1/p}D)$ intersects ∂D if and only if there exists $r, -1 < r < 1$, so that $1 = a + r|b|R^{1/p}$; by solving for $|b|R^{1/p}$, it is clear that this is equivalent to (b). □

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

Remark 2.12. When the assumption (H1) holds, Theorem 2.3 implies that (a) and (b) of Corollary 2.11 are equivalent to:

(c) $(a + bL)$ has a non-trivial periodic point; and

(d) $\sum_{k=1}^{\infty}((1 - |a|)/|b|)^{pk}\beta(k) < \infty$.

When $a = 0$, the equivalence of (a), (c) and (d) holds without the extra hypothesis on $\vec{\beta}$, at least when $b = 1$ (see [**10**, Theorem 2.4]). This remark illustrates how unstable chaos is under perturbations. For $\beta(k) \equiv 1 (k \in \mathbb{N})$, $(a + L)$ is chaotic if and only if $(a + L)$ has a non-trivial periodic point if and only if $0 < |a| < 2$. For $\beta(k) \equiv 1/k^2$ $(k \in \mathbb{N})$, $(a + L)$ is chaotic if and only if $|a| < 2$ and $(a + L)$ has a non-trivial periodic point if and only if $|a| \leq 2$.

It also seems of interest that for any non-zero complex b and any $\vec{\beta}$, $(bL \pm 1)$ is chaotic. The special case of a, b real, $b > -a \geq 1$ appears in [**14**].

Now we turn to functions of L generating a chaotic semigroup.

THEOREM 2.13. *Assume (H1). The following are equivalent, for g holomorphic and constant on a neighborhood of $\|L\|\overline{D}$:*

(a) *$g(L)$ generates a chaotic semigroup;*

(b) *$g(R^{1/p}D) \cap i\mathbb{R}$ is non-empty;*

(c) *$e^{g(L)}$ has a non-trivial periodic point;*

(d) *$\{e^{tg(L)}\}_{t\geq 0}$ has a non-trivial periodic point;*

(e) *$e^{g(L)}$ is chaotic.*

Proof. Let $f(z) := e^{g(z)}$. By Theorem 2.3, since $f(L) = e^{g(L)}$, (b) is equivalent to (c) and (e). It is clear that (e) implies (a) and (a) implies (d). (d) $\Longrightarrow$ (b). As in the proof of Theorem 2.3, (d) implies that there exists $\mu \in R^{1/p}D$ such that

$$e^{tg(\mu)} = 1,$$

so that $g(\mu) \in i\mathbb{R}$. □

THEOREM 2.14. *Suppose Q is a polynomial. Then the following are equivalent:*

(a) *$Q(L)$ generates a chaotic semigroup;*

(b) *$Q(R^{1/p}D) \cap i\mathbb{R}$ is non-empty;*

(c) *$e^{Q(L)}$ is chaotic.*

Proof. Let $f(z) \equiv e^{Q(z)}$. Then (b) is equivalent to $f(R^{1/p}D) \cap \partial D$ being non-empty.

As with the proof of Theorem 2.8, the equivalence of (b) and (c) will follow once we show that f satisfies the hypotheses of Lemma 2.7; this is equivalent to showing that $Q(R^{1/p}\partial D) \cap i\mathbb{R}$ is finite. To this end, suppose $z = x + iy \in R^{1/p}\partial D$ $(x, y \in \mathbb{R})$ and $Q(z) \in i\mathbb{R}$. Then $\mathrm{Re}(Q(z)) = 0$, while $x^2 + y^2 = R^{2/p}$ and the fact that $\mathrm{Re}(Q(z))$ contains only even powers of y implies that $\mathrm{Re}(Q(z))$ is a polynomial in x; hence $\mathrm{Re}(Q(z))$ has only finitely many zeroes for $z \in R^{1/p}\partial D$, as desired. This proves the equivalence of (b) and (c). (a) $\Longrightarrow$ (b) follows from Lemma 2.2(3) and the fact, just shown, that $Q(R^{1/p}\partial D) \cap i\mathbb{R}$ is finite, since $P\sigma(L) \subseteq R^{1/p}\overline{D}$. (c) $\Longrightarrow$ (a) is obvious from the definition of chaos. □

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

Remark 2.15. When $Q(z) \equiv a + bz$ $(a, b \in \mathbb{C})$, so that $Q(L) = a + bL$, it is not hard to see that (b) of Theorem 2.14 is equivalent to

$$|\mathrm{Re}(a)| < |b| R^{1/p}.$$

Thus Theorem 2.14 generalizes [**3**, Theorem 5.3]. Special cases also appear in [**12**], where it is shown that L generates a chaotic semigroup on $\ell^2(\mathbb{N})$, and in [**14**], where it is shown that $(a + bL)$ generates a chaotic semigroup when $b > -a > 0$.

It is interesting that bL generates a chaotic semigroup, for any complex b.

The equivalence of (a) and (d) in Theorem 2.13 fails in general, even for first-degree polynomials as in [**3**, Theorem 5.3]. Take $f(z) \equiv z - 1$. If $\beta(k) \equiv (1/k^2)$ $(k \in \mathbb{N})$, then $\{e^{tf(L)}\}$ has the periodic point $\vec{h}_1$, but since $f(P\sigma(L)) \cap i\mathbb{R} = \{0\}$, Lemma 2.2(3) implies that $\{e^{tf(L)}\}_{t\geq 0}$ is not chaotic. If $\beta(k) \equiv 1$, then $\{e^{tf(L)}\}_{t\geq 0}$ no longer has any periodic points.

Here is another statement about the abundance of chaos, when any chaos at all exists.

COROLLARY 2.16. *The following are equivalent for* $1 \leq p < \infty, r > \|L\|$*:*

(a) *there exists* f *holomorphic on* rD *such that* $f(L)$ *is chaotic;*

(b) $R > 0$*;*

(c) $P\sigma(L)$ *is non-empty;*

(d) *for any non-constant* f *holomorphic on* rD*, there exists a constant* $c > 0$ *so that* $cf(L)$ *is chaotic.*

Proof. The equivalence of (b) and (c) is clear from the fact that $\vec{h}_\mu$ is an eigenvector if and only if $\sum_{k=1}^{\infty} \beta(k)|\mu|^{kp} < \infty$. The remaining equivalences are a consequence of Lemma 2.2(1). □

Remark 2.17. A statement similar to (d), except that 'chaotic' is replaced with 'hypercyclic', for generalized backward shifts, appears in [**5**, Theorem 3.6].

3. *Discrete weighted bilateral shifts*

It is not hard to show that by straightforward modifications of §2 we can obtain the above results for weighted bilateral shifts.

Let us denote by $\ell^p(\mathbb{Z}, \vec{\beta})$ $(1 \leq p < \infty)$ the Banach space X of complex-valued sequences $\vec{x} := \{x(k)\}_{k=-\infty}^{\infty}$ such that

$$\|\vec{x}\|^p := \sum_{k=-\infty}^{\infty} |x(k)|^p \beta(k)$$

is finite. Here the sequence $\vec{\beta}$ satisfies

$$\sup_{k\in\mathbb{Z}} \frac{\beta(k)}{\beta(k+1)} < \infty.$$

We will denote by $\ell^p(\mathbb{Z})$, $\ell^p(\mathbb{Z}, \vec{\beta})$ with $\beta(k) := 1$.

Denote by L the left-shift on $\ell^p(\mathbb{Z}, \vec{\beta})$,

$$(L\vec{x})(k) := x(k+1) \quad (k \in \mathbb{Z}).$$

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

For a, b extended real non-negative numbers, $b > a$, denote by $A(a, b)$ the annulus

$$A(a, b) := \{z \in \mathbb{C} \mid a < |z| < b\}.$$

If $b \leq a$, $A(a, b)$ will be the empty set.

As in §2, $f(L)$ is defined by the Riesz functional calculus when f is holomorphic in a neighborhood of $\sigma(L)$, and we denote by

$$\vec{h}_\mu := (\ldots, \mu^{-2}, \mu^{-1}, 1, \mu, \mu^2, \ldots),$$

that is, $\vec{h}_\mu(k) := \mu^k (k \in \mathbb{Z})$, the eigenvector for L corresponding to the eigenvalue μ.

Denote

$$R_1 := \underline{\lim}_{k\to\infty}(\beta(k))^{-1/k} \quad \text{and} \quad R_2 := \overline{\lim}_{k\to\infty}(\beta(-k))^{1/k}.$$

Note that $\vec{h}_\mu \in \ell^p(\mathbb{Z}, \vec{\beta})$ if and only if $\sum_{k=-\infty}^{\infty} \beta(k)|\mu|^k < \infty$. Thus $P\sigma(L) \subseteq \overline{A(R_2, R_1)}$ and it equals $A(R_2, R_1)$ if $\sum_{k=-\infty}^{\infty} \beta(k) R_i^k = \infty$ for $i = 1, 2$.

The following lemma can be obtained just by mimicking the proof of Lemma 2.2.

LEMMA 3.1. *Suppose f is holomorphic and non-constant in a neighborhood of $\overline{P\sigma(L)}$ and $0 < R_2 < R_1$.*

(1) *If $f(A(R_2^{1/p}, R_1^{1/p})) \cap \partial D$ is non-empty, then $f(L)$ is chaotic.*

(2) *If $f(L)$ is chaotic, then $f(P\sigma(L)) \cap \partial D$ is infinite.*

(3) *If $f(L)$ generates a chaotic semigroup, then $f(P\sigma(L)) \cap i\mathbb{R}$ is infinite.*

In this context the hypothesis (H1) of the previous section is modified as follows

$$0 < R_2 < R_1 \quad \text{and} \quad \sum_{k=-\infty}^{\infty} \beta(k) R_i^k \text{ diverges for } i = 1, 2. \tag{H2}$$

Under this assumption the analogue of Theorem 2.3 is the following.

THEOREM 3.2. *Suppose (H2) holds. Then the following are equivalent, for f holomorphic and non-constant in a neighborhood of $\sigma(L)$:*

(a) *$f(L)$ is chaotic;*

(b) *$f(A(R_2^{1/p}, R_1^{1/p})) \cap \partial D$ is non-empty;*

(c) *$f(L)$ has a non-trivial periodic point.*

Remark 3.3. (i) Corollary 2.11 is modified as follows.

Suppose a and b are complex numbers. Then the following are equivalent:

(a) $(a + bL)$ is chaotic;

(b)
$$R_2 < \left(\frac{|1 - |a||}{|b|}\right)^p < R_1.$$

If we impose the assumption (H2) the above assertions are equivalent to:

(c) $(a + bL)$ has a non-trivial periodic point;

(d)
$$\sum_{k=-\infty}^{\infty} \left(\frac{|1 - |a||}{|b|}\right)^{pk} \beta(k) < \infty.$$

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

(ii) Analogously to Theorem 2.13, we get the following.

Assume (H2) holds. The following are equivalent, for g holomorphic and non-constant on a neighborhood of $\sigma(L)$:

(a) $g(L)$ generates a chaotic semigroup;
(b) $g(A(R_2^{1/p}, R_1^{1/p})) \cap i\mathbb{R}$ is non-empty;
(c) $e^{g(L)}$ has a non-trivial periodic point;
(d) $\{e^{tg(L)}\}_{t\geq 0}$ has a non-trivial periodic point;
(e) $e^{g(L)}$ is chaotic.

(iii) Here is the analogue of Theorem 2.14.

Suppose Q is a polynomial. Then the following are equivalent:

(a) $Q(L)$ generates a chaotic semigroup;
(b) $Q(A(R_2^{1/p}, R_1^{1/p})) \cap i\mathbb{R}$ is non-empty;
(c) $e^{Q(L)}$ is chaotic.

(iv) Analogously to Corollary 2.16, we get the following.

If $0 < R_2 < R_1$, then for any non-constant f holomorphic on $\sigma(L)$, there exists $c > 0$ such that $cf(L)$ is chaotic.

4. *Continuous weighted unilateral shifts*

Throughout this section, we will denote by $L^p([0, \infty), \rho)$ the Banach space X of complex-valued functions f on $[0, \infty)$ such that

$$\|f\|^p \equiv \int_0^\infty |f(s)|^p \rho(s)\, ds$$

is finite. Here $\rho : [0, \infty) \to (0, \infty)$ is measurable and satisfies

$$\sup_{s\geq 0} \frac{\rho(s)}{\rho(s+t)} < \infty, \tag{4.0}$$

for all $t \geq 0$.

Let B be d/ds, the generator of

$$(e^{tB} f)(s) \equiv f(s+t) \quad (s, t \geq 0)$$

on $L^p([0, \infty), \rho)$. Note that (4.0) is precisely what is needed to make $\{e^{tB}\}_{t\geq 0}$ a strongly continuous semigroup.

Let *LHP* denote the open left half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$. The continuous analogue of the radius of convergence of a power series will be

$$\omega \equiv \sup \left\{ \mu \in \mathbb{R} \;\middle|\; \int_0^\infty e^{\mu s} \rho(s)\, ds < \infty \right\}.$$

If the set we are trying to take a supremum of is empty, we will say that $\omega = -\infty$.

Denote by

$$h_\mu(s) \equiv e^{\mu s} \quad (s \geq 0)$$

the eigenvector for B corresponding to the eigenvalue μ.

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

Note that $P\sigma(B)$ is equal either to $(\omega/p) + LHP$ if $\int_0^\infty e^{\omega s}\rho(s)\,ds = \infty$ or its closure if not. We adopt the custom that

$$(\infty + LHP) \equiv \mathbb{C} \quad \text{and} \quad (-\infty + LHP) \equiv \text{the empty set.}$$

Here is the continuous analogue of Lemma 2.1.

LEMMA 4.1. *Suppose $\Omega \subseteq (\omega/p) + LHP$. If Ω has an accumulation point, then $H_\Omega := \{h_\mu \mid \mu \in \Omega\}$ is dense in $L^p([0,\infty),\rho)$.*

Proof. Suppose ϕ annihilates H_Ω. Define

$$g(\mu) \equiv \int_0^\infty e^{\mu s}\phi(s)\,ds.$$

Then g is analytic on $(\omega/p) + LHP$, and since ϕ annihilates H_Ω, it follows that $g(\mu) = 0$ in Ω. Since Ω has an accumulation point, $g \equiv 0$, so that by uniqueness of the Laplace transform, $\phi \equiv 0$. □

LEMMA 4.2. *Suppose $g \mapsto g(B)$ is a linear map from a vector space $\mathcal{A}$ of complex-valued functions holomorphic on a neighborhood of $\sigma(B)$ into $B(L^p([0,\infty),\rho))$ such that*

$$P\sigma(g(B)) \subseteq g(P\sigma(B)) \cup \{0\}, \quad \forall g \in \mathcal{A}. \tag{4.3}$$

Then for $f \in \mathcal{A}$ we have the following.

(1) *If $f((\omega/p) + LHP) \cap \partial D$ is non-empty, then $f(B)$ is chaotic.*

(2) *If $f(B)$ is chaotic, then $f(P\sigma(B)) \cap \partial D$ is infinite.*

Proof. The proof of (1) is the same as the proof of Lemma 2.2(1), with $(\omega/p) + LHP$ replacing $R^{1/p}D$ and Lemma 4.1 replacing Lemma 2.1. The proof of (2) is the same as the proof of Lemma 2.2(2), with (4.3) replacing [**9**, Theorem 5.12.2]. □

The continuous analogue of the hypothesis (H1) on the weight function ρ will be

$$\omega < \infty \quad \text{and} \quad \int_0^\infty e^{\omega s}\rho(s)\,ds \text{ diverges.} \tag{H3}$$

As with Theorem 2.3 one can prove the following.

THEOREM 4.4. *Under the assumptions of Lemma 4.2 together with (H3) for $f \in \mathcal{A}$, the following are equivalent:*

(a) *$f(B)$ is chaotic;*

(b) *$f((\omega/p) + LHP) \cap \partial D$ is non-empty;*

(c) *$f(B)$ has a non-trivial periodic point.*

Proof. (c) $\Longrightarrow$ (b). By the same proof as Lemma 4.2(2), $f(P\sigma(L)) \cap \partial D$ is non-empty. Since (H3) implies $P\sigma(L) = (\omega/p) + LHP$, (b) follows. (b) $\Longrightarrow$ (a) is Lemma 4.2(1). □

We will dispense with a detailed discussion of which g satisfy the hypotheses of Lemma 4.2; in a future paper [**2**] it will be shown that (4.3) holds when $g(z) \to 0$ for $|z| \to \infty$ in $\sigma(B)$. Our interest in this paper is g equal to exponentials of polynomials, so that we may address semigroups generated by differential operators.

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

For Q a polynomial, define the differential operator $Q(B)$ as follows. If

$$Q(z) \equiv \sum_{k=0}^{n} a_k z^k,$$

for some complex $a_0, a_1, \ldots, a_n$, with $a_n \neq 0$, define

$$Q(B) \equiv \sum_{k=0}^{n} a_k B^k, \quad \mathcal{D}(Q(B)) \equiv \mathcal{D}(B^n). \tag{4.5}$$

THEOREM 4.6. *Suppose that the weight function ρ satisfies the assumption (H3) and Q is a polynomial such that $Q(B)$ generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$. Then the following are equivalent:*

(a) $\{T(t)\}_{t\geq 0}$ *is chaotic;*
(b) $Q((\omega/p) + LHP) \cap i\mathbb{R}$ *is non-empty;*
(c) $\{T(t)\}_{t\geq 0}$ *has a non-trivial periodic point;*
(d) $T(1)$ *is chaotic;*
(e) $T(1)$ *has a non-trivial periodic point.*

Proof. For any $t \geq 0$, let $f_t(z) \equiv e^{tQ(z)}$, and let $\mathcal{A}$ be the span of $\{f_t \mid t \geq 0\}$, with

$$f_t(B) \equiv T(t) \quad (t \geq 0).$$

To see that f_t satisfies (4.3), note first that, since $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup generated by $Q(B)$,

$$P\sigma(f_t(B)) \equiv P\sigma(T(t)) \subseteq e^{tP\sigma(Q(B))} \cup \{0\}.$$

[**11**, Theorem 2.2.4] Since Q is a polynomial,

$$P\sigma(Q(B)) = Q(P\sigma(B)).$$

The equivalence of (d) and (e) now follows from Theorem 4.4. Theorem 4.4 now also implies the equivalence of (d) and (b), since (b) is equivalent to

$$\exp\left[Q\left(\frac{\omega}{p} + LHP\right)\right] \cap \partial D$$

being non-empty.

Clearly (d) $\Longrightarrow$ (a) $\Longrightarrow$ (c). Since we have shown that f_t satisfies (4.3), the same argument as in the proof of Theorem 2.3 (c) $\Longrightarrow$ (b) shows that (c) implies that there exists $\mu \in (\omega/p) + LHP$ such that

$$e^{tQ(\mu)} = 1,$$

which implies (b). This completes the proof. □

COROLLARY 4.7. *Under the assumptions of Lemma 4.2 and (H3) the following are equivalent, for $a, b \in \mathbb{C}$, $b > 0$, if we define the strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $L^p([0,\infty), \rho)$ as follows:*

$$(T(t)f)(s) \equiv e^{at} f(s + bt) \quad (s, t \geq 0).$$

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

(a) $\{T(t)\}_{t\geq 0}$ *is a chaotic semigroup.*
(b) $\int_0^\infty \exp\{-(p/b)[\mathrm{Re}(a)]s\}\rho(s)\,ds < \infty$.
(c) $\mathrm{Re}(a) + (b\omega/p) > 0$.
(d) *There exists* $t > 0$ *and non-trivial* $f \in L^p([0,\infty), \rho)$ *so that*

$$f(s+bt) = e^{-at} f(s), \quad \forall s \geq 0.$$

(e) $T(1)$ *has a non-trivial periodic point.*
(f) $T(1)$ *is chaotic.*

Proof. Apply Theorem 4.6 with $Q(z) \equiv a + bz$.

(a) $\Longleftrightarrow$ (c). By Theorem 4.6, (a) is equivalent to

$$\left[\left(\mathrm{Re}(a) + \frac{b\omega}{p} + LHP\right) \cap i\mathbb{R}\right] = \left[a + b\left(\frac{\omega}{p} + LHP\right)\right] \cap i\mathbb{R}$$

being non-empty. This is equivalent to (c).

It is clear from the definition of ω that (c) is equivalent to (b). The remaining equivalences are a special case of Theorem 4.6. □

COROLLARY 4.8. *Under the assumptions of Lemma 4.2 and (H3) the following are equivalent, if*

$$(T(t)f)(s) \equiv f(s+t) \quad (s,t \geq 0).$$

(a) $\{T(t)\}_{t\geq 0}$ *is a chaotic strongly continuous semigroup.*
(b) $\int_0^\infty \rho(s)\,ds < \infty$.

Remark 4.9. It is interesting to compare Corollary 4.8 to [**3**, Theorem 4.7], where it is shown that $\{e^{tB}\}_{t\geq 0}$ is hypercyclic if and only if $\underline{\lim}_{s\to\infty}\rho(s) = 0$. For example, if $\rho(s) \equiv 1/s$, then $\{e^{tB}\}_{t\geq 0}$ is hypercyclic but not chaotic.

COROLLARY 4.10. *The following are equivalent, for* $1 \leq p < \infty$*:*
(a) *there exists a polynomial* Q *such that* $Q(B)$ *generates a chaotic semigroup;*
(b) $\omega > -\infty$*;*
(c) *there exists real* μ *such that* $\int_0^\infty e^{\mu s}\rho(s)\,ds < \infty$*;*
(d) ρ *is exponentially bounded;*
(e) $P\sigma(B)$ *is non-empty;*
(f) *whenever* Q *is a polynomial such that* $Q(B)$ *generates a strongly continuous semigroup, then there exists a real constant* c *such that* $(Q(B) + c)$ *generates a chaotic semigroup* $\{e^{t(Q(B)+c)}\}_{t\geq 0} = \{e^{tc}e^{tQ(B)}\}_{t\geq 0}$.

5. *Continuous weighted bilateral shifts*

It is straightforward to modify the results and proofs in §4.

Throughout this section, we will denote by $L^p(\mathbb{R}, \rho)$ the Banach space X of complex-valued functions f on $\mathbb{R}$ such that

$$\|f\|^p \equiv \int_{-\infty}^{\infty} |f(s)|^p \rho(s)\,ds$$

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

is finite. Here $\rho : \mathbb{R} \to (0, \infty)$ is measurable and satisfies

$$\sup_{s \in \mathbb{R}} \frac{\rho(s)}{\rho(s+t)} < \infty,$$

for all $t \geq 0$.

Let B be d/ds, the generator of the strongly continuous semigroup

$$(e^{tB} f)(s) \equiv f(s+t) \quad (s \in \mathbb{R}, t \geq 0)$$

on $L^p(\mathbb{R}, \rho)$.

For $b > a$ extended real numbers, denote by $V_{a,b}$ the vertical strip

$$V_{a,b} \equiv \{z \mid a < \mathrm{Re}(z) < b\}.$$

If $b \leq a$, $V_{a,b}$ will be the empty set.

Define

$$\omega_1 \equiv \sup \left\{ \mu \in \mathbb{R} \;\middle|\; \int_0^\infty e^{\mu s} \rho(s)\, ds < \infty \right\};$$

$$\omega_2 \equiv \inf \left\{ \mu \in \mathbb{R} \;\middle|\; \int_{-\infty}^0 e^{\mu s} \rho(s)\, ds < \infty \right\}.$$

As in the previous section, denote by

$$h_\mu(s) \equiv e^{\mu s} \quad (s \in \mathbb{R})$$

the eigenvector for B corresponding to the eigenvalue μ.

Note that for $\omega_1 > \omega_2$

$$P\sigma(B) \subset \frac{1}{p} \overline{V_{\omega_2, \omega_1}}.$$

In order to have

$$P\sigma(B) = \frac{1}{p} V_{\omega_2, \omega_1},$$

we will sometimes need the hypotheses analogous to (H3):

$$\omega_1 > \omega_2 > -\infty, \quad \int_0^\infty e^{\omega_1 s} \rho(s)\, ds \text{ and } \int_{-\infty}^0 e^{\omega_2 s} \rho(s)\, ds \text{ divergent.} \qquad \text{(H4)}$$

For Q a polynomial, define $Q(B)$ as in (4.5).

THEOREM 5.1. *Assume (H4) and suppose Q is a polynomial such that $Q(B)$ generates a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$. Then the following are equivalent:*

(a) *$\{T(t)\}_{t \geq 0}$ is chaotic;*

(b) *$Q(p^{-1} V_{\omega_2, \omega_1}) \cap i\mathbb{R}$ is non-empty;*

(c) *$\{T(t)\}_{t \geq 0}$ has a non-trivial periodic point;*

(d) *$T(1)$ is chaotic;*

(e) *$T(1)$ has a non-trivial periodic point.*

COROLLARY 5.2. *Assume (H4). The following are equivalent, for $a, b \in \mathbb{C}$, $b > 0$, if we define the strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ on $L^p(\mathbb{R}, \rho)$ as follows:*

$$(T(t) f)(s) \equiv e^{at} f(s + bt) \quad (t \geq 0, s \in \mathbb{R}).$$

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

(a) $\{T(t)\}_{t\geq 0}$ *is a chaotic semigroup.*
(b) $\int_{-\infty}^{\infty} \exp\{-(p/b)[\mathrm{Re}(a)]s\}\rho(s)\,ds < \infty$.
(c) $b\omega_2/p < -\mathrm{Re}(a) < b\omega_1/p$.
(d) *There exists* $t > 0$ *and non-trivial* $f \in L^p(\mathbb{R}, \rho)$ *so that*

$$f(s+bt) = e^{-at} f(s), \quad \forall s \in \mathbb{R}.$$

(e) $T(1)$ *has a non-trivial periodic point.*
(f) $T(1)$ *is chaotic.*

COROLLARY 5.3. *Assume (H4). The following are equivalent, if*

$$(T(t)f)(s) \equiv f(s+t) \quad (t \geq 0, s \in \mathbb{R}).$$

(a) $\{T(t)\}_{t\geq 0}$ *is a chaotic strongly continuous semigroup.*
(b) $\int_{-\infty}^{\infty} \rho(s)\,ds < \infty$.

Remark 5.4. In [**3**, Theorem 4.8], it is shown that $\{e^{tB}\}_{t\geq 0}$ is hypercyclic if and only if for each $s \in \mathbb{R}$ there is a sequence $\{t_j\}_{j=1}^{\infty}$ of positive numbers such that

$$\lim_{j\to\infty} \rho(s+t_j) = \lim_{j\to\infty} \rho(-t_j+s) = 0.$$

As in Remark 4.9, it is easy to construct weight functions ρ such that $\{e^{tB}\}_{t\geq 0}$ is hypercyclic but not chaotic.

As in Corollary 4.10, if we have one chaotic semigroup, we are guaranteed many. Indeed, Lemma 4.2 is valid if we replace $(\omega/p) + LHP$ by $(1/p)V_{\omega_2,\omega_1}$ and this result implies the following corollaries.

COROLLARY 5.5. *Suppose* $\omega_1 > \omega_2$ *and* Q *is a polynomial such that* $Q(B)$ *generates a strongly continuous semigroup. Then there exists* $c \in \mathbb{R}$ *such that* $(Q(B)+c)$ *generates a chaotic semigroup* $\{e^{t(Q(B)+c)}\}_{t\geq 0} = \{e^{tc}e^{tQ(B)}\}_{t\geq 0}$.

Theorem 5.1 and Corollary 5.5 are of more interest than Theorem 4.6 and Corollary 4.10 because there are many more polynomials Q for which $Q(B)$ generates a strongly continuous semigroup. The heat semigroup is only one example, but may be of particular interest.

COROLLARY 5.6. *Suppose* $\omega_1 > \omega_2$ *and*

$$\sup_{s\in\mathbb{R}} \frac{\rho(s)}{\rho(s-t)} < \infty, \quad \forall t \geq 0. \tag{$*$}$$

Then B^2 *generates a chaotic strongly continuous semigroup.*

Proof. Hypothesis $(*)$ guarantees that $\{e^{tB}\}_{t\geq 0}$ extends to a strongly continuous group. This implies [**6**, 2.8.7, 2.8.11] that B^2 generates a strongly continuous semigroup.

By the proof of Theorem 5.1 and the analogue of Lemma 4.2(1), it is sufficient to show that

$$\{z^2 \mid z \in V_{\omega_2,\omega_1}\}$$

intersects the imaginary axis. But this is clear when we write $z = u + iv$, with u and v real and note that the lines $|v| = |u|$ are mapped to the imaginary axis by $z \mapsto z^2$. □

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press

Acknowledgements. We are extremely indebted to Alfredo Peris who read the first version of the paper with care for detail and made us aware of the necessity of the assumption (H1) in some of our results by giving the counterexamples cited in Remark 2.4. The authors also appreciate the helpful discussions with V. Protopopescu.

REFERENCES

[1] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey. On Devaney's definition of chaos. *Amer. Math. Monthly* **99** (1992), 332–334.

[2] R. deLaubenfels. Spectral mapping theorems for operators with polynomially bounded resolvent. Submitted.

[3] W. Desch, W. Schappacher and G. F. Webb. Hypercyclic and chaotic semigroup of linear operators. *Ergod. Th. & Dynam. Sys.* **17** (1997), 793–819.

[4] R. L. Devaney. *An Introduction to Chaotic Dynamical Systems*. Addison-Wesley, New York, 1989.

[5] G. Godefroy and J. H. Shapiro. Operators with dense, invariant cyclic vector manifolds. *J. Funct. Anal.* **98** (1991), 229–269.

[6] J. A. Goldstein. *Semigroups of Linear Operators and Applications*. Oxford University Press, New York, 1985.

[7] K.-G. Grosse-Erdmann. Universal families and hypercyclic operators. *Bull. Amer. Math. Soc.* **36** (1999), 345–381.

[8] K.-G. Grosse-Erdmann. Hypercyclic and chaotic weighted shifts. *Studia Math.* **139** (2000), 47–68.

[9] E. Hille and R. S. Phillips. *Functional Analysis and Semi-Groups (Amer. Math. Soc. Coll. Publ., 31)*. American Mathematical Society, Providence, RI, 1957.

[10] F. Martinez-Gimenez and A. Peris. Chaos for backward shift operators. Submitted.

[11] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, 1983.

[12] C. R. MacCluer. Chaos in linear distributed systems. *J. Dyn. Systems, Measurement, and Control* **114** (1992), 322–324.

[13] V. Protopopescu. Chaotic behaviour of linear orbits. *Proc. Conf. on Mathematical Models of Nonlinear Excitation, Transport, Dynamics, Control in Condensed Systems and Other Mediums*. Tver, July 1996, pp. 207–215.

[14] V. Protopopescu and Y. Y. Azmy. Topological chaos for a class of linear models. *Math. Models Methods Appl. Sci.* **2** (1992), 79–90.

https://doi.org/10.1017/S0143385701001675 Published online by Cambridge University Press