FUNCTIONS REGULAR IN THE UNIT CIRCLE

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1.1. Let $$f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$$

be a function regular for $|z| < 1$. With the hypotheses $f(0) = 0$ and

$$u(r,\theta) < (1-r)^{-\alpha} \quad (0 \leqslant r < 1), \tag{1}$$

for some positive constant α, Cartwright (1) has deduced upper bounds for $|f(z)|$ in the unit circle. Three cases have arisen and according as (1) holds with $\alpha < 1$, $\alpha = 1$ or $\alpha > 1$, the bounds on each circle $|z| = r$ are given respectively by

$$(a) \quad |f(z)| < K(\alpha)(1-r)^{-1},$$

$$(b) \quad |f(z)| < \frac{K(\alpha)}{1-r}\left(\log\frac{1}{1-r}\right)^2,$$

$$(c) \quad |f(z)| < K(\alpha)(1-r)^{-\alpha},$$

$K(\alpha)$ being a constant which depends only on the corresponding value of α which occurs in (1). We shall always use the symbols K and A to represent constants dependent on certain parameters such as α, not necessarily having the same value at each occurrence.

1.2. In the present paper we replace the hypothesis (1) by a similar inequality in which the index α is not necessarily independent of r. We write this index as $\sigma(1-r)$ and assume that it has a finite limit value σ as r tends to 1.

The case in which $\sigma < 1$ is covered by the results of § 1·1 and the inequality (a) holds. This paper is concerned mainly with the case in which $\sigma = 1$, and two theorems are proved. In the first we relax the sufficient conditions of § 1.1 which imply (a).

Theorem 1. *Let $\chi(x)$ be a positive continuous function for which*

$$\int_{x_0}^{\infty} \frac{d\xi}{\xi\chi(\xi)}$$

is finite. Suppose also that there is a finite constant d such that $\chi(x)$ and $(\log x)^d[\chi(x)]^{-1}$ increase to infinity with x for $x > x_0$. Then if $f(z)$ is regular for $|z| < 1$, $f(0) = 0$ and

$$u(r,\theta) < \frac{1}{1-r}\left[\chi\left(\frac{1}{1-r}\right)\right]^{-2} \quad (r_0 < r < 1),$$

there is a constant K which depends only on r_0 and the particular function $\chi(x)$ such that when $|z| \leqslant r$,

$$|f(z)| < K(1-r)^{-1} \quad (0 \leqslant r < 1).$$

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The second theorem corresponds to (*b*) and gives a generalization of that particular result. In the statement and proof we make use of the function

$$L(t;\, m, n;\, \alpha_m, \ldots, \alpha_n) \equiv \prod_{\nu=m}^{n} (\log_\nu 1/t)^{\alpha_\nu}, \tag{2}$$

where $1 \leqslant m \leqslant n$ and $\log_m x \equiv \log(\log_{m-1} x)$, $\log_1 x \equiv \log x$.

It will always be assumed that $\alpha_m, \ldots, \alpha_n$ are real and $\alpha_m \neq 0$. Thus according as α_m is positive or negative the function (2) will tend to infinity or zero as t tends to zero. In particular, we write

$$L(t;\, m) \equiv L(t;\, 1, m;\, -1, \ldots, -1),$$

and thus state

THEOREM 2. *Suppose that $f(z)$ is regular for $|z| < 1$, $f(0) = 0$ and*

$$u(r,\theta) < (1-r)^{-1}\{L(1-r;\, m)\}^2 L(1-r;\, m, n;\, \alpha_m, \ldots, \alpha_n),$$

where $\alpha_m > 0$ and $r_0 \leqslant r < 1$. Then there is a constant K which depends only on the set of values $r_0, m, n, \alpha_m, \ldots, \alpha_n$ such that when $|z| \leqslant r$, $r_0 \leqslant r < 1$,

$$|f(z)| < K(1-r)^{-1} L(1-r;\, m, n;\, \alpha_m, \ldots, \alpha_n).$$

We note that the condition $\alpha_m > 0$ is essential. For if $\alpha_m < 0$, $f(z)$ satisfies the hypotheses of Theorem 1. The only instance in which $\alpha_m = 0$ may arise is when the hypothesis on $u(r,\theta)$ is of the form

$$u(r,\theta) < (1-r)^{-1}\{L(1-r;\, m)\}^2.$$

But this may be rewritten as

$$u(r,\theta) < (1-r)^{-1}\{L(1-r;\, m+1)\}^2 L(1-r;\, m+1, m+1;\, 2),$$

and we have the conclusion that

$$|f(z)| < K(1-r)^{-1} L(1-r;\, m+1,\, m+1;\, 2).$$

The result (*b*) is obtained by taking $m = 0$ in this case.

When the limit value σ is greater than 1, the conclusion of the corresponding theorem here is similar to (*c*). In fact, by introducing proximate orders, only minor modifications of the proof of (*c*) in (1) are necessary to give this generalization. Consequently in this paper we shall define the proximate orders and state the theorem without proof.

A function $\rho(x)$ is said to be a Lindelöf proximate order if it satisfies the following conditions ((3), pp. 64–7):

$$\left.\begin{aligned} &\rho(x) \text{ is real, continuous and differentiable in adjacent intervals for } x > x_0;\\ &\text{as } x \to \infty,\ \rho(x) \to \rho \text{ where } 0 < \rho < \infty;\\ &\rho'(x)\, x \log x \to 0 \text{ as } x \to \infty. \end{aligned}\right\} \tag{3}$$

A Lindelöf proximate order $\rho(x)$ being given there always exists a function $V(z)$ which is regular for

$$|\theta| \leqslant \min(\pi - \delta, \pi/\rho - \delta) \quad (r > r_0)$$

for each positive value δ, and such that

$$V(re^{i\theta}) \sim r^{\rho(r)} e^{i\theta\rho}, \tag{4}$$

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as $r \to \infty$ in this sector. As an example of such a function when $\rho < 1$, Valiron ((3), pp. 129–30) considered

$$V(z) = \frac{\sin \pi\rho}{\pi} \int_{x_0}^{\infty} \frac{z\xi^{\rho(\xi)}}{\xi(\xi+z)} d\xi, \tag{5}$$

and we shall examine this particular function later in more detail. We now state the theorem mentioned above as

THEOREM 3. *Let $\sigma(1/x) = \rho(x)$ be a Lindelöf proximate order for which $\rho(x) \to \rho > 1$ as $x \to \infty$. Then if $f(z)$ is a function regular for $|z| < 1, f(0) = 0$ and*

$$u(r, \theta) < (1-r)^{-\sigma(1-r)} \quad (r_0 \leqslant r < 1),$$

there is a constant K which depends only on r_0 and the particular function $\sigma(1-r)$ such that, if $|z| \leqslant r$,

$$|f(z)| < K(1-r)^{-\sigma(1-r)} \quad (r_0 \leqslant r < 1).$$

1.3. The proofs of Theorems 1 and 2 are based on the proof of (*b*) given in (1). Some of the lemmas now required are simple extensions of those which appeared there. To avoid undue repetition of methods, we merely state without proof a few results that will be needed later.

LEMMA 1.† *Let $\sigma(1/x) = \rho(x)$ be a Lindelöf proximate order and*

$$|u(r, \theta)| < (1-r)^{-\sigma(1-r)},$$

for $r_0 \leqslant r < 1$. Then there is a constant K which depends only on r_0 and the particular function $\sigma(1-r)$ such that

$$|f(z)| < K(1-r)^{-\sigma(1-r)},$$

when $|z| \leqslant r$ and $r_0 \leqslant r < 1$.

LEMMA 2‡. *Let $P(x)$ be a function positive and continuous for $x > x_0$ and such that*

$$P_1(x) = \int_{x_0}^{x} \frac{P(\xi)}{\xi} d\xi$$

increases to infinity with x. Suppose further that $F(z)$ is a function regular for $|\theta| \leqslant \frac{1}{2}\pi$, $r \geqslant r_0$, and such that in this region

$$\Re F(re^{i\theta}) < AP(r), \quad |F(r_0 e^{i\theta})| < B.$$

Then for each positive value δ there is a constant K depending on δ, r_0 and the particular function $P(x)$ such that

$$\Re F(re^{i\theta}) > -K(B + AP_1(r)),$$

when $|\theta| \leqslant \frac{1}{2}\pi - \delta$, $r > r_0$.

LEMMA 3‡. *Suppose that $F(z)$ is regular and*

$$\Re F(re^{i\theta}) < A[\chi(r)]^{-1},$$

for $|\theta| \leqslant \frac{1}{2}\pi$, $r \geqslant r_0$, where $\chi(r)$ is an increasing continuous function of r for which

$$\int_{x_0}^{\infty} \frac{d\xi}{\xi\chi(\xi)}$$

† This lemma corresponds to Theorem 1 of (1).

‡ Lemmas 2 and 3 may both be obtained by the method used to establish Lemma 2 of (1). See also (2), pp. 95–6.

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is finite. If also $|F(r_0 e^{i\theta})| < B$ *when* $|\theta| \leqslant \frac{1}{2}\pi$, *then for each positive value* δ *there is a constant* K *depending on* δ, r_0 *and the particular function* $\chi(r)$ *such that*

$$\Re F(r e^{i\theta}) > -K(B + Ar),$$

when $|\theta| \leqslant \frac{1}{2}\pi - \delta$, $r > r_0$.

2.1. In (1) the inequality (*b*) was established by mapping a certain subregion of the unit circle on to a region in which a lemma corresponding to Lemma 2 could be applied. For the proofs of Theorems 1 and 2 we follow a similar procedure, but now we require a more delicate type of transformation. We must consider two different types of auxiliary functions, and, since the simpler of these occurs in the proof of Theorem 2, we deal with this first.

We consider the transformation

$$1 - z = (1 - w)\{\log_m 1/(1 - w)\}^\beta \quad (\beta > 0) \tag{6}$$

applied to the region

$$|\psi| \leqslant \tfrac{1}{2}\pi, \quad s < s_0, \tag{7}$$

where

$$1 - w = s e^{i\psi}, \quad 1 - z = t e^{i\phi}.$$

Each branch of the many-valued function on the right-hand side of (6) is regular and uniform in (7), and we shall examine that branch which makes some interval of the real axis $r_0 < z < 1$ correspond to an interval of the real axis $w(r_0) < w < 1$. The relation (6) may be rewritten as

$$\frac{t}{s} e^{i(\phi - \psi)} = \{\log_m (s e^{i\psi})^{-1}\}^\beta, \tag{8}$$

which is a form more appropriate for an examination of the transformation near the point $w = 1$.

If ψ is bounded it can be shown that

$$\log_m (s e^{i\psi})^{-1} = (\log_m 1/s)\{1 - i\psi L(s; m)(1 + o(1))\}, \tag{9}$$

as s tends to zero, where $L(s; m)$ tends to zero with s by definition. By means of a Taylor expansion it follows that

$$\{\log_m (s e^{i\psi})^{-1}\}^\beta = (\log_m 1/s)^\beta \{1 - i\beta\psi L(s; m)(1 + o(1))\},$$

and from (8) we have the asymptotic properties of the transformation (6) in the region (7) near the point $w = 1$;

$$t/s \sim (\log_m 1/s)^\beta, \tag{10}$$

$$\phi - \psi = -\beta\psi L(s; m)(1 + o(1)).$$

The latter gives

$$\phi = \psi - \beta\psi L(s; m)(1 + o(1)),$$

and if s_0 is small enough ϕ is an increasing function of ψ in the region (17). Thus when $|\psi| \leqslant \frac{1}{2}\pi$,

$$|\phi| \leqslant \tfrac{1}{2}\pi - \tfrac{1}{2}\beta\pi L(s; m)(1 + o(1)),$$

that is

$$\tfrac{1}{2}\pi - |\phi| \geqslant \tfrac{1}{2}\beta\pi L(s; m)(1 + o(1)). \tag{11}$$

Now if $|z| < 1$, then $|\phi| < \frac{1}{2}\pi$ and

$$1 - r \leqslant t, \tag{12}$$

$$\tfrac{1}{2}\pi - |\phi| \leqslant \tfrac{1}{2}\pi \cos\phi = \frac{\pi}{4}\frac{1 - r^2 + t^2}{t}. \tag{13}$$

Also we have, by (10) and the definition of $L(t; m)$,

$$L(t; m) \sim L(s; m), \tag{14}$$

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as s tends to zero. Hence (11), (13) and (14) give

$$1-r^2>\beta tL(t;\, m)-t^2 \quad (t<t_0),$$

from which

$$1-r>\tfrac{1}{2}\beta tL(t;\, m) \quad (t<t_0), \tag{15}$$

if t_0 is sufficiently small. The inequalities (15) represent a subregion of the unit circle which has the point $z = 1$ on its boundary. We have shown that this region is transformed by (6) into a region which contains (7), for some sufficiently small positive value s_0 depending on β, m and t_0.

2·2. The quantities $\alpha_{m+1}, \ldots, \alpha_n$ which occur in the expression $L(t;\, m,\, n;\, \alpha_m, \ldots, \alpha_n)$ will be the same throughout the present section, and for brevity we write

$$L(t;\, m, n;\, \alpha_m, \ldots, \alpha_n) = L^*(t;\, m;\, \alpha_m).$$

The inequality (12) then gives

$$\{L(1-r;\, m)\}^2 L^*(1-r;\, m;\, \alpha_m) \sim \{L(t;\, m)\}^2 L^*(t;\, m;\, \alpha_m),$$

as $t \to 0$ in the region (15). But by hypothesis

$$u(r,\theta) < (1-r)^{-1}\{L(1-r;\, m)\}^2 L^*(1-r;\, m;\, \alpha_m),$$

for $|z| < 1$; and so, in (15),

$$u(r,\theta) < 4(\beta t)^{-1} L(t;\, m)\, L^*(t;\, m;\, \alpha_m), \tag{16}$$

if t_0 is small enough.

The function

$$\Phi(w) = f\left[1-(1-w)\left\{\log_m \frac{1}{1-w}\right\}^{\beta}\right]$$

is regular in the region (7), and by (6), (10) and (16) we have

$$\Re\Phi(w) < 4(\beta s)^{-1} L(s;\, m)\, L^*(s;\, m;\, \alpha_m - \beta).$$

We now choose $\beta = \frac{1}{2}\alpha_m$ and show that the function $\Psi(W) = \Phi(1/W)$ satisfies the hypotheses of Lemma 2. For if $W = R e^{i\Theta}$, then $\Psi(W)$ is regular in a sector $|\Theta| \leqslant \frac{1}{2}\pi$, $R > R_0$, and

$$\Re\Psi(W) < 4\beta^{-1} R L(R^{-1};\, m)\, L^*(R^{-1};\, m;\, \tfrac{1}{2}\alpha_m).$$

With the notation of Lemma 2 we write $A = 4\beta^{-1}$,

$$P(x) = L(x^{-1};\, m)\, L^*(x^{-1};\, m;\, \tfrac{1}{2}\alpha_m),$$

and therefore

$$\begin{aligned}
P_1(x) &= \int_{x_0}^{x} \frac{P(\xi)}{\xi}\, d\xi \\
&= \int_{x_0}^{x} \xi^{-1} L(\xi^{-1};\, m)\, L^*(\xi^{-1};\, m;\, \tfrac{1}{2}\alpha_m)\, d\xi \\
&= \int_{x_0}^{x} (\xi \log\xi \ldots \log_m \xi)^{-1} (\log_m \xi)^{\frac{1}{4}\alpha_m} L^*(\xi^{-1};\, m;\, \tfrac{1}{4}\alpha_m)\, d\xi \\
&< L^*(x^{-1};\, m;\, \tfrac{1}{4}\alpha_m) \int_{x_0}^{x} (\xi \log\xi \ldots \log_m \xi)^{-1} (\log_m \xi)^{\frac{1}{4}\alpha_m}\, d\xi \\
&< 4(\alpha_m)^{-1} L^*(x^{-1};\, m;\, \tfrac{1}{4}\alpha_m)\, (\log_m x)^{\frac{1}{4}\alpha_m} \\
&= 4(\alpha_m)^{-1} L^*(x^{-1};\, m;\, \tfrac{1}{2}\alpha_m),
\end{aligned}$$

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since $\alpha_m > 0$, and $L^*(\xi^{-1}; m; \frac{1}{4}\alpha_m) \to \infty$ as $\xi \to \infty$. From the inequality (c) we deduce that there is a constant B depending on $R_0, m, n, \alpha_m, \ldots, \alpha_n$ such that

$$|\Psi(R_0 e^{i\Theta})| < B,$$

when $|\Theta| \leqslant \frac{1}{2}\pi$. An application of Lemma 2 then gives

$$\Re\Psi(W) > -KRP_1(R),$$

and so

$$\Re\Phi(w) > -Ks^{-1}L^*(s; m; \tfrac{1}{2}\alpha_m),$$

in some region $s < s_0$, $|\psi| \leqslant \frac{1}{2}\pi - \delta$, $0 < \delta < \frac{1}{2}\pi$. Hence by (6) and (10),

$$\begin{aligned} u(r,\theta) &> -Kt^{-1}L^*(t; m; \tfrac{1}{2}\alpha_m + \beta) \\ &= -Kt^{-1}L^*(t; m; \alpha_m), \end{aligned}$$

in a region which includes a portion of the real axis $\theta = 0$, $t < t_0$. In particular, when $\theta = 0$ we have $1 - r = t$ and

$$u(r,0) > -K(1-r)^{-1}L^*(1-r; m; \alpha_m),$$

for $r_1 < r < 1$, where $r_1 = 1 - t_0$.

Now it is evident that the preceding argument could have been applied to any function $f(ze^{i\theta_0})$ and the same result obtained. Thus

$$u(r,\theta) > -K(1-r)^{-1}L^*(1-r; m; \alpha_m) \quad (0 \leqslant \theta < 2\pi),$$

which, together with the hypothesis of Theorem 2, gives

$$|u(r,\theta)| < K(1-r)^{-1}L^*(1-r; m; \alpha_m)$$

for $r_1 \leqslant |z| < 1$. Since $f(z)$ is regular for $|z| < 1$ an application of the maximum modulus principle shows that the same inequality, possibly with a different value of K, holds in the annulus $r_0 \leqslant |z| < 1$.

Finally, we observe that the relation

$$(1-r)^{-1}L^*(1-r; m; \alpha_m) = (1-r)^{-\sigma(1-r)}$$

defines a Lindelöf proximate order $\rho(x) = \sigma(1/x)$. We therefore apply Lemma 1 and obtain the inequality

$$|f(z)| < K(1-r)^{-1}L^*(1-r; m; \alpha_m),$$

thus completing the proof of Theorem 2.

3.1. The proof of Theorem 1 given here is similar to that of Theorem 2, but, because of the generality of the hypothesis which is based on the convergence of a certain integral, particular functions such as the logarithm occurring in the transformation (6) are not suitable. We require a function with behaviour similar to that of $V(z)$ in (4), but with modulus much smaller than is possible there if we restrict ourselves to the condition that $\rho(x)$ tends to a positive limit as x tends to infinity. Such a function may be obtained from (5) by dividing by an appropriate power of z. By proceeding in this way the resulting function may have its modulus as small as we please, and we avoid the difficulties that arise if we allow the function $\rho(\xi)$ occurring in (5) to tend to zero, although this is effectively what happens in the relation corresponding to (4).

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We define a *zero proximate order* as a function which satisfies all the conditions (3) with the exception that its limit as x tends to infinity is zero. If $\rho(x)$ is a zero proximate order, the function $\frac{1}{2}+\rho(x)$ is a Lindelöf proximate order which tends to the value $\frac{1}{2}$ as x tends to infinity, and

$$V_1(z) = \int_{x_0}^{\infty} \frac{z\xi^{\frac{1}{2}+\rho(\xi)}}{\xi(\xi+z)}\,d\xi \tag{17}$$

is a function regular in the sector

$$|\theta| \leqslant \pi-\delta, \quad r > r_0, \tag{18}$$

for each positive δ. By (4) we have

$$V_1(r\,e^{i\theta}) \sim \pi r^{\frac{1}{2}+\rho(r)} e^{i\frac{1}{2}\theta},$$

for each value θ in (18). The function

$$U(z) = z^{-\frac{1}{2}}V_1(z) \tag{19}$$

is also regular in (18) and $\qquad U(r\,e^{i\theta}) \sim \pi r^{\rho(r)},$

as r tends to infinity. It follows that

$$\Re U(r\,e^{i\theta}) = \pi r^{\rho(r)}(1+o(1)), \quad \Im U(r\,e^{i\theta}) = o(r^{\rho(r)}); \tag{20}$$

that is, the magnitudes of the real and imaginary parts of $U(r\,e^{i\theta})$ are appreciably different for large values of r in (18). In the following paragraphs we suppose that $|U(z)|$ decreases to zero as $|z|$ increases to infinity in (18). By (20) we have that $\Re U(z)$ decreases at approximately the same rate along each line for which θ is constant in (18). It will be shown later that $\Im U(z)$ possesses a similar property.

3·2. From each function $\chi(x)$ defined in Theorem 1, we must be able to construct a corresponding proximate order. The next lemma shows that this is possible.

LEMMA 4. *Suppose that $\chi(x)$ satisfies the hypotheses of Theorem* 1 *and write*

$$F(x) = \int_x^{\infty} \frac{d\xi}{\xi\chi(\xi)}. \tag{21}$$

Then the function

$$\rho(x) = \frac{\log F(x)}{\log x} \tag{22}$$

is a zero proximate order.

By hypothesis, the function $F(x)$ is real-valued and differentiable for sufficiently large values of x and so also is $\rho(x)$. We must prove further that as $x\to\infty$ both $\rho(x)$ and $\rho'(x)\,x\log x$ tend to zero.

From the hypothesis of Theorem 1 it can be deduced that there exist constants K_1 and d such that

$$\chi(x) < K_1(\log x)^d,$$

for $x > x_0$. We may assume $d > 1$. Since $F(x)$ is bounded for $x > x_0$, there is also a constant K_2 such that

$$K_2 > F(x) > \frac{1}{K_1}\int_x^{\infty} \frac{d\xi}{\xi(\log\xi)^d} = \frac{(\log x)^{1-d}}{K_1(d-1)}.$$

Thus

$$\frac{\log K_2}{\log x} > \frac{\log F(x)}{\log x} > \frac{-\log K_1(d-1)-(d-1)\log_2 x}{\log x},$$

and $\rho(x)\to 0$ as $x\to\infty$, by (22).

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Now from (21) we have
$$F'(x) = -[x\chi(x)]^{-1}, \tag{23}$$
and by the differentiation of (22)
$$\rho'(x) = \frac{F'(x)}{F(x)\log x} - \frac{\log F(x)}{x(\log x)^2}.$$
Hence
$$\begin{aligned}\rho'(x)\,x\log x &= \frac{xF'(x)}{F(x)} - \frac{\log F(x)}{\log x}\\ &= -[\chi(x)\,F(x)]^{-1} - \rho(x).\end{aligned}$$
Since $\chi(x)\,(\log x)^{-d}$ decreases as x increases to infinity, we have
$$\begin{aligned}F(x) &= \int_x^\infty \frac{(\log\xi)^d}{\chi(\xi)}\frac{d\xi}{\xi(\log\xi)^d}\\ &> \frac{(\log x)^d}{\chi(x)}\int_x^\infty \frac{d\xi}{\xi(\log\xi)^d}\\ &= (\log x)\,[(d-1)\,\chi(x)]^{-1}.\end{aligned}$$
It follows that as $x\to\infty$, $[F(x)\,\chi(x)]^{-1}\to 0$ and, since $\rho(x)\to 0$, we have $\rho'(x)\,x\log x\to 0$. Thus $\rho(x)$ is a zero proximate order.

3·3. In §3·1 it was shown how, a zero proximate order $\rho(x)$ being given, it is possible to construct a function $U(z)$ regular in the sector (18) and satisfying (20). By (17), (19) and (22) we can write
$$\begin{aligned}U(z) &= \int_{x_0}^\infty \left(\frac{z}{\xi}\right)^{\frac{1}{2}} \frac{\xi^{\rho(\xi)}}{\xi+z}\,d\xi\\ &= \int_{x_0}^\infty \left(\frac{re^{i\theta}}{\xi}\right)^{\frac{1}{2}} \frac{(\xi+re^{-i\theta})\,F(\xi)}{r^2+2r\xi\cos\theta+\xi^2}\,d\xi,\end{aligned}$$
and thus
$$U(z) = \int_{x_0}^\infty \left(\frac{r}{\xi}\right)^{\frac{1}{2}} \frac{[(r+\xi)\cos\frac{1}{2}\theta - i(r-\xi)\sin\frac{1}{2}\theta]\,F(\xi)}{r^2+2r\xi\cos\theta+\xi^2}\,d\xi. \tag{24}$$
By separating the real and imaginary parts of this function, we have
$$\Im U(z) = -\int_{x_0}^\infty \left(\frac{r}{\xi}\right)^{\frac{1}{2}} \frac{(r-\xi)\sin\frac{1}{2}\theta F(\xi)}{r^2+2r\xi\cos\theta+\xi^2}\,d\xi. \tag{25}$$
The two following lemmas enable us to estimate the magnitude of this latter integral for large values of r. The first is concerned with a particular form of the integral, and the more general case is then deduced.

LEMMA 5. *Suppose that $\chi(x)$ satisfies the hypotheses of Theorem* 1 *and $F(x)$ is defined by* (21). *Then*
$$\int_{x_0}^\infty \frac{\xi^{-\frac{1}{2}}(r-\xi)\,F(\xi)}{(r+\xi)^2}\,d\xi = \frac{A(r)\,r^{-\frac{1}{2}}}{\chi(r)},$$
where the upper and lower bounds of $A(r)$ are both finite and positive for $r > r_0$.

We observe that, by integrating by parts,
$$\int_{x_0}^\infty \frac{\xi^{-\frac{1}{2}}F(\xi)}{(r+\xi)}\,d\xi = \left[\frac{2\xi^{\frac{1}{2}}F(\xi)}{r+\xi}\right]_{x_0}^\infty + 2\int_{x_0}^\infty \frac{\xi^{\frac{1}{2}}F(\xi)}{(r+\xi)^2}\,d\xi - 2\int_{x_0}^\infty \frac{\xi^{\frac{1}{2}}F'(\xi)}{r+\xi}\,d\xi,$$

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which, by a rearrangement of terms and (23), gives

$$\int_{x_0}^{\infty} \frac{\xi^{-\frac{1}{2}}(r-\xi)\,F(\xi)}{(r+\xi)^2}\,d\xi = 2\int_{x_0}^{\infty} \frac{\xi^{-\frac{1}{2}}}{(r+\xi)\,\chi(\xi)}\,d\xi + O\left(\frac{1}{r}\right). \tag{26}$$

We divide the range of integration of the right-hand integral and, since $\chi(\xi)$ and $\xi^{\frac{1}{4}}[\chi(\xi)]^{-1}$ both increase to infinity with ξ, the following inequalities hold for sufficiently large values of r:

$$\int_{x_0}^{r} \frac{\xi^{-\frac{1}{2}}}{(r+\xi)\,\chi(\xi)}\,d\xi < \frac{r^{-\frac{3}{4}}}{\chi(r)}\int_{x_0}^{r} \xi^{-\frac{3}{4}}d\xi \quad < \frac{4r^{-\frac{1}{2}}}{\chi(r)},$$

$$\int_{r}^{\infty} \frac{\xi^{-\frac{1}{2}}}{(r+\xi)\,\chi(\xi)}\,d\xi < \frac{1}{\chi(r)}\int_{r}^{\infty} \xi^{-\frac{3}{2}}d\xi \quad < \frac{2r^{-\frac{1}{2}}}{\chi(r)},$$

$$\int_{x_0}^{r} \frac{\xi^{-\frac{1}{2}}}{(r+\xi)\,\chi(\xi)}\,d\xi > \frac{r^{-\frac{3}{2}}}{2\chi(r)}\int_{x_0}^{r} d\xi \quad = \frac{r^{-\frac{3}{2}}(r-x_0)}{2\chi(r)},$$

$$\int_{r}^{\infty} \frac{\xi^{-\frac{1}{2}}}{(r+\xi)\,\chi(\xi)}\,d\xi > \frac{r^{\frac{1}{2}}}{2\chi(r)}\int_{r}^{\infty} \xi^{-2}d\xi = \frac{r^{-\frac{1}{2}}}{2\chi(r)}.$$

Together with (26) these inequalities complete the proof of the lemma. In particular, as $r\to\infty$ we have

$$2 \leqslant \underline{\lim}\, A(r) \leqslant \overline{\lim}\, A(r) \leqslant 12.$$

As a corollary we have

LEMMA 6. *Let us define the functions $F(x)$ and $\chi(x)$ as in Lemma 5 and*

$$\tau(x_0, x;\, r, \theta) = \int_{x_0}^{x} \frac{\xi^{-\frac{1}{2}}(r-\xi)\,F(\xi)}{r^2+2r\xi\cos\theta+\xi^2}\,d\xi.$$

Then for each positive value δ,

$$\tau(x_0, \infty;\, r, \theta) = A(r, \theta)\, r^{-\frac{1}{2}}[\chi(r)]^{-1},$$

where $A(r,\theta)$ has upper and lower bounds which are finite and positive for $|\theta| \leqslant \pi-\delta$, $r > r_0$.

Without loss of generality we may assume that $x_0 = 1$. Since $F(\xi)$ tends to zero as ξ increases to infinity, we have, as r tends to infinity,

$$|\tau(r^2, \infty;\, r, \theta)| < F(r)\int_{r^2}^{\infty} \frac{\xi^{-\frac{1}{2}}}{\xi - r}\,d\xi = O\left(\frac{1}{r}\right). \tag{27}$$

For the remainder of the range of integration we use successively the substitutions

$$\xi = ry, \quad y = 1/\eta,$$

and so have

$$\begin{aligned}
\tau(1, r^2;\, r, \theta) &= r^{-\frac{1}{2}}\int_{r^{-1}}^{r} \frac{y^{-\frac{1}{2}}(1-y)\,F(ry)}{1+2y\cos\theta+y^2}\,dy \\
&= r^{-\frac{1}{2}}\int_{r^{-1}}^{1} \frac{y^{-\frac{1}{2}}(1-y)\,F(ry)}{1+2y\cos\theta+y^2}\,dy - r^{-\frac{1}{2}}\int_{r^{-1}}^{1} \frac{\eta^{-\frac{1}{2}}(1-\eta)\,F(r/\eta)}{1+2\eta\cos\theta+\eta^2}\,d\eta \\
&= r^{-\frac{1}{2}}\int_{r^{-1}}^{1} \frac{y^{-\frac{1}{2}}(1-y)\,(F(ry)-F(r/y))}{1+2y\cos\theta+y^2}\,dy.
\end{aligned}$$

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The integrand of the latter integral is positive, since $F(x)$ is a decreasing function of x. Further, when $|\theta| \leqslant \pi - \delta$, $\cos\theta \geqslant -\cos\delta$ and

$$\tfrac{1}{2}(1-\cos\delta)(1+\xi)^2 \leqslant 1 + 2\xi\cos\theta + \xi^2 \leqslant (1+\xi)^2.$$

Therefore if r_0 is large enough,

$$\tau(1, r^2;\ r, 0) \leqslant \tau(1, r^2;\ r, \theta) \leqslant \operatorname{cosec}^2(\tfrac{1}{2}\delta)\,\tau(1, r^2;\ r, 0) \quad (r > r_0)$$

and the lemma now follows from Lemma 5 and (27).

3·4. The behaviour of the function $U(z)$ is summarized by the two relations

$$\Re U(r e^{i\theta}) = \pi F(r)(1 + o(1)), \tag{28}$$

$$\Im U(r e^{i\theta}) = A(r, \theta)\sin\tfrac{1}{2}\theta\,[\chi(r)]^{-1}, \tag{29}$$

as $r \to \infty$ in the sector (18). We now consider the transformation

$$1 - z = (1-w)\{1 - U(1/(1-w))\}, \tag{30}$$

applied to the region (7).

With the notation of §2·1, (30) may be written as

$$\frac{t}{s} e^{i(\phi-\psi)} = 1 - U(1/(s e^{i\psi})); \tag{31}$$

and since, by (28) and (29), $U(r e^{i\theta}) \to 0$ as $r \to \infty$, we have that in the region (7) near the point $w = 1$

$$t = s(1 + o(1)), \quad t < s. \tag{32}$$

Further, if w belongs to this latter region we have, by (28), (29) and (31), that

$$\begin{aligned} \phi - \psi &= \tan^{-1}\left\{\frac{-\Im U(1/(s e^{i\psi}))}{1 - \Re U(1/(s e^{i\psi}))}\right\} \\ &= -A_1(s, \psi)\,\psi[\chi(1/s)]^{-1}, \end{aligned} \tag{33}$$

where the upper and lower bounds of $A_1(s, \psi)$ in (7) are positive and finite if s_0 is sufficiently small. Consequently

$$\phi = \psi - A_1(s, \psi)\,\psi[\chi(1/s)]^{-1},$$

and there is a finite positive constant A such that

$$|\phi| \leqslant \tfrac{1}{2}\pi - \tfrac{1}{2}\pi A[\chi(1/s)]^{-1},$$

when $s e^{i\psi}$ is a point of the region (7). It follows that

$$\tfrac{1}{2}\pi - |\phi| \geqslant \tfrac{1}{2}\pi A[\chi(1/s)]^{-1},$$

and so by (13) and (32)

$$1 - r > At[\chi(1/t)]^{-1} \quad (t < t_0). \tag{34}$$

Hence the transformation (30) maps the region (34) on to some region which contains (7) for an appropriate positive value s_0.

In the region (34), the hypothesis of Theorem 1 may be written, with (12), as

$$u(r, \theta) < [At\chi(1/t)]^{-1}.$$

Then if

$$\Phi_1(w) = f\{w + (1-w)\,U(1/(1-w))\},$$

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we have by (32) $$\Re\Phi_1(w) < K[s\chi(1/s)]^{-1},$$

for some finite constant K. It can be verified, therefore, that the function

$$\Psi_1(W) = \Phi_1(1/W)$$

satisfies the hypotheses of Lemma 3. Consequently there exists a constant K depending only on δ, s_0 and the particular function $\chi(x)$ such that

$$\Re\Phi_1(w) > -Ks^{-1}$$

in the region $|\psi| \leqslant \frac{1}{2}\pi - \delta$, $s < s_0$. The remainder of the proof of Theorem 1 follows in the same way as that of Theorem 2 at the end of § 2·2. In fact we can deduce

$$|u(r,\theta)| < K(1-r)^{-1},$$

and subsequently, by Lemma 1,

$$|f(z)| < K(1-r)^{-1},$$

when $|z| \leqslant r$, $0 \leqslant r < 1$.

4. We conclude with an example showing that Theorem 2 gives a best possible conclusion with the given hypotheses in certain particular cases. Further examples may be constructed to show that the general result is also best possible, but in this case the theory becomes unduly complicated.

Let $$g(z) = -(1-z)^{-1}\log_m(b/(1-z)), \tag{35}$$

where b is a positive constant large enough to ensure the regularity of $g(z)$ for $|z| < 1$. If we write $M(r) = \max_{|z|=r} |g(z)|$, then

$$M(r) \sim (1-r)^{-1}\log_m(1/(1-r)), \tag{36}$$

as $r \to 1$.

Using (9) we may write

$$g(z) = -(te^{i\phi})^{-1}\{\log_m(b/t)\}\{1 - i\phi L(t/b;\, m)(1+o(1))\},$$

and so

$$\Re g(z) = -(bT)^{-1}(\log_m 1/T)\{\cos\phi - \phi\sin\phi\, L(T;\, m)(1+o(1))\},$$

as $t = bT \to 0$. Thus $\Re g(z)$ is negative in the region

$$\cos\phi \geqslant \pi L(T;\, m), \quad T < T_0, \tag{37}$$

for a sufficiently small value of T_0. Since $\cos\phi$ is positive when $|z| < 1$, it remains to consider $\Re g(z)$ in the region defined by

$$0 < \cos\phi < \pi L(T;\, m), \quad r < 1, \quad T < T_0. \tag{38}$$

In this region we have

$$1 - r < 1 - r^2 + t^2 = 2t\cos\phi < 2\pi bTL(T;\, m),$$

that is $$1 - r < T_1, \tag{39}$$

where $$T_1 = 2\pi bTL(T;\, m). \tag{40}$$

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It follows that when (38) holds,

$$\begin{aligned}\Re g(z) &< \pi(bT)^{-1}(\log_m 1/T)\,L(T;\,m)\\ &= 2\pi^2 T_1^{-1}(\log_m 1/T)\,(L(T;\,m))^2\\ &< 4\pi^2 T_1^{-1}(\log_m 1/T_1)\,(L(T_1;\,m))^2,\end{aligned}$$

by (40). Since the last term of this series of inequalities increases to infinity with T_1^{-1}, we have, from (39),

$$\begin{aligned}\Re g(z) &< 4\pi^2(1-r)^{-1}(\log_m 1/(1-r))\,(L(1-r;\,m))^2\\ &= 4\pi^2(1-r)^{-1}(L(1-r;\,m))^2\,L(1-r;\,m,m;\,1),\end{aligned}$$

which holds for $r<1$, $T<T_0$. In the remainder of the unit circle $\Re g(z)$ has an upper bound depending only on m and T_0. Hence there are constants K and r_0 such that

$$\Re g(z) < K(1-r)^{-1}(L(1-r;\,m))^2\,L(1-r;\,m,m;\,1),$$

when $r_0<r<1$. This example indicates that the conclusion of Theorem 2 is best possible when $m=n$, $\alpha_m=1$.

The function $g(z)$ also shows that the result of Theorem 1 is more or less best possible. In that theorem we deduced the inequality (a) when a certain integral was convergent. The corresponding integral, associated with the function $g(z)$ above, is

$$\int_{x_0}^{x} \xi^{-1}L(\xi^{-1};\,m)\,L(\xi^{-1};\,m,\,m;\,\tfrac{1}{2})\,d\xi \sim 2(\log_m x)^{\frac{1}{2}},$$

as $x\to\infty$. Thus by (36) we have that this integral may increase to infinity as slowly as we please without necessarily implying (a).

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