Relativistic motion 7

The special theory of relativity rests on just two axioms: the identity of the laws of physics in all inertial reference frames, and the invariance of the speed of light. After briefly reviewing the basics of special relativity, including the Lorentz transformations and time dilation, we look at the effects of relativity during an accelerated journey. Time dilation has implications even for a spacecraft moving at low velocities. We examine the relativistic rocket equation and then rockets that are propelled by photons or a combination of matter and photons. General relativistic effects are not considered. In a later chapter we look at the impact of relativistic motion on optical phenomena.

7.1 Basic relations

Throughout this and later chapters I adopt the standard notion of $\beta = v/c$ to indicate the velocity of an object moving at velocity v as a fraction of the speed of light c, and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \text{where} \quad 0 \leq |\beta| < 1, \tag{7.1}$$

which maps β to γ as shown in figure 7.1. At low velocities, then using a Taylor expansion and neglecting terms of the order of β^3 and higher, we see that

$$\gamma \simeq 1 + \frac{\beta^2}{2} \quad \text{when} \quad |\beta| \ll 1. \tag{7.2}$$

From equation 7.1 we can derive the following useful relations:

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}\,, \tag{7.3}$$

$$\gamma\beta = \sqrt{\gamma^2 - 1}\,, \tag{7.4}$$

$$\gamma(1-\beta) = \sqrt{\frac{1-\beta}{1+\beta}}\,, \tag{7.5}$$

$$\gamma(1+\beta) = \sqrt{\frac{1+\beta}{1-\beta}}\,. \tag{7.6}$$

A body of mass m moving at a velocity v has a total energy relative to the observer of

$$E = \gamma mc^2 = \gamma E_0, \tag{7.7}$$

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

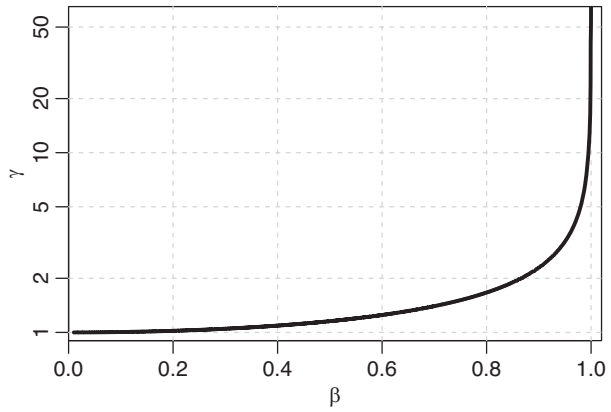


Fig. 7.1 The relationship between $\gamma = (1 - \beta^2)^{-1/2}$ and $\beta = v/c$.

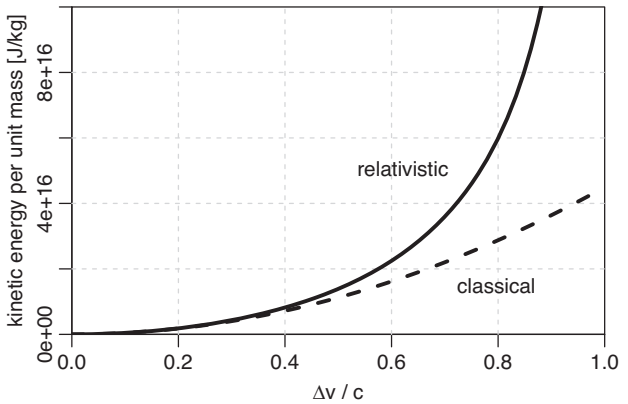


Fig. 7.2 The kinetic energy per unit mass of a relativistic rocket (solid line), $(\gamma - 1)c^2$, and the classical Newtonian case (dashed line), $(1/2)v^2$.

where $E_0 = mc^2$ is the rest energy. The mass m is invariant. The momentum of the body is

$$p = \gamma m v. \tag{7.8}$$

Combining the previous two equations and using equation 7.4 we can write

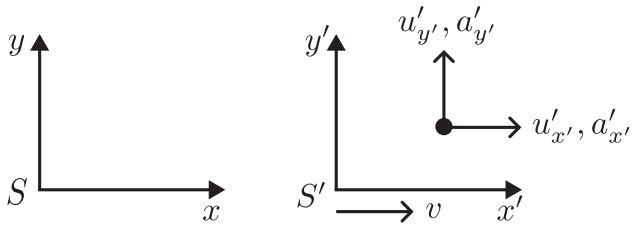
$$E^2 = p^2c^2 + m^2c^4. \tag{7.9}$$

The kinetic energy per unit mass of the body is $(\gamma - 1)c^2$. This is plotted in figure 7.2 in SI units to show the large amounts of energy involved at relativistic speed. For comparison, a 1 GW power station running continuously for one year produces 3×10^{16} J of energy, which is the energy of a 1 kg mass moving at $0.66\,c$, or a 1000 kg mass moving at $0.026\,c$.

7.2 Lorentz transformations

We will need to transform time, position, velocity, and acceleration between inertial reference frames. Consider motion in two reference frames, S and S', whereby S' is moving with a

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008



Reference frames for Lorentz transformations. The reference frame S' moves with a velocity v along the positive x-axis relative to the reference frame S. Fig. 7.3

velocity v along the x-axis relative to S, as shown in figure 7.3. The x'-axis is parallel to the x-axis. The time in S is t and the time in S' is t'.

The frames coincide at $t = t' = 0$. Using the axioms of special relativity one can derive the following Lorentz transformations between the coordinate systems:

$$x = \gamma(x' + vt'), \tag{7.10a}$$

$$y = y', \tag{7.10b}$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right). \tag{7.10c}$$

The transformations for the z-axis, being perpendicular to the x-axis, behave in the same way as y. The same is true for its velocity and acceleration transformations (below).

We can rearrange equations 7.10 to get the primed quantities in terms of the unprimed ones. As S is moving with a velocity $-v$ relative to S', those rearranged equations can be attained more simply by taking equations 7.10 and swapping prime and non-prime symbols and changing the sign of v.

Consider two events taking place at the same position $x'_1 = x'_2$ but at different times t'_1 and t'_2 in S'. What is the time between these events in S? Let their corresponding times be t_1 and t_2 in S. We can relate these to the times in S' using equation 7.10c. Defining $\Delta t = t_2 - t_1$ and $\Delta t' = t'_2 - t'_1$, this gives

$$\Delta t = \gamma \Delta t'. \tag{7.11}$$

Thus any time interval in S' will appear longer in S. If we have two identical clocks at rest in the two frames, then as seen from S, the clock in S' takes longer for its hands to move. In other words, the moving clock appears to tick more slowly. This is known as *time dilation*. Time measured within the rest frame is called the *proper time*.

Note that $\Delta x = v\Delta t$ from equation 7.10a. The two events are not recorded at the same position in S. Time dilation is not caused by light travel time effects.

One consequence of time dilation is that if a spacecraft covers some distance Δx in the Earth's rest frame S at an arbitrarily large velocity (below c, of course), then it can cover this in an arbitrarily short time as measured in its frame S'. But this time will not be short in the Earth's frame: that will always be at least $\Delta x/c$.

Consider now a rod of length $\Delta x'$ as measured in S' lying parallel to the x'-axis (and therefore also the x-axis). What is the length of the rod in S? To determine this we measure the positions of the two ends of the rod at the same time $t_1 = t_2$ in S. The inversion of equation 7.10a is $x' = \gamma(x - vt)$. With $\Delta x' = x'_2 - x'_1$ and $\Delta x = x_2 - x_1$, then taking the difference of this equation for the two events gives

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

$$\Delta x = \frac{1}{\gamma}\Delta x'. \tag{7.12}$$

Thus an object in S' will be measured to be shorter in S in its direction of travel. This is known as *length contraction*. Distance measured within the rest frame is called the *proper distance*. Note that lengths perpendicular to the direction of travel are unaffected because $y = y'$.

We see there is an asymmetry in the factor γ in the time dilation and length contraction equations. This arises because for the former we consider two events at the same position in S', whereas for the latter we consider two events at the same time in S. In both cases the effects are independent of the sign of the velocity: it doesn't matter whether the source is moving towards or away from the observer.

We could equally well consider what an observer in frame S' would measure for a clock and ruler at rest in frame S. The result is identical in the sense that they too would see a clock and ruler moving at a speed v, and so they too would measure a time dilation and length contraction. Naturally, when we consider the situation from S' instead of S, we have to change also that the events are then at the same position in S (time dilation) and same time in S' (length contraction).

Consider now an object moving with a velocity $(u'_{x'}, u'_{y'})$ in S' (figure 7.3). To find the velocities that an observer at rest in S observes, we differentiate the Lorentz transformations to get

$$u_x = \frac{dx}{dt} = \frac{u'_{x'} + v}{1 + \frac{vu'_{x'}}{c^2}}, \tag{7.13a}$$

$$u_y = \frac{dy}{dt} = \frac{u'_{y'}}{\gamma\left(1 + \frac{vu'_{x'}}{c^2}\right)}, \tag{7.13b}$$

where $u'_{x'} = dx'/dt'$ and $u'_{y'} = dy'/dt'$.

Differentiating again we get the accelerations

$$a_x = \frac{du_x}{dt} = \frac{a'_{x'}}{\gamma^3\left(1 + \frac{vu'_{x'}}{c^2}\right)^3}, \tag{7.14a}$$

$$a_y = \frac{du_y}{dt} = \frac{1}{\gamma^2\left(1 + \frac{vu'_{x'}}{c^2}\right)^3}\left[\left(1 + \frac{vu'_{x'}}{c^2}\right)a'_{y'} - \frac{vu'_{y'}}{c^2}a'_{x'}\right], \tag{7.14b}$$

where $a'_{x'} = du'_{x'}/dt'$ and $a'_{y'} = du'_{y'}/dt'$. We see that acceleration is not absolute, but depends on the observer. Furthermore, a constant acceleration in S' does not imply a constant acceleration in S, as $u'_{x'}$ may be varying as a result of the acceleration. Note also that the acceleration a_x parallel to the relative direction of travel of the two observers (in S' and S) depends only on the velocities and acceleration in that parallel direction. In contrast, the perpendicular acceleration depends on the velocities and accelerations in both the parallel and perpendicular directions.

Below we will use one particular case of this acceleration equation, namely when we consider the acceleration of the frame S' itself along the x-axis. We get this when $u'_{x'} = 0$, which gives

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

$$a_x = \frac{1}{\gamma^3} a'_{x'}. \tag{7.15}$$

When an acceleration $a'_{x'}$ is measured within frame S', the acceleration measured by an observer moving at a velocity v relative to it and in the same direction as the acceleration is smaller by a factor of γ^3.

7.3 Constantly accelerating spacecraft

Consider a spacecraft initially at $x = x_0$ at time $t = t_0$ moving at speed $v = v_0$ in reference frame S. The spacecraft then starts to accelerate in its direction of travel. The acceleration has a constant value a' in its own instantaneous rest frame. Even though the spacecraft is accelerating, and so no longer provides an inertial reference frame over a non-zero time duration, at any instant we can define a frame that is momentarily at rest relative to the spacecraft. We want to find the motion of the spacecraft – acceleration, velocity, position – in S as a function of time in S. It is important to remember that in this section, because the velocity of the spacecraft in S is not constant, γ is not constant either.

The acceleration in S is given by equation 7.15. Because γ increases with time, this acceleration will decrease with time.

The velocity in S is obtained by integrating equation 7.15 over t to give

$$v = \frac{a't + \gamma_0 v_0}{\sqrt{1 + (a't + \gamma_0 v_0)^2/c^2}} \quad \text{where} \quad \gamma_0 = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2}, \tag{7.16}$$

where I have dropped the x subscript because the velocity and acceleration are parallel. Integrating again over t we get the position of the spacecraft in S:

$$x = \frac{c^2}{a'}\left[\sqrt{1 + \left(\frac{a't + \gamma_0 v_0}{c}\right)^2} - \gamma_0^2\right] + x_0. \tag{7.17}$$

It will also be useful to express v and x in terms of the spacecraft time t'. The relation between the two times is given by the equation for time dilation, which in differential form is

$$dt = \gamma dt' . \tag{7.18}$$

If we substitute for v (via γ) from equation 7.16, and then integrate, we get the relationship between the two times:

$$t = \frac{c}{a'}\left[\sinh\left(\frac{a't'}{c} + \operatorname{arcsinh}\left[\frac{\gamma_0 v_0}{c}\right]\right) - \frac{\gamma_0 v_0}{c}\right] . \tag{7.19}$$

When the spacecraft starts at $x_0 = 0$ and $v_0 = 0$, these expressions are simplified; they are summarized in Table 7.1 in terms of both t and t'. Hyperbolic tangent functions occur frequently in special relativity so they are plotted in figure 7.4.

The expressions are plotted in figure 7.5 for two accelerations: 1 g (solid black line) and 0.5 g (dashed black line), where g is the standard acceleration due to gravity at the Earth's surface. The top two plots show the relation between the times in the two reference frames. This difference is not a simple time dilation, because one frame has accelerated with respect to the other, and so the speed has varied. After 10 years in the initial rest frame, the clock on

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

Table 7.1 Expressions for the time t, position x, velocity v, and acceleration a of a spacecraft as observed from inertial frame S, in terms of the time t in S in the left column, and in terms of the time t' in the spacecraft's instantaneous rest frame in the right column. The spacecraft starts at $t = t' = 0$ at $x = 0$ and $v = 0$ and undergoes constant acceleration a' in its instantaneous rest frame.

	t	t'
t	t	$\frac{c}{a'}\sinh\left(\frac{a't'}{c}\right)$
x	$\frac{c^2}{a'}\left[\sqrt{1+\left(\frac{a't}{c}\right)^2}-1\right]$	$\frac{c^2}{a'}\left[\cosh\left(\frac{a't'}{c}\right)-1\right]$
v	$a't\left[1+\left(\frac{a't}{c}\right)^2\right]^{-1/2}$	$c\tanh\left(\frac{a't'}{c}\right)$
a	$a'\left[1+\left(\frac{a't}{c}\right)^2\right]^{-3/2}$	$a'\,\mathrm{sech}^3\left(\frac{a't'}{c}\right)$

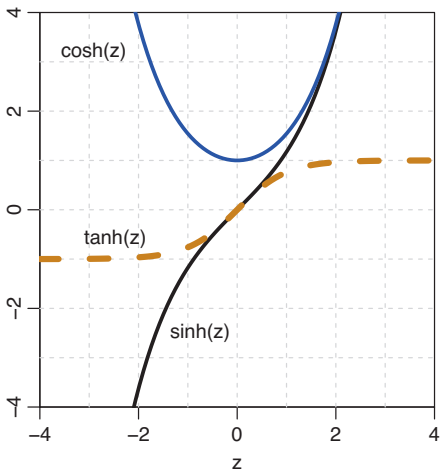
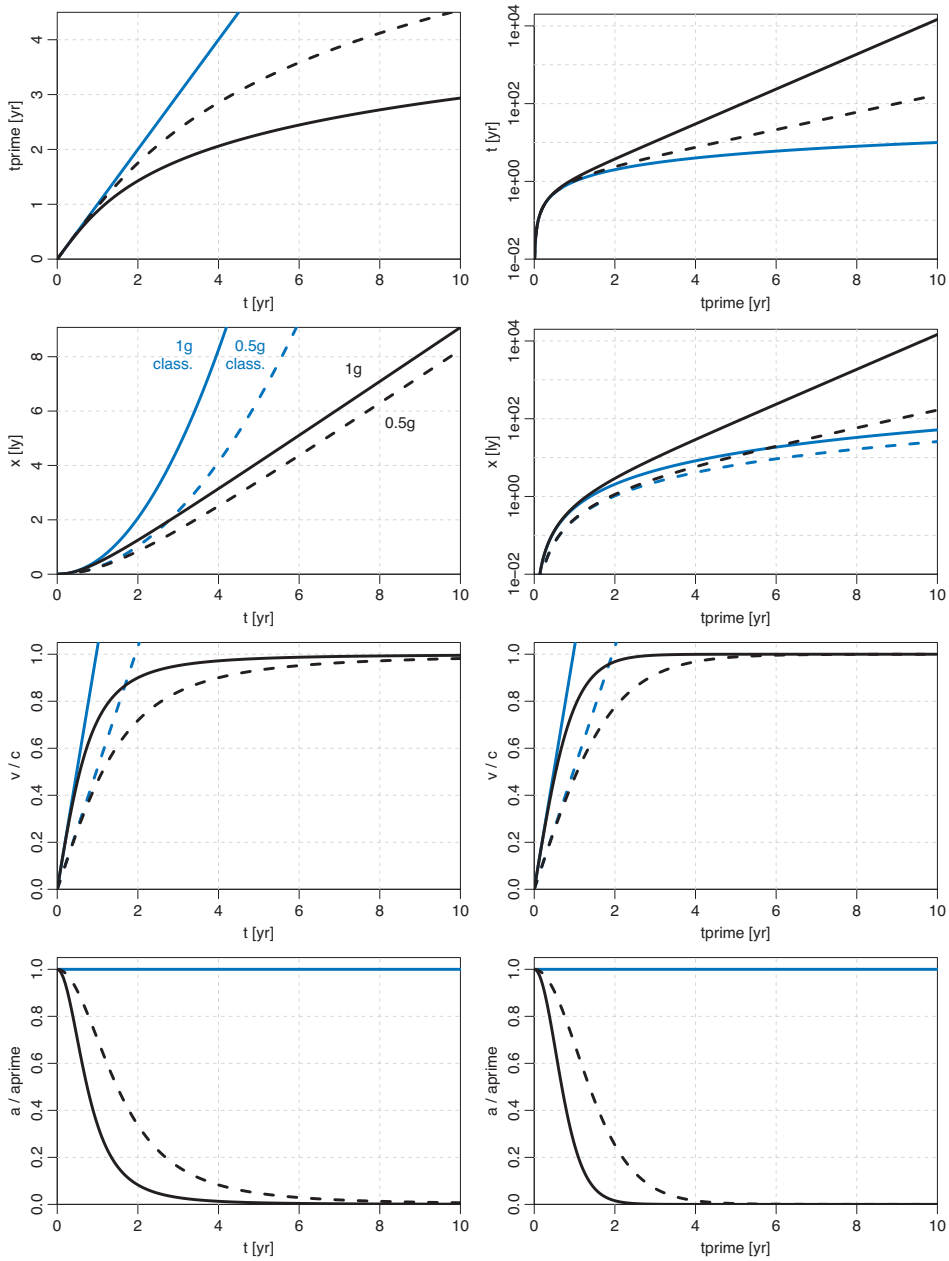


Fig. 7.4 The three main hyperbolic functions ($\mathrm{sech}(z) = 1/\cosh(z)$).

the spacecraft accelerating at 1 g has advanced by only 2.9 years, whereas after 10 years of spacecraft time, nearly 15 000 years have passed in the initial rest frame (note the log scale in the right panel).

The second row shows the distance travelled by the spacecraft. After 10 years in the initial rest frame, the spacecraft accelerating at 1 g has travelled 9.08 ly. At half the acceleration it has travelled nearly as far, 8.25 ly. The Newtonian relation is shown by the orange lines in figure 7.5, and yields distances of 51.6 ly and 25.8 ly, respectively, which are in the same ratio as the two accelerations (because $x = (1/2)at^2$), a factor of two. In special relativity that is not the case because, as shown in the third row, the spacecraft is travelling close to the speed of light after just a few years as measured in either reference frame. The relationship

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008



The time t, position x, velocity v, and acceleration a of a spacecraft observed from inertial frame S in terms of the time t in that frame t (left column) and the time t' in the spacecraft's instantaneous rest frame (right column). The spacecraft starts at $t = t' = 0$ at $x = 0$ and $v = 0$ and undergoes constant acceleration a' in its instantaneous rest frame. The solid black line is for an acceleration of 1 g, the dashed black line is for an acceleration of 0.5 g. The orange lines are the corresponding results for classical Newtonian mechanics. Other than the top-left panel, which is just an inversion of the top-right panel, the panels correspond to the equations in Table 7.1.

Fig. 7.5

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

between distance and acceleration then becomes very nonlinear (bottom row). In particular, the closer the spacecraft gets to the speed of light, the slower its velocity increases.

For 1 g acceleration, the Newtonian model predicts velocities larger than the speed of light after about a year (it so happens that $c \simeq g \times \text{yr}$). It therefore predicts far larger distances travelled at a given initial rest frame time than the relativistic calculation. But in terms of spacecraft time, the Newtonian model predicts shorter distances because it assumes that time is universal.

Returning to the relativistic calculations, we saw that because of the extreme difference between the timescales in the two reference frames after several years, the spacecraft travels far further in 10 years of spacecraft time than in 10 years of initial rest frame time: the right plot in the second row shows that at 1 g, it travels nearly 15 000 ly. Yet at 0.5 g it travels only 167 ly within the same time. In both cases the spacecraft has been travelling close to the speed of light (in the initial rest frame) for most of its travel time. But for the higher acceleration spacecraft, those 10 years of spacecraft time correspond to a far longer period of time in the initial rest frame (15 000 years vs 169 years), and so the former travels much further.

Once the spacecraft has reached nearly the speed of light, it moves at nearly 1 ly per year, where both time and distance are measured in the initial rest frame. This is the *coordinate velocity*, which we have been denoting with v. But for each year of spacecraft time, the spacecraft will cover far more than 1 ly in the initial rest frame. This quantity, $w = \Delta x/\Delta t'$, where Δx is measured in S and $\Delta t'$ is measured in S', is called the *proper velocity*. More formally,

$$w = \frac{dx}{dt'} = \gamma v, \tag{7.20}$$

the relation to the coordinate velocity following from the equation for time dilation (equation 7.11).[1] The proper velocity is numerically larger than the speed of light once $\beta > 1/\sqrt{2}$. Nothing physically moves at this velocity in any inertial frame, however, so it is not in conflict with special relativity. It just tells us how far the spacecraft moves per unit of spacecraft time.

The dimensionless version of the proper velocity is $\omega = w/c = \gamma\beta$. Other useful conversions are

$$\beta = \frac{\omega}{\sqrt{1+\omega^2}} \quad \text{and} \tag{7.21a}$$

$$\gamma^2 = 1 + \omega^2 . \tag{7.21b}$$

When $|\beta| \ll 1$, $\omega \simeq \beta$. When $\beta \simeq 1$, $\omega \simeq \gamma$.

7.4 Force and Newton's second law

In special relativity, the momentum of a body of mass m moving at velocity v in frame S is $p = \gamma mv$. If force F, as measured in S, acts on the body, its rate of change of momentum is given by Newton's second law:

$$F = \frac{dp}{dt} = \frac{d(\gamma mv)}{dt} = \frac{d(mw)}{dt} = \frac{1}{\gamma}\frac{d(mw)}{dt'}, \tag{7.22}$$

[1] Unfortunately, the attribution of the descriptor 'proper' in special relativity is not really consistent.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

where in the penultimate expression I have used the proper velocity from equation 7.20, and in the last one the proper time, $dt = \gamma dt'$. If the mass is constant, the above expression becomes

$$F = m\frac{d(\gamma v)}{dt} = m\frac{1}{\gamma}\frac{dw}{dt'} . \tag{7.23}$$

With $a = dv/dt$, we find by differentiation that

$$\frac{dw}{dt} = \frac{d(\gamma v)}{dt} = \gamma^3 a = a', \tag{7.24}$$

with the last step given by the Lorentz transformation from equation 7.15. Thus we can write

$$F = \gamma^3 ma = ma' . \tag{7.25}$$

The acceleration a' is called the *proper acceleration*. It equals the derivate of the proper velocity with respect to the coordinate time. Newton's second law for constant mass looks like its classical counterpart when we write it in terms of the proper acceleration. In terms of the proper time, it follows from equation 7.24 that

$$a' = \frac{1}{\gamma}\frac{dw}{dt'} . \tag{7.26}$$

We saw at the end of section 7.2 that the proper acceleration is the acceleration experienced within the instantaneous rest frame. This is the physical acceleration that an astronaut feels in a rocket when it is firing its engines, for example. This is conceptually different from the *coordinate acceleration a*, which is the rate of change of velocity of an object as measured by an observer in their rest frame on their clock.

7.5 The relativistic rocket equation

Once a rocket starts to move at relativistic speeds, the derivation of the classical rocket equation using Newtonian mechanics in section 3.1 is no longer valid. An equivalent derivation in special relativity can be performed (I'll outline one in section 7.8). This again gives the change in velocity of a rocket of initial mass M_i after it has ejected $M_i - M_f$ of propellant at velocity v_e relative to the rocket, but the expression is now

$$\Delta v = c \tanh\left[\frac{v_e}{c}\ln\left(\frac{M_i}{M_f}\right)\right] . \tag{7.27}$$

As the tanh function is limited to the range ± 1 (figure 7.4), and we are only dealing with positive quantities, Δv lies in the range 0 to c. This function is shown in figure 7.6. The inverse of this is

$$\frac{M_i}{M_f} = \exp\left[\frac{c}{v_e}\operatorname{atanh}\left(\frac{\Delta v}{c}\right)\right] . \tag{7.28}$$

With $v_e = 0.01\,c$, achieving a velocity of $0.2\,c$ requires a mass ratio of 640 million. When v_e is increased to $0.05\,c$, the required mass ratio drops to 58, and when $v_e = 0.1\,c$, it is just 7.6. To achieve relativistic speeds clearly requires a large v_e.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

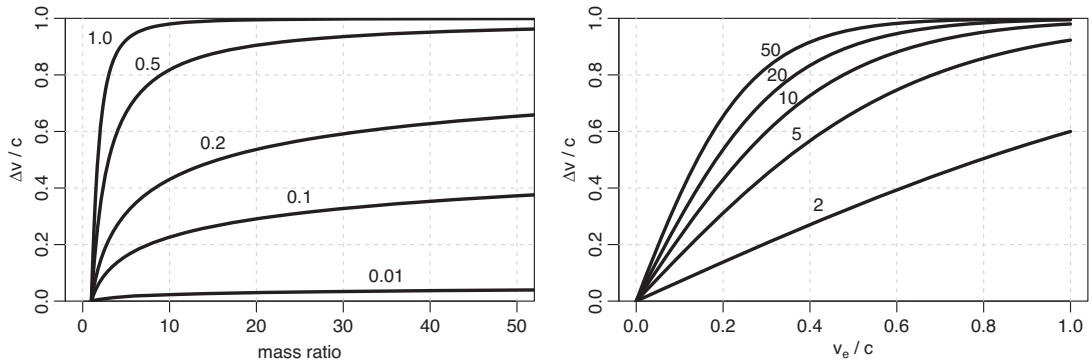


Fig. 7.6 The relativistic rocket equation (equation 7.27) as function of mass ratio for several values of v_e/c (left), and as function of v_e/c for several mass ratios (right).

The tanh function can be written as a Taylor expansion, $\tanh z = z - z^3/3 + \mathcal{O}(z^5)$. When z is small, $\tanh z \simeq z$. Thus for $v_e \ll c$ and/or small mass ratios, the relativistic rocket equation becomes the classical rocket equation. The ratio of Δv from the relativistic to classical rocket equations is $\tanh(z)/z$ with $z = (v_e/c)\ln(M_i/M_f)$. Only once $z > 0.4$ do the two velocities differ by more than 5%.

With the classical rocket equation, we found that in order to accelerate from zero to Δv and then decelerate back to zero again required the square of the mass ratio needed for the acceleration part (see section 3.2). From equation 7.27 we see that this is still the case with the relativistic rocket equation, because

$$\frac{\Delta v}{c} = \tanh\left[\frac{v_e}{c}\ln\left(\frac{M_i}{M_m}\right)\right] = \tanh\left[\frac{v_e}{c}\ln\left(\frac{M_m}{M_f}\right)\right], \tag{7.29}$$

where M_m is the mass after the first boost, from which it again follows that $M_i/M_f = (M_m/M_f)^2$. However, unlike with the classical rocket equation, squaring the mass ratio does not generally double Δv. We will see an example of this in the next section.

Given that tanh is defined in terms of exponentials, the relativistic rocket equation can also be written as

$$\Delta v = c\,\frac{Q^2-1}{Q^2+1}, \quad \text{where} \quad Q = \left(\frac{M_i}{M_f}\right)^{(v_e/c)}, \quad \text{or} \tag{7.30a}$$

$$\frac{M_i}{M_f} = \left(\frac{1+\Delta v/c}{1-\Delta v/c}\right)^{(c/2v_e)}. \tag{7.30b}$$

We tend to think of the rocket equation as describing a situation in which a mass $M_i - M_f$ is ejected at velocity v_e. However, once nuclear reactions are involved that convert a non-negligible fraction of the initial mass into energy, not all of that mass is ejected at the speed of the propellant, so we need to adopt a different *effective* exhaust velocity to accommodate this. We will address this in section 7.8.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

7.6 A trip to Alpha Centauri

Let us consider a trip from the Earth to Alpha Centauri, a distance we take to be 4.4 ly in this section. For simplicity, we will assume that the origin and target are stationary relative to one another, and refer to this as the Earth's rest frame.

Consider first a flyby mission. Starting at rest, a spacecraft accelerates continuously at 1 g in its instantaneous rest frame until it reaches Alpha Cen. Using the expressions in Table 7.1, we can compute that the spacecraft time required for this mission is 2.32 years, the Earth time is 5.28 years, and the velocity at encounter is 0.984 c.

We can compare this flyby to a rendezvous mission, in which the spacecraft stops accelerating at the halfway point (2.2 ly) and then decelerates at 1 g in its instantaneous rest frame, arriving with zero velocity in the Earth's rest frame. The spacetime diagram for this mission is shown as the solid line in figure 7.7 (ignore the return journey for now). We can compute that the Earth time to the halfway point is 3.02 years (spacecraft time 1.80 years), when the velocity is 0.952 c. The second half of the outward journey is identical to the first half, just played out backwards, so the total journey time to Alpha Cen is double these values. The top three rows of Table 7.2 summarize these results.

For a rocket with fixed effective exhaust velocity, we can use the relativistic rocket equation (section 7.5) to determine the mass ratios required for the two types of journey. Suppose $v_e = 0.2\,c$. The rocket's mass decreases with time as it exhausts propellant, so in order to maintain constant acceleration in its instantaneous rest frame, the mass flow rate will have to

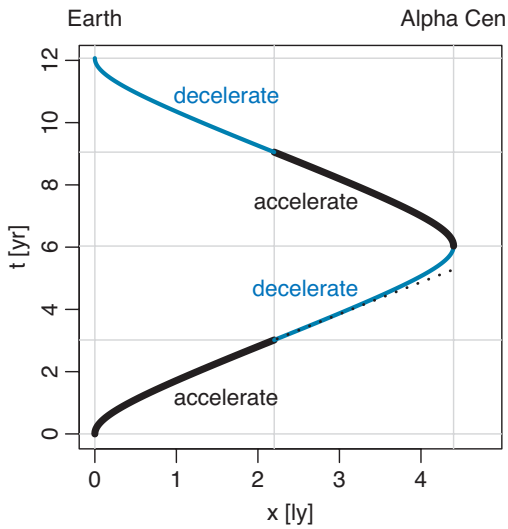


Fig. 7.7 The relativistic distance–time diagram for a round trip mission covering the 4.4 ly from the Earth to Alpha Centauri and back. The solid line shows the profile for a spacecraft that accelerates continuously at 1g in its instantaneous rest frame, changing the direction of the acceleration (i.e. to decelerate) at the halfway point: the thick parts of the line are acceleration and the thin deceleration. The return journey is the mirror image of the outward trip. The dotted line shows how the spacecraft would continue its outward journey if it continued to accelerate rather than decelerate at the halfway point, that is, a flyby rather than a rendezvous at Alpha Centauri.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

Table 7.2 Parameters of various relativistic trips for a spacecraft (S/C) with constant acceleration/deceleration. The mass ratios in the final column are for $v_e = 0.2\,c$.

Trip profile	Distance travelled [ly]	Duration Earth t [yr]	Duration S/C t' [yr]	Total $\Delta v/c$	Mass ratio M_i/M_f
1 g accel (flyby)	4.4	5.28	2.32	0.984	160 000
1 g accel (to halfway point)	2.2	3.02	1.80	0.952	10 600
1 g accel + decel (rendezvous)	4.4	6.03	3.59	0.952 + 0.952	110×10^6
1 g accel	18.8	19.73	3.59	0.9988	110×10^6
0.1 g accel (flyby)	4.4	10.23	8.92	0.726	100
0.1 g accel (to halfway point)	2.2	6.89	6.41	0.579	27
0.1 g accel + decel (rendezvous)	4.4	13.78	12.83	0.579 + 0.579	750
0.1 g accel	9.8	16.91	12.83	0.868	750

decrease with time too (equation 3.10). Note, though, that the (relativistic) rocket equation makes no assumption about the mass flow rate.

For the flyby mission we can simply plug the Δv value into equation 7.28 to find a mass ratio of about 160 000. This is a huge value, unachievable in practice. This shows how infeasible it is to get a rocket close to the speed of light, even if v_e is high (higher than we could achieve even with fusion engines, as we shall see in chapter 8).

For the rendezvous mission, the mass ratio required to accelerate at 1 g to the halfway point is, using the required $\Delta v = 0.952\,c$, 10 600. As shown in section 7.5, the mass ratio required for the entire mission is the square of this, about 110 million. This is 700 times larger than the mass ratio for the flyby mission.

Why does the rendezvous mission require so much more propellant than the flyby mission? It is not because of decelerating rather than accelerating: the spacecraft does not 'know' in which direction it is firing its rocket; nor can this matter. It is also not because of the difference in maximum velocity: the flyby mission, which uses less propellant, attains a higher velocity. Instead, it is because for the rendezvous mission more spacecraft time elapses, and so the rockets have to be fired for longer. It is only about 1.55 times longer (3.59 years vs 2.32 years), but this actually implies a lot more than 1.55 times as much propellant.

It must be the case that if we continuously accelerate the rocket for 3.59 years of spacecraft time, it will require the same mass ratio as the rendezvous mission. From the $v(t')$ equation in Table 7.1, we see that a continuous acceleration at 1 g for this time gives $\Delta v = 0.9988\,c$. Putting this into the relativistic rocket equation shows that we require a mass ratio of exactly the same value as for the rendezvous mission (110 million). This can be seen directly by substituting $\Delta v = v(t')$ from the equation in Table 7.1 into equation 7.28, to give $M_i/M_f = \exp(a't'/v_e)$. The mass ratio required to maintain constant proper acceleration depends exponentially on the spacecraft time.

The results of the above calculations are summarized in Table 7.2, which also shows the results for a lower acceleration of 0.1 g. If we adopt more plausible values of $v_e = 0.08\,c$ and $a' = 0.005\,g$, then we find that a flyby mission to Alpha Cen takes 41.5 years of Earth time, requires a mass ratio of 14.3, and achieves an encounter velocity of 0.21 c. A rendezvous

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

mission with the same v_e and a' would take 58.6 years and require a mass ratio of 43. Its velocity at the turn-around point is $0.15\,c$. As we shall see in chapter 8, $v_e = 0.08\,c$ is around the theoretical maximum for a fusion rocket.

7.6.1 The twin paradox

We saw in the previous section that the time elapsed onboard the spacecraft for the trip to Alpha Cen is less than the time elapsed on the Earth for the same trip. This can be see as a consequence of time dilation, or more generally the Lorentz transformations (we have not used anything else). To get back to the Earth, the spacecraft just repeats the same journey in reverse (figure 7.7), and so the durations in both the spacecraft and Earth frames are twice as long.

Suppose one of two twins – let's call her Sabina – travelled on the spacecraft, and the journey was done with constant 1 g acceleration as in Table 7.2. By the time Sabina gets back to Earth, $2 \times 3.59 = 7.18$ years of time have passed on the spacecraft; she has aged by this amount. Her twin sister Emily, who worked at ground control on Earth for the entire journey, has instead aged by $2 \times 6.03 = 12.06$ years. So Emily is older. From Emily's perspective (in the Earth's rest frame), Sabina flew to a point 4.4 ly away and then turned around and came back. But coordinate velocity and acceleration are relative. So from Sabina's perspective (in the spacecraft rest frame), Emily also flew to a point 4.4 ly away and then turned around and came back. So wouldn't Emily be the younger one? Their perspectives of the journey appear to be symmetric – same distances, velocities, and coordinate accelerations – so why is there a difference in their ages at all?

This is the notorious twin paradox, and like most (all?) paradoxes in physics, it isn't one. The resolution is that the situation is not symmetric for the twins: Emily has been in the same inertial frame all of the time, whereas Sabina has undergone various proper accelerations to move between a continuous set of inertial frames before finally returning to Emily's frame where the age comparison is done. Once Sabina arrived at Alpha Cen (and came to rest), she would have been $6.03 - 3.59 = 2.44$ years younger than Emily (both twins are in the same rest frame, just physically separated). If Emily then got in a spacecraft and did the same journey to Alpha Cen, then when she arrived, she would again be the same age as her sister.

A lot of time and ink has been spent trying to resolve the paradox, and the reader will have no problem finding many discussions on the internet. Some explanations are not really correct, for example the claim that the twin paradox is a consequence of general relativity (perhaps understandable because gravity also introduces ageing differences).[2] Some explanations use modifications of the above scenario to overcome plausible objections or to show that some claimed solutions do not work. Many identify acceleration as the key, although in some sense it is only indirectly responsible as a way of inducing changes in the inertial frame of one twin. Different people will be more or less satisfied by different explanations.

[2] Having said that, the twin paradox can also be understood as a consequence of the strong equivalence principle. The accelerating twin can be considered as experiencing the effect of a strong gravitational field which, through gravitational time dilation, causes that twin's clock to tick more slowly than the non-accelerating twin.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

7.7 Photon rockets

The rocket equation suggests we should aim to use a propellant with a high exhaust velocity. The highest possible is the speed of light, achievable only with photons. Although they have no mass, photons do have momentum, so can propel a spacecraft. Let us work out the rocket equation for this case.

Consider the situation shown in figure 7.8. The rocket initially has a mass of $M_f + M_p$ and is at rest in some inertial reference frame S. The propellant mass M_p is converted into photons that propel the rocket to a velocity Δv in S. When we derive the classical or relativistic rocket equations, we normally consider differentials of mass and velocity because the velocity of the propellant in the initial rest frame varies. But now the propellant is photons, which always have a velocity c, so we can consider just the initial and final states. In S, the initial mass–energy is $(M_f + M_p)c^2$ and the final mass–energy of just the rocket is $\gamma M_f c^2$, where $\gamma = \gamma(\Delta v)$. Suppose that the total energy of the photons emitted (as measured in frame S) is E_p. By conservation of energy,

$$(M_f + M_p)c^2 = \gamma M_f c^2 + E_p. \tag{7.31}$$

In S, the final momentum of the rocket is $\gamma M_f \Delta v$ and the momentum of the photons is E_p/c. Thus by conservation of momentum,

$$0 = \gamma M_f \Delta v - \frac{E_p}{c}. \tag{7.32}$$

Eliminating E_p between these we get the photon rocket equation,

$$\frac{M_p}{M_f} = \gamma(1 + \beta) - 1, \tag{7.33}$$

where $\beta = \Delta v/c$. Writing $(M_p + M_f)/M_f = M_i/M_f$ and then rearranging equation 7.33, using also the relation in equation 7.6, we find that the photon rocket equation is just a special case of the relativistic rocket equation (equation 7.30b) when $v_e = c$.

Matter can in principle be converted entirely into photons via electron–positron annihilation (equation 8.11 in the next chapter). If anything other than photons is produced, then the above derivation has retained this 'waste' mass–energy on board the spacecraft as part of M_f. (We will modify this approach in the next section.)

Regardless of how the photons are produced, we have assumed that all of the mass M_p is converted into photons. Note, however, that $E_p < M_p c^2$, as can be seen from the energy conservation equation above. In the final rest frame of the rocket, the energy expended is indeed $M_p c^2$. But in frame S, some of the rest energy from M_p has gone into the kinetic

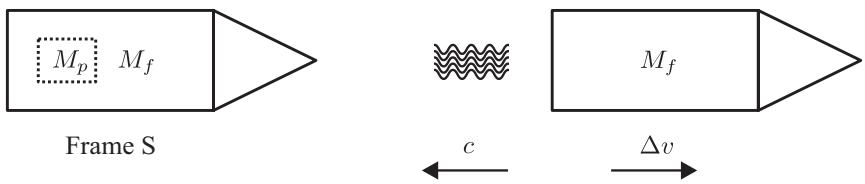


Fig. 7.8 The photon rocket. Left: Initially the rocket of mass $M_p + M_f$ is at rest in frame S. Right: The mass M_p is converted into photons that leave the rocket at the speed of light c, accelerating the rocket to velocity Δv.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

energy of the rocket. That is the whole point of a rocket, after all. If we write $E_p = \alpha M_p c^2$, then solving for α using the above equations gives

$$\alpha = \frac{\beta}{1+\beta-1/\gamma}\,. \tag{7.34}$$

This decreases monotonically from 1 at $\beta = 0$ to 0.5 at $\beta = 1$. This means that at very low velocities, almost all of the propellant rest energy goes into the photons, and so virtually none into the rocket. But as the rocket approaches the speed of light, half of the energy goes into the photons, and the other half into the rocket.

If at $\beta = 0$, all of the energy goes into photons and none into the rocket, how can the rocket ever get going? The answer is that the velocity in all these expressions is the velocity of the rocket after it has released at least one photon. Equation 7.34 therefore does not make sense when $\beta = 0$ exactly, because this would correspond to not having emitted a photon. So at the very beginning, only a tiny fraction of the energy is put into the rocket, but it is not zero. The same applies for a matter rocket too.

Another way to look at the energy partition between the rocket and the exhaust is to consider the energy efficiency, the ratio of the rocket's kinetic energy to the energy expended. The energy E_r that the photon rocket expends in its rest frame is $M_p c^2$. Substituting for M_p from equation 7.33 we get

$$E_r = [\gamma(1+\beta)-1]M_f c^2. \tag{7.35}$$

The kinetic energy of the rocket, $(\gamma-1)M_f c^2$, divided by E_r is the efficiency. We will look at this in detail in the next section when we consider a more general rocket, but one may peek ahead to see the efficiency for this photon rocket in figure 7.11 (right panel, curve marked $\epsilon = 1$). The maximum efficiency is 0.5, occurring at $\beta = 1$, and the efficiency goes to zero at $\beta = 0$.

7.8 Mixed matter–photon rockets

A nuclear reaction used to power a rocket often produces massive particles and massless photons, both of which have momentum and could potentially be used to propel a spacecraft. As the reaction must conserve energy and momentum, only a fraction of the energy released by the reaction can go into the kinetic energy of the spacecraft. Here we use those conservation laws to compute the effective exhaust velocity and energy efficiency of a mixed matter–photon rocket.

Consider a small mass of fuel/propellant dm on board a rocket. This undergoes a reaction that converts a fraction $(1-\epsilon)$ of its mass–energy into massive particles that can be used for propulsion ('propellant mass'); see figure 7.9. Of the remaining part ϵdm, a fraction η is converted into useable energy, and the other fraction $1-\eta$ is lost. This lost fraction may include heat (via photons) to the spacecraft, which is later radiated isotropically, or neutral particles resulting from the reactions, such as neutrinos or neutrons that cannot form part of the propellant (discussed below in section 7.8.2). Of the useable energy portion η, a fraction ζ goes into the kinetic energy of the massive particles (the propellant mass), and the rest is emitted as collimated photons that also propel the spacecraft. We can therefore think of four

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

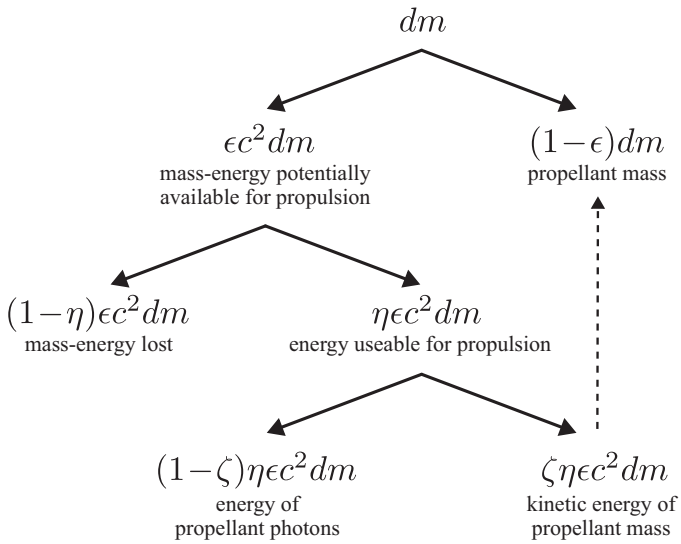


Fig. 7.9 Partition of mass–energy derived from an initial fuel mass *dm* in a mixed matter–photon rocket. Adapted from Westmoreland (2010).

distinct mass–energy end products: the mass of the propellant particles; the kinetic energy of the propellant particles; the energy of the propelling photons; and mass–energy losses.

Consider the instantaneous reference frame of the spacecraft defined at the moment before the reaction. The spacecraft has mass M. After the reaction it has mass $M + dM$, where $dM = -dm$ and $dm > 0$, velocity dV, and therefore momentum[3] $(M + dM)dV$. This must balance the momentum of the particles and photons in the propellant, which we assume have been collimated so that they are ejected in the same direction as one another. Assuming that the particles are moving at common velocity v_m, they have momentum $\gamma_m v_m(1-\epsilon)dm$, where $\gamma_m = (1 - v_m^2/c^2)^{-1/2}$. The photons have momentum E/c, where $E = (1 - \zeta)\eta\epsilon c^2 dm$ is their energy (figure 7.9). Equating the momentum before and after the reaction we get

$$0 = (M + dM)dV - \gamma_m v_m(1 - \epsilon)\, dm - (1 - \zeta)\eta\epsilon c\, dm \tag{7.36a}$$

$$= (M + dM)dV + \gamma_m v_m(1 - \epsilon)\, dM + (1 - \zeta)\eta\epsilon c\, dM. \tag{7.36b}$$

Rearranging this and neglecting the higher-order term $dMdV$ we get

$$dV = -v_e \frac{dM}{M}, \quad \text{where} \tag{7.37a}$$

$$v_e = (1 - \epsilon)\gamma_m v_m + (1 - \zeta)\eta\epsilon c. \tag{7.37b}$$

Equation 7.37a has the same form as the classical rocket equation (equation 3.3), except now the velocity V is relative to the instantaneous reference frame of the rocket. This velocity is sometimes called the *proper speed*,[4] denoted by σ. The proper speed is the integral of the proper acceleration over the proper time, and is related to the coordinate velocity (in the

[3] Strictly this is the relativistic momentum $\gamma_{dV}(M + dM)dV$ where $\gamma_{dV} = (1 - (dV/c)^2)^{-1/2}$. But in the limit as $dV \to 0$, $\gamma_{dV} \to 1$.

[4] The proper speed divided by the speed of light, σ/c, is called the *rapidity*. The proper speed should not be confused with proper velocity defined in section 7.3.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

original rest frame) by $\beta = \tanh(\sigma/c)$. Using this expression (with V in place of σ) in equation 7.37a, we get the relativistic rocket equation (see exercises).

The above derivation has given us an expression for the effective exhaust velocity v_e, although here it is better referred to as a specific impulse because it involves a mixture of particles moving at velocity v_m and photons at velocity c. Note that even when we have no propellant photons ($\zeta = 1$), the effective exhaust velocity v_e is not equal to the velocity of the particles v_m. Even in the classical limit ($\gamma_m = 1$) with no losses ($\eta = 0$), we have $v_e = (1 - \epsilon)v_m$. The factor $(1 - \epsilon)$ arises because some of the fuel mass has been converted into energy and so cannot be ejected as propellant mass.

7.8.1 Effective exhaust velocity

We get a deeper insight into v_e when we use energy conservation to equate the kinetic energy of the particles to the mass–energy converted and used to accelerate them:

$$(\gamma_m - 1)(1 - \epsilon)c^2\, dm = \zeta\eta\epsilon c^2 dm, \tag{7.38a}$$

$$\gamma_m = \frac{\zeta\eta\epsilon}{1 - \epsilon} + 1. \tag{7.38b}$$

As we would expect, the velocity of the particles depends on the three conversion factors.[5] Using equation 7.4 to rewrite $\gamma_m\beta_m$ in terms of γ_m in equation 7.37b, and then using equation 7.38b, we get an expression for the effective exhaust velocity in terms of just the three conversion factors:

$$\frac{v_e}{c} = \sqrt{\zeta\eta\epsilon(\zeta\eta\epsilon - 2\epsilon + 2)} + (1 - \zeta)\eta\epsilon. \tag{7.39}$$

Rockets with only matter as their propellant have $\zeta = 1$. If there are no mass-energy losses ($\eta = 1$),

$$\frac{v_e}{c} = \sqrt{\epsilon(2 - \epsilon)} \quad \text{(matter rocket with } \eta = 1\text{)}, \tag{7.40}$$

where ϵ is now the fraction of the mass converted into energy that is then used to propel the remaining fraction $(1 - \epsilon)$ of the mass. In principle, if $\epsilon = 1$, the effective exhaust velocity is the speed of light, but then no mass is available for ejection. The other limiting case of $\epsilon = 0$ is also impractical because then no energy is available to accelerate the propellant. In the limit of small ϵ, equation 7.40 reduces to[6]

$$\frac{v_e}{c} \simeq \sqrt{2\epsilon} \quad (\zeta = 1, \eta = 1, \epsilon \ll 1). \tag{7.41}$$

For a photon rocket, $\zeta = 0$, so

$$\frac{v_e}{c} = \eta\epsilon \quad \text{(photon rocket)}. \tag{7.42}$$

Such a rocket still produces a 'propellant' mass of $(1 - \epsilon)dm$, but this mass receives no kinetic energy. This waste product is simply dumped at zero velocity relative to the rocket. When $\epsilon = 1$ there is no propellant mass at all. The propellant photons are generated by 100%

[5] As we shall see in section 8.5.2, a nuclear reaction generally produces more than one type of massive particle, which move with different velocities. In that case v_m is the velocity corresponding to their combined kinetic energy.

[6] This is also what we would derive using classical mechanics and equating the energy released, $\epsilon M_p c^2$, with the kinetic energy of the remaining mass, $(1/2)(1 - \epsilon)M_p v_e^2$, for $\epsilon \ll 1$.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

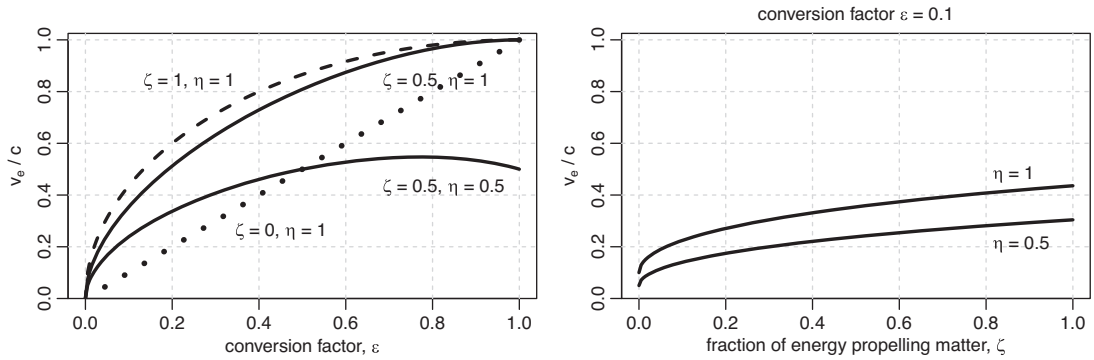


Fig. 7.10 Variation of the effective exhaust velocity for a mixed matter–photon rocket with ϵ (left) and with ζ (right), according to equation 7.39. In the left panel, the dashed line shows the variation for a matter rocket ($\zeta = 1$) with no energy losses ($\eta = 1$) (equation 7.40), and the dotted line the variation for a photon rocket ($\zeta = 0$) also with no energy losses (equation 7.42).

conversion of mass into energy, so $v_e/c = \eta$, where η is the fraction of the energy which is converted into photons that propel the spacecraft. I refer to this as the *pure* photon rocket. If we further have $\eta = 1$ (no energy losses), then $v_e = c$, which is the situation we derived in section 7.7.

Figure 7.10 shows how the effective exhaust velocity varies with some of the parameters for the general case of a mixed matter–photon rocket. The two solid lines in the left panel show how v_e varies with ϵ for two different values of η at fixed $\zeta = 0.5$. When there are no losses ($\eta = 1$), v_e increases monotonically with increasing ϵ, or equivalently with a decreasing amount of propellant mass. But when there are losses ($\eta = 0.5$), v_e reaches a maximum before decreasing. This panel also shows that the matter rocket (dashed line) gives a larger v_e than the photon rocket (dotted line) at all ϵ. The right panel shows the variation of v_e with ζ when 90% of the initial mass is converted to propellant mass. As the proportion of energy going into the particles, as opposed to the photons, increases, the larger the value of v_e. In terms of effective exhaust velocity, a matter rocket is preferred over a photon rocket.

7.8.2 Mass losses

In deriving equation 7.39 for the effective exhaust velocity, we assumed that the momentum gained by the spacecraft was equal and opposite to the momentum gained by the particles and the photons, excluding the losses. In reality, the products of numerous nuclear reactions used to propel a spacecraft move in random directions in all three dimensions. By conserving momentum in one dimension above, we assumed that all the useable products of the nuclear reaction were narrowly collimated in one direction. This could be achieved with strong magnetic fields for the charged particles, and with reflectors for the photons, but it may not be achieved for neutrons, which are neutral, or for neutrinos, which barely interact.[7] Yet both these particles can carry away significant amounts of energy and momentum. How has our derivation taken these losses into account?

[7] It might be possible to absorb some of the neutrons through collisions in the engine, but this can cause structural problems.

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

The important thing to realize is that the portion $(1-\epsilon)dm$ is the mass of the *useable* particles that emerge from the nuclear reaction, that is, those that can be part of the massive propellant. This is usually just the charged particles. The other portion, the mass ϵdm, or equivalently the mass–energy $\epsilon c^2 dm$, comprises everything else, some of which is useable and some which is not (see figure 7.9). The useable part is the kinetic energy of the massive propellant particles plus the energy of the photons we can collimate. The unusable part is the mass and kinetic energy of particles that are not part of the propellant (typically neutrinos and neutrons), as well as photons we cannot collimate.

There is a potential source of confusion here, because the symbol ϵ is sometimes used in the context of nuclear reactions to indicate just the fraction of the initial mass that is converted into kinetic energy. We are instead using ϵ to include also the mass of any unusable particles, such as neutrinos or neutrons. The fraction of initial rest energy available for propulsion is $\eta\epsilon$, of which a fraction ζ goes into accelerating the massive particles.

We will use the theory developed above in sections 8.5.2 and 8.8 to compute ϵ, η, and ζ and therefore v_e for some real fusion and antimatter reactions.

7.8.3 Energy efficiency

Let us now consider how efficiently energy is used in a mixed matter–photon rocket. Consider a rocket of initial mass M_i initially at rest in frame S. Once it has attained a velocity v in S, its mass is M_f, and some of the energy released has gone into the rocket and some into the propellant. There are different ways in which we could define an efficiency. I will define it as

$$\xi = \frac{\text{kinetic energy of rocket}}{\text{energy put into propellant}}, \tag{7.43}$$

which parallels the definition used in section 3.6 for classical rockets. The numerator is $(\gamma_v - 1)M_f c^2$ and the denominator is $\eta\epsilon M_p c^2$ (see figure 7.9), where $M_p = M_i - M_f$. Thus

$$\xi = \frac{(\gamma_v - 1)M_f}{\eta\epsilon M_p} = \frac{(\gamma_v - 1)}{\eta\epsilon\left(\frac{M_i}{M_f} - 1\right)}. \tag{7.44}$$

The velocity and the mass ratio M_i/M_f are related via the relativistic rocket equation, equation 7.28. Using this we can write the efficiency as

$$\xi = \frac{(\gamma_v - 1)}{\eta\epsilon\left(\exp\left[\frac{c}{v_e}\operatorname{atanh}\beta_v\right] - 1\right)}, \tag{7.45}$$

where v_e/c is given by equation 7.39. The energy efficiency depends on the three conversion factors – ϵ, η, ζ – as well as the final speed of the rocket β_v in the initial rest frame (the one in which we are measuring the kinetic energy of the rocket).

The above energy efficiency is for the rocket as a whole. It should not be confused with η, which is an 'internal' efficiency of the conversion of mass into energy available for propelling the spacecraft. Note also that the 'energy put into propellant' in the denominator of equation 7.43 is the energy as measured in the rest frame of the rocket. This makes sense, because this is the energy that the rocket has to provide. The kinetic energy in the numerator, in contrast, is measured in the initial rest frame. This also makes sense, because we are interested in how fast the rocket is moving with respect to that initial rest frame (the kinetic energy in its own rest frame is zero).

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

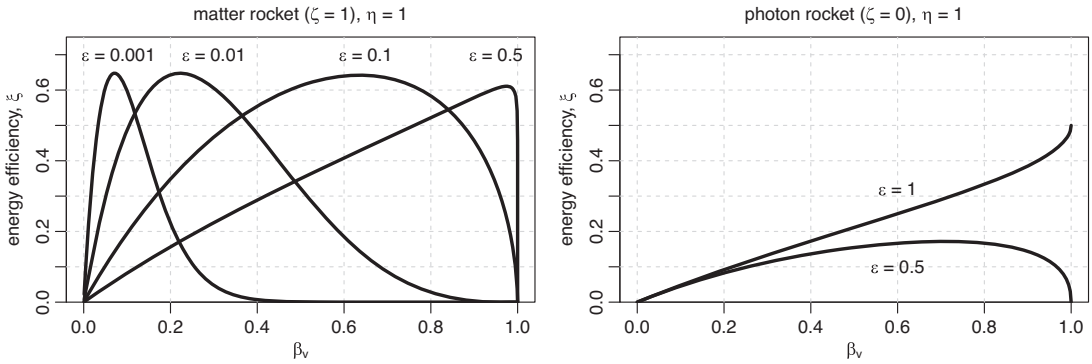


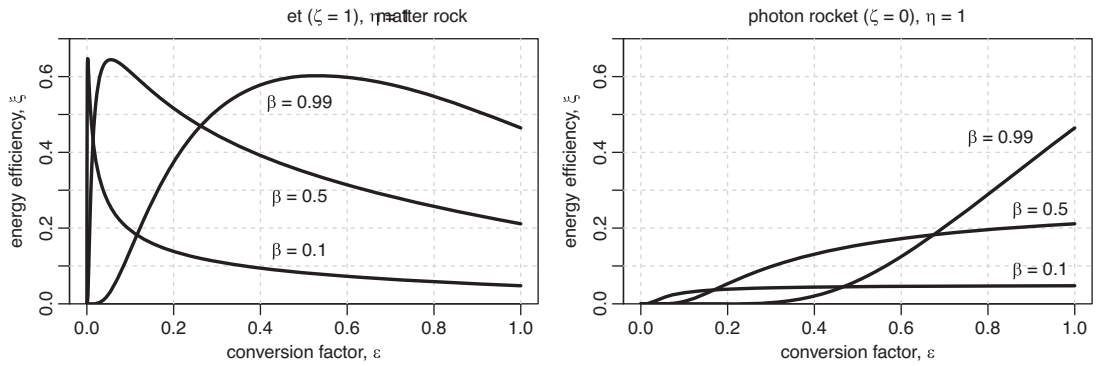
Fig. 7.11 The energy efficiency of a relativistic matter rocket (left), and a photon rocket (right), as defined by equation 7.43 and expressed by equation 7.45, as a function of the final velocity of rocket. In both panels we assume there are no energy losses when mass is converted into energy ($\eta = 1$). The matter rocket is shown for four different values of the fraction of the mass that is converted into energy to propel the remaining propellant. The photon rocket is shown for a pure photon rocket ($\epsilon = 1$), in which all of the mass is converted into photons, and one in which only half the mass is converted into useful photons ($\epsilon = 0.5$). In the latter case the remaining mass receives no kinetic energy, so contributes nothing to the propulsion.

Consider a matter rocket, $\zeta = 1$, with perfect conversion efficiency, $\eta = 1$. The left panel of figure 7.11 shows how the energy efficiency ξ varies with the final velocity of the spacecraft for four different values of ϵ. As $\eta = 1$, these values of ϵ correspond to the fraction of the initial mass that is converted into kinetic energy of the propellant. We see that for a wide range of ϵ, an energy efficiency of around 0.65 can in principle be achieved. That is, 65% of the energy available for propulsion goes into the rocket, with the rest going into the propellant. For all values of ϵ, the efficiency drops to zero as the velocity tends to 0 or c. Thus as a spacecraft accelerates, the energy efficiency will initially be very low. If the cruise velocity is low, say around 0.05 to 0.1 c, then the plot shows that to maximize efficiency we should use a small ϵ, of the order of 0.001: it is more efficient to convert only a small fraction of the propellant mass into energy and use that to accelerate the rest. But with this value of ϵ it is very inefficient to accelerate the spacecraft much beyond $0.3\,c$. At larger velocities, a larger ϵ is more efficient. Thus to accelerate efficiently from zero to large velocities, we would ideally use different reactions with different values of ϵ.

Figure 7.12 plots the efficiency against ϵ for three values of β. This shows more clearly that at low velocities (e.g. $\beta = 0.1$) there is a narrow range of small values of ϵ that maximize the efficiency for the matter rocket. To maximize the energy efficiency with a single reaction, we should try to match ϵ to the velocity we wish to achieve. There is only limited freedom to do this in practice, however, because ϵ is determined by the specific nuclear reactions available for energy conversion. For some of these, η is less than unity due to the production of non-interacting particles, as we shall see in chapter 8.

Let us turn now to the photon rocket, again for $\eta = 1$. The variation of its energy efficiency with velocity is shown in the right panel of figure 7.11. The line labelled $\epsilon = 1$ is a pure photon rocket, in which all of the mass–energy of the propellant is converted into photons. In contrast to the matter rocket, the efficiency of the pure photon rocket increases monotonically with the velocity of the rocket. The maximum efficiency, only achieved at the speed of light,

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008



As figure 7.11, but now showing the efficiencies as a function of ϵ for three different values of the final rocket velocity β. **Fig. 7.12**

is 0.5, which is less than the maximum efficiency of the matter rocket. The expression for the efficiency simplifies for a pure photon rocket as already found at the end of section 7.7.

The lower line in the right panel of figure 7.11 labelled $\epsilon = 0.5$ corresponds to only half of the initial mass being converted into photons. The other half is massive particles, but they receive no kinetic energy (by definition of being a photon rocket). As there is no point keeping this on board and wasting energy accelerating it, it is simply dumped from the spacecraft at zero relative velocity. In this case there is a maximum efficiency at a velocity below the speed of light and, as we would expect, this efficiency is lower than for the pure photon rocket. The right panel of figure 7.12 shows how the efficiency varies with ϵ. The efficiency increases monotonically with ϵ, in stark contrast to the matter rocket (shown in the left panel). The efficiency of the photon rocket only increases slowly with ϵ at low velocities, however.

In the classical Newtonian limit, $\gamma_v \simeq 1 + v^2/2c^2$. For a matter rocket with $\eta = 1$ and $\epsilon \ll 1$, which is also the classical limit, we get $\epsilon = v_e^2/2c^2$ (equation 7.41). This allows us to write equation 7.44 as

$$\xi = \frac{v^2/v_e^2}{\left(\frac{M_i}{M_f} - 1\right)} \quad \text{(classical limit)}, \tag{7.46}$$

which is the same expression for the energy efficiency of the classical rocket in equation 3.21 and figure 3.7.

7.9 Summary

When dealing with large relative velocities, the classical laws of motion have to be replaced by those derived from special relativity. The velocity and distance travelled by a constantly accelerating spacecraft as seen by an external observer are less in special relativity than in classical mechanics. Time dilation means that the time measured by a clock on a moving spacecraft is less than that measured on a clock in the initial rest frame (equation 7.11). The 'proper' acceleration experienced by the spacecraft is greater than the 'coordinate' acceleration measured in the initial rest frame (equation 7.24).

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008

The rocket equation must be modified to accommodate special relativity (equation 7.29). The difference with respect to the classical rocket equation only becomes large when the rocket velocity is a significant fraction of the speed of light.

As photons have momentum, they can be used as a means of propulsion: the photon rocket. In later chapters we will look at nuclear and antimatter reactions for propulsion, which in general provide a mix of particles and photons. These reactions convert a variable fraction of their rest mass–energy into kinetic energy, some of which is lost, and the rest is divided among photons and particles (figure 7.9). We derived a general expression for the effective exhaust velocity (equation 7.39). Examining the energy efficiency (equation 7.45), we found that a single reaction does not have optimal efficiency over a wide range of spacecraft velocities.

7.10 Exercises

1. Show that integrating equation 7.15 over t yields equation 7.16.
2. Plot the ratio of Δv from the classical rocket equation (Δv_{cl}) to that from the relativistic rocket equation (Δv_{rel}) as a function of $\Delta v_{\text{rel}}/c$. For what values of Δv_{rel} is this ratio greater than 1.05? Can the classical rocket equation ever apply if v_e is near the speed of light? If so, what condition would have to apply? If not, why not?
3. In section 7.6 we looked at rather extreme relativistic journeys involving a large v_e ($0.2\,c$) and large accelerations ($1\,g$ and $0.1\,g$). Compute the quantities in Table 7.2 for a flyby and rendezvous to Alpha Cen assuming an effective exhaust velocity of $v_e = 0.05\,c$ and a constant acceleration of $0.005\,g$.
4. In section 7.8 just after equation 7.37, I introduced the proper speed. From its definition, and using equation 7.15, show that the proper speed equals $c\,\text{atanh}\,\beta$. Then show that equation 7.37a leads to the relativistic rocket equation (equation 7.27).
5. Why does the energy efficiency of a rocket depend on an external observer?

Downloaded from https://www.cambridge.org/core. UNIVERSIDAD AUTONOMA METROPOLITANA, on 16 Nov 2026 at 05:13:46, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/9781009700429.008