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# A NOTE ON SOME QUADRATICS AND CUBICS OVER FINITE FIELDS

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**Abstract**

We determine the conditions for the reducibility of some parametrised families of quadratic and cubic polynomials over finite fields, and count the number of irreducible trinomials. The existence of a factorisation of these polynomials plays an important role in studying the finite groups of exceptional Lie types.

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## 1. Introduction

The reducibility of polynomials over finite fields is important for many applications, including coding theory and cryptography. The well-known Berlekamp Algorithm [2] provides a method for factorising such polynomials. A more challenging problem is to decide reducibility for parametrised families of polynomials. For example, Dickson [5] determined for which nonzero parameters $\alpha, \beta$, in the finite field $\mathbb{F}_{p^n}$ with $p > 3$ a prime, the polynomial $x^3 + \alpha x + \beta$ is reducible over $\mathbb{F}_{p^n}$. The analogous characterisation was obtained by Williams [9] for finite fields of characteristic 2 and 3. Polynomials of this form are trinomials as they have exactly three nontrivial terms. Von zur Gathen [8] has considered the irreducibility of trinomials over finite fields and formulated some conjectures on their distribution. This line of research was continued by Ahmadi in [1], who solved some of these conjectures and also proved irreducibility results for trinomials of the form $x^d + \alpha x^k + \beta$, where $\alpha, \beta \in \mathbb{F}_q^\times$ with even degree $d$.

Results of this flavour have applications in group theory. For example, in the course of classifying the conjugacy classes of the simple group of Lie type $G_2(q)$, Chang [4]

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required knowledge of the parameters $\zeta \in \mathbb{F}_q$ for which $x^3 - 3x - \zeta$ is irreducible over $\mathbb{F}_q$. Similarly, the determination of the conjugacy classes in $F_4(2^n)$ required Shinoda [7] to classify those parameters $\zeta \in \mathbb{F}_{2^n}$ for which $x^3 + x + \zeta$ is irreducible and, at the same time, $x^2 + \zeta x + \zeta^2 + 1$ is reducible.

This note is also motivated by a problem in group theory (see [6]): the suborbit classification of the (primitive) actions of $G_2(q)$ requires a detailed analysis of the reducibility of $x^2 + \zeta x - \zeta$ and $x^3 - 3x^2 - \zeta$ with parameter $\zeta \in \mathbb{F}_q$. We consider a slightly more general case and state our main result as follows.

THEOREM 1.1. *Let $q$ be a prime power, $\gamma \in \mathbb{F}_q, \zeta \in \mathbb{F}_q^\times$, and*

$$P = x^2 + \zeta x - \zeta \quad \text{and} \quad Q = x^3 - \gamma x^2 - \zeta.$$

*For $q \equiv 0, 1 \bmod 3$, if $\gamma = 3$ or $\gamma^2 + 3\gamma + 9 = 0$, then at least one of $P$ and $Q$ is reducible. If $q \equiv 2 \bmod 3$ and $\gamma = 3$, then there are $\frac{1}{3}(q+1)$ elements $\zeta \in \mathbb{F}_q^\times$ such that both $P$ and $Q$ are irreducible.*

## 2. Proof of Theorem 1.1

Throughout, let $q = p^n$ be a prime power. If $p = 3$, then, by assumption, we have $\gamma = 0$ and $Q$ always has a root since $c \mapsto c^3$ is a Frobenius automorphism of $\mathbb{F}_{3^n}$. For $p \neq 3$, by construction, $P$ and $Q$ are trinomials over $\mathbb{F}_q$.

If $p > 3$, then rewriting $x = X + 3^{-1}\gamma$ yields $Q = X^3 - 3^{-1}\gamma^2 X - (\zeta + 2 \cdot 27^{-1}\gamma^3)$. Thus, in this case, [5, Theorems 1 and 3] applies, and $P$ and $Q$ are both irreducible only if we have $\zeta^2 + 4\zeta$ is a nonsquare and $-\zeta(4\gamma^3 + 27\zeta)$ is a square in $\mathbb{F}_q$, and $2^{-1}(\mu\sqrt{-3} + \zeta) + 27^{-1}\gamma^3$ is a noncube in the field extension $\mathbb{F}_q(\sqrt{-3})$, where $\mu \in \mathbb{F}_q$ with $81\mu^2 = -\zeta(4\gamma^2 + 27\zeta)$. This answers the irreducibility question, but it remains to count. Unfortunately, a similar result cannot be easily deduced from Williams' work [9] for $p = 2$. In the following, we present a uniform proof for all $p$ (excluding $p = 3$ as mentioned above). Since $P$ and $Q$ are reducible if and only if they have a root, for a fixed $\gamma \in \mathbb{F}_q^\times$, we consider

$$\mathcal{S}_1 = \{\zeta \in \mathbb{F}_q^\times \mid P \text{ has roots in } \mathbb{F}_q^\times\} \quad \text{and} \quad \mathcal{S}_2 = \{\zeta \in \mathbb{F}_q^\times \mid Q \text{ has roots in } \mathbb{F}_q^\times\}.$$

We will determine the size of these sets separately; then determine $|\mathcal{S}_1 \cup \mathcal{S}_2|$ from $|\mathcal{S}_1 \cup \mathcal{S}_2| = |\mathcal{S}_1| + |\mathcal{S}_2| - |\mathcal{S}_1 \cap \mathcal{S}_2|$.

The polynomial $P$ is reducible if and only if there are $u, v \in \mathbb{F}_q^\times$ such that $P = (x - u)(x - v)$, which is equivalent to $uv = -\zeta$ and $u + v = -\zeta$. Thus, $P$ is reducible if and only if $\zeta = v^2(1 - v)^{-1}$ for some $v \in \mathbb{F}_q \backslash \{0, 1\}$. If we define $f\colon \mathbb{F}_q \backslash \{0, 1\} \to \mathbb{F}_q^\times$ by $f(c) = c^2(1 - c)^{-1}$, then $|\mathcal{S}_1| = |\mathrm{Im}\, f|$. Note that $f(s) = f(t)$ with $s, t \notin \{0, 1\}$ if and only if $s = t$ or $t = s(s - 1)^{-1}$. Since $s = s(s - 1)^{-1}$ if and only if $q$ is odd and $s = 2$,

$$|\mathcal{S}_1| = |\mathrm{Im}\, f| = \begin{cases} \frac{1}{2}(q - 2) & \text{if } q \equiv 0 \bmod 2, \\ \frac{1}{2}(q - 1) & \text{otherwise.} \end{cases}$$

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Now, we consider $\mathcal{S}_2$. For a fixed $\gamma \in \mathbb{F}_q^\times$, we know that $Q$ has a root $u \in \mathbb{F}_q$ if and only if $\zeta = u^3 - \gamma u^2$. Since $\zeta \neq 0$, it follows that $|\mathcal{S}_2| = |\mathrm{Im}\, g|$, where $g\colon \mathbb{F}_q \setminus \{0, \gamma\} \to \mathbb{F}_q^\times$ is the map $g(c) = c^3 - \gamma c^2$. Note that $g(s) = g(t)$ for $s, t \notin \{0, \gamma\}$ if and only if $s = t$ or $t = ks$ for some $k \in \mathbb{F}_q \backslash \{0, 1\}$ such that

$$g(ks) - g(s) = s^2(k-1)(k^2 s + ks - \gamma k + s - \gamma) = 0.$$

Since $s^2(k-1) \neq 0$, the latter is equivalent to $\ell_s(k) = 0$, where

$$\ell_s(k) = k^2 + r_s k + r_s \quad \text{with} \quad r_s = 1 - \gamma s^{-1} \neq 0.$$

Note that any such $k$ satisfies $ks \notin \{0, \gamma\}$ (for otherwise, $r_s = 0$ or $\ell_s = 1$, which is a contradiction). Thus, we are interested in

$$\kappa(s) = |\{k \in \mathbb{F}_q \setminus \{0, 1\} \mid \ell_s(k) = 0\}|,$$

which informs us of Im $g$. Suppose $\ell_s(k)$ has roots $u, v$; note that $u, v \notin \{0, -1\}$. Then, $u + v = -r_s$ and $uv = r_s$, so $r_s = -v^2(1+v)^{-1}$. Moreover, $k = 1$ is a root of $\ell_s(k)$ if and only if $q$ is odd and $r_s = -2^{-1}$. We have $u = v$ if and only if $p = 3$ and $r_s = 1$, or $q \equiv \pm 1 \bmod 6$ and $s \in \{-3^{-1}\gamma, \gamma\}$. With such notation, since $s \neq \gamma$ by assumption,

$$\kappa(s) = \begin{cases} 0 & \text{if } r_s \notin \{-v^2(1+v)^{-1} \mid v \in \mathbb{F}_q \backslash \{0, -1\}\}; \\ 1 & \text{if } q \equiv \pm 1 \bmod 6 \text{ and } r_s \in \{4, -2^{-1}\}; \\ 2 & \text{if } r_s \in \{-v^2(1+v)^{-1} \mid v \in \mathbb{F}_q \backslash \{0, -1\}\} \setminus \{1, 4, -2^{-1}\}; \end{cases}$$

recall that $p = 2$ is allowed, in which case $-2^{-1}$ does not occur. Since the map $g$ restricted to the subset $\mathcal{K}_1 = \{s \in \mathbb{F}_q \backslash \{0, \gamma\} \mid \kappa(s) = 0\}$ is injective, restricted on $\mathcal{K}_2 = \{s \in \mathbb{F}_q \backslash \{0, \gamma\} \mid \kappa(s) = 1\}$ is 2-to-1, restricted to $\mathcal{K}_3 = \{s \in \mathbb{F}_q \backslash \{0, \gamma\} \mid \kappa(s) = 2\}$ is 3-to-1, and $\mathcal{K}_1 \sqcup \mathcal{K}_2 \sqcup \mathcal{K}_3 = \mathbb{F}_q \backslash \{0, \gamma\}$,

$$|\mathrm{Im}\, g| = |\mathcal{K}_1| + \tfrac{1}{2}|\mathcal{K}_2| + \tfrac{1}{3}|\mathcal{K}_3|.$$

More specifically, observe that for any $c \neq 1$, there exists $s = \gamma(1-c)^{-1}$ such that $r_s = c$. Also, by construction, $r_s \neq 1$ and $1 \in \{-v^2(1+v)^{-1} \mid v \in \mathbb{F}_q \backslash \{0, -1\}\}$ if and only if $q \equiv 0, 1 \bmod 3$. Moreover, by symmetry,

$$|\mathcal{K}_1| = |\{-v^2(1+v)^{-1} \mid v \in \mathbb{F}_q \backslash \{0, -1\}\}| = |\mathrm{Im}\, f|$$

and

$$|\mathcal{K}_3| = |\{-v^2(1+v)^{-1} \mid v \in \mathbb{F}_q \backslash \{0, -1\}\} \setminus \{1, 4, -2^{-1}\}|$$
$$= \begin{cases} \frac{1}{2}(q-2) & \text{if } q \equiv 2 \bmod 6, \\ \frac{1}{2}(q-2) - 1 & \text{if } q \equiv 4 \bmod 6, \\ \frac{1}{2}(q-1) - 3 & \text{if } q \equiv 1 \bmod 6, \\ \frac{1}{2}(q-1) - 2 & \text{if } q \equiv 5 \bmod 6. \end{cases}$$

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We therefore deduce that

$$|\mathcal{S}_2| = |\mathrm{Im}\ g| = \begin{cases} q-2-\frac{1}{2}(q-2)+\frac{1}{6}(q-2) = \frac{2}{3}(q-2) & \text{if } q \equiv 2 \bmod 6, \\ q-2-\frac{1}{2}(q-2)+1+\frac{1}{6}(q-4) = \frac{2}{3}(q-1) & \text{if } q \equiv 4 \bmod 6, \\ q-2-\frac{1}{2}(q-1)+1+\frac{2}{2}+\frac{1}{6}(q-7) = \frac{2}{3}(q-1) & \text{if } q \equiv 1 \bmod 6, \\ q-2-\frac{1}{2}(q-1)+\frac{2}{2}+\frac{1}{6}(q-5) = \frac{2}{3}(q-2) & \text{if } q \equiv 5 \bmod 6. \end{cases}$$

In summary,

$$|\mathcal{S}_2| = |\mathrm{Im}\ g| = \begin{cases} \frac{2}{3}(q-1) & \text{if } q \equiv 1 \bmod 3, \\ \frac{2}{3}(q-2) & \text{if } q \equiv 2 \bmod 3. \end{cases}$$

It remains to examine $\mathcal{S}_1 \cap \mathcal{S}_2$. With the same setup as above, $\zeta \in \mathcal{S}_1 \cap \mathcal{S}_2$ if and only if there exist $s \in \mathbb{F}_q \backslash \{0, \gamma\}$ and $t \in \mathbb{F}_q \backslash \{0, 1\}$ such that $\zeta = f(t) = g(s)$, where $f, g$ are as defined above. The latter equality is equivalent to $h(t) = 0$, where

$$h(t) = t^2 + (s^3 - \gamma s^2)t - (s^3 - \gamma s^2).$$

When $p \neq 2$, the quadratic equation $h(t) = 0$ in $t$ holds for some $t \in \mathbb{F}_q \backslash \{0, 1\}$ if and only if

$$c(s) = s^2(s-\gamma)(s^3 - \gamma s^2 + 4)$$

is a square in $\mathbb{F}_q$. In particular, given $s \notin \{0, \gamma\}$, we see $c(s) = 0$ if and only if $-4 = s^3 - \gamma s^2$. If $\gamma^2 + 3\gamma + 9 = 0$, then $q \equiv 1 \bmod 3$ and $\gamma = (-3 \pm 3w)2^{-1}$ for $w \in \mathbb{F}_q^\times$ such that $-3 = w^2$, and

$$c(s) = s^2(s-\gamma)(s+3^{-1}\gamma)(s-2\cdot 3^{-1}\gamma)^2.$$

Note that when $\gamma = 3$, the same factorisation exists; that is,

$$c(s) = s^2(s-\gamma)(s+1)(s-2)^2.$$

For $p \neq 2$, if $q \equiv 1 \bmod 3$, then let $\gamma \in \{3, (-3+3w)2^{-1}, (-3-3w)2^{-1}\}$; if $q \equiv 2 \bmod 3$, then let $\gamma = 3$. Recall that if $p \neq 2$, then $\kappa(s) = 2$ if and only if the discriminant $s^{-2}(\gamma - s)(\gamma + 3s)$ of $\ell_s(k)$ is a square in $\mathbb{F}_q^\times$, and $\kappa(s) = 0$ if and only if $(\gamma - s)(\gamma + 3s)$ is a nonsquare in $\mathbb{F}_q^\times$. Since $-3$ is a square in $\mathbb{F}_q^\times$ if $q \equiv 2 \bmod 3$, and is a nonsquare if $q \equiv 1 \bmod 3$ and $-3(s-\gamma)(s+3^{-1}\gamma) = (\gamma - s)(3s+\gamma)$, it follows that

$$\begin{aligned} \mathcal{S}_1 \cap \mathcal{S}_2 &= \{-4\} \cup \{s^3 - \gamma s^2 \mid s \in \mathbb{F}_q \backslash \{0, \gamma\} \text{ and } c(s) \text{ is a square in } \mathbb{F}_q^\times\} \\ &= \{-4\} \cup \begin{cases} \{s^3 - \gamma s^2 \mid s \in \mathbb{F}_q \backslash \{0, \gamma\} \text{ and } \kappa(s) = 2\} & \text{if } q \equiv 1 \bmod 6, \\ \{s^3 - \gamma s^2 \mid s \in \mathbb{F}_q \backslash \{0, \gamma\} \text{ and } \kappa(s) = 0\} & \text{if } q \equiv 5 \bmod 6. \end{cases} \end{aligned}$$

Now, consider the case where $p = 2$ and $q = 2^n$. The trace map over $\mathbb{F}_{2^n}$ is defined by

$$\mathrm{tr}\colon \mathbb{F}_{2^n} \to \{0, 1\}, \quad s \mapsto \left( \sum_{i=0}^{n-1} s^{(2^i)} \right) \bmod 2.$$

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It follows that $\mathrm{tr}(x) = \mathrm{tr}(x^2)$ and $\mathrm{tr}(x) + \mathrm{tr}(y) = \mathrm{tr}(x+y)$ for all $x, y \in \mathbb{F}_{2^n}$. We now consider $\gamma = 1$ or $\gamma^2 + \gamma + 1 = 0$. Note that there exists $\gamma \in \mathbb{F}_q^\times$ such that $\gamma^2 + \gamma + 1 = 0$ if and only if $q \equiv 1 \bmod 3$ and $\gamma^3 = 1$. Moreover, it is well known that $x^2 + \alpha x + \beta$ for $\alpha, \beta \in \mathbb{F}_q^\times$ is reducible over $\mathbb{F}_q$ if and only if $\mathrm{tr}(a^{-2}b) = 0$ (see [3, Theorem 6.69]). Thus, $h(t) = 0$ has:

- two solutions if and only if $\mathrm{tr}(s^{-2}(s+\gamma)^{-1}) = 0$ and
- no solution if and only if $\mathrm{tr}(s^{-2}(s+\gamma)^{-1}) = 1$.

However, when $p = 2$, the quadratic $\ell_s(k) = 0$ has two solutions if and only if $\mathrm{tr}(s(s+\gamma)^{-1}) = 0$ and no solution if and only if $\mathrm{tr}(s(s+\gamma)^{-1}) = 1$. Observe that if $\gamma^3 = 1$, then $1 + s^3 = \gamma^3 + s^3 = (\gamma + s)(\gamma^2 + s^2 + \gamma s)$ and so

$$\mathrm{tr}(s^{-2}(s+\gamma)^{-1}) + \mathrm{tr}(s(s+\gamma)^{-1}) = \mathrm{tr}(\gamma^2 s^{-2}) + \mathrm{tr}(\gamma s^{-1}) + \mathrm{tr}(1) = \mathrm{tr}(1).$$

Since $\mathrm{tr}(1) = 0$ if and only if $q \equiv 1 \bmod 3$, and $\mathrm{tr}(1) = 1$ if and only if $q \equiv 2 \bmod 3$, it follows that

$$\begin{aligned}\mathcal{S}_1 \cap \mathcal{S}_2 &= \{s^3 + \gamma s^2 \mid s \in \mathbb{F}_q \backslash \{0, \gamma\} \text{ and } \mathrm{tr}(s^{-2}(s+1)^{-1}) = 0\} \\ &= \begin{cases} \{s^3 + \gamma s^2 \mid s \in \mathbb{F}_q \backslash \{0, \gamma\} \text{ and } \kappa(s) = 2\} & \text{if } q \equiv 4 \bmod 6, \\ \{s^3 + \gamma s^2 \mid s \in \mathbb{F}_q \backslash \{0, \gamma\} \text{ and } \kappa(s) = 0\} & \text{if } q \equiv 2 \bmod 6. \end{cases}\end{aligned}$$

Together with our discussion for odd $q$ above, we have shown that

$$|\mathcal{S}_1 \cap \mathcal{S}_2| = \begin{cases} \frac{1}{6}(q-1) & \text{if } q \equiv 1 \bmod 6, \\ \frac{1}{2}(q-1) & \text{if } q \equiv 5 \bmod 6, \\ \frac{1}{6}(q-4) & \text{if } q \equiv 4 \bmod 6, \\ \frac{1}{2}(q-2) & \text{if } q \equiv 2 \bmod 6. \end{cases}$$

In summary, applying the Principle of Inclusion-Exclusion, we see that

$$|\mathcal{S}_1 \cup \mathcal{S}_2| = \begin{cases} q-1 & \text{if } q \equiv 1 \bmod 3 \text{ and } \gamma = 3 \text{ or } \gamma^2 + 3\gamma + 9 = 0, \\ \frac{2}{3}(q-2) & \text{if } q \equiv 2 \bmod 3 \text{ and } \gamma = 3. \end{cases}$$

This completes the proof. □

## 3. Further discussion

Although our proof for the main result excludes the case $p = 3$, our discussion for $\mathcal{S}_2$ remains valid for any $\gamma \in \mathbb{F}_q^\times$ in all characteristics. More specifically, from the discussion above, we also obtain the following result.

COROLLARY 3.1. *Let $q$ be a prime power, $\zeta \in \mathbb{F}_q^\times$ and*

$$Q = x^3 - \gamma x^2 - \zeta.$$

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*Then, for each* $\gamma \in \mathbb{F}_q^\times$,

$$|\{\zeta \in \mathbb{F}_q^\times \mid Q \text{ is reducible}\}| = \begin{cases} \frac{1}{3}(2q-3) & \text{if } q \equiv 0 \bmod 3, \\ \frac{2}{3}(q-1) & \text{if } q \equiv 1 \bmod 3, \\ \frac{2}{3}(q-2) & \text{if } q \equiv 2 \bmod 3. \end{cases}$$

The next result is a corollary of [9, Theorem 1] that turns out to be useful in finding suborbit representatives in primitive $G_2(q)$-actions, where $G_2(q)$ is the finite simple group of exceptional Lie type $G_2$ over $\mathbb{F}_q$. In the case where $p = 2$, setting $x = y + 1$ yields the depressed form of $Q$, namely, $D = y^3 + \gamma y + \zeta$, which has the same reducibility as $Q$. In particular, by assumption, we have $\gamma^3 = 1$ if $q \equiv 1 \bmod 3$ and $\gamma = 1$ if $q \equiv 2 \bmod 3$. For the sake of completeness, we include a restatement of [9, Theorem 1].

THEOREM 3.2 [9, Theorem 1]. *Let* $q = 2^n$ *for some positive integer* $n$. *Let* $D = x^3 + \gamma x + \zeta$, *where* $\gamma \in \mathbb{F}_q, \zeta \in \mathbb{F}_q^\times$. *Let* $t_1, t_2$ *denote the roots of* $t^2 + \zeta t + \gamma^3 = 0$. *Then,* $t_1, t_2$ *lie in* $\mathbb{F}_q$ *if* $\operatorname{tr}(\gamma^3\zeta^{-2}) = 0$ *and in* $\mathbb{F}_{q^2}$ *otherwise.*

(a) *D has three distinct roots in* $\mathbb{F}_q$ *if and only if* $\operatorname{tr}(\gamma^3\zeta^{-2}) = \operatorname{tr}(1)$ *and* $t^2 + \zeta t + \gamma^3$ *has roots* $t_1, t_2$ *that are cubes in* $\mathbb{F}_q$ *(n even) or in* $\mathbb{F}_{q^2}$ *(n odd).*
(b) *D has precisely one root in* $\mathbb{F}_q$ *if and only if* $\operatorname{tr}(\gamma^3\zeta^{-2}) \neq \operatorname{tr}(1)$.
(c) *D has no root in* $\mathbb{F}_q$ *if and only if* $\operatorname{tr}(\gamma^3\zeta^{-2}) = \operatorname{tr}(1)$ *and* $t^2 + \zeta t + \gamma^3$ *has roots* $t_1, t_2$ *that are noncubes in* $\mathbb{F}_q$ *(n even) or in* $\mathbb{F}_{q^2}$ *(n odd).*

COROLLARY 3.3. *Let* $q = 2^n$ *for some positive integer* $n$ *and let* $\zeta \in \mathbb{F}_q^\times$. *Let* $\xi$ *be a primitive element of* $\mathbb{F}_{q^2}^\times$ *and*

$$\epsilon = \begin{cases} 1 & \text{if } q \equiv 1 \bmod 3, \\ -1 & \text{if } q \equiv 2 \bmod 3. \end{cases}$$

*Let* $D = x^3 + \gamma x + \zeta$, *where* $\zeta \in \mathbb{F}_q^\times$ *and* $\gamma = 1$ *if* $n$ *is odd, or* $\gamma \in \mathbb{F}_q$ *such that* $\gamma^3 = 1$ *if* $n$ *is even. Then, the following hold.*

(a) *D is irreducible over* $\mathbb{F}_q$ *if and only if* $\zeta = \eta^{3j+1} + \eta^{-3j-1}$ *for some* $j$ *with* $0 \le j < \frac{1}{3}(q - \epsilon)$, *where* $\eta = \xi^{q+\epsilon}$.
(b) *D has precisely one root in* $\mathbb{F}_q$ *if and only if* $\zeta = \nu^k + \nu^{-k}$ *for some* $k$ *with* $0 < k \le \lfloor \frac{1}{2}(q + \epsilon) \rfloor$, *where* $\nu = \xi^{q-\epsilon}$.
(c) *If D is irreducible over* $\mathbb{F}_q$, *then* $x^2 + \zeta(\zeta + 1)^{-1}x + \zeta(\zeta + 1)^{-1}$ *is reducible over* $\mathbb{F}_q$.
(d) *If* $r \in \mathbb{F}_q^\times$ *is a root of D, then* $x^2 + \zeta x + \zeta$ *and* $x^2 + rx + r$ *have the same reducibility over* $\mathbb{F}_q$.

PROOF. (a) It follows from Theorem 3.2(c) that $D$ is irreducible if and only if $\operatorname{tr}(\zeta^{-2}) = \operatorname{tr}(1)$ and the roots of $x^2 + \zeta x + 1$ (in $\mathbb{F}_q$ if $n$ is even or in $\mathbb{F}_{q^2}$ if $n$ is odd)

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are noncubes. Moreover, $t \neq 0$ is a root of $x^2 + \zeta x + 1$ if and only if $t^{-1}$ is also a root and $t + t^{-1} = \zeta$. Since $\zeta \in \mathbb{F}_q^\times$, it follows that $t^q + t^{-q} = t + t^{-1}$, and so either $t \in \mathbb{F}_q^\times$ or $t \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$ and $t^{q+1} = 1$. Such $t$ is a noncube if and only if $t = \eta^{3j\pm1}$ for some $j \in \mathbb{Z}$, where $\eta = \xi^{q+\epsilon}$. However, if $t = \eta^{3j-1}$, then $t^{-1} = \eta^{-3j+1} = \eta^{3j'+1}$ for some $j' = -j \in \mathbb{Z}$. Thus, the irreducibility criteria in Theorem 3.2(c) is equivalent to $\zeta = \eta^{3j+1} + \eta^{-3j-1}$ for some $j \in \mathbb{Z}$.

Note that if $j - k = \frac{1}{3}(q - \epsilon)$, then $\eta^{3j+1} + \eta^{-3j-1} = \eta^{3k+1} + \eta^{-3k-1}$; thus, for $\zeta \in \mathbb{F}_q^\times$, the trinomial $D$ is irreducible if and only if $\zeta \in \{\eta^{3j+1} + \eta^{-3j-1} \mid 0 \leq j < \frac{1}{3}(q - \epsilon)\}$; there are precisely $\frac{1}{3}(q - \epsilon)$ such irreducible polynomials.

(b) From Theorem 3.2(b), we see that $D$ has precisely one root in $\mathbb{F}_q^\times$ if and only if $\mathrm{tr}(\zeta^{-2}) \neq \mathrm{tr}(1)$. Since $\mathrm{tr}(1) \equiv n - 1 \bmod 2$, it follows that $\mathrm{tr}(\zeta^{-2}) \neq \mathrm{tr}(1)$ if and only if the quadratic $x^2 + \zeta x + 1$ has two roots $t, t^{-1} \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$ if $q \equiv 1 \bmod 3$, or two roots $t, t^{-1} \in \mathbb{F}_q^\times$ if $q \equiv 2 \bmod 3$; in both cases, $t + t^{-1} = \zeta$. As seen above, such roots $t$ satisfy $t^{q+\epsilon} = 1$. That is, $D$ has precisely one root in $\mathbb{F}_q$ if and only if $\zeta = \nu^k + \nu^{-k}$, where $\nu = \xi^{q-\epsilon}$, for some $0 < k < q + \epsilon$. Moreover, if $i + j = q + \epsilon$, then $\nu^i + \nu^{-i} = \nu^j + \nu^{-j}$. From this, we can conclude that $x^2 + \zeta x + 1$ is irreducible over $\mathbb{F}_q$ if and only if $\zeta \in \{\nu^k + \nu^{-k} \mid 0 < k \leq \lfloor (q + \epsilon)/2 \rfloor\}$.

(c) The quadratic $x^2 + \zeta(\zeta + 1)^{-1}x + \zeta(\zeta + 1)^{-1}$ is reducible if and only if $\mathrm{tr}(1 + \zeta^{-1}) = 0$. Since $\mathrm{tr}(\zeta^{-2}) = \mathrm{tr}(1)$ by Theorem 3.2(c) and $\mathrm{tr}(\zeta^{-1}) = \mathrm{tr}(\zeta^{-2})$, the claim follows.

(d) By assumption, we have $r(r + 1)^2 = \zeta$. Since

$$\mathrm{tr}(r^{-1}(r + 1)^{-2}) + \mathrm{tr}(r^{-1}) = \mathrm{tr}((r + 1)^{-1}) + \mathrm{tr}((r + 1)^{-2}) = 0,$$

it follows that $\mathrm{tr}(\zeta^{-1}) = \mathrm{tr}(r^{-1})$, namely, $x^2 + x + r^{-1}$ is reducible over $\mathbb{F}_q$ if and only if $x^2 + x + \zeta^{-1}$ is reducible over $\mathbb{F}_q$, which proves the claim. □

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