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Discrete subadditive functions as Gomory functions

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1. *Introduction*

Our aim, in this paper, is to study a class of functions which occurs in pure integer programming, and to investigate conditions under which discrete subadditive functions belong to that class. The inspiration for the paper was the problem of classifying discrete metrics used in pattern recognition, while the methods of proof of the main theorem are those of pure integer programming.

Chvátal functions are rational functions of n rational coordinates which are built up from rational coordinate functions by a finite number of two types of operations: taking linear combinations with non-negative rational coefficients and rounding up to integer values. Gomory functions are maxima of finite collections of Chvátal functions. These functions are used in integer programming for generating cutting planes. It is known that the value function (the optimal objective value as a function of the right-hand-side coefficients) of each pure integer programme is the restriction of a Gomory function to a suitable domain. This domain itself can be represented by a non-positivity condition on another Gomory function known as the Consistency Tester. The relationship between Chvátal and Gomory functions and integer programming is described in Schrijver[**14**]. Chvátal functions are so named as a result of a paper by Chvátal[**4**] and Gomory functions from a paper by Gomory[**6**].

Gomory functions are subadditive and are non-decreasing in each variable. Each Chvátal function is shift-periodic in each of its variables and its values throughout its domain are determined by its values in a bounded region. Each Gomory function is eventually shift-periodic in each of its variables. If a Gomory function is the maximum of a collection of Chvátal functions, each of which uses only integer coefficients, then the values of the Gomory function at all the integer points are determined by the values of the function at the points of the Hilbert bases of a collection of rational polyhedral cones. Moreover, a shift-periodic integral Gomory function is a Chvátal function.

Some of the periodicity properties of Chvátal and Gomory functions can be inferred from known periodicity properties of value functions of integer programmes. Gomory[**8**] showed that the solution of any integer programme (regarded as a minimization with '$\geqslant$' constraints) in rational coefficients has a periodic property so long as the right-hand-side coefficients are sufficiently large (the result is also described in [**5**]). Blair and Jeroslow[**1**] showed that the value function of such an integer programme is a Gomory function. This result can also be deduced from Wolsey's[**18**] characterization of the b-hull of an integer programme. In [**1**] it is also shown that any Gomory function is the value function of some integer programme. Williams[**16**] shows that the value function of an integer programme over a cone is

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a Chvátal function. Some of these results are covered in Nemhauser and Wolsey [**10**]. However, these scattered references do not lead to the range of results on shift-periodicity and eventual shift-periodicity given here. The general results for rational Chvátal and Gomory functions are given in Section 2 while the special results for integral Chvátal and Gomory functions are given in Section 3.

The main result of Section 4 is that each integral function which is non-negative, non-decreasing and subadditive on a region $[0,N]^n$ in $\mathbb{N}^n$ is the restriction of a Gomory function to that region. It is shown also that if a function is non-negative, non-decreasing and subadditive throughout $\mathbb{N}^n$, and if the function is determined by its values on a finite set of points either by shift-periodicity or by a Hilbert basis for the function, then the function is the restriction of a Gomory function to $\mathbb{N}^n$. The method of proof is to construct an integer programme for which the given function is the value function. The results are illustrated in Section 5.

The results of this paper have been used in the study of discrete metrics. If a discrete metric on the integer lattice $\mathbb{Z}^n$ is invariant under translations and also under reflections in the coordinate axes then it is determined by a function which is subadditive in the first octant. If the metric is increasing in each coordinate and is determined by its values on a Hilbert basis then it can be presented as a Gomory function. Discrete metrics as Gomory functions are studied in [**11**]. Explicit use of the form of the spheres of metrics presented by Gomory functions allows one to investigate properties of some families of discrete metrics designed to be used in pattern recognition. Chaudhuri *et al.* [**13**] introduced a family of metrics on $\mathbb{Z}^n$, determined by a parameter depending on n, and calculated bounds for the proportional errors in the approximation of Euclidean distances by these functions. The results of this paper show that these metrics can be presented as Gomory functions, leading to calculation of the maximum proportional errors for metrics using their parameters, and to optimal values of the parameters which give much better approximations of Euclidean distance [**12**]. Also, it is now clear that chamfer metrics [**2**] can be represented by Gomory functions for which the Hilbert basis is the set of prime integer points in the mask for the metric. This leads to a characterization of metric subgraphs for the chamfer metrics [**13**].

2. *Chvátal and Gomory functions*

The n-dimensional *Chvátal functions* are functions from $\mathbb{Q}^n$ to $\mathbb{Q}$ which are built up from the coordinate functions by a finite number of operations of taking linear combinations with non-negative rational coefficients and of rounding up to integer values. Precisely, the class $\mathscr{C}_n$ of n-dimensional Chvátal functions is the smallest class of functions with the following properties:

(i) the class contains the functions $(x_1, x_2, \ldots, x_n) \to x_j$, $1 \leqslant j \leqslant n$;

(ii) if α, β are rational and non-negative, and if f, g are in the class then so also is $\alpha f + \beta g$;

(iii) if f is in the class then so also is $\lceil f \rceil$.

The notation $\lceil f \rceil$ is used for the function $\lceil f \rceil(v) = \lceil f(v) \rceil$ where $\lceil x \rceil$ is the integer n such that $n-1 < x \leqslant n$. Similarly, $\lfloor x \rfloor$ is the integer n such that $n \leqslant x < n+1$.

The n-dimensional *Gomory functions* are functions from $\mathbb{Q}^n$ to $\mathbb{Q}$ which are built in

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a similar way to Chvátal functions, the additional operation of taking maxima being allowed. The class $\mathscr{G}_n$ of n-dimensional Gomory functions is the smallest class of functions with properties (i), (ii), (iii), and also

(iv) if f and g are in the class then so also is $\max\{f, g\}$.

Presentations of functions as Chvátal or Gomory functions are not unique. We note first that in a presentation of a Gomory function the use of maximization can be left to a final step.

LEMMA 2·1. *Every Gomory function can be presented as a maximum of a finite number of Chvátal functions.*

Proof. If $g = \max\{g_i \mid i \in I\}$ and $h = \max\{h_j \mid j \in J\}$ then $\alpha g + \beta h = \max\{\alpha g_i + \beta h_j \mid i \in I,\ j \in J\}$, and $\lceil g \rceil = \max\{\lceil g_i \rceil \mid i \in I\}$, while $\max\{g, h\} = \max\{g_i, h_j \mid i \in I,\ j \in J\}$. The proof follows by induction on the number of steps in the presentation of the function [1]. ∎

A presentation of a Chvátal function can sometimes be simplified by the suppression of some of the roundings.

LEMMA 2·2. *Let* $(a, b) = 1$. *Then* $\lceil a\lceil x \rceil / b \rceil = \lceil ax/b \rceil$ *for all* x *in* $\mathbb{Q}$ *if and only if* $a = 1$.

Proof. If $\lceil x/b \rceil = n$ then $(n-1)b < x \leqslant nb$. Since nb is an integer it follows that $(n-1)b < \lceil x \rceil \leqslant nb$, and so that $\lceil \lceil x \rceil / b \rceil = n$. On the other hand, if $(a, b) = 1$ with $a > 1$ then there exists a positive integer n and a rational number x such that $(n-1)a < xa < b < na$. Then $\lceil x \rceil = n$, so that $\lceil a\lceil x \rceil / b \rceil > 1$, while $\lceil ax/b \rceil \leqslant 1$. ∎

LEMMA 2·3. *Let* $(a, b) = (a, b) = 1$. *Then*

$$\left\lceil \frac{a}{b}\left\lceil \frac{cn}{d} \right\rceil \right\rceil = \left\lceil \frac{acn}{bd} \right\rceil$$

for each integer n *if and only if* $(a, d) = 1$ *and* $a/b \leqslant \delta$, *where* δ *is the least positive value of*

$$\left\lceil \frac{acn}{bd} \right\rceil - \frac{a}{b}\left\lfloor \frac{cn}{d} \right\rfloor$$

for $n \in \mathbb{N}$ *and* $n/d \notin \mathbb{N}$.

Proof. Suppose that $(a, d) \neq 1$. Take $n' = (a/(a, d))(c/(b, c))$. Then $acn'/bd \in \mathbb{N}$ while $cn'/d \notin \mathbb{N}$. Thus

$$\left\lceil \frac{acn'}{bd} \right\rceil = \frac{acn'}{bd} < \frac{a}{b}\left\lceil \frac{cn'}{d} \right\rceil \leqslant \left\lceil \frac{a}{b}\left\lceil \frac{cn'}{d} \right\rceil \right\rceil.$$

Hence the condition $(a, d) = 1$ is necessary.

Now suppose that $(a, d) = 1$. If $n/d \in \mathbb{N}$ then $cn/d = \lceil cn/d \rceil$ and $\lceil acn/bd \rceil = \lceil a\lceil cn/d \rceil / b \rceil$. Thus it remains to be shown that when $(a, d) = 1$, $\lceil acn/bd \rceil = \lceil a\lceil cn/d \rceil / b \rceil$ for all $n \in \mathbb{N}$ such that $n/d \notin \mathbb{N}$ if and only if $a/b \leqslant \delta$. Observe that $n/d \notin \mathbb{N}$ implies that $cn/d \notin \mathbb{N}$ and $acn/bd \notin \mathbb{N}$. If $a/b \leqslant \delta$ then

$$\frac{acn}{bd} < \frac{a}{b}\left(\left\lfloor \frac{cn}{d} \right\rfloor + 1\right) = \frac{a}{b}\left(\lceil cn \rceil / d\right) \leqslant \left\lceil \frac{acn}{bd} \right\rceil,$$

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so that $\lceil acn/bd\rceil = \lceil a/\lceil cn/d\rceil/b\rceil$ whenever $n\in\mathbb{N}$ and $a/d\notin\mathbb{N}$. Now suppose that $a/b > \delta$. There exists an integer $n''\in\mathbb{N}$ such that $n''/d\notin\mathbb{N}$ and

$$\left\lceil\frac{acn''}{bd}\right\rceil - \frac{a}{b}\left\lfloor\left(\frac{cn''}{d}\right)\right\rfloor = \delta.$$

Then

$$\frac{a}{b}\left\lceil\left(\frac{cn''}{d}\right)\right\rceil = \frac{a}{b}\left(\left\lfloor\left(\frac{cn''}{d}\right)\right\rfloor + 1\right) > \left\lceil\left(\frac{acn''}{bd}\right)\right\rceil,$$

so that $\lceil acn''/bd\rceil = \lceil a\lceil cn''/d\rceil/b\rceil$. ∎

We have not been able to find an expression for δ in terms of a, b, c and d. The example with $a/b = 2/3$ and $c/d = 4/5$ shows that δ can be greater than the sum of the least positive values of

$$\frac{acn}{bd} - \frac{a}{b}\left\lfloor\left(\frac{cn}{d}\right)\right\rfloor \quad\text{and}\quad \left\lceil\left(\frac{acn}{bd}\right)\right\rceil - \frac{acn}{bd}.$$

for $n\in\mathbb{N}$ and $n/d\notin\mathbb{N}$, which are a/bd and $(b,c)/bd$, respectively. In this example $\delta = 5/15$ which is attained for $n = 13\,(\mathrm{mod}\,15)$. The least positive value $a/bd = 2/15$ is attained for $n = 4\,(\mathrm{mod}\,5)$ while the least positive value $(b,c)/bd = c/bd = 1/15$ is attained for $n = 13\,(\mathrm{mod}\,15)$. Nevertheless, δ is not less than $a/bd + (b/c)/bd$ so that we have the following corollary.

COROLLARY 2·1. *Let* $(a,b) = (c,d) = (a,d) = 1$, *and* $ad < a + (c,d)$. *Then*

$$\left\lceil\frac{a}{b}\left\lceil\left(\frac{cn}{d}\right)\right\rceil\right\rceil = \left\lceil\left(\frac{acn}{bd}\right)\right\rceil$$

for each integer n.

The corollary ensures, for example, that $\lceil 2\lceil 7n/3\rceil/7\rceil = \lceil 2n/3\rceil$ for each integer n. One should note that Lemmas 2·2 and 2·3 are consistent with each other since $a/b \leqslant \delta$ when $a = 1$.

Chvátal and Gomory functions are *non-decreasing* in each coordinate in the sense that $f(x_1,\ldots,x_j,\ldots,x_n) \leqslant f(x_1,\ldots,x_j+a,\ldots,x_n)$ for all $(x_1,\ldots,x_n)$ and all $a \geqslant 0$. They are subadditive and also repetitive in the senses defined below.

A function $f:\mathbb{Q}^n\to\mathbb{Q}$ is said to be *subadditive* if $f(v+w) \leqslant f(v)+f(w)$ for all $v,w\in\mathbb{Q}^n$. If α, $\beta \geqslant 0$ and f and g are subadditive then so also are $\alpha f+\beta g$, $\max\{f,g\}$ and $\lceil f\rceil$. Thus since the coordinate functions are subadditive, all the Chvátal and Gomory functions are subadditive.

A function $f:\mathbb{Q}^n\to\mathbb{Q}$ will be said to be *shift-periodic* in x_j uniformly on a region D if there exist positive integers p and q such that for each $(x_1,\ldots,x_n)$ in D and each integer λ, $f(x_1,\ldots,x_j+\lambda p,\ldots,x_n) = f(x_1,\ldots,x_n)+\lambda q$. The function will also be said to be (p,q)-shift-periodic in x_j uniformly on D. If D contains the origin then $q = f(0,\ldots,p,\ldots,0)$. In the context of Chvátal and Gomory functions the shift-period constant q is required to be an integer in order that shift-periodicity is preserved on rounding up; and it is then convenient to require the shift-period constant p also to be an integer. A function will be said to be *eventually positively shift-periodic* in x_j uniformly in a region D if there exist positive integers p, q and Λ such that for each $(x_1,\ldots,x_n)$ in D and each positive integer λ,

$$f(x_1,\ldots,x_j+\Lambda p+\lambda p,\ldots,x_n) = f(x_1,\ldots,x_j+\Lambda p,\ldots,x_n)+\lambda q.$$

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A function will be said to be *eventually negatively shift-periodic* in x_j uniformly in a region D if there exist positive integers p, q and Λ such that for each $(x_1, \ldots, x_n)$ in D and each positive integer λ,

$$f(x_1, \ldots, x_j - \Lambda p - \lambda p, \ldots, x_n) = f(x_1, \ldots, x_j - \Lambda p, \ldots, x_n) - \lambda q.$$

LEMMA 2·4. *If $\alpha_1, \alpha_2 \in \mathbb{Q}$ and f_1, f_2 are shift-periodic in x_j uniformly on D then so also is $\alpha_1 f_1 + \alpha_2 f_2$.*

Proof. Let $\alpha_j = a_j/b_j$ and $g = \alpha_1 f_1 + \alpha_2 f_2$. Let f_1 be (p_1, q_1)-shift-periodic and f_2 be (p_2, q_2)-shift-periodic and let $p = b_1 b_2 p_1 p_2$. Then for $(x_1, \ldots, x_n) \in A$

$$\begin{aligned} g(x_1, \ldots, x_j + \lambda p, \ldots, x_n) &= \alpha f_1(x_1, \ldots, x_j + \lambda p, \ldots, x_n) + \beta f_2(x_1, \ldots, x_j + \lambda p, \ldots, x_n) \\ &= \alpha f_1(x_1, \ldots, x_n) + \lambda a_1 b_2 p_2 q_1 + \beta f_2(x_1, \ldots, x_n) + \lambda a_2 b_1 p_1 q_2 \\ &= g(x_1, \ldots, x_n) + \lambda a_1 b_2 p_2 q_1 + \lambda a_2 b_1 p_1 q_2. \end{aligned}$$ ∎

LEMMA 2·5. *If f is shift-periodic in x_j uniformly on D then so also is $\lceil f \rceil$.*

Proof. If $f(x_1, \ldots, x_j + \lambda p, \ldots, x_n) = f(x_1, \ldots, x_n) + \lambda q$, then

$$\lceil f(x_1, \ldots, x_j + \lambda p, \ldots, x_n) \rceil = \lceil f(x_1, \ldots, x_n) + \lambda q \rceil = \lceil f(x_1, \ldots, x_n) \rceil + \lambda q.$$ ∎

Proposition 2·1. *All Chvátal functions are shift-periodic in each coordinate uniformly on $\mathbb{Q}^n$.*

Proof. Since the coordinate functions are shift-periodic uniformly on $\mathbb{Q}^n$ the proposition follows from Lemmas 2·4 and 2·5. ∎

Gomory functions need not be shift-periodic uniformly on $\mathbb{Q}^n$ but they are eventually positively and eventually negatively shift-periodic uniformly on each finite subset of $\mathbb{Q}^n$. It is shown in Proposition 3·2 that integral Gomory functions which are shift-periodic in each coordinate are Chvátal functions.

Before stating the next proposition we give an example from pure integer programming.

Example 2·1. The Gomory function

$$\max \begin{cases} \lceil x_1/6 + 2\lceil x_3/2 \rceil/3 \rceil, \\ 5\lceil x_1/2 \rceil + 2\lceil x_2 \rceil, \\ \lceil x_1/2 \rceil + 2\lceil x_2/5 + 2\lceil x_1/6 + 2\lceil x_3/2 \rceil/3 \rceil/5 \rceil \end{cases}$$

$x_1, x_2 \in \mathbb{Q}$, is the value function of the integer programme

$$\begin{aligned} \text{Minimize} \quad & y_2 \\ \text{subject to} \quad & 4y_1 + 2y_2 \geqslant x_1 \\ & -9y_1 - 2y_2 \geqslant x_2 \\ & -2y_1 + 2y_2 \geqslant x_3 \\ & \text{with } y_1 \text{ and } y_2 \text{ integers.} \end{aligned}$$

The component Chvátal functions are respectively $(6, 1)$, $(2, 5)$ and $(30, 19)$-shift-periodic in x_1, $(1, 0)$, $(1, 2)$ and $(5, 2)$-shift-periodic in x_2, and $(6, 2)$, $(1, 0)$, and $(30, 8)$-shift-periodic in x_3, all uniformly in $\mathbb{Q}^3$. The Gomory function itself is eventually

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positively $(2,5)$-shift-periodic in x_1, $(1,2)$-shift-periodic in x_2, and $(6,2)$-shift-periodic in x_3. It is eventually negatively $(6,1)$-shift-periodic in x_1, $(1,0)$-shift-periodic in x_2, and $(1,0)$-shift-periodic in x_3, all uniformly on each finite subset.

PROPOSITION 2·2. *Let $f^i:\mathbb{Q}^n\to\mathbb{Q}$ be (p^i,q^i)-shift-periodic in x_j uniformly on a set D, and let $g=\max\{f^i\mid 1\leqslant i\leqslant k\}$. Set $q^i/p^i=\rho^i$.*

(a) If $\rho^i=\rho$ whenever $1\leqslant i\leqslant k$, then g is (p,q)-shift-periodic in x_j uniformly on D with $q/p=\rho$.

(b) The function $g(x_1,\ldots,x_n)$ is eventually positively (p,q)-shift-periodic in x_j uniformly on each finite subset of D with $q/p=\max\{\rho^i\mid 1\leqslant i\leqslant k\}$, and eventually negatively (p,q)-shift-periodic in x_j uniformly on each finite subset of D with $q/p=\min\{\rho^i\mid 1\leqslant i\leqslant k\}$.

(c) If g is (p,q)-shift-periodic in x_j uniformly on D with $q/p=\rho$ then $\rho=\rho^i$, $1\leqslant i\leqslant k$.

Proof. (a) Let $P=p^1p^2,\ldots,p^k$. Then for $(x_1,\ldots,x_n)$ in D,

$$\begin{aligned}g(x_1,\ldots,x_j+P,\ldots,x_n)&=\max\{f_1(x_1,\ldots,x_n)+P\rho^i\mid 1\leqslant i\leqslant k\}\\&=g(x_1,\ldots,x_2)+P\rho.\end{aligned}$$

Thus g is $(P,P\rho)$-shift-periodic in x_j uniformly on D.

(b) It follows from part (a) that for the rest of the proof we may suppose that $\rho^1>\rho^2>\ldots>\rho^k$.

Let F be a finite subset of D. Then there exists $\Lambda^+>0$ such that $\Lambda^+P(\rho^1-\rho^i)\geqslant f^i(x_1,\ldots,x_j+\nu p^1,\ldots,x_n)-f^1(x_1,\ldots,x_j+\nu p^1,\ldots,x_n)$ whenever $0\leqslant\nu<P/q^1$, $2\leqslant i\leqslant k$, and $(x_1,\ldots,x_n)$ is in F.

Now if $\lambda=\mu P/p^1+\nu$ with $\mu\geqslant 0$, and $0\leqslant\nu<P/p^1$, and if $(x_1,\ldots,x_n)$ is in F, then

$$\begin{aligned}g(x_1,\ldots,x_j+\Lambda^+P+\lambda p^1,\ldots,x_n)&=\max\{f^i(x_1,\ldots,x_j+\Lambda^+P+\lambda p^1,\ldots,x_n)\mid 1\leqslant i\leqslant k\}\\&=\max\{f^i(x_1,\ldots,x_j+\nu p^1,\ldots,x_n)+(\Lambda^++\mu)P\rho^i\mid 1\leqslant i\leqslant k\}\\&=f^1(x_1,\ldots,x_j+\nu p^1,\ldots,x_n)+(\Lambda^++\mu)P\rho^1\\&=f^1(x_1,\ldots,x_j+\Lambda^+P,\ldots,x_n)+\lambda q^1.\end{aligned}$$

These equations are valid, in particular, when $\lambda=0$. Thus for $\lambda\geqslant 0$,

$$g(x_1,\ldots,x_j+\Lambda^+P+\lambda p^1,\ldots,x_n)=g(x_1,\ldots,x_j+\Lambda P,\ldots,x_n)+\lambda q^1.$$

Hence g is eventually positively (p^1,q^1)-shift-periodic in x_j uniformly on F.

Similarly, there exists $\Lambda^->0$ such that $\Lambda^-P(\rho^i-\rho^k)\geqslant f^i(x_1,\ldots,x_j+\nu p^1,\ldots,x_n)-f^k(x_1,\ldots,x_j+\nu p^1,\ldots,x_n)$ whenever $0\leqslant\nu<P/q^k$, $1\leqslant i\leqslant k-1$, and $(x_1,\ldots,x_n)$ is in F. Now if $\lambda=\mu P/p^k+\nu$ with $\mu\geqslant 0$, and $0\leqslant\nu<P/p^k$, and if $(x_1,\ldots,x_n)$ is in F, then

$$\begin{aligned}g(x_1,\ldots,x_j-\Lambda^-P-\lambda p^k,\ldots,x_n)&=\max\{f^i(x_1,\ldots,x_j-\Lambda^-P-\lambda p^k,\ldots,x_n)\mid 1\leqslant i\leqslant k\}\\&=\max\{f^i(x_1,\ldots,x_j-\nu p^k,\ldots,x_n)-(\Lambda^-+\mu)P\rho^i\mid 1\leqslant i\leqslant k\}\\&=f^1(x_1,\ldots,x_j-\nu p^k,\ldots,x_n)-(\Lambda^-+\mu)P\rho^k\\&=f^1(x_1,\ldots,x_j-\Lambda^-P,\ldots,x_n)-\lambda q^1.\end{aligned}$$

These equations are valid, in particular, when $\lambda=0$. Thus for $\lambda\geqslant 0$,

$$g(x_1,\ldots,x_j-\Lambda^-P-\lambda p^k,\ldots,x_n)=g(x_1,\ldots,x_j-\Lambda^-P,\ldots,x_n)-\lambda q^k.$$

Hence g is eventually negatively (p^1,q^1)-shift-periodic in x_j uniformly on F.

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(*c*) The arguments in the proof of part (*b*) show that for all $(x_1, \ldots, x_n)$ in a finite subset F of D and for each positive integer λ

$$g(x_1, \ldots, x_j + \Lambda^+ pP + \lambda pp^1, \ldots, x_n) = g(x_1, \ldots, x_j + \Lambda^+ pP, \ldots, x_n) + \lambda pq^1.$$

If g is (p, q)-shift-periodic in x_j uniformly on D then

$$g(x_1, \ldots, x_j + \Lambda^+ pP + \lambda pp^1, \ldots, x_n) = g(x_1, \ldots, x_j + \Lambda^+ pP, \ldots, x_n) + \lambda p^1 q.$$

Thus $q/p = q^1/p^1$. Similarly, for all $(x_1, \ldots, x_n)$ in a finite subset of F of D and for each positive integer λ

$$g(x_1, \ldots, x_j - \Lambda^- pP - \lambda pp^k, \ldots, x_n) = g(x_1, \ldots, x_j - \Lambda^- pP, \ldots, x_n) - \lambda pq^k.$$

If g is (p, q)-shift-periodic in x_j uniformly on D then

$$g(x_1, \ldots, x_j - \Lambda^- pP - \lambda pp^k, \ldots, x_n) = g(x_1, \ldots, x_j - \Lambda^- pP, \ldots, x_n) - \lambda p^k q.$$

Thus $q/p = q^k/p^k$. Hence $\rho^1 = \rho^k$, so that $\rho^i = \rho$ whenever $1 \leqslant i \leqslant k$. ∎

COROLLARY 2·2. *Each Gomory function is eventually positively and eventually negatively shift-periodic in each variable uniformly on each finite subset of* $\mathbb{Q}^n$. *If the Gomory function is shift-periodic in* x_j *uniformly on a set* D *then all the component Chvátal functions are shift periodic with the same shift periodicity ratio.*

If the functions are defined only for positive values of the coordinates then the concept of eventually negatively shift-periodic does not apply, while the definition of shift-periodicity has to be modified to require $f(x_1, \ldots, x_j + \lambda p, \ldots, x_n) = f(x_1 \ldots, x_n) + \lambda q$ when both sides of the equation are defined. The proof of Proposition 2·2 (*c*) is no longer valid, and the statement has to be modified.

PROPOSITION 2·3. *Let* $f^i : (\mathbb{Q}^+)^n \to \mathbb{Q}$ *be* (p^i, q^i)*-shift-periodic in* x_j *uniformly on a set* D, *and let* $g = \max\{f^i \mid 1 \leqslant i \leqslant k\}$. *Set* $q^i/p^i = \rho^i$.

(*a*) *If* $\rho^i = \rho$ *whenever* $1 \leqslant i \leqslant k$, *then* g *is* (p, q)*-shift-periodic in* x_j *uniformly on* D *with* $q/p = \rho$.

(*b*) *The function* $g(x_1, \ldots, x_n)$ *is eventually positively* (p, q)*-shift-periodic in* x_j *uniformly on each finite subset of* D *with* $q/p = \max\{\rho^i \mid 1 \leqslant i \leqslant k\}$.

(*c*) *If* g *is* (p, q)*-shift-periodic in* x_j *uniformly on* D *with* $q/p = \rho$ *then* $\rho = \max\{\rho^i \mid 1 \leqslant i \leqslant k\}$ *and* $g(x_1, \ldots, x_n) = g^i(x_1, \ldots, x_n) \geqslant f^i(x_1, \ldots, x_n)$ *for all* $(x_1, \ldots, x_n)$ *in* D *and all* i, $1 \leqslant i \leqslant k$, *where* $g^1 = \max\{f^i \mid \rho^i = \rho\}$.

Proof. The proofs of parts (*a*) and (*b*) are the same as those for Proposition 2·2 (*a*) and the first statement of (*b*). For part (*c*) the former proof that $q/p = q^1/p^1$ applies in this case also. Now suppose that there exists $i > 1$ and $(x_1, \ldots, x_n)$ in D such that $g(x_1, \ldots, x_n) = f^i(x_1, \ldots, x_n) > f^1(x_1, \ldots, x_n)$. Then

$$\begin{aligned} f^1(x_1, \ldots, x_n) + \Lambda^+ pP\rho^1 + \lambda pq^1 &= f^1(x_1, \ldots, x_j + \Lambda^+ pP, \ldots, x_n) + \lambda pq^1 \\ &= g(x_1, \ldots, x_j + \Lambda^+ pP + \lambda pp^1, \ldots, x_n) \\ &= g(x_1, \ldots, x_j + \Lambda^+ pP, \ldots, x_n) + \lambda pq^1 \\ &= g(x_1, \ldots, x_n) + \Lambda^+ Pq + \lambda pq^1 \\ &= f^i(x_1, \ldots, x_n) + \Lambda^+ Pq + \lambda pq^1. \end{aligned}$$

The contradiction proves part (*c*) of the proposition. ∎

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COROLLARY 2·3. *A function which is the restriction of a Gomory function to* $(\mathbb{Q}^+)^n$ *is eventually positively shift-periodic in each coordinate. If it is* (p,q)*-shift-periodic in* x_j *uniformly on a set* D *then it is dominated in* x_j *by component Chvátal functions whose shift-periodicity ratios are* q/p.

It is worth remarking that Corollary 2·3 can also be shown to be true through results of Blair and Jeroslow, Gomory, and Williams. In [1] it is shown that any Gomory function is the value function of some integer programme. In [7] the following 'asymptotic' result is proved: if the right-hand-side values of an integer programme are sufficiently large then the optimal solution can be obtained by solving an integer programme over a cone (sometimes referred to as the 'group problem'). The value function for this problem is shown in [16] to be a Chvátal function. It is that Chvátal function within the Gomory function which becomes dominant for sufficiently large arguments so determining the eventual positive shift-periodicity of the Gomory function.

Let f be a Gomory function obtained through the composition of a finite sequence of operations of linear combinations, rounding up and taking maxima. Then the function obtained by taking the same sequence of operations of linear combinations and maxima, omitting the rounding up operations, is another Chvátal or Gomory function f'. It is called the *carrier* of f [1]. The shift-period constants of a Gomory function and its carrier are related.

PROPOSITION 2·4. *If a Gomory function* f *is eventually positively or eventually negatively* (p,q)*-shift-periodic and its carrier* f' *is* (p',q')*-shift-periodic in* x_j *uniformly on a set* D *then* $q/p = q'/p'$.

Proof. It is proved in [1] that there exists a constant $K \geqslant 0$ such that $0 \leqslant f(x_1,\ldots,x_n) - f'(x_1,\ldots,x_n) \leqslant K$ for all $(x_1,\ldots,x_n)$ in $\mathbb{Q}^n$. Suppose that f is eventually positively (p,q)-shift-periodic and $f(x_1,\ldots,x_j+\Lambda p+\lambda p,\ldots,x_n) = f(x_1,\ldots,x_j+\Lambda p,\ldots,x_n)+\lambda q$ for all points $(x_1,\ldots,x_n)$ in D and all $\lambda > 0$. If $p'q > pq'$ then given $(a_1,\ldots,a_n)$ in D there exists $\lambda > 0$ such that $\lambda(p'q-pq') > f'(a_1,\ldots,a_j+\Lambda pp',\ldots,a_n) - f(a_1,\ldots,a_j+\Lambda pp',\ldots,a_n)+K$. Thus

$$\begin{aligned} f'(a_1,\ldots,a_j+\Lambda pp'+\lambda pp',\ldots,a_n)+K &= f'(a_1,\ldots,a_j+\Lambda pp',\ldots,a_n)+\lambda pq'+K \\ &< f(a_1,\ldots,a_j+\Lambda pp',\ldots,a_n)+\lambda p'q \\ &= f(a_1,\ldots,a_j+\Lambda pp'+\lambda pp',\ldots,a_n). \end{aligned}$$

But $f'(a_1,\ldots,a_j+\Lambda pp'+\lambda pp',\ldots,a_n)+K \geqslant f(a_1,\ldots,a_j+\Lambda pp'+\lambda pp',\ldots,a_n)$. Similarly, the assumption $pq' < p'q$ leads to a contradiction to $f' \leqslant f$. Similar contradictions arise from the assumption that f is eventually negatively shift-periodic. ∎

The carrier of a Chvátal function is a linear function $\sum \alpha_j x_j$ of the coordinates with non-negative coefficients. It is (p_j,q_j)-shift-periodic in x_j uniformly on $\mathbb{Q}^n$ with $q_j/p_j = \alpha_j$. The carrier of a Gomory function is the maximum of a finite number of these linear functions. It is eventually (p_j,q_j)-shift-periodic in x_j uniformly on each finite subset of $\mathbb{Q}^n$. Although the values of p_j and q_j may depend on the finite subset, in each case q_j/p_j is the maximum of the coefficients α_j for the component linear functions.

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3. *Integral Chvátal and Gomory functions*

The restriction to $\mathbb{Z}^n$ of a Chvátal or Gomory functions which takes integer values for all these values of the variables is said to be an *integral Chvátal* or *Gomory function*. The values of an integral function are determines by sequences of differences of values. For each fixed base point $(a_1, \ldots, a_n)$ and each coefficient x_j the values of an integral valued function f at the points $(a_1, \ldots, a_{j-1}, x_j, a_{j+1}, \ldots, a_n)$ are determined by the sequence $s_k = f(a_1, \ldots, a_{j-1}, k+1, a_{j+1}, \ldots, a_n) - f(a_1, \ldots, a_{j-1}, k, a_{j+1}, \ldots, a_n)$. A function f is (p, q)-shift-periodic in x_j uniformly on a set D if and only if for each point $(a_1, \ldots, a_n)$ in D the corresponding sequence s_k is periodic with period p and $s_k + \ldots + s_{k+p} = q$ for each integer k. Suppose that for a given point $(a_1, \ldots, a_n)$ the function $f(a_1, \ldots, a_{j-1}, x_j, a_{j+1}, \ldots, a_n)$ is (p', q')-shift-periodic and (p'', q'')-shift-periodic. Then the corresponding sequence s_k is periodic of period p' and periodic of period p''. Let $p = (p', p'')$ with $p' = \mu p$ and $p'' = \nu p$. Then the sequence is periodic of period p, and $s_k + \ldots + s_{k+p} = q$ for each integer k, where $q' = \mu q$ and $q'' = \nu q$. Thus the function is (p, q)-shift-periodic. It follows that if f is (p', q')-shift-periodic and (p'', q'')-shift-periodic in x_j uniformly on a set D then f is (p, q)-shift-periodic in x_j uniformly on D.

The integral Gomory functions which are $(1, q)$-shift-periodic in some variable take a special form.

PROPOSITION 3·1. *If f is an n-dimensional integral Chvátal or Gomory function which is $(1, q)$-shift-periodic in x_j uniformly on $\mathbb{Z}^n$ then it is the sum of the 1-dimensional Chvátal function $(x_1, \ldots, x_n) \to qx_j$ with an $(n-1)$-dimensional integral Chvátal or Gomory function.*

Proof. By assumption $f(x_1, \ldots, x_{j-1}, \lambda, x_{j+1}, \ldots, x_n) = f(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n) + \lambda q$ and the right-hand side is the sum of an n-dimensional function with the 1-dimensional function qx_j. ∎

By Proposition 2·2 (*c*) a Gomory function $g: \mathbb{Q}^n \to \mathbb{Q}$ which is shift-periodic in each coordinate can be presented as the maximum of a finite collection of Chvátal functions each of which has the same shift-periodicity ratio as g. A one-dimensional example is given by the two Chvátal functions $f^1(x) = \lceil 5\lceil x/2 \rceil / 3 \rceil$ and $f^2(x) = \lceil 5\lceil x/3 \rceil / 2 \rceil$, both of which are $(6, 5)$-shift-periodic with carrier $5x/6$. Since $f^1(1) = 2 < 3 = f^2(1)$ while $f^1(3) = 4 > 3 = f^2(3)$, the function $g(x) = \max\{f^1(x), f^2(x)\}$ is neither $f^1(x)$ nor $f^2(x)$, but it is $(6, 5)$-shift-periodic with carrier $5x/6$. However, this function $g(x)$ can also be presented as the Chvátal function $\lceil (f^1(x) + f^2(x))/2 \rceil$. We shall now show that every integral Gomory function which is shift-periodic in each coordinate is a Chvátal function.

PROPOSITION 3·2. *If the integral Gomory function g is (p_j, q_j)-shift-periodic in each coordinate x_j uniformly in $\mathbb{Q}^n$ then it can be presented as a Chvátal function.*

Proof. By Proposition 2·1 (*c*) $g = \max\{f^i \mid 1 \leqslant i \leqslant k\}$ for some set of Chvátal functions f^i each of which is (p_j^i, q_j^i)-shift-periodic in each coordinate x_j uniformly in $\mathbb{Q}^n$, with $q_j^i / p_j^i = q_j / p_j$ whenever $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n$. Then g is determined by its values on the set of points with $0 \leqslant x_j \leqslant \max\{p_j^i \mid 1 \leqslant i \leqslant k\}$, $1 \leqslant j \leqslant n$. Thus the differences between values of the functions f^i at points in $\mathbb{Q}^n$ are bounded.

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Since g is integral, $\max\{f^i\} = \max\{\lceil f^i \rceil\}$, so there is no loss of generality in supposing that the functions f^i are integral. Consider first the case $k = 2$. Suppose that $-b < f^1(\mathbf{v}) - f^2(\mathbf{v}) \leqslant a$ for all $\mathbf{v}$ in $\mathbb{Q}^n$. Set $g^1(\mathbf{v}) = \lceil \lambda f^1(\mathbf{v}) + (1-\lambda) f^2(\mathbf{v}) \rceil$, with $\lambda = 1/b$. If $f^1(\mathbf{v}) = f^2(\mathbf{v})$ then $g^1(\mathbf{v}) = f^1(\mathbf{v})$. If $f^1(\mathbf{v}) < f^2(\mathbf{v})$ then $f^2(\mathbf{v}) - \lambda f^1(\mathbf{v}) - (1-\lambda) f^2(\mathbf{v}) = \lambda[f^2(\mathbf{v}) - f^1(\mathbf{v})] < 1$, so that again $g^1(\mathbf{v}) = f^1(\mathbf{v})$. If $f^2(\mathbf{v}) < f^1(\mathbf{v})$ then $f^2(\mathbf{v}) < \lambda f^1(\mathbf{v}) + (1-\lambda) f^2(\mathbf{v}) < f^1(\mathbf{v})$ so that $f^2(\mathbf{v}) + 1 \leqslant g^1(\mathbf{v})$. Thus $\max\{f^1(\mathbf{v}), f^2(\mathbf{v})\} = \max\{f^1(\mathbf{v}), g^1(\mathbf{v})\}$ and $-b < f^1(\mathbf{v}) - g^1(\mathbf{v}) \leqslant a - 1$ for all points $\mathbf{v}$ in $\mathbb{Q}^n$. Define a sequence of functions recursively by $g^r(\mathbf{v}) = \lceil \lambda f^1(\mathbf{v}) + (1-\lambda) g^{r-1}(\mathbf{v}) \rceil$ and assume inductively that $-b < f^1(\mathbf{v}) - g^{r-1}(\mathbf{v}) \leqslant a - r + 1$ for all $\mathbf{v}$ in $\mathbb{Q}^n$. Then $\max\{f^1(\mathbf{v}), f^2(\mathbf{v})\} = \max\{f^1(\mathbf{v}), g^r(\mathbf{v})\}$ and $-b < f^1(\mathbf{v}) - g^r(\mathbf{v}) \leqslant a - r$ for all $\mathbf{v}$ in $\mathbb{Q}^n$. Hence $g^a(\mathbf{v}) = \max\{f^1(\mathbf{v}), f^2(\mathbf{v})\}$. The proof is completed by induction on k since $\max\{f^1, \ldots, f^k\} = \max\{\max\{f^1, f^2\}, f^3, \ldots, f^k\} = \max\{g^a, f^3, \ldots, f^k\}$. ∎

The following example shows that not all shift-periodic Gomory functions can be presented as Chvátal functions.

Example 3·1. The Gomory function $g(x) = \max\{\lceil 2x \rceil, x + \lceil x \rceil\}$ is $(1,2)$-shift-periodic in x uniformly in $\mathbb{Q}$, with

$$g(x) = \begin{cases} 1+x, & 0 < x \leqslant 1/2, \\ 2, & 1/2 < x \leqslant 1. \end{cases}$$

Every Chvátal function $f(x)$ can be written in the form $\sum a_i \lceil f_i(x) \rceil + bx$, where the functions $f_i(x)$ are Chvátal functions and a_i and zb are non-negative rationals. Then the function $f(x)$ is piecewise monotone with gradient b except at a finite number of points of discontinuity in each bounded interval. Since the function $g(x)$ has gradients 1 and 0 in alternate half-unit intervals it is not a Chvátal function.

Chvátal functions need not satisfy the positive linear homogeneity condition $f(\lambda \mathbf{v}) = \lambda f \mathbf{v})$ for $\lambda \geqslant 0$. However, all Gomory functions satisfy a limiting version of the condition.

PROPOSITION 3·3. *If g is a Gomory function then, for each $\mathbf{v}$ in $\mathbb{Z}^n$, $g(\lambda \mathbf{v})/\lambda g(\mathbf{v})$ tends to 1 as λ tends to infinity.*

Proof. The carrier g' of the function g is the maximum of a finite number of linear functions. The domain of g' is the union of a pointed polyhedral cones, in each of which g' takes the values of one of the linear functions. Thus for each $\mathbf{v}$ in $\mathbb{Z}^n$ and λ in $\mathbb{N}$, $g'(\lambda \mathbf{v}) = \lambda g'(\mathbf{v})$. The result now follows from the observation that there exists K such that $|g(\lambda \mathbf{v}) - g'(\lambda \mathbf{v})| < K$ for all $\mathbf{v}$ and λ. ∎

Integral Gomory functions which are the maxima of Chvátal functions using only integer coefficients are their own carrier functions and so enjoy the homogeneity property and a type of additive property described below. The domain of definition of such a function is the union of a finite number of pointed rational polyhedral cones.

A set of vectors $\{\mathbf{v}^i : 1 \leqslant i \leqslant k\}$ is a Hilbert basis for a cone if each integral vector in the cone is a non-negative integral combination of the vectors $\mathbf{v}^i$. The set of integer points in a rational polyhedral cone is generated by an integral Hilbert basis, and if the cone is pointed then there is a unique minimal integral Hilbert basis which generates the integer points in the cone [7] [14]. An integral Gomory function takes the values of one of the component Chvátal functions in a pointed cone. In each cone

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the Chvátal function is determined by its values on an integral Hilbert basis. Thus the Gomory function is determined by its values on the union of these Hilbert bases. We say that a finite set $\{\mathbf{v}^i : 1 \leqslant i \leqslant k\}$ of vectors in $\mathbb{N}^n$ is a *Hilbert basis* for a function g is given $\mathbf{v}$ in $\mathbb{N}^n$ there exist coefficients α^i in $\mathbb{N}$ such that $\mathbf{v} = \sum \alpha^i \mathbf{v}^i$ and $g(\mathbf{v}) = \sum \alpha^i g(\mathbf{v}^i)$.

PROPOSITION 3·4. *If the Chvátal functions $\{f_j : 0 \leqslant j \leqslant k\}$ use only integer coefficients and $g = \max\{f_j : 0 \leqslant j \leqslant k\}$ then $g(\lambda \mathbf{v}) = \lambda g(\mathbf{v})$ for each $\mathbf{v}$ in $\mathbb{Z}^n$ and λ in $\mathbb{N}$. Moreover, there exists a Hilbert basis for g.*

Proof. The conditions guarantee that the restrictions of the Gomory function g and its carrier g' are equal to $\mathbb{Z}^n$. For each j, $0 \leqslant j \leqslant k$, the carrier f'_j of the Chvátal function f_j is a linear function with non-negative integer coefficients, there exists a polyhedral cone C_j such that $g'(\mathbf{v}) = f'_j(\mathbf{v})$ for all $\mathbf{v}$ in $C_j \cap \mathbb{Z}^n$. Each polyhedral cone C_j, $1 \leqslant j \leqslant k$, has a finite basis, say $\{\mathbf{v}^i_j : 1 \leqslant i \leqslant k_j\}$. Thus for $\mathbf{v}$ in C_j there exist coefficients α^i such that $\mathbf{v} = \sum \alpha^i \mathbf{v}^i_j$ and $g(\mathbf{v}) = f'_j(\mathbf{v}) = \sum \alpha^i f'_j(\mathbf{v}^i_j) = \sum \alpha^i g(\mathbf{v}^i_j)$, and if λ is a non-negative integer then $g(\lambda \mathbf{v}) = f'_j(\lambda \mathbf{v}) = \lambda f'_j(\mathbf{v}) = \lambda g(\mathbf{v})$. ∎

4. *Subadditive functions*

We now consider non-decreasing subadditive functions $f : \mathbb{N}^n \to \mathbb{N}$. Points in space will be represented here by vectors $\mathbf{v}$. The unit vectors in the n coordinates will be denoted by $\mathbf{i}_1, \ldots, \mathbf{i}_n$. Thus $\mathbf{v} = (x_1, \ldots, x_n) = x_1 \mathbf{i}_1 + \ldots + x_n \mathbf{i}_n$.

PROPOSITION 4·1. *For each subadditive function $f : \mathbb{N}^n \to \mathbb{N}$, each base point $\mathbf{v}$ and each unit vector $\mathbf{i}_j$ the function $f(\mathbf{v} + \lambda \mathbf{i}_j)/\lambda$ tends to a non-negative finite limit as λ tends to infinity.*

Proof. For a fixed base point $\mathbf{v}$ set $f(\mathbf{v} + \lambda \mathbf{i}_j)/\lambda = \rho(\lambda)$. Then $\liminf \rho(\lambda) \geqslant 0$. Moreover, $\limsup \rho(\lambda) \leqslant f(\mathbf{i}_j)$ since $f(\mathbf{v} + \lambda \mathbf{i}_j) \leqslant f(\mathbf{v}) + \lambda f(\mathbf{i}_j)$ when λ is an integer. Suppose $\limsup \rho(\lambda) > L > \liminf \rho(\lambda)$. Choose $\Lambda > f(\mathbf{v})/L$ such that $\rho(\Lambda) < L$. There exists K such that $f(\mathbf{v}) + f(\mu \mathbf{i}_j) < K$ whenever $0 \leqslant \mu < \Lambda$. Let $\lambda = k\Lambda + \mu$ with $k \geqslant 0$ and $0 \leqslant \mu < \Lambda$. Then $f(\mathbf{v} + \lambda \mathbf{i}_j) \leqslant kf(\mathbf{v} + \Lambda \mathbf{i}_j) - kf(\mathbf{v}) + f(\mathbf{v}) + f(\mu \mathbf{i}_j) < k\Lambda L - kf(\mathbf{v}) + K$. Thus $f(\mathbf{v} + \lambda \mathbf{i}_j)/\lambda < L + K/k\Lambda$. We can make $K/k\Lambda$ arbitrarily small by choosing k large enough. Hence $\limsup \rho(\lambda) \leqslant L$ which contradicts our supposition. ∎

Note that for the function $f : \mathbb{N} \to \mathbb{N}$, $f(n) = n$ if $n < N$ and $f(n) = N$ if $n \geqslant N$, the ratio $f(n)/n$ tends to zero as n tends to infinity. Thus for a subadditive function the limit L can be zero, even when the function is non-decreasing in each coordinate, whereas for a Gomory function which depends on the jth coordinate the corresponding limit must be positive. Note also that the function $f : \mathbb{N} \to \mathbb{N}$, $f(n) = r$ if $2^{r-1} \leqslant n < 2^r$, $r \geqslant 1$, is non-decreasing and subadditive, though in contrast to Gomory functions it is not shift-periodic or even eventually shift-periodic. Thus the class of non-decreasing subadditive functions $f : \mathbb{N}^n \to \mathbb{N}$ is much wider than the class of restrictions of Gomory functions. Nevertheless, the restriction of each non-decreasing subadditive function to a bounded domain can be represented by the restriction of a Gomory function to that domain. If a subadditive function is non-decreasing and is either shift-periodic in each coordinate or determined by its values on a Hilbert basis then it can be represented by the restriction of a Gomory function

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to $\mathbb{N}^n$. The Gomory function is the value function of the following integer programme.

$$\begin{aligned} &\text{Minimize} && \sum f(\mathbf{v})\, y(\mathbf{v}) \\ &\text{subject to} && \sum v_1\, y(\mathbf{v}) \geqslant x_1 \\ & && \vdots \\ & && \sum v_n\, y(\mathbf{v}) \geqslant x_n \\ & && y(\mathbf{v}) \geqslant 0, \end{aligned} \tag{I}$$

the sums being taken over all vectors $\mathbf{v}$ in a non-empty finite subset S of $\mathbb{Z}^n$.

Here the numbers $f(\mathbf{v})$ are determined by the function f, the rational numbers x_j are independent variables, and $y(\mathbf{v})$ is a function from S into $\mathbb{N}$. The minimum value of the sum $\sum f(\mathbf{v})\, y(\mathbf{v})$ is a function $g(\mathbf{x}) = g(x_1, \ldots, x_n)$ which is called the value function of the integer programme. It is known that the value function of each integer programme is the restriction of a Gomory function to a suitable domain [1]. It is shown here under various conditions that the value function of the pure integer programme (I) is the function f. First it is shown that the value function is not less than f. The following lemma is related to Proposition 2·13 of [1].

LEMMA 4·1. *Let the function $f: \mathbb{N}^n \to \mathbb{N}$ be subadditive and non-decreasing in each coordinate with $f(0) = 0$. Let g be the value function of the integer programme (I). Then $g(\mathbf{v}) \geqslant f(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{N}^n$.*

Proof. For a given vector $\mathbf{x}$ in $\mathbb{N}^n$ let $y(\mathbf{v})$ be a function satisfying the conditions of the integer programme (I) and let $\mathbf{v}^1, \ldots, \mathbf{v}^k$ be a sequence of vectors which contains $y(\mathbf{v})$ copies of each vector $\mathbf{v}$ in S. The function $y(\mathbf{v})$ is required to satisfy the conditions that $\sum_{j=1}^{k} v_i^j \geqslant x_i$ for each i, $0 \leqslant i \leqslant n$. Some coordinates in some of these vectors can be reduced to generate a set of vectors $\tilde{\mathbf{v}}^j$ such that $0 \leqslant \tilde{v}_i^j \leqslant v_i^j$ for each i, j with $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$, and $\sum_{j=1}^{k} \tilde{v}_i^j = x_i$ for each i, i.e. $\sum_{j=1}^{k} \tilde{\mathbf{v}}^j = \mathbf{x}$. Thus

$$\sum f(\mathbf{v})\, y(\mathbf{v}) = \sum_{j=1}^{k} f(\mathbf{v}^j) \geqslant \sum_{j=1}^{k} f(\tilde{\mathbf{v}}^j) \geqslant f(\mathbf{x}),$$

the first inequality because f is non-decreasing in each coordinate and the second because f is subadditive. Hence $g(\mathbf{x}) \geqslant \min \sum f(\mathbf{v})\, y(\mathbf{v}) \geqslant f(\mathbf{x})$. ∎

THEOREM 4·1. *Let the function $f: \mathbb{N}^n \to \mathbb{N}$ be subadditive and non-decreasing in each coordinate with $f(\mathbf{0}) = 0$. Then given any positive number N, there exists a Gomory function $g: \mathbb{Q}^n \to \mathbb{Q}$ such that $f(\mathbf{v}) = g(\mathbf{v})$ whenever $\mathbf{v} \in [0, N]^n$.*

Proof. Let $\mathbf{v} = v_1 \mathbf{i}_1 + \ldots + v_n \mathbf{i}_n$. Let $g(\mathbf{x})$ be the value function of the integer programme (I) with $S = [0, N]^n$. It follows from Lemma 4·1 that $g(\mathbf{x}) \geqslant f(\mathbf{x})$ for all $\mathbf{x}$ in $[0, N]^n$. For a particular vector $\mathbf{x}$ in $[0, N]^n$ let $y(\mathbf{x}) = 1$ and $y(\mathbf{v}) = 0$ when $\mathbf{v} \neq \mathbf{x}$. Then $\sum f(\mathbf{v})\, y(\mathbf{v}) = f(\mathbf{x})$. Thus $g(\mathbf{x}) \leqslant f(\mathbf{x})$. Hence $g(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x}$ in $[0, N]^n$ and g is a Gomory function. ∎

THEOREM 4·2. *Let the function $f: \mathbb{N}^n \to \mathbb{N}$ be subadditive, non-decreasing in each coordinate, shift-periodic in each coordinate uniformly on $\mathbb{N}^n$, with $f(\mathbf{0}) = 0$. Then there exists a Chvátal function $g: \mathbb{Q}^n \to \mathbb{Q}$ such that $f(\mathbf{v}) = g(\mathbf{v})$ for all $\mathbf{v}$ in $\mathbb{N}^n$.*

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Proof. Suppose that f is (p_j, q_j)-shift-periodic in x_j uniformly on $\mathbb{N}^n$. Then $f(\mathbf{x}+p_j\mathbf{i}_j) = f(\mathbf{x})+q_j = f(\mathbf{x})+f(p_j\mathbf{i}_j)$ if $0 \leqslant j \leqslant n$. Let $\mathbb{P}$ be the region $\{\mathbf{v} = (v_1, \ldots, v_n) : 0 \leqslant v_j \leqslant p_j,\ 0 \leqslant j \leqslant n\}$. Let $g(\mathbf{x})$ be the value function of the integer programme (I) with $S = \mathbb{P}$. It follows from Lemma 4·1 that $g(\mathbf{x}) \geqslant f(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{N}^n$. Given $\mathbf{x}$ in $\mathbb{N}^n$ there exist $\mathbf{x}'$ in $\mathbb{P}$ and positive integers α_j such that $x_j = x'_j + a_j p_j$ whenever $0 \leqslant j \leqslant n$, and $f(\mathbf{x}) = f(\mathbf{x}')+\alpha_j q_j$. Let $y(\mathbf{x}') = 1$, $y(p_j\mathbf{i}_j) = \alpha_j$ for $0 \leqslant j \leqslant n$, and $y(\mathbf{v}) = 0$ for other values of $\mathbf{v}$. The $\sum f(\mathbf{v})\, y(\mathbf{v}) = f(\mathbf{x}')+\sum \alpha_j f(p_j\mathbf{i}_j) = f(\mathbf{x}'+\sum \alpha_j p_j) = f(\mathbf{x})$. Thus $g(\mathbf{x}) \leqslant f(\mathbf{x})$. Hence $g(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{N}^n$, and g is a Gomory function and by Proposition 3·2 g is a Chvátal function. ∎

THEOREM 4·3. *Let the function $f: \mathbb{N}^n \to \mathbb{N}$ be subadditive and non-decreasing in each coordinate with $f(\mathbf{0}) = 0$, and let $\mathbb{V} = \{\mathbf{v}^i : 1 \leqslant i \leqslant k\}$ be a Hilbert basis for f. Then there exists a Gomory function g such that $g(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{N}^n$.*

Proof. Let g be the value function of the integer programme (I) with $\mathbb{S} = \mathbb{V}$. It follows from Lemma 4·1 that $g(\mathbf{x}) \geqslant f(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{N}^n$. Given $\mathbf{x} = \sum \alpha^i \mathbf{v}^i$ in $\mathbb{N}^n$, let $y(\mathbf{v}^i) = \alpha^i$ and $y(\mathbf{v}) = 0$ for $\mathbf{v} \notin \mathbb{V}$. Then $\sum f(\mathbf{v})\, y(\mathbf{v}) = \sum \alpha^i f(\mathbf{v}^i) = f(\mathbf{x})$ so that $g(\mathbf{x}) \leqslant f(\mathbf{x})$. Hence $g(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbb{N}^n$ and g is a Gomory function. ∎

5. *Examples*

We give examples of calculations of Gomory function presentations for some integral subadditive function. It may be possible to extend a function which is subadditive on a finite set in more than one way to a function which is subadditive on a larger set. So a Gomory function of a function defined on a finite set need not be unique. The first example illustrates this point.

Example 5·1. The function $f(x_1, x_2)$ defined on the set of points $\mathcal{S} = \{(0,0), (1,0), (2,0), (0,1), (1,1), (2,1)\}$ and taking values $0, 3, 5, 2, 4, 5$ at those points is subadditive and non-decreasing in each coordinate. It can be extended to $\mathbb{N}^n$ in many ways. We can obtain Gomory functions which are equal to this function on $\mathcal{S}$ by considering the following pure integer programme.

$$\begin{aligned}
&\text{Minimize} && 3y(1,0)+5y(2,0)+2y(0,1)+4y(1,1)+5y(2,1) && \\
&\text{subject to} && y(1,0)+2y(2,0)+\quad y(1,1)+2y(2,1) \geqslant x_1 && (1)\\
& && y(0,1)+y(1,1)+\ y(2,1) \geqslant x_2 && (2)\\
& && y(1,0) \geqslant 0 && (3)\\
& && y(2,0) \geqslant 0 && (4)\\
& && y(0,1) \geqslant 0 && (5)\\
& && y(1,1) \geqslant 0 && (6)\\
& && y(2,1) \geqslant 0. && (7)
\end{aligned}$$

The sum of the first two inequalities gives

$$y(1,0)+2y(2,0)+y(0,1)+2y(1,1)+3y(2,1) \geqslant x_1+x_2.$$

Since all the values $y(i,j)$ are non-negative, all the coefficients on the left can be raised to 5. If we now divide by 5 and observe that the left-hand side is an integer, we obtain

$$y(1,0)+y(2,0)+y(0,1)+y(1,1)+y(2,1) \geqslant \lceil (x_1+x_2)/5 \rceil. \qquad (8)$$

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Similarly, twice the sum of the first two inequalities with the coefficients raised to multiples of 3 gives

$$3y(1,0)+6y(2,0)+3y(0,1)+6y(1,1)+6y(2,1) \geqslant 2(x_1+x_2).$$

If we now divide by 3 and observe that the left-hand side is an integer, we obtain

$$y(1,0)+2y(2,0)+y(0,1)+2y(1,1)+2y(2,1) \geqslant \lceil 2(x_1+x_2)/3 \rceil. \tag{9}$$

Adding inequalities (1), (8) and (9) gives

$$3y(1,0)+5y(2,0)+2y(0,1)+4y(1,1)+5y(2,1) \geqslant x_1+\lceil (x_1+x_2)/5 \rceil+\lceil 2(x_1+x_2)/3 \rceil.$$

Thus the value function $g(x_1,x_2)$ of the integer programme is not less than this Chvátal function, and one can check that this Chvátal function does agree with f on the six points of its domain. There are many other Chvátal functions which agree with f. Any Chvátal function ϕ of $s=x_1+x_2$ such that $\phi(0)=0$, $\phi(1)=2$ and $\phi(2)=\phi(3)=3$ when added to the function $\psi(x_1+x_2)=x_1$ is such a function. For example, $\phi(s)=\lceil 3\lceil 2s/3 \rceil/2 \rceil$ has the desired properties.

Example 5·2. Consider the function $f(x_1,x_2)$ defined on $\mathbb{N}^2$ which is $(3,3)$-shift-periodic in x_1 and $(2,2)$-shift-periodic in x_2 and takes the values shown in Figure 1 for $0 \leqslant x_1 \leqslant 3$ and $0 \leqslant x_2 \leqslant 2$.

2	2	4	5	5
1	2	3	4	5
0	0	2	3	3
	0	1	2	3

Fig. 1. Values of a shift-periodic subadditive function in an initial block.

Since the function is shift-periodic in each variable it is sufficient to check subadditivity for pairs of points in the initial block. This function is subadditive and non-decreasing in each variable. By Theorem 4·2, it is a Chvátal function which is the value function of the following pure integer programme.

Minimize
$$\begin{aligned} &2y(1,0)+3y(2,0)+3y(3,0)\\ &+2y(0,1)+3y(1,1)+4y(2,1)+5y(3,1)\\ &+2y(0,2)+4y(1,2)+5y(2,2)+5y(3,2) \end{aligned}$$

subject to
$$\begin{aligned} &y(1,0)+2y(2,0)+3y(3,0)\\ &+\ y(1,1)+2y(2,1)+3y(3,1)\\ &+\ y(1,2)+2y(2,2)+3y(3,2) \geqslant x_1 \qquad (10)\\ &y(0,1)+\ y(1,1)+\ y(2,1)\ +y(3,1)\\ &+2y(0,2)+2y(1,2)+2y(2,2)+2y(3,2) \geqslant x_2 \qquad (11)\\ &y(i,j) \geqslant 0. \end{aligned}$$

Since the value function of this integer programme is known to be $(3,3)$-shift-periodic in x_1 and $(2,2)$-shift-periodic in x_2 we consider Chvátal functions whose carriers are x_1+x_2. The inequality $2\times(10)+3\times(11)$ is

$$\begin{aligned} &2y(1,0)+\ 4y(2,0)+\ 6y(3,0)\\ &+3y(0,1)+5y(1,1)+\ 7y(2,1)+\ 9y(3,1)\\ &+6y(0,2)+8y(1,2)+10y(2,2)+12y(3,2) \geqslant 2x_1+3x_2. \end{aligned}$$

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Raising each coefficient to a multiple of 6 and dividing by 6 gives

$$\begin{aligned} &y(1,0)+\ y(2,0)+\ y(3,0)\\ +&y(0,1)+\ y(1,1)+2y(2,1)+2y(3,1)\\ +&y(0,2)+2y(1,2)+2y(2,2)+2y(3,2)\geqslant \lceil(2x_1+3x_2)/6\rceil. \end{aligned}$$

Similarly, the inequality $4\times(10)+3\times(11)$ with coefficients raised to multiples of 6, on dividing by 6 gives

$$\begin{aligned} &y(1,0)+2y(2,0)+2y(3,0)\\ +&y(0,1)+2y(1,1)+2y(2,1)+3y(3,1)\\ +&y(0,2)+2y(1,2)+3y(2,2)+3y(3,2)\geqslant \lceil(4x_1+3x_2)/6\rceil. \end{aligned}$$

Thus the value function of the integer programme is not less than $\lceil(2x_1+3x_2)/6\rceil+\lceil(4x_1+3x_2)/6\rceil$. This function is shift-periodic as required and it agrees with the function f on the block $0\leqslant x_1\leqslant 3, 0\leqslant x_2\leqslant 2$, so that it agrees with f throughout $\mathbb{N}^2$.

Example 5·3. Consider the function f which is subadditive on $\mathbb{N}^2$ and is determined by its values at the points of a Hilbert basis

$$S=\{(1,0),(2,1)\}\cup\{(2,1),(1,1),(1,2)\}\cup\{(1,2),(0,1)\}:$$
$$f(0,0)=0, f(1,0)=f(0,1)=5, f(1,1)=8, f(2,1)=f(1,2)=12.$$

The restriction of f to each of the three polyhedral cones bounded by the axes and the lines $x_1=2x_2, x_2=2x_1$ is a linear function of its values at a set of points which generate the cone. In this case the points are those of the Hilbert basis for the cone, so that the restriction of f to each cone is linear. Thus

$$\begin{aligned} f(x_1,x_2)&=\begin{cases} 5x_1+2x_2, & x_2\leqslant x_1/2,\\ 4x_1+4x_2, & x_1/2\leqslant x_2\leqslant 2x_1,\\ 2x_1+5x_2, & 2x_1\leqslant x_2 \end{cases}\\ &=\ \max\{5x_1+2x_2, 4x_1+4x_2, 2x_1+5x_2\}. \end{aligned}$$

This is the value function for the integer programme

$$\begin{aligned} \text{Minimize}\quad & 5y(1,0)+5y(0,1)+8y(1,1)+12y(2,1)+12y(1,2)\\ \text{subject to}\quad & y(1,0)+\ y(1,1)+\ 2y(2,1)+\ y(1,2)\geqslant x_1,\\ & y(0,1)+\ y(1,1)+\ y(2,1)+\ 2y(1,2)\geqslant x_2,\\ & y(1,0),\ y(0,1),\ y(1,1),\ y(2,1),\ y(1,2)\geqslant 0. \end{aligned}$$

The proof given in [**1**] that the value function of a pure integer programme is a Gomory function does not lead to an effective algorithm for finding Gomory function presentations of value functions. The examples above illustrate methods which can be used to find candidate expressions for subadditive functions, which then have to be checked. Effective algorithms for expressing value functions of pure integer programmes in closed form as Gomory functions would be of considerable help in applications of the subject. Wolsey [**18**] shows how different integer programming algorithms imply value functions in different forms, the Gomory function being that resulting from Gomory's algorithm. Williams [**15**] gives a method of deriving another form based on an extension of Fourier–Motzkin elimination. In [**16**] he gives a

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method of deriving the value function for an integer programme over a cone as a Chvátal function. The cone arises from the set of binding constraints in the linear programming relaxation in a solution space which includes the objective value. For a general integer programme many more constraints may be binding leading to a Gomory function involving many Chvátal functions.

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