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ON RANDOMLY SPACED OBSERVATIONS AND CONTINUOUS-TIME RANDOM WALKS

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Abstract

We consider random variables observed at arrival times of a renewal process, which possibly depends on those observations and has regularly varying steps with infinite mean. Due to the dependence and heavy-tailed steps, the limiting behavior of extreme observations until a given time t tends to be rather involved. We describe the asymptotics and extend several partial results which appeared in this setting. The theory is applied to determine the asymptotic distribution of maximal excursions and sojourn times for continuous-time random walks.

Keywords: Extreme value theory; point process; renewal process; continuous-time random walk; excursion; sojourn time

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1. Introduction

A sequence of observations collected at arrival times of a renewal process represents a popular modelling framework in applied probability. It underlies many standard stochastic models, ranging from risk theory and engineering to theoretical physics. The asymptotic distribution of the maximum of such observations until a given time t is well understood in the case of the renewal process with finite-mean interarrival times; see [4] and the references therein. In the infinite mean case, the asymptotic theory for partial maxima tends to be more complicated, we particularly refer the reader to Shanthikumar and Sumita [26], Anderson [1] or a recent paper by Barczyk and Kern [2] who allow a certain degree of dependence between the observations and the interarrival times.

Here we aim to characterize the limiting behavior of all upper-order statistics in a sequence (X_n) up to a time $\tau(t)$, where $(\tau(t))$ represents the renewal process generated by a sequence of nonnegative random variables (Y_n), i.e.

$$\tau(t) = \inf\{k\colon Y_1 + \cdots + Y_k > t\} \quad \text{for } t \geq 0. \tag{1}$$

Throughout, we shall assume that (X_n, Y_n), $n \geq 1$, form independent and identically distributed (i.i.d.) pairs. The interarrival times are assumed to be regularly varying with index $\alpha \in (0, 1)$ (as in [1], [2], [16], and [19]). In particular, the Y_i have infinite mean and belong to the maximum domain of attraction (MDA) of a Fréchet distribution. Moreover, in contrast to [2], the X_i can belong to the MDA of any of the three extreme value distributions, denoted by G.

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Our analysis is based on the theory of point processes. A related approach was used by Barczyk and Kern [2]. However, we succeed in applying simple continuous mapping arguments which were explicitly avoided in [2], essentially because sequences (X_n) and (Y_n) can have simultaneous extremes. We also relax some restrictions used in the literature concerning dependence between the observations and interarrival times.

We apply our results to study the continuous-time random walk (CTRW). It is essentially a random walk subordinated to a renewal process; see the original paper by Montroll and Weiss [18] or [24] for an overview of the literature on various applications. Under mild assumptions, excursions of such a walk are regularly varying with index $\frac{1}{2}$. Hence, our theory applies, and one can determine the limiting distribution of extremely long excursions and sojourn times at level 0 of a CTRW.

The paper is organized as follows. Notation and auxiliary results are introduced in Section 2. In Section 3 we present the limiting distribution of all extreme observations until a given time. Our main result extends and complements several previous results in this context, including [2], [3], [13], [17], and [19]. In Section 4 we apply the main result to study sojourn times and excursions of a CTRW. In particular, in this section we extend the analysis of the asymptotic distribution for the ranked lengths of excursions in the simple symmetric random walk given by Csáki and Hu [6].

2. Preliminaries

We assume that the observations X_i, $i \geq 1$, belong to an MDA of some extreme value distribution, denoted by G. Because of the correspondence between MDAs of Fréchet and Weibull distributions, we discuss only observations in the Gumbel (which is denoted by $G = \Lambda$) and Fréchet ($G = \Phi_\beta$ for $\beta > 0$) MDAs in detail (see [10, Section 3.3.2]). In particular, $X_1 \in \mathrm{MDA}(G)$ implies that there exist functions $a(t)$ and $b(t)$ such that

$$t\mathbb{P}(X_1 > a(t)x + b(t)) \to -\log G(x) \quad \text{as } t \to \infty \tag{2}$$

(see Resnick [22]).

Since we assumed that the renewal steps Y have regularly varying distribution of infinite mean with index $\alpha \in (0, 1)$ (we denote this by $Y \sim \mathrm{reg\,var}(\alpha)$). In such a case; it is well known (see Feller [12]) that there exists a strictly positive sequence (d_n) such that

$$d_n^{-1}(Y_1 + \cdots + Y_n) \xrightarrow{\mathrm{D}} S_\alpha,$$

where S_α has the stable law with the index α, scale parameter $\sigma = 1$, skewness parameter $\beta = 1$, and shift parameter $\mu = 0$. Convergence in distribution is denoted by '$\xrightarrow{\mathrm{D}}$'. In particular, S_α is strictly positive almost surely. The sequence (d_n) can be chosen such that $n(1 - F_Y(d_n)) \to 1$ as $n \to \infty$, where F_Y denotes the cumulative distribution function of Y_1. Denote $d(t) = d_{\lfloor t \rfloor}$ for $t \geq 0$, with $d_0 = 0$, it is well known that d is regularly varying with index $1/\alpha$. Consider the partial sum process $T(t) = \sum_{i=1}^{\lfloor t \rfloor} Y_i$, with $T(t) = 0$ for $0 \leq t < 1$. It is well known that

$$\left(\frac{T(tc)}{d(t)}\right)_{c \geq 0} \xrightarrow{\mathrm{D}} (S_\alpha(c))_{c \geq 0} \quad \text{as } t \to \infty, \tag{3}$$

in a space of càdlàg functions $D[0, \infty)$ endowed with Skorokhod J_1-topology (see, e.g. Resnick [23]). The limiting process $(S_\alpha(c))_{c \geq 0}$ is an α-stable process with strictly increasing sample paths.

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According to Seneta [25] (see also [16] and [23]), there exists a function $\widetilde{d}$ which is an asymptotic inverse of d, i.e.

$$d(\widetilde{d}(t)) \sim \widetilde{d}(d(t)) \sim t \quad \text{as } t \to \infty. \tag{4}$$

Moreover, $\widetilde{d}$ is known to be a regularly varying function with index α.

For a function $z \in D([0,\infty),[0,\infty))$, by '$z^{\leftarrow}$' we denote its right-continuous generalized inverse, i.e. $z^{\leftarrow}(u) = \inf\{s \in [0,\infty) \colon z(s) > u\}$, $u \geq 0$. Denote by

$$W_\alpha(c) = \inf\{x \colon S_\alpha(x) > c\} = S_\alpha^{\leftarrow}(c), \qquad c \geq 0, \tag{5}$$

the first hitting-time process of the α-stable subordinator $(S_\alpha(t))_{t\geq 0}$. The continuity of the mapping $z \mapsto z^{\leftarrow}$, together with (3) and (4) implies that

$$\left(\frac{\tau(tc)}{\widetilde{d}(t)}\right)_{c\geq 0} \xrightarrow{\mathrm{D}} (W_\alpha(c))_{c\geq 0} \quad \text{in } D([0,\infty),[0,\infty)) \tag{6}$$

endowed with J_1-topology. This limiting result is well known, see, e.g. [16, Corollary 3.6], or, alternatively, the proof of Theorem 1 below. For an α-stable process S_α and fixed $c \geq 0$, the hitting-time $W_\alpha(c)$ has the Mittag-Leffler distribution.

3. Extremes of randomly spaced observations

Here, we again assume that $X_1 \in \mathrm{MDA}(G)$ and $Y_1 \sim \mathrm{reg\,var}(\alpha)$, $0 < \alpha < 1$. For $t \geq 0$, we define the point process N_t as

$$N_t = \sum_{i\geq 1} \delta_{(i/\widetilde{d}(t),\widetilde{X}_{t,i},\widetilde{Y}_{t,i})} \tag{7}$$

with $N_0 = 0$, where $\widetilde{Y}_{t,i} = Y_i/t$, and $\widetilde{X}_{t,i} = (X_i - \widetilde{b}(t))/\widetilde{a}(t)$ with $\widetilde{a}(t) := a(\widetilde{d}(t))$ and $\widetilde{b}(t) := b(\widetilde{d}(t))$, where $a(t)$ and $b(t)$ satisfy (2), and $\widetilde{d}(t)$ is defined in (4).

The state space of N_t depends on the MDA of the observations (see (2)), and it can be written as $[0,\infty) \times \mathbb{E}$, where

$$\mathbb{E} = \begin{cases} [-\infty,\infty] \times [0,\infty] \setminus \{(-\infty,0)\}, & X_1 \in \mathrm{MDA}(\Lambda), \\ [0,\infty] \times [0,\infty] \setminus \{(0,0)\}, & X_1 \in \mathrm{MDA}(\Phi_\beta). \end{cases}$$

Throughout this paper, we use the standard vague topology on the space of point measures $M_p([0,\infty) \times \mathbb{E})$; see Resnick [22].

By [22, Proposition 3.21], vague convergence of measures in

$$\widetilde{d}(t)\mathbb{P}((\widetilde{X}_{t,i},\widetilde{Y}_{t,i}) \in \cdot) \xrightarrow{\mathrm{v}} \mu_0(\cdot) \quad \text{as } t \to \infty \tag{8}$$

is equivalent to $N_t \xrightarrow{\mathrm{D}} N$ as $t \to \infty$, where N is a Poisson random measure (PRM) with mean measure $\lambda \times \mu_0$, denoted by $\mathrm{PRM}(\lambda \times \mu_0)$, but in our setting, even more can be said.

Theorem 1. *Condition (8) is equivalent to*

$$\left(N_t, \frac{T(\widetilde{d}(t)\cdot)}{t}, \frac{\tau(t)}{\widetilde{d}(t)}\right) \xrightarrow{\mathrm{D}} (N, S_\alpha(\cdot), W_\alpha) \quad \text{as } t \to \infty \tag{9}$$

in the product space $M_p \times D[0,\infty) \times \mathbb{R}$ and the corresponding product topology (of vague, J_1, and Euclidean topologies), where N is a $\mathrm{PRM}(\lambda \times \mu_0)$, while S_α and $W_\alpha = W_\alpha(1)$ denote the α-stable subordinator and the first-passage time from (3) and (6), respectively.

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If one denotes the limiting point process above as $N = \sum_i \delta_{(T_i, P_i, Q_i)}$, then S_α has representation $S_\alpha(t) = \sum_{T_i \le t} Q_i$, while $W_\alpha(c)$ is the inverse of the increasing process S_α as in (5). Actually, due to continuity of the mapping $z \mapsto z^{\leftarrow}$, we prove a stronger result than the theorem above claims. Namely, under the same assumption, (9) holds jointly with (6).

Proof of Theorem 1. The first part of the proof follows the lines of the proof of Resnick [23, Theorem 7.1]. Observe that, for $s, t \ge 0$, the partial sum process $T(\cdot)$ satisfies

$$\frac{T(\widetilde{d}(t)s)}{t} = \sum_{i=1}^{\lfloor \widetilde{d}(t)s \rfloor} \widetilde{Y}_{t,i}.$$

From (3), we know that $T(\widetilde{d}(t)\cdot)/t \xrightarrow{\mathrm{D}} S_\alpha(\cdot)$ as $t \to \infty$ in $D([0,\infty),[0,\infty))$ with J_1-topology. The limiting process $S_\alpha(\cdot)$ has the same distribution as in (3). Note that the generalized inverse of $T(\widetilde{d}(t)\cdot)/t$ is equal to

$$\left(\frac{T(\widetilde{d}(t)\cdot)}{t}\right)^{\leftarrow}(u) = \inf\left\{s\colon \sum_{i=1}^{\lfloor \widetilde{d}(t)s \rfloor} Y_i > tu\right\} = \frac{\inf\{k \in \mathbb{N}\colon \sum_{i=1}^{k} Y_i > tu\}}{\widetilde{d}(t)} = \frac{\tau(tu)}{\widetilde{d}(t)}$$

for every $u \ge 0$, where $\tau(\cdot)$ is defined in (1). Using limiting theory for the partial sum process $T(\cdot)$ and continuity of the mapping $z \mapsto z^{\leftarrow}$, one can show that

$$\frac{\tau(t\cdot)}{\widetilde{d}(t)} \xrightarrow{\mathrm{D}} W_\alpha(\cdot) \quad \text{in } D([0,\infty),[0,\infty)) \tag{10}$$

with J_1-topology, where $W_\alpha \overset{\mathrm{D}}{=} S_\alpha^{\leftarrow}$. Finally, the continuous mapping argument yields the joint convergence in (9). □

Abusing the notation somewhat, for any time period $A \subseteq [0,\infty)$ and an arbitrary point measure $n \in M_p([0,\infty)\times\mathbb{E})$, we introduce the restricted point measure $n|_A$ as $n|_{A\times\mathbb{E}}$. Since the distribution of point processes N_t contains the information about all upper-order statistics in the sequence (X_n), it is useful to study the limit of point processes N_t restricted to time intervals determined by the renewal process $(\tau(t))$. For an illustration, denote by $M^\tau(t)$, $t \ge 0$, the maximum of observations arriving before time t, i.e. $M^\tau(t) = \sup\{X_i : i \le \tau(t)\}$. Note that

$$\frac{M^\tau(t) - \widetilde{b}(t)}{\widetilde{a}(t)} \le x \quad \Longleftrightarrow \quad N_t|_{[0,\tau(t)/\widetilde{d}(t)]\times(x,\infty]\times[0,\infty]} = 0$$

for all suitably chosen x. Using similar arguments, the following theorem yields the limiting distribution of all upper-order statistics until a given time t.

Theorem 2. *Assume that (8) holds, then*

$$N_t|_{[0,\tau(t)/\widetilde{d}(t)]} \xrightarrow{\mathrm{D}} N|_{[0,W_\alpha]} \quad \text{and} \quad N_t|_{[0,\tau(t)/\widetilde{d}(t))} \xrightarrow{\mathrm{D}} N|_{[0,W_\alpha)} \quad \text{as } t \to \infty, \tag{11}$$

where N is a PRM$(\lambda \times \mu_0)$*, and $W_\alpha = W_\alpha(1)$ denotes the first-passage time from (6).*

Proof. Observe that the limiting point process N in general has a point exactly at the right endpoint of the interval $[0, W_\alpha]$. This forbids direct application of the continuous mapping argument; see the discussion in Barczyk and Kern [2]. However, it turns out that one can adapt the argument from the proof of [3, Theorem 4.1].

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Denote by $n, n_t, t > 0$, arbitrary Radon point measures in $M_p([0,\infty)\times\mathbb{E})$. One can always write

$$n_t = \sum_i \delta_{(v_i^t, x_i^t, y_i^t)}, \qquad n = \sum_i \delta_{(v_i, x_i, y_i)}$$

for some sequences (v_i^t, x_i^t, y_i^t) and (v_i, x_i, y_i) with values in $[0,\infty)\times\mathbb{E}$. Further, denote their corresponding cumulative sum functions by

$$s_t(u) = \int_{[0,u]\times\mathbb{E}} y n_t(\mathrm{d}v, \mathrm{d}x, \mathrm{d}y) \qquad s(u) = \int_{[0,u]\times\mathbb{E}} y n(\mathrm{d}v, \mathrm{d}x, \mathrm{d}y), \qquad u \geq 0.$$

Suppose that they take finite value for each $u > 0$, but tend to ∞ as $u \to \infty$. This makes s_t and s well defined, unbounded, and nondecreasing elements of the space of càdlàg functions $D[0,\infty)$.

Assume that $(n_t, s_t) \to (n, s)$ in the product topology (of vague and J_1 topologies) as $t \to \infty$. If

$$0 < s(s^{\leftarrow}(1)-) < 1 < s(s^{\leftarrow}(1)) \quad \text{and} \quad n(\{v\}\times(0,\infty)\times\mathbb{E}) \leq 1 \quad \text{for all } v \geq 0, \tag{12}$$

by the straightforward adaptation of [3, Theorem 4.1] to the three-dimensional case

$$n_t|_{[0,s_t^{\leftarrow}(1)]} \xrightarrow{\mathrm{v}} n|_{[0,s^{\leftarrow}(1)]} \quad \text{and} \quad n_t|_{[0,s_t^{\leftarrow}(1))} \xrightarrow{\mathrm{v}} n|_{[0,s^{\leftarrow}(1))}.$$

We can now apply the continuous mapping theorem. Observe that, by Theorem 1,

$$\left(N_t, \frac{T(\widetilde{d}(t)\cdot)}{t}\right) \xrightarrow{\mathrm{D}} (N, S_\alpha(\cdot)).$$

On the other hand, the limiting Poisson process N and α-stable subordinator S_α satisfy the regularity assumptions in (12) with probability 1, and, therefore, (11) holds. □

The limiting distribution of all upper-order statistics until a given time t, determined by the theorem above, can be quite complicated depending on the joint distribution of N and W_α in (9). Therefore, we examine two important particular types of dependence between the X_n and the Y_n in detail. The first is called *the asymptotic tail independence* (see, e.g. De Haan and Ferreira [7] and Resnick [22]). Roughly speaking, it requires that when Y_n is large, there is negligible probability of X_n being large. The second type of dependence we consider in detail is called *the asymptotic full tail dependence*. Intuitively, it implies that the X_n and Y_n are highly tail-dependent in the sense that if one of them is large, then the other one is also large, asymptotically with probability 1 (see De Haan and Resnick [8] or Resnick [22, pp. 296–298]).

3.1. Asymptotic tail independence

Recall that $((X_n, Y_n))$ is an i.i.d. sequence of random vectors such that $X_1 \in \mathrm{MDA}(G_1)$, where $G_1 = \Lambda$ or Φ_β with $\beta > 0$ and $Y_1 \sim \mathrm{reg\,var}(\alpha)$ for $\alpha \in (0,1)$. Hence, $Y_1 \in \mathrm{MDA}(G_2)$, where $G_2 = \Phi_\alpha$. By $F_{X,Y}$ denote the joint cumulative distribution function of (X_1, Y_1), and set $U_X = 1/(1-F_X)$ and $U_Y = 1/(1-F_Y)$. Note that $U_X(X)$ and $U_Y(Y)$ are $\mathrm{reg\,var}(1)$ at ∞.

It is known, see De Haan and Resnick [8], that for the so-called tail independent X_1 and Y_1, the measure μ_0 in (8) is concentrated on the axes. In particular, for $X_1 \in \mathrm{MDA}(\Lambda)$ and $(x, y) \in [-\infty,\infty]\times[0,\infty]\setminus\{(-\infty, 0)\}$, we have

$$\mu_0(([-\infty, x]\times[0, y])^c) = -\log G_1(x) - \log G_2(y) = \exp(-x) + y^{-\alpha}. \tag{13}$$

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Similarly, if $X_1 \in \mathrm{MDA}(\Phi_\beta)$ and $(x, y) \in [0, \infty]^2 \setminus \{(0, 0)\}$, then

$$\mu_0(([0, x] \times [0, y])^c) = -\log G_1(x) - \log G_2(y) = x^{-\beta} + y^{-\alpha}. \tag{14}$$

Let $\{N_t : t \geq 0\}$ be point processes from (7). It is known that $N_t \xrightarrow{\mathrm{D}} N$, where N is a $\mathrm{PRM}(\lambda \times \mu_0)$, is equivalent to $F_{X,Y} \in \mathrm{MDA}(H)$ (we refer the reader to Resnick [22, Section 5.4] for a definition of a multivariate MDA) with

$$H(x, y) = \begin{cases} \exp\{-\mu_0(([-\infty, x] \times [0, y])^c)\}, & X_1 \in \mathrm{MDA}(\Lambda), \\ \exp\{-\mu_0(([0, x] \times [0, y])^c)\}, & X_1 \in \mathrm{MDA}(\Phi_\beta), \end{cases} \tag{15}$$

and μ_0 defined in (13) and (14). The measure μ_0 is often called the exponent measure.

Under our assumptions, necessary and sufficient condition for (8) can be inferred from the literature; namely, for a measure μ_0 as in (13) and (14), (8) is equivalent to

$$\lim_{x \to \infty} \mathbb{P}(X_1 > U_X^{\leftarrow}(x) \mid Y_1 > U_Y^{\leftarrow}(x)) = 0. \tag{16}$$

(see De Haan and Ferreira [7, Theorem 6.2.3]).

To show this, observe that (8) is equivalent to $F_{X,Y} \in \mathrm{MDA}(H)$ for H given by (15). Since $U_X(X)$ and $U_Y(Y)$ are reg var(1) at ∞, by Sibuya's theorem (see [8, Theorem 5]), $F_{X,Y} \in \mathrm{MDA}(H)$ is further equivalent to $\lim_{x \to \infty} \mathbb{P}(U_X(X_1) > x \mid U_Y(Y_1) > x) = 0$. It remains to prove the equivalence between this condition and (16). This can be done by an application of Leadbetter *et al.* [15, Theorem 1.7.13].

Remark 1. If (16) holds then X_1 and Y_1 are called asymptotically tail independent. Under this condition, the limiting Poisson process N in (9) can be decomposed into two independent parts, e.g. for $X_1 \in \mathrm{MDA}(\Phi_\beta)$ and $N = \sum_i \delta_{(T_i, P_i, 0)} + \sum_i \delta_{(T_i', 0, Q_i)}$. This makes the restriction of N to the first two coordinates $N^{(2)} = \sum_{i \geq 1} \delta_{(T_i, P_i)}$ independent of $S_\alpha = \sum_{T_i' \leq t} Q_i$, and W_α for the same reason.

Condition (16) does not seem straightforward to verify directly in general. In some examples one can verify a simpler sufficient condition introduced in the following lemma (see Anderson [1, Condition A]).

Lemma 1. *If*

$$\lim_{x \to \infty} \limsup_{y \to \infty} \mathbb{P}(X_1 > x \mid Y_1 > y) = 0 \tag{17}$$

then (16) holds.

Proof. Assuming (17) holds, for every $\varepsilon > 0$, there exists $x_1 \in \mathbb{R}$ such that, for all $x \geq x_1$, we have $\limsup_{y \to \infty} \mathbb{P}(X_1 > x \mid Y_1 > y) < \varepsilon$. In particular,

$$\limsup_{y \to \infty} \mathbb{P}(X_1 > x_1 \mid Y_1 > y) < \varepsilon.$$

Moreover, there exists $y_0 > 0$ such that, for all $y > y_0$, we have $\mathbb{P}(X_1 > x_1 \mid Y_1 > y) < \varepsilon$. If we denote $x_0 = \inf\{x > 0 \colon U_X^{\leftarrow}(x) > x_1, U_Y^{\leftarrow}(x) > y_0\}$ then, for all $x > x_0$,

$$\mathbb{P}(X_1 > U_X^{\leftarrow}(x) \mid Y_1 > U_Y^{\leftarrow}(x)) \leq \mathbb{P}(X_1 > x_1 \mid Y_1 > U_Y^{\leftarrow}(x)) < \varepsilon. \qquad \square$$

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Example 1. Let $M_k^\tau(t), t \geq 0$, represent the kth upper-order statistics in $\{X_1, \ldots, X_{\tau(t)}\}$. Under the assumption of asymptotic tail independence, from (11) we obtain, for all x such that $0 < G(x) < \infty$,

$$\begin{aligned}\mathbb{P}(M_k^\tau(t) \leq \widetilde{a}(t)x + \widetilde{b}(t)) &= \mathbb{P}\left(N_t\left(\left[0, \frac{\tau(t)}{\widetilde{d}(t)}\right] \times (x, \infty] \times \mathbb{R}_+\right) \leq k-1\right) \\ &\to \mathbb{P}(N([0, W_\alpha] \times (x, \infty] \times \mathbb{R}_+) \leq k-1) \\ &= \mathbb{E}\left(\frac{\Gamma_k(-W_\alpha \log G(x))}{\Gamma(k)}\right) \quad \text{as } t \to \infty,\end{aligned}$$

where $\Gamma_k(x)$ is the incomplete gamma function. For $k = 1$, i.e. for the partial maxima of the first $\tau(t)$ observations, the result first appeared in Berman [5].

3.2. Asymptotic full tail dependence

In the case when observations and interarrival times are exactly equal, the limiting behavior of the maximum can be found in Lamperti [14]. We keep the assumptions and notation from Section 3.1. The main difference is that the limiting measure μ_0 in (8) will be concentrated on a line, i.e. on the set

$$C = \begin{cases}\{(u, v) \in (-\infty, \infty) \times (0, \infty) \colon \exp(-u) = v^{-\alpha}\} & \text{if } G_1 = \Lambda, \\ \{(u, v) \in (0, \infty) \times (0, \infty) \colon u^{-\beta} = v^{-\alpha}\} & \text{if } G_1 = \Phi_\beta.\end{cases}$$

More precisely, for $y > 0$ and $C_{(y,\infty)} = \{(u, v) \in C \colon v > y\}$, the measure μ_0 is determined by $\mu_0(C_{(y,\infty)}) = y^{-\alpha}$.

In this setting, a necessary and sufficient condition for (8) is the full tail dependence condition known from the literature (see De Haan and Resnick [8, Theorem 6]), i.e. for the measure μ_0 above, (8) is equivalent to

$$\lim_{x \to \infty} \mathbb{P}(X > U_X^{\leftarrow}(x) \mid Y > U_Y^{\leftarrow}(x)) = 1. \tag{18}$$

4. Excursions and sojourn times of a continuous-time random walk

In this section we use results from Section 3 to obtain the limiting distribution of extremely long sojourn times at level 0 of a CTRW. We consider a CTRW which is a simple symmetric random walk subordinated to a certain renewal process. Hence, jumps are always of magnitude 1, while waiting times between jumps correspond to interarrival times of the subordinating renewal process.

Let $(E_n)_{n \geq 1}$ be an i.i.d. sequence of nonnegative random variables with finite expectation. Suppose $E_1 \in \mathrm{MDA}(G)$, where $G = \Lambda$ or $G = \Phi_\beta$ for $\beta > 1$. Denote the partial sum of the sequence (E_n) by $T(n) = \sum_{i=1}^n E_i$ and set $T(0) = 0$. Additionally, let $(N(t))_{t \geq 0}$ denote a renewal process generated by the sequence (E_n), i.e.

$$N(t) = \max\left\{k \geq 0 \colon T(k) = \sum_{i=1}^k E_i \leq t\right\}.$$

Let $(\varepsilon_n)_{n \geq 1}$ be an i.i.d. sequence of Rademacher random variables which is independent of the sequence (E_n). In particular, $\mathbb{P}(\varepsilon_1 = 1) = \mathbb{P}(\varepsilon_1 = -1) = \frac{1}{2}$. Set $S_0 = 0$ and let $S_n = \sum_{i=1}^n \varepsilon_i$

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for $n \geq 1$, then define continuous-time random walk as the process

$$Z(t) = S_{N(t)} = \sum_{i=1}^{N(t)} \varepsilon_i, \qquad t \geq 0.$$

Our goal is to determine the limiting distribution of the longest time interval during which the CTRW $Z(\cdot)$ remains at level 0 before time t, including the last possibly incomplete part.

4.1. The longest sojourn time at level 0

Let X_1 denote the time spent during the first visit to the origin. Clearly, $X_1 = E_1$. Since $Z(t) = 0$ for all $t \in [0, E_1)$, and $Z(E_1) \neq 0$, the time of the first return to the origin is defined as

$$A_1 = \inf\{t \geq X_1 \colon Z(t) = 0\}.$$

Further, if we set $Y_1 = A_1$ then Y_1 represents the sum of the first sojourn time at the origin and the time spent in the first excursion away from the origin. Clearly, the duration of the first excursion, denoted by R_1, satisfies $R_1 = Y_1 - X_1$. If we set $A_0 = 0$ then for $i \geq 1$, the time spent during the ith visit at the origin or the time spent on the ith excursion can be defined recursively as

$$\begin{aligned} X_i &= \inf\{t \geq A_{i-1} \colon Z(t) \neq 0\} - A_{i-1} = E_{N(A_{i-1})+1}, \\ A_i &= \inf\{t \geq A_{i-1} + X_i \colon Z(t) = 0\}, \end{aligned}$$

and note that $Y_i = A_i - A_{i-1}$ and $R_i = Y_i - X_i$. Clearly, the sequence $(X_n)_{n\geq 1}$ defined so is i.i.d. Moreover, X_1 belongs to the same MDA as E_1. An illustration of these random variables is given in Figure 1.

To apply Theorem 2 we need to determine the joint tail behavior of random variables Y_i and R_i. Note that $(R_n)_{n\geq 1}$ is an i.i.d. sequence of nonnegative random variables, and for $u > 0$, it satisfies $R_i \stackrel{\mathrm{D}}{=} \sum_{i=2}^{K} E_i$, where $K = \inf\{m \geq 1 \colon S_m = 0\}$, and $S_n = \sum_{i=1}^{n} \varepsilon_i$. It is well known (see, e.g. [9, Section 4.3]) that $K \sim \operatorname{reg\,var}(\frac{1}{2})$. Therefore, e.g. by [11, Proposition 4.3], we have $\mathbb{P}(R_1 > u) \sim \mathbb{E}(\sqrt{E_1})\mathbb{P}(K > u)$ as $u \to \infty$. In particular, $R_1 \sim \operatorname{reg\,var}(\frac{1}{2})$, and, consequently, $Y_1 \sim \operatorname{reg\,var}(\frac{1}{2})$.

It is not difficult now to verify (17), just observe that $Y_1 = R_1 + X_1$ with R_1 and X_1 independent. Now, using Theorem 1 and the analysis in Section 3.1, we obtain

$$\left(N_t, \frac{\tau(t)}{\widetilde{d}(t)}\right) \stackrel{\mathrm{D}}{\to} (N, W_{1/2}) \quad \text{as } t \to \infty, \tag{19}$$

where $N \sim \mathrm{PRM}(\lambda \times \mu_0)$. Since by the assumption that $X_1 \in \mathrm{MDA}(G)$, where $G = \Lambda$ or $G = \Phi_\beta$ for $\beta > 1$, μ_0 has the form given in (13) or (14), respectively. Moreover, in this case, by Remark 1, $N^{(2)}$ is independent of the random variable $W_{1/2}$.

In the following theorem we describe the asymptotic distribution of the longest sojourn time of a CTRW at level 0. Denote the longest sojourn time at level 0 up to time t by $Q(t)$. First observe that

$$Q(t) = \begin{cases} M^{\tau}(t) = \max\{X_1, \ldots, X_{\tau(t)}\} & \text{if } \sum_{i=1}^{\tau(t)-1} Y_i + X_{\tau(t)} < t, \\ \max\left\{M^{\tau-1}(t), t - \sum_{i=1}^{\tau(t)-1} Y_i\right\} & \text{if } t \leq \sum_{i=1}^{\tau(t)-1} Y_i + X_{\tau(t)}. \end{cases}$$

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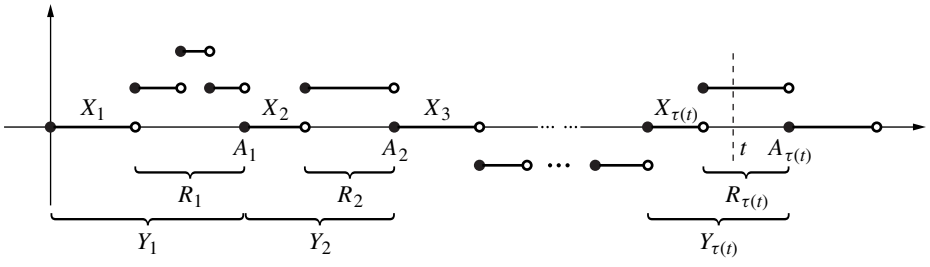


FIGURE 1: A sketch of a CTRW and associated random variables X_i, A_i, Y_i, and R_i.

In either case $M^{\tau-1}(t) \le Q(t) \le M^{\tau}(t)$. Further, observe that the random events

$$\left\{ \frac{M^{\tau}(t) - \widetilde{b}(t)}{\widetilde{a}(t)} \le x \right\} \quad \text{and} \quad \left\{ \frac{M^{\tau-1}(t) - \widetilde{b}(t)}{\widetilde{a}(t)} \le x \right\}$$

correspond to

$$\{N_t|_{[0,\tau(t)/\widetilde{d}(t)]\times(x,\infty]\times[0,\infty]} = 0\} \quad \text{and} \quad \{N_t|_{[0,\tau(t)/\widetilde{d}(t))\times(x,\infty]\times[0,\infty]} = 0\}.$$

Theorem 3. *Under the above assumptions*

$$\mathbb{P}\left(\frac{Q(t) - \widetilde{b}(t)}{\widetilde{a}(t)} \le x \right) \to \mathbb{E}(G(x)^{W_{1/2}}) \quad \text{as } t \to \infty. \tag{20}$$

Proof. Using (19) and Theorem 2, by the discussion before the theorem, for an arbitrary x, we obtain

$$\mathbb{P}\left(\frac{M^{\tau}(t) - \widetilde{b}(t)}{\widetilde{a}(t)} \le x \right) \to \mathbb{P}(N|_{[0,W_{1/2}]\times(x,\infty]\times[0,\infty]} = 0) \quad \text{as } t \to \infty,$$

and

$$\mathbb{P}\left(\frac{M^{\tau-1}(t) - \widetilde{b}(t)}{\widetilde{a}(t)} \le x \right) \to \mathbb{P}(N|_{[0,W_{1/2})\times(x,\infty]\times[0,\infty]} = 0).$$

By the independence between $N^{(2)}$ and $W_{1/2}$, see Remark 1, both limiting probabilities above are equal to

$$\begin{aligned} \mathbb{P}(N^{(2)}|_{[0,W_{1/2}]\times(x,\infty]} = 0) &= \int_0^{+\infty} \exp(-s\mu_0(([-\infty, x] \times [0, \infty])^c))\, \mathrm{d}F_{W_{1/2}}(s) \\ &= \mathbb{E}(G(x)^{W_{1/2}}). \end{aligned}$$

Since $M^{\tau-1}(t) \le Q(t) \le M^{\tau}(t)$ the left-hand side in (20) has the same limit as $t \to \infty$. □

4.2. The longest excursion

Similarly, one can determine the limiting distribution of the length of the longest excursion completed until time t, i.e. the longest time interval during which CTRW $Z(t)$ is not equal to 0, completed until time t. In contrast to Section 4.1, here we only assume that the E_i have finite expectation. Now, keeping the same notation, we are interested in the limiting distribution of

$$M^{\tau-1}(t) = \sup\{R_i : i \le \tau(t) - 1\}, \tag{21}$$

which corresponds to the length of the longest completed excursion.

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In this setting, the point processes N_t are constructed using sequences (R_n) (instead of (X_n)) and (Y_n). Recall that $U_R = 1/(1 - F_R)$, and observe that $U_R^{\leftarrow}(x) \le U_Y^{\leftarrow}(x)$ for all x, since $Y_1 = R_1 + X_1$. To show (18), observe that $x\mathbb{P}(Y_1 > U_Y^{\leftarrow}(x)) \to 1$, and, since Y_1 is reg var$(1/2)$, it is well known that the same holds for $R_1 = Y_1 - X_1$ because $\mathbb{E}X_1 < \infty$. Moreover, one can show that $x\mathbb{P}(R_1 > U_Y^{\leftarrow}(x)) = x\mathbb{P}(Y_1 - X_1 > U_Y^{\leftarrow}(x)) \to 1$. Clearly,

$$\begin{aligned}\mathbb{P}(R_1 > U_R^{\leftarrow}(x) \mid Y_1 > U_Y^{\leftarrow}(x)) &= \frac{\mathbb{P}(R_1 > U_R^{\leftarrow}(x),\, R_1 + X_1 > U_Y^{\leftarrow}(x))}{\mathbb{P}(Y_1 > U_Y^{\leftarrow}(x))} \\ &\ge \frac{\mathbb{P}(R_1 > U_Y^{\leftarrow}(x))}{\mathbb{P}(Y_1 > U_Y^{\leftarrow}(x))}.\end{aligned}$$

Since the last ratio above tends to 1 as $x \to \infty$, (18) is proved.

Using Theorem 2 and Section 3.2, we obtain

$$N_t|_{[0,\tau(t)/\widetilde{d}(t))} \xrightarrow{\mathrm{D}} N|_{[0,W_{1/2})} \quad \text{as } t \to \infty. \tag{22}$$

But, in this case, $N = \sum_{i\ge1}\delta_{(T_i,P_i,P_i)}$, while $W_{1/2} = W_{1/2}(1) = S_{1/2}^{\leftarrow}(1)$ with $S_{1/2}$ equal to the $\frac{1}{2}$-stable subordinator $S_{1/2}(t) = \sum_{T_i\le t} P_i$, which makes $W_{1/2}$ completely dependent on $N^{(2)} = \sum_{i\ge1}\delta_{(T_i,P_i)}$; see Remark 1.

Clearly (22) identifies the limiting distribution of all upper-order statistics in the sequence of excursions. It follows that rescaled excursions behave asymptotically as completed jumps of the subordinator $S_{1/2}$ until the passage of level 1. The distribution of those jumps is well understood; see Perman [20] or Pitman and Yor [21]. Alternatively, one could include the last, possibly incomplete excursion in the analysis. It turns out that the convergence result still holds, but one needs to add the last almost sure incomplete jump of the subordinator $S_{1/2}$ to the limiting distribution. All those jumps have the distribution of the Pitman–Yor point processes, see [20] and [21].

To illustrate our claims, the next theorem gives the limiting distribution of $M^{\tau-1}(t)$. It can be proved in a similar way to that of Theorem 3, since the R_i are regularly varying it can be deduced from the results in [2] as well. Observe that for the point process N in (22), the event $\{N|_{[0,W_{1/2})\times(x,\infty]\times[0,\infty]} = 0\}$ corresponds to $\{\sup_{T_i<W_{1/2}} P_i \le x\}$, i.e. to the event that the largest jump of $\frac{1}{2}$-stable subordinator, before it hits $[1,\infty)$, is less than or equal to x.

Theorem 4. *Under the assumptions above, $M^{\tau-1}(t)$ in (21) satisfies*

$$\frac{M^{\tau-1}(t)}{t} \xrightarrow{\mathrm{D}} V \quad \text{as } t \to \infty,$$

where V has the distribution of the largest jump of $\frac{1}{2}$-stable subordinator completed before the passage of level 1.

The limiting random variable V in Theorem 4 has a continuous distribution with support on $(0, 1)$, see Perman [20]. This is also the distribution of the longest completed excursion of the standard Brownian motion during time interval $[0, 1]$. Its density is given in [20, Corollary 9]. Interestingly, the closed-form expression for the density is known only on the interval $(\frac{1}{3}, 1)$.

In the special case when all waiting times of a CTRW have unit length, the model above boils down to the simple symmetric random walk on integers. In particular, from (22) one can deduce the asymptotic distribution of all upper-order statistics for the length of excursions of the simple symmetric random walk given in Csáki and Hu [6].

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