NOWHERE TRIVIAL AUTOMORPHISMS OF $P(\lambda)/[\lambda]^{<\lambda}$ FOR λ INACCESSIBLE

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Abstract. For λ inaccessible, we show that "there is a nowhere trivial automorphism of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$" follows (in ZFC) from $2^\lambda = \lambda^+$, and is consistent (via forcing) with $2^\lambda > \lambda^+$.

§1. Introduction. We investigate the rigidity of the Boolean algebra $P(\lambda)/[\lambda]^{<\lambda}$ for λ inaccessible.

For $\lambda = \omega$ there is extensive literature on this topic (see, e.g., the survey [3]); some general results on $P(\lambda)/[\lambda]^{<\kappa}$ can be found in [5]. In [4] it was shown, for λ inaccessible and $2^\lambda = \lambda^{++}$, that consistently every automorphism is densely trivial.

In this article we show:

(Theorem 5.3)

> If λ is (strongly) inaccessible and $2^\lambda = \lambda^+$, then there is a nowhere trivial automorphism of the Boolean algebra $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.

Note that the weaker variant "there is a *nontrivial* automorphism" follows from [7, Lemma 3.2] (the proof there was faulty, and fixed in [8]); and for λ measurable, a proof (again only for "nontrivial") was given in [4].

We also show:

(Theorem 6.1)

> It is consistent that λ is inaccessible, 2^λ an arbitrary regular cardinal, and that there is a nowhere trivial automorphism of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.

§2. Notation. We assume throughout the article that λ is inaccessible.

For $A \subseteq \lambda$, $[A]^{<\lambda}$ denotes the subsets of A of size less than λ; and $[A]^\lambda$ those of size λ. With $[A]$ we denote the equivalence class of A modulo $[\lambda]^{<\lambda}$. We write $A \subseteq^* B$ for $|A \setminus B| < \lambda$, and $A =^* B$ for $A \subseteq^* B \,\&\, B \subseteq^* A$, or equivalently $[A] = [B]$.

However, we also use $f[A] := \{f(a) : a \in A\}$. So, for example, $[f[A]]$ is the equivalence class of the f-image of A.

$\mathrm{Sym}(X)$ is the set of the permutations of X, i.e., of the bijections from X to X.

We consider $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ as Boolean algebra. A (Boolean algebra) automorphism π of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ is called trivial on A (for $A \in [\lambda]^\lambda$) if there is an $f \in \mathrm{Sym}(\lambda)$ such

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that $\pi([B]) = [f[B]]$ for all $B \subseteq A$. π is called nowhere trivial, if there is no such pair (f, A).

For $\delta \leq \lambda$, $C \subseteq \delta$ closed and nonempty, and $\alpha \in C$, we set

$$I^*(C, \delta, \alpha) := \big\{\beta : \alpha \leq \beta < \min\big((C \cup \{\delta\}) \setminus (\alpha + 1)\big)\big\}.$$

So the $I^*(C, \delta, \alpha)$, for $\alpha \in C$, form an increasing interval partition of $\delta \setminus \min(C)$.

§3. Approximations.

3.1. Definition and final limits. In this section, we define the set AP of "approximations." An approximation $\mathbf{a}$ will induce a "partial monomorphism" $\tilde{\boldsymbol{\pi}}^{\mathbf{a}}$ defined on some sub-Boolean Algebra $\tilde{\mathbf{B}}^{\mathbf{a}}$ of $P(\lambda)/[\lambda]^{<\kappa}$ which is trivial, i.e., generated by some $\boldsymbol{\pi}^{\mathbf{a}} \in \mathrm{Sym}(\lambda)$. We will use such approximations to build a nowhere trivial automorphism $\tilde{\phi}$ of $P(\lambda)/[\lambda]^{<\kappa}$ as limit (i.e., $\tilde{\phi} \restriction \tilde{\mathbf{B}}^{\mathbf{a}} = \tilde{\boldsymbol{\pi}}^{\mathbf{a}}$) (cf. Fact 3.7).

Definition 3.1. AP is the set of objects $\mathbf{a}$ consisting of $\mathbf{C}^{\mathbf{a}}$, $\boldsymbol{\pi}^{\mathbf{a}}$, and $\mathbf{B}^{\mathbf{a}}$, such that:

- $\boldsymbol{\pi}^{\mathbf{a}} \in \mathrm{Sym}(\lambda)$;
- $\mathbf{C}^{\mathbf{a}} \subseteq \lambda$ club such that $\boldsymbol{\pi}^{\mathbf{a}} \restriction \varepsilon \in \mathrm{Sym}(\varepsilon)$ for all $\varepsilon \in \mathbf{C}^{\mathbf{a}}$;
- $\mathbf{B}^{\mathbf{a}}$ is a subset of $\mathcal{P}(\lambda)$.

$\mathbf{C}^{\mathbf{a}}$ gives us a partition of λ.

Definition 3.2. For $\mathbf{a} \in \mathrm{AP}$ and $\varepsilon \in \mathbf{C}^{\mathbf{a}}$ we set $I^{\mathbf{a}}_{\varepsilon} := I^*(\mathbf{C}^{\mathbf{a}}, \lambda, \varepsilon)$.

So the $I^{\mathbf{a}}_{\varepsilon}$ form an increasing interval partition of $\lambda \setminus \min(\mathbf{C}^{\mathbf{a}})$; and $\boldsymbol{\pi}^{\mathbf{a}} \restriction I^{\mathbf{a}}_{\varepsilon} \in \mathrm{Sym}(I^{\mathbf{a}}_{\varepsilon})$.

$\boldsymbol{\pi}^{\mathbf{a}}$ induces a (trivial) automorphism of $P(\lambda)/[\lambda]^{<\lambda}$, and $\tilde{\boldsymbol{\pi}}^{\mathbf{a}}$ is the restriction of this automorphism to $\mathbf{B}^{\mathbf{a}}$.

Definition 3.3.

- $\tilde{\mathbf{B}}^{\mathbf{a}} := \mathbf{B}^{\mathbf{a}}/[\lambda]^{<\lambda} = \{[A] : A \in \mathbf{B}^{\mathbf{a}}\}$.
- $\tilde{\boldsymbol{\pi}}^{\mathbf{a}} : \tilde{\mathbf{B}}^{\mathbf{a}} \to P(\lambda)/[\lambda]^{<\lambda}$ is defined by $[A] \mapsto [\boldsymbol{\pi}^{\mathbf{a}}[A]]$.

Definition 3.4. $\mathbf{b} \geq_{\mathrm{AP}} \mathbf{a}$, if $\mathbf{a}, \mathbf{b} \in \mathrm{AP}$ and

(1) $\mathbf{C}^{\mathbf{b}} \subseteq^* \mathbf{C}^{\mathbf{a}}$;
(2) $\boldsymbol{\pi}^{\mathbf{b}} \restriction I^{\mathbf{a}}_{\varepsilon} = \boldsymbol{\pi}^{\mathbf{a}} \restriction I^{\mathbf{a}}_{\varepsilon}$ for all but boundedly many $\varepsilon \in \mathbf{C}^{\mathbf{b}}$;
(3) $\mathbf{B}^{\mathbf{b}} \supseteq \mathbf{B}^{\mathbf{a}}$, and $\tilde{\boldsymbol{\pi}}^{\mathbf{b}}$ extends[1] $\tilde{\boldsymbol{\pi}}^{\mathbf{a}}$.

Clearly, $\leq_{\mathrm{AP}}$ is a (nonempty) quasiorder.

Lemma 3.5. *If $(\mathbf{a}_i)_{i<\delta}$ is a $\leq_{\mathrm{AP}}$ increasing chain such that $\bigcup_{i<\delta} \tilde{\mathbf{B}}^{\mathbf{a}_i} = P(\lambda)/[\lambda]^{<\lambda}$, then $\tilde{\phi} := \bigcup_{i<\delta} \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}$ is a Boolean algebra monomorphism of $P(\lambda)/[\lambda]^{<\lambda}$.*

If additionally $\bigcup_{i<\delta} \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}[\tilde{\mathbf{B}}^{\mathbf{a}_i}] = P(\lambda)/[\lambda]^{<\lambda}$, then $\tilde{\phi}$ is an automorphism.

Proof. We use $\vee$ and ^{c} for the Boolean-algebra-operations, i.e., $[A \cup B] = [A] \vee [B]$, and $[A]^c = [\lambda \setminus A]$. It is enough to show that $\tilde{\phi}$ is injective, honors $\vee$ and ^{c}, and maps $[\emptyset]$ to itself.

[1] I.e., if $A \in \mathbf{B}^{\mathbf{a}}$, then $\boldsymbol{\pi}^{\mathbf{a}}[A] =^* \boldsymbol{\pi}^{\mathbf{b}}[A]$ or equivalently $\tilde{\boldsymbol{\pi}}^{\mathbf{a}}([A]) = \tilde{\boldsymbol{\pi}}^{\mathbf{b}}([A])$.

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For X_1, X_2 in $P(\lambda)/[\lambda]^{<\lambda}$ there is an $i < \delta$ and some $A_1, A_2, A_{\rm union}$ in $\mathbf{B}^{\mathbf{a}_i}$, such that $[A_j] = X_j$ for $j = 1, 2$ and $[A_{\rm union}] = [A_1 \cup A_2] = X_1 \vee X_2$. Then

$$\boldsymbol{\pi}^{\mathbf{a}_i}[A_{\rm union}] =^* \boldsymbol{\pi}^{\mathbf{a}_i}[A_1 \cup A_2] = \boldsymbol{\pi}^{\mathbf{a}_i}[A_1] \cup \boldsymbol{\pi}^{\mathbf{a}_i}[A_2],$$

and

$$\tilde{\phi}(X_1 \vee X_2) = \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}([A_{\rm union}]) = [\boldsymbol{\pi}^{\mathbf{a}_i}[A_{\rm union}]] =$$
$$= \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}([A_1]) \vee \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}([A_2]) = \tilde{\phi}(X_1) \vee \tilde{\phi}(X_2).$$

If $X_1 \neq X_2$, i.e., $A_1 \neq^* A_2$, then $\boldsymbol{\pi}^{\mathbf{a}_i}[A_1] \neq^* \boldsymbol{\pi}^{\mathbf{a}_i}[A_2]$, i.e., $\tilde{\phi}(X_1) \neq \tilde{\phi}(X_2)$.

Similarly we can show $\tilde{\phi}([\lambda \setminus A_1]) = \tilde{\phi}([A_1])^c$ and $\tilde{\phi}([\emptyset]) = [\emptyset]$. ⊣

DEFINITION 3.6. For a pair (f, A) with $A \in [\lambda]^\lambda$ and $f \in \mathrm{Sym}(\lambda)$, we say $\mathbf{a} \in \mathrm{AP}$ "spoils (f, A)," if there is an $A' \in [A]^\lambda \cap \mathbf{B}^{\mathbf{a}}$ such that $|\boldsymbol{\pi}^{\mathbf{a}}[A'] \cap f[A']| < \lambda$.

If $\tilde{\phi}$ is an automorphism extending such a $\tilde{\boldsymbol{\pi}}^{\mathbf{a}}$, then f cannot witness that $\tilde{\phi}$ is trivial on A. Therefore, we have the following fact.

FACT 3.7. *If $(\mathbf{a}_i)_{i<\delta}$ is a $\leq_{\rm AP}$ increasing chain such that*:
- $\bigcup_{i<\delta} \tilde{\mathbf{B}}^{\mathbf{a}_i} = \bigcup_{i<\delta} \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}[\tilde{\mathbf{B}}^{\mathbf{a}_i}] = P(\lambda)/[\lambda]^{<\lambda}$, *and*
- *for every (f, A) there is an $i < \delta$ such that $\mathbf{a}_i$ spoils (f, A),*

then $\tilde{\phi} := \bigcup_{i<\delta} \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}$ is a nowhere trivial Boolean algebra automorphism of $P(\lambda)/[\lambda]^{<\lambda}$.

We will use this fact to get a nowhere trivial automorphism, both in the case $2^\lambda = \lambda^+$ in Section 5, as well as in the forcing construction of Section 6.

3.2. Short sequences and their limits and good ordinals. We will often modify an $\mathbf{a} \in \mathrm{AP}$ by replacing $\mathbf{B}^{\mathbf{a}}$ with another $\mathbf{B}' \subseteq P(\lambda)$. Let the result be $\mathbf{b}$. We call $\mathbf{b}$ "$\mathbf{a}$ with $\mathbf{B}^{\mathbf{a}}$ replaced by $\mathbf{B}'$," or "$\mathbf{a}$ with X added to $\mathbf{B}^{\mathbf{a}}$" in case $\mathbf{B}' = \mathbf{B}^{\mathbf{a}} \cup \{X\}$. Obviously $\mathbf{b} \in \mathrm{AP}$. If[2] $\mathbf{B}' \supseteq \mathbf{B}^{\mathbf{a}}$ then $\mathbf{b} \geq_{\rm AP} \mathbf{a}$.

Similarly we can get a stronger approximation by thinning out $\mathbf{C}$. To summarize, we have the following fact.

FACT 3.8. *If $\mathbf{a} \in \mathrm{AP}$, $D \subseteq \mathbf{C}^{\mathbf{a}}$ club, and $B \subseteq P(\lambda)$ with $B \supseteq \mathbf{B}^{\mathbf{a}}$. Then $\mathbf{b} \geq_{\rm AP} \mathbf{a}$, for the $\mathbf{b}$ defined by $\boldsymbol{\pi}^{\mathbf{b}} := \boldsymbol{\pi}^{\mathbf{a}}$, $\mathbf{C}^{\mathbf{b}} := D$ and $\mathbf{B}^{\mathbf{b}} := B$.*

In the definition of $\leq_{\rm AP}$ we require that some things hold "apart from a bounded set," or equivalently, "above some α." We say that α is good for an increasing sequence of $\mathbf{a}_i$, if the requirements for each pair are met above α. We will generally only be able to find such an α for "short sequences."

DEFINITION 3.9.
(1) AP_λ is the set of $\mathbf{a} \in \mathrm{AP}$ such that $|\mathbf{B}^{\mathbf{a}}| \leq \lambda$. Analogously, $\mathrm{AP}_{<\lambda}$ is the set of $\mathbf{a} \in \mathrm{AP}$ such that $|\mathbf{B}^{\mathbf{a}}| < \lambda$.
(2) $(\mathbf{a}_i)_{i\in J}$ is a "short sequence," if $J < \lambda$ (or more generally, J is a set of ordinals with $|J| < \lambda$), each $\mathbf{a}_i \in \mathrm{AP}_{<\lambda}$, and the sequence is $\leq_{\rm AP}$-increasing, i.e., $j > i$ in J implies $\mathbf{a}_j \geq_{\rm AP} \mathbf{a}_i$.
(3) Let $\bar{\mathbf{a}} := (\mathbf{a}_i)_{i\in J}$ be short. We say that α is good for $\bar{\mathbf{a}}$, if for all $i \leq k$ in J:
   (a) $\alpha \in \mathbf{C}^{\mathbf{a}_i}$.
   (b) $\mathbf{C}^{\mathbf{a}_k} \subseteq \mathbf{C}^{\mathbf{a}_i}$ above α. (I.e., $\beta \geq \alpha$ and $\beta \in \mathbf{C}^{\mathbf{a}_k}$ implies $\beta \in \mathbf{C}^{\mathbf{a}_i}$.)

[2] But we will also use the case $\mathbf{B}' \subseteq \mathbf{B}^{\mathbf{a}}$, e.g., in the proof of Lemma 5.1.

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(c) $\boldsymbol{\pi}^{\mathbf{a}_k} \restriction I^{\mathbf{a}_i}_{\varepsilon} = \boldsymbol{\pi}^{\mathbf{a}_i} \restriction I^{\mathbf{a}_i}_{\varepsilon}$ for all $\varepsilon \geq \alpha$ in $\mathbf{C}^{\mathbf{a}_k}$.
(d) $\boldsymbol{\pi}^{\mathbf{a}_k}[A] \setminus \alpha = \boldsymbol{\pi}^{\mathbf{a}_i}[A] \setminus \alpha$, for all $A \in \mathbf{B}^{\mathbf{a}_i}$.

(4) For $\mathbf{a}, \mathbf{b}$ in $\mathrm{AP}_{<\lambda}$, we say $\mathbf{b} >_\zeta \mathbf{a}$, if ζ is good for the sequence $(\mathbf{a}, \mathbf{b})$ (so in particular $\mathbf{a} < \mathbf{b}$).

FACT 3.10.

(1) *If* $\mathbf{a} \in \mathrm{AP}_{<\lambda}$, *then* $\mathbf{b} \geq_{\mathrm{AP}} \mathbf{a}$ *iff* $(\exists \zeta \in \lambda)\, \mathbf{b} >_\zeta \mathbf{a}$.
(2) *If* $\bar{\mathbf{a}} = (\mathbf{a}_i)_{i \in J}$ *is short, then* $G := \{\alpha \in \lambda : \alpha \text{ good for } \bar{\mathbf{a}}\}$ *is club, more concretely* $G = \bigcap_{i<\delta} \mathbf{C}^{\mathbf{a}_i} \setminus \alpha^*$ *for some* $\alpha^* < \lambda$.
(3) *If* $\mathbf{b}$ *is the result of enlarging* $\mathbf{B}^{\mathbf{a}}$ *in* $\mathbf{a}$, *then* $\mathbf{b} >_\zeta \mathbf{a}$ *for all* $\zeta \in \mathbf{C}^{\mathbf{a}}$.

LEMMA 3.11. *If* $\bar{\mathbf{a}}$ *is short, then it has an* $\leq_{\mathrm{AP}}$*-upper-bound* $\mathbf{b} \in \mathrm{AP}_{<\lambda}$.

PROOF. Set $D := \bigcap_{i \in J} C^{\mathbf{a}_i}$, and ζ_0 be the smallest $\bar{\mathbf{a}}$-good ordinal. So in particular $\zeta_0 \in D$; and any $\zeta \geq \zeta_0$ is in D iff it is $\bar{\mathbf{a}}$-good.

Fix for now some $\zeta \in D \setminus \zeta_0$. Let ζ^+ be the D-successor of ζ, i.e., $\min(D \setminus (\zeta + 1))$.

For $i \in J$, set $\gamma(\zeta, i)$ to be the $C^{\mathbf{a}_i}$-successor of ζ. Then the sequence $\gamma(\zeta, i)$ is weakly increasing with $i \in J$ and has limit ζ^+. If $\alpha < \gamma(\zeta, i)$ (we also say "α is stable at i"), then $\boldsymbol{\pi}^{\mathbf{a}_i}(\alpha) = \boldsymbol{\pi}^{\mathbf{a}_j}(\alpha)$ for all $j > i$ in J.

We define $\pi^{\lim}(\alpha)$ for all $\alpha \geq \zeta_0$ as $\boldsymbol{\pi}^{\mathbf{a}_i}(\alpha)$ for some i stable for α.

To summarize: Whenever $I := \zeta^+ \setminus \zeta$ for some $\zeta \in D \setminus \zeta_0$ with ζ^+ the D-successor, we get:

(1) $(\forall \alpha \in I)\, (\exists i \in J)\, (\forall j > i)\, \pi^{\lim}(\alpha) = \boldsymbol{\pi}^{\mathbf{a}_j}(\alpha)$.
(2) $\pi^{\lim} \restriction I \in \mathrm{Sym}(I)$.
(3) If $i \in J$ and $A \in \mathbf{B}^{\mathbf{a}_i}$, then $\pi^{\lim}[A'] = \boldsymbol{\pi}^{\mathbf{a}_i}[A']$ where $A' := A \cap I$.

For (2), note that $\boldsymbol{\pi}^{\mathbf{a}_i} \in \mathrm{Sym}(I)$ for all $i \in J$. If $\alpha_1 \neq \alpha_2 \in I$, then there is an i in J stable for both, and $\boldsymbol{\pi}^{\mathbf{a}_i}(\alpha_1) \neq \boldsymbol{\pi}^{\mathbf{a}_i}(\alpha_2)$. So $\boldsymbol{\pi}^{\lim}$ is injective. And if $\alpha_1 \in I$ and i in J stable for α_1, then there is an $\alpha_2 \in I^{\mathbf{a}_i}_{\zeta}$ with $\boldsymbol{\pi}^{\lim}(\alpha_2) = \boldsymbol{\pi}^{\mathbf{a}_i}(\alpha_2) = \alpha_1$, so $\pi^{\lim}$ is surjective.

For (3): Set $B := \boldsymbol{\pi}^{\mathbf{a}_i}[A']$. As I is above the good ζ_0, we have $B = \boldsymbol{\pi}^{\mathbf{a}_j}[A']$ for all $j \in J$ with $j > i$. So for $\alpha \in A'$, all $\boldsymbol{\pi}^{\mathbf{a}_j}(\alpha)$ are in B, and also stabilize to $\pi^{\lim}(\alpha)$, which therefore has to be in B. Analogously, we get If $\alpha \in I \setminus A$, then $\boldsymbol{\pi}^{\mathbf{a}_j}(\alpha) \notin B$ stabilizes to $\pi^{\lim}(\alpha)$, which therefore is not in B. As $\pi^{\lim}[I] = I$, we get $\pi^{\lim}[I \cap A] = B$.

We can now define $\mathbf{b}$ as

$$\mathbf{C}^{\mathbf{b}} := D \setminus \zeta_0; \quad \boldsymbol{\pi}^{\mathbf{b}}(\alpha) = \begin{cases} \alpha & \text{if } \alpha < \zeta_0 \\ \pi^{\lim}(\alpha) & \text{otherwise;} \end{cases} \quad \mathbf{B}^{\mathbf{b}} := \bigcup_{i \in J} \mathbf{B}^{\mathbf{a}_i}. \qquad \dashv$$

§4. Initial segments. We will work with initial segments of approximations (without the $\mathbf{B}$ part).

DEFINITION 4.1.

- An "initial segment" b consists of a "height" δ^b, a closed $C^b \subseteq \delta^b$ (possibly empty), and a $\pi^b \in \mathrm{Sym}(\delta^b)$ such that $\pi^b \restriction \zeta \in \mathrm{Sym}(\zeta)$ for all $\zeta \in C^b$.
- The set of initial segments is called IS.
- $b >_{\mathrm{IS}} a$, if $\delta^b > \delta^a$, $\delta^a \in C^b$, $C^b \cap \delta^a = C^a$, and $\pi^b \restriction \delta^a = \pi^a$.
- $b \geq_{\mathrm{IS}} a$ if $b >_{\mathrm{IS}} a$ or $b = a$.
- For $\zeta \in C^b$, we set $I^b_\zeta := I^*(C^b, \delta^b, \zeta)$.

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So the I^b_ζ form an increasing interval partition of $\delta^b \setminus \min(C^b)$, and $\pi^b \restriction I^b_\zeta \in \mathrm{Sym}(I^b_\zeta)$.

It is easy to see that $\leq_{\mathrm{IS}}$ is a partial order.

Some trivialities are as follows.

FACT 4.2. *Assume that $\bar{b} = (b_i)_{i<\xi}$, with $\xi \leq \lambda$ limit, is a $<_{\mathrm{IS}}$-increasing*[3] *sequence.*

(1) *If $\xi < \lambda$, then the following $b_\xi \in \mathrm{IS}$ is the $\leq_{\mathrm{IS}}$-supremum of $\bar{b}$, and we call it "the limit" of $\bar{b}$: $\delta^{b_\xi} := \bigcup_{i<\xi} \delta^{b_i}$, $C^{b_\xi} := \bigcup_{i<\xi} C^{b_i}$, and $\pi^{b_\xi} := \bigcup_{i<\xi} \pi^{b_i}$.*

(2) *If $\xi = \lambda$, then to each $B \subseteq P(\lambda)$ there is a $\mathbf{b} \in \mathrm{AP}$ as follows, which we call "a limit" of $\bar{b}$: $\mathbf{C}^{\mathbf{b}} := \bigcup_{i<\lambda} C^{b_i}$, $\boldsymbol{\pi}^{\mathbf{b}} := \bigcup_{i<\lambda} \pi^{b_i}$, and $\mathbf{B}^{\mathbf{b}} := B$.*

Let us call an $<_{\mathrm{IS}}$-increasing sequence $\bar{b}$ "continuous" if b_γ is the limit of $(b_\alpha)_{\alpha<\gamma}$ for all limits $\gamma < \delta$. We will only use continuous sequences.

DEFINITION 4.3. Let $\mathbf{a} \in \mathrm{AP}_{<\lambda}$ and $b \in \mathrm{IS}$ with $\delta^b \in \mathbf{C}^{\mathbf{a}}$. We say $c >_{\mathbf{a}} b$, if the following holds:

- $c >_{\mathrm{IS}} b$.
- $(C^c \cup \{\delta^c\}) \setminus \delta^b \subseteq \mathbf{C}^{\mathbf{a}}$.
- For all $\zeta \in C^c \setminus \delta^b$, $\pi^c \restriction I^{\mathbf{a}}_\zeta = \boldsymbol{\pi}^{\mathbf{a}} \restriction I^{\mathbf{a}}_\zeta$.
- For all $A \in \mathbf{B}^{\mathbf{a}}$, $\pi^c[A'] = \boldsymbol{\pi}^{\mathbf{a}}[A']$ where we set $A' := A \cap \delta^c \setminus \delta^b$.

For a short sequence $\bar{\mathbf{a}} = (\mathbf{a}_i)_{i\in J}$ we say $c >_{\bar{\mathbf{a}}} b$ if $c >_{\mathbf{a}_i} b$ for all $i \in J$.

LEMMA 4.4. *Let $\mathbf{a}, \mathbf{b}$ in $\mathrm{AP}_{<\lambda}$.*

(1) *$<_{\mathbf{a}}$ is a partial order*

(2) *If $\zeta < \lambda$, $c \in \mathrm{IS}$, and $(d_i)_{i\in\zeta}$ is a $>_{\mathrm{IS}}$-increasing sequence such that $d_i >_{\mathbf{a}} c$ for all $i < \zeta$, then also the limit d_ζ satisfies $d_\zeta >_{\mathbf{a}} c$.*

(3) *Let c, d be in IS. If $\mathbf{b} >_{\delta^c} \mathbf{a}$, then $d >_{\mathbf{b}} c$ implies $d >_{\mathbf{a}} c$.*

(4) *Assume $\bar{c} := (c_i)_{i\in\lambda}$ is a continuous increasing sequence in IS such that for some $i_0 < \lambda$ we have $c_i <_{\mathbf{a}} c_{i+1}$ for all $i > i_0$.*
Then any limit $\mathbf{c} \in \mathrm{AP}$ of the $\bar{c}$ with $\mathbf{B}^{\mathbf{c}} \supseteq \mathbf{B}^{\mathbf{a}}$ satisfies $\mathbf{c} >_{\mathrm{AP}} \mathbf{a}$.

(5) *Let $\bar{\mathbf{a}}$ be short, $b \in \mathrm{IS}$, δ^b good for $\bar{\mathbf{a}}$, and $E \subseteq \lambda$ club.*
Then there is a $c >_{\bar{\mathbf{a}}} b$ with $\delta^c \in E$ and $C^c = C^b \cup \{\delta^b\}$

PROOF. For (5), use (the proof of) Lemma 3.11: Pick any $\delta^c \in \bigcap_{i\in J} C^{\mathbf{a}_i} \cap E \setminus (\delta^b + 1)$ and set $C^c = C^b \cup \{\delta^b\}$ and $\pi^c = \pi^{\mathrm{lim}} \restriction \delta^c$.

The rest is straightforward. ⊣

We now turn to spoiling (f, A).

DEFINITION 4.5. Given $f \in \mathrm{Sym}(\lambda)$ and $A \in [\lambda]^\lambda$, we define $c >^{f,A} b$ by: $c >_{\mathrm{IS}} b$, $f \restriction \delta^c \in \mathrm{Sym}(\delta^c)$, and there is a $\xi^* \in A \cap \delta^c \setminus \delta^b$ with $f(\xi^*) \neq \pi^c(\xi^*)$.

We write $c >^{f,A}_{\bar{\mathbf{a}}} b$ for: $c >_{\bar{\mathbf{a}}} b$ & $c >^{f,A} b$.

Obviously we have the following fact.

FACT 4.6. *If $b' >_{\mathrm{IS}} b$ and $c >^{f,A} b'$, then $c >^{f,A} b$.*

[3] I.e., strictly increasing.

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LEMMA 4.7. *Assume* $(b_i)_{i\in\lambda}$ *is* $<_I$ *S-increasing such that unboundedly often* $b_{i+1} >^{f,A} b_i$. *Then for some* $A' \in [A]^\lambda$, *every limit* $\mathbf{b}$ *of* $(b_i)_{i\in\lambda}$ *with* $A' \in \mathbf{B}^{\mathbf{b}}$ *spoils* (f, A).

PROOF. By taking a subsequence, we can assume that for all odd i (i.e., $i = \delta + 2n + 1$ with δ limit or 0 and $n \in \omega$) $b_{i+1} >^{f,A} b_i$.

For i odd, set $I_i := \delta^{b_{i+1}} \setminus \delta^{b_i}$ and let $\xi_i \in I_i$ satisfy $f(\xi_i) \neq \pi^{b_{i+1}}(\xi_i) = \boldsymbol{\pi}^{\mathbf{b}}(\xi_i)$.

If i is odd, then $\boldsymbol{\pi}^{\mathbf{b}} \restriction I_i \in \mathrm{Sym}(I_i)$ and $f \restriction \delta^{b_{i+1}} \in \mathrm{Sym}(\delta^{b_{i+1}})$.

So if $i < j$ are both odd, then $f(\zeta_j) > \delta^{b_{i+1}} > \boldsymbol{\pi}^{\mathbf{b}}(\zeta_i)$; and if $j < k$ are both odd then $f(\zeta_j) < \delta^{b_j} \leq \boldsymbol{\pi}^{\mathbf{b}}(\zeta_k)$. This means that $f(\zeta_j)$ is different from all $\boldsymbol{\pi}^{\mathbf{b}}(\zeta_i)$ for i odd.

So we can set $A' = \{\zeta_j : j \text{ odd}\}$ and get that $f[A']$ is disjoint to $\boldsymbol{\pi}^{\mathbf{b}}[A']$. So $\mathbf{b}$ with A' added to $\mathbf{B}$ spoils (f, A). ⊣

LEMMA 4.8. *If* $\bar{\mathbf{a}}$ *is short,* $b \in \mathrm{IS}$, δ^b *good for* $\bar{\mathbf{a}}$, $f \in \mathrm{Sym}(\lambda)$, *and* $A \in [\lambda]^\lambda$, *then there is some* $d >^{f,A}_{\bar{\mathbf{a}}} b$.

PROOF. Let J be the index set of $\bar{\mathbf{a}}$.

Set $\mathbf{B} := \bigcup_{i\in J} \mathbf{B}^{\mathbf{a}_i}$. Recall that $\bar{\mathbf{a}}$ being short implies that each $\mathbf{a}_i$ is in $\mathrm{AP}_{<\lambda}$, so $|\mathbf{B}^{\mathbf{a}_i}| < \lambda$ and thus $|\mathbf{B}| < \lambda$.

Let $\zeta_0 < \lambda$ be the supremum of all $\mathbf{C}^{\mathbf{a}_i}$-successors of δ^b.

Set $E := \{\zeta \in \lambda : f \restriction \zeta \in \mathrm{Sym}(\zeta)\}$ (a club-set). Pick $\zeta_1 \in E$ such that $|A \cap (\zeta_1 \setminus \zeta_0)| > |2^{\mathbf{B}}|$. Pick $c >_{\bar{\mathbf{a}}} b$ with $\delta^c \in E \setminus \zeta_1$ and such that $C^c = C^b \cup \{\delta^b\}$.

For α, β in $A \cap (\zeta_1 \setminus \zeta_0)$ set $\alpha \sim \beta$ iff $(\forall X \in \mathbf{B})\,(\alpha \in X \leftrightarrow \beta \in X)$. As there are at most $|2^{\mathbf{B}}|$ many equivalence classes, there have to be $\beta_0 \neq \beta_1$ in $A \cap (\zeta_1 \setminus \zeta_0)$ with $\beta_0 \sim \beta_1$.

If $\pi^c(\beta_i) \neq f(\beta_i)$ for $i = 0$ or $i = 1$, set $d := c$. Otherwise, defines d as follows:

$\delta^d = \delta^c$, $C^d = C^c$, and $\pi^d(\alpha) := \begin{cases} \pi^c(\beta_1) & \text{if } \alpha = \beta_0, \\ \pi^c(\beta_0) & \text{if } \alpha = \beta_1, \\ \pi^c(\alpha) & \text{otherwise.} \end{cases}$ Note that in any case, $d >_{\bar{a}} b$:

Set $I := \delta^d \setminus \delta^b$. As $\beta_0 \sim \beta_1$ we have $\pi^d[A \cap I] = \pi^c[A \cap I] = \boldsymbol{\pi}^{\mathbf{a}_i}[A \cap I]$ for all $i \in J$ and $A \in \mathbf{B}^{\mathbf{a}_i}$ (as $c >_{\bar{\mathbf{a}}} b$).

And as the β_0, β_1 are above ζ_0, and $I^{\mathbf{a}_i}_{\delta^b} \leq \zeta_0$, we have $\pi^d \restriction I^{\mathbf{a}_i}_{\delta^b} = \pi^c \restriction I^{\mathbf{a}_i}_{\delta^b} = \boldsymbol{\pi}^{\mathbf{a}_i} \restriction I^{\mathbf{a}_i}_{\delta^b}$. ⊣

§5. $2^\lambda = \lambda^+$ for λ inaccessible implies a nowhere trivial automorphism.

LEMMA 5.1. *Every increasing sequence in* AP_λ *of length* $<\lambda^+$ *has an upper bound.*

PROOF. We can assume without loss of generality that the increasing sequence is $\bar{a} := (\mathbf{a}_i)_{i\in\xi}$ with $\xi \leq \lambda$.

For $i < \xi$, enumerate[4] $\mathbf{B}^{\mathbf{a}_i}$ as $\{x^j_i : j \leq \lambda\}$, and set $B^j_i := \{x^k_i : k \leq j\}$ for $j < \lambda$. We enumerate in a way so that the B^j_i are increasing with $i < \xi$. Let $\mathbf{a}^j_i$ be $\mathbf{a}_i$ with $\mathbf{B}^{\mathbf{a}_i}$ replaced by B^j_i, and for $\ell < \lambda$, we set $\bar{\mathbf{a}}^\ell := (\mathbf{a}^\ell_k)_{k<\min(\ell,\xi)}$.

[4] with lots of repetitions

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Note the following:

- For all $i < \ell < \lambda$ and $\alpha \in C^{\mathbf{a}^\ell_j} = C^{\mathbf{a}^i_j} = C^{\mathbf{a}_j}$ we have $\mathbf{a}^i_j <_\alpha \mathbf{a}^\ell_j$.
- $\bar{\mathbf{a}}^\ell$ is short.
- $\mathbf{c} \in \mathrm{AP}$ is an upper bound of $\bar{\mathbf{a}}$ iff it is an upper bound of all $\mathbf{a}^\ell_k$ for $\ell < \lambda$ and $k < \min(\ell, \xi)$.

We now construct by induction on $\ell < \lambda$ an $<_{\mathrm{IS}}$-increasing continuous sequence $(c^\ell)_{\ell\in\lambda}$, such that δ^{c^ℓ} is $\bar{\mathbf{a}}^\ell$-good:

- At limits γ we let c^γ be the limit of the $(c^k)_{k<\gamma}$, and note that (by induction) its height is $\bar{\mathbf{a}}^\gamma$-good.
- For $j = \ell + 1$, let E be the club set of $\bar{\mathbf{a}}^{\ell+1}$-good ordinals, and choose, as in Lemma 4.4(5) $c^{\ell+1} >_{\bar{\mathbf{a}}^\ell} c^\ell$ with $\delta^{c^{\ell+1}} \in E$.

Let $\mathbf{c}$ be the limit of the c^ℓ with $\mathbf{B}^{\mathbf{c}} := \bigcup_{i<\xi} \mathbf{B}^{\mathbf{a}_i}$.

We claim that $\mathbf{c} \geq_{\mathrm{AP}} \mathbf{a}^\ell_j$ for all $\ell < \lambda$ and $j < \min(\ell, \xi)$:

Assume that $k > \ell$. As δ^{c^k} is $\bar{\mathbf{a}}^k$-good, we have $\delta^{c^k} \in C^{\mathbf{a}_j}$. Therefore $\mathbf{a}^\ell_j < \mathbf{a}^k_j$. Also, $c^{k+1} >_{\bar{\mathbf{a}}^k} c^k$, so (by definition) $c^{k+1} >_{\mathbf{a}^k_j} c^k$, and so, by Lemma 4.4(3), $c^{k+1} >_{\mathbf{a}^\ell_j} c^k$. Then Lemma 4.4(4) gives us $\mathbf{c} >_{\mathrm{AP}} \mathbf{a}^\ell_j$, as required. ⊣

LEMMA 5.2. *Given* $\mathbf{a} \in \mathrm{AP}_\lambda$, $f \in \mathrm{Sym}(\lambda)$, *and* $A \in [\lambda]^\lambda$, *there is a* $\mathbf{b} \geq_{\mathrm{AP}} \mathbf{a}$ *which is in* AP_λ *and spoils* (f, A).

PROOF. Enumerate $\mathbf{B}^{\mathbf{a}}$ as $\{x^j : j \in \lambda\}$ and let $\mathbf{a}^j$ be $\mathbf{a}$ with $\mathbf{B}$ replaced by $\{x^i : i < j\}$. So $\mathbf{a}^j \in \mathrm{AP}_{<\lambda}$. We construct a continuous increasing sequence b^i $(i < \lambda)$ in IS such that δ^{b^i} is $\mathbf{a}^i$-good: Given b^i, we find $b^{i+1} >^{f,A}_{\mathbf{a}^i} b^i$ as in Lemma 4.8. Let $\mathbf{b}$ be the limit of the b^i with $\mathbf{B}^{\mathbf{b}} = \mathbf{B}^{\mathbf{a}} \cup \{A'\}$ as in Lemma 4.7.

And $\mathbf{b} >_{\mathrm{AP}} \mathbf{a}^j$ for all $j < \lambda$ and therefore $\mathbf{b} >_{\mathrm{AP}} \mathbf{a}$. ⊣

We can now easily show the following theorem.

THEOREM 5.3. *If* λ *is* (*strongly*) *inaccessible and* $2^\lambda = \lambda^+$, *then there is a nowhere trivial automorphism of the Boolean algebra* $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.

PROOF. We construct, by induction on $i \in \lambda^+$, an increasing chain of $\mathbf{a}_i$ in AP_λ as follows:

- For limit i, we take limits according to Lemma 5.1.
- For odd successors $i = j + 1 = \delta + 2n + 1$ (δ limit or 0, $n \in \omega$), pick by bookkeeping some X_j and let $\mathbf{a}_{j+1}$ be the same as $\mathbf{a}_j$ but with X_j and $(\boldsymbol{\pi}^{\mathbf{a}_j})^{-1}[X_j]$ added to $\mathbf{B}^{\mathbf{a}_j}$.
- For even successors $i = j + 1 = \delta + 2n + 2$, we pick by bookkeeping an $f_j \in \mathrm{Sym}(\lambda)$ and an $A_j \in [\lambda]^\lambda$. Then we choose $\mathbf{a}_{j+1} \geq_{\mathrm{AP}} \mathbf{a}_j$ spoiling (f_j, A_j), using Lemma 5.2.

Then $\tilde{\phi} := \bigcup_{i<\lambda} \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}$ is a nowhere trivial automorphism according to Fact 3.7. ⊣

§6. Forcing a nowhere trivial automorphism with $2^\lambda > \lambda^+$, λ inaccessible.

THEOREM 6.1. *Assume* λ *is inaccessible,* $2^\lambda = \lambda^+$, *and* $\mu > \lambda^+$ *is regular. Then there is a cofinality preserving* ($<\lambda$*-closed and* λ^+*-cc*) *poset which forces*: $2^\lambda = \mu$, *and there is a nowhere trivial automorphism of* $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.

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For the rest of this section we fix λ and μ as in the lemma.

We will construct a $<\lambda$-support iteration $(P_\alpha, Q_\alpha)_{\alpha<\mu}$. We call the final limit P. We denote the P_α-extension $V[G_\alpha]$ by V_α.

Each Q_α and therefore also each P_α will be $<\lambda$-closed.

So the followings statements are absolute between P_α-extensions: $x \in \mathrm{AP}$, $x <_{\mathrm{AP}} y$, as well as IS (as set, of size λ).

Each Q_α will add an $\mathbf{a}^*_\alpha \in \mathrm{AP}$, such that the $\mathbf{a}^*_\alpha$ are $<_{\mathrm{AP}}$-increasing in α.

By induction we assume we live in the P_α-extension V_α where we already have the increasing sequence $(\mathbf{a}^*_i)_{i<\alpha}$. (We do not claim that this sequence has an upper bound in V_α.)

We now define Q_α, which we will just call Q to improve readability.

Definition 6.2. $q \in Q$ consists of:

(1) A $b^q \in \mathrm{IS}$, also called "trunk of q."
   We also write δ^q, π^q, C^q, and I^q_β instead of δ^{b^q}, etc.

(2) A set $X^q \in [\alpha]^{<\lambda}$, and for $\beta \in X^q$, a set $\mathbf{B}^q_\beta \in [\mathbf{B}^{\mathbf{a}^*_\beta}]^{<\lambda}$, such that the $\mathbf{B}^q_\beta$ are increasing in β.

For $\beta \in X^q$, we set $\mathbf{a}^q_\beta$ to be $\mathbf{a}^*_\beta$ with $\mathbf{B}$ replaced by $\mathbf{B}^q_\beta$. We set $\bar{\mathbf{a}}^q := (\mathbf{a}^q_\beta)_{\beta\in X^q}$ (which is short). To be a valid condition in Q, we require δ^{b^q} to be good for $\bar{\mathbf{a}}^q$.

("Short" and "good" are defined in Definition 3.9.) As we use Q as forcing poset, we follow the notation that $r \leq_Q q$ means that r is stronger than q (whereas in $<_{\mathrm{AP}}$ and $<_{\mathrm{IS}}$ the stronger object is the larger one).

Definition 6.3. $r \leq_Q q$ if:

(1) $b^r \geq_{\bar{\mathbf{a}}^q} b^q$ (see Definition 4.3).
(2) $X^r \supseteq X^q$, and $\mathbf{B}^r_\beta \supseteq \mathbf{B}^q_\beta$ for $\beta \in X^q$.

The following follows immediately from by Lemma 4.4(3).

Fact 6.4. *Assume that $r \leq_Q q$, $b \in \mathrm{IS}$ and that δ^b is good for $\bar{\mathbf{a}}^r$. Then $c \geq_{\bar{\mathbf{a}}^r} b$ implies $c \geq_{\bar{\mathbf{a}}^q} b$.*

This implies that $\leq_Q$ is transitive. (It even is a partial order.)

Lemma 6.5. *For $q \in Q$, the following holds (in V_α): Let $E \subseteq \lambda$ be club.*

(1) *For $\beta < \alpha$ and $A \in \mathbf{B}^{\mathbf{a}^*_\beta}$ there is an $r <_Q q$ with $\delta^r \in E$, $\beta \in X^r$, and $A \in \mathbf{B}^r_\beta$.*
(2) *For any $A \in [\lambda]^\lambda$ and $f \in \mathrm{Sym}(\lambda)$ (both in V_α), the following set is dense: $D_{q,f,A} := \{r \perp q\} \cup \{r \leq q : b^r <^{f,A} b^q\}$.*
(3) *Q is λ-centered, witnessed by the function that maps q to its trunk, b^q.*
   (Actually, even $<\lambda$ many conditions with the same trunk have a lower bound.)
(4) *Q_α is $<\lambda$-closed.*
   Moreover, a descending sequence $(q_i)_{i\in\xi}$ $(\xi < \lambda)$ has a canonical limit r, and the trunk of r is the union of the trunks of the q_i.

Proof. (1): Extend $\bar{\mathbf{a}}^q$ in the obvious way to $\bar{\mathbf{a}}^r$: Add β to the index set, set $\mathbf{B}^r_\beta := \{A\} \cup \bigcup_{\zeta\in X^q\cap(\beta+1)} \mathbf{B}^q_\zeta$, and add A to all $\mathbf{B}^q_\zeta$ for $\zeta \in X^q \setminus \beta$. Let $E' := \{\zeta \in \lambda : \zeta \text{ good for } \bar{\mathbf{a}}^r\}$. Then E' is club according to Fact 3.10, so we can use Lemma 4.4(5) to find $b^r >_{\bar{\mathbf{a}}^q} b^q$ with $\delta^r \in E \cap E'$.

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(2) Assume that $s \leq q$ (so $b^s > b^q$). Then according to Lemma 4.8 we can find an $r \leq_Q s$ with $b^r >^{f,A} b^s$ and therefore $b^r >^{f,A} b^q$ (according to Fact 4.6).

(3) Let $(q_i)_{i\in\mu}$, $\mu < \lambda$ all have the same trunk b. Then the following r is a condition in Q: $b^r = b$, $X^r = \bigcup_{i<\mu} X^{q_i}$, and $B^r_\zeta = \bigcup_{i<\mu \ \&\ \zeta\in X^{q_i}} B^{q_i}_\zeta$.

(4) Let $(q_i)_{i<\zeta}$ with $\zeta < \lambda$ be $<_Q$-decreasing. Then the obvious union r is an element of Q and stronger than each q_i:

b^r is the union of the b^{q_i}, as in Fact 4.2, and $X^r := \bigcup_{i<\zeta} X^{q_i}$ and $\mathbf{B}^r_\beta := \bigcup_{i<\zeta,\beta\in X^{q_i}} \mathbf{B}^{q_i}_\beta$ for each $\beta \in X^r$.

Then δ^r is good for $\mathbf{a}^r_\beta$ for $\beta \in X^r$: It is enough to show that δ^r is good for all $\mathbf{a}^{q_i}_\beta$ (for sufficiently large i). Fix such an i. If $j > i$, then δ^{q_j} is good for $\bar{\mathbf{a}}^{q_j}$ and therefore for $\mathbf{a}^{q_j}_\beta$ and therefore for $\mathbf{a}^{q_i}_\beta$. So the limit δ^r is good as well.

Similarly one can argue that $b^r >_{\bar{\mathbf{a}}^{q_i}} b^{q_i}$ for all $i < \zeta$. ⊣

DEFINITION 6.6. Let $G(\alpha)$ be Q_α-generic. We define $\mathbf{a}^*_\alpha$ (in $V_{\alpha+1}$) as follows: $\mathbf{C}^{\mathbf{a}^*_\alpha} := \bigcup_{q\in G(\alpha)} C^q$, $\boldsymbol{\pi}^{\mathbf{a}^*_\alpha} := \bigcup_{q\in G(\alpha)} \pi^q$, and $\mathbf{B}^{\mathbf{a}^*_\alpha} := P(\lambda)$.

LEMMA 6.7. *$P_{\alpha+1}$ forces*:

(1) *$\mathbf{a}^*_\alpha >_{\mathrm{AP}} \mathbf{a}^*_\beta$ for all $\beta < \alpha$.*

(2) *$\mathbf{a}^*_\alpha$ spoils (f, A) for all $(f, A) \in V_\alpha$.*

The proof consists of straightforward density arguments.

PROOF. For (1) we know that by there is some $q \in G(\alpha)$ with $\beta \in X^q$. This implies that $\mathbf{C}^{\mathbf{a}^*_\alpha} \subseteq \mathbf{C}^{\mathbf{a}^*_\beta}$ above δ^q and that $\boldsymbol{\pi}^{\mathbf{a}^*_\alpha} \restriction I^{\mathbf{a}^*_\beta}_\zeta = \boldsymbol{\pi}^{\mathbf{a}^*_\beta} \restriction I^{\mathbf{a}^*_\beta}_\zeta$ for all $\zeta \in \mathbf{C}^{\mathbf{a}^*_\beta} \setminus \delta^q$. We can also assume that a given $A \in \mathbf{B}^{\mathbf{a}^*_\beta}$ is in $\mathbf{B}^q_\beta$, which implies that $\boldsymbol{\pi}^{\mathbf{a}^*_\alpha}[A] = \boldsymbol{\pi}^{\mathbf{a}^*_\beta}[A]$ above δ^q.

For (2) and $(f, A) \in V_\alpha$ we know by Lemma 6.5(2) that for each $q \in G(\alpha)$ there is an $r < q$ in $G(\alpha)$ such that $b^r >^{f,A} b^q$. I.e, in $V_{\alpha+1}$, $\mathbf{a}^*_\alpha$ is a limit of an $<_{\mathrm{IS}}$-increasing sequence as in Lemma 4.7, therefore $\mathbf{a}^*_\alpha$ spoils (f, A) (as A' certainly is in $\mathbf{B}^{\mathbf{a}^*_\alpha} = P(\lambda)$). ⊣

So P adds a sequence $(\mathbf{a}^*_\alpha)_{\alpha<\mu}$ that we can use in Fact 3.7 to get a nowhere trivial automorphism. We will now show that P is λ^+-cc, which finishes the proof of Theorem 6.1.

LEMMA 6.8. *Set $t(p) := (b^{p(\alpha)})_{\alpha\in\mathrm{dom}(p)}$ (i.e., the sequence of trunks). Then the following set D is dense: p in D if there is an $x \in V$ such that the empty condition forces $t(p) = x$.*

PROOF. We claim that the lemma holds for P_α, by induction on P_α. Successors and limits of cofinality $\geq\lambda$ are clear.

Let α be a limit with cofinality $\kappa < \lambda$, and $(\alpha_i)_{i\in\kappa}$ cofinal in α, $\alpha_0 = 0$. Set $D_j := D \cap P_{\alpha_j}$ (by induction dense in P_{α_j}). We construct by induction on $j \in \kappa$ a decreasing sequence $p_j \in P_\alpha$ such that $p_0 = p$ and $p_j \restriction \alpha_j \in D$:

Successors: Given p_j, we find $r \leq p_j \restriction \alpha_{j+1}$ in D_{j+1} and set $p_{j+1} := r \wedge p_j$ (which is the same as $r \wedge p$).

Limits: Given $(p_i)_{i<\xi}$ with $\xi \leq \kappa$, let p_ξ be the pointwise canonical limit. Note that we can calculate (in V) each $p_\xi(\beta)$ from the sequence $(p_i(\beta))_{i<\xi}$ (it is just the union). ⊣

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LEMMA 6.9. (*Assuming $2^\lambda = \lambda^+$ in the ground model.*) *P is λ^+-cc.*

PROOF. Assume $(a_i)_{i\in\lambda^+}$ is a sequence in P. For every a_i find an $a_i' \leq a$ in D. By Fodor (or the Delta-system lemma) there is an $X \subseteq \lambda^+$ of size λ^+ such that $\{\mathrm{dom}(a_i') : i \in X\}$ form a Delta system with heart Δ, and furthermore we can assume that $t(a_i') \restriction \Delta$ (the sequence of trunks restricted to Δ) is the same for all $i \in X$. (There are $\lambda^{|\Delta|} = \lambda < \lambda^+$ many such restrictions.) Then for i, j in X, the conditions a_i' and a_j' (and therefore also a_i and a_j) are compatible. ⊣

REMARK 6.10. Generally, preserving λ^+-cc for $\lambda > \omega_1$ is much more cumbersome than for $\lambda = \omega$, as there is no obvious universal theorem analogous to "the finite support iteration of ccc forcings is ccc." In our case, it was very easy to show λ^+-cc manually. However, we could have used existing iteration theorems. We give two examples (but there surely are many more). Note that the following theorems do not require λ to be inaccessible.

(1) From [9] (generalising the $\lambda = \aleph_1$ case from [1, Lemma 4.1]):
   - Definition [9, p. 237]: Q is λ-centered closed, if a centered subset D of Q of size $<\lambda$ has a lower bound.
   - Lemma [9, p. 237]: Assume $2^{<\lambda} = \lambda$. Let P be a $<\lambda$-support iteration such that each iterand is (forced to be) λ-linked and λ-centered closed. Then P is λ^+-cc.

   It is easy to see that our Q satisfies the requirements (Q is even λ-centered and "λ-linked closed").

(2) From [2] (generalizing the $\lambda = \aleph_1$ case from [6]):
   - [2, Definition 2.2.2]: Q is "stationary λ^+-Knaster," if for every sequence $(p_i)_{i<\lambda^+}$ in Q there exists a club $E \subseteq \lambda^+$ and a regressive function f on $E \cap S^{\lambda^+}_\lambda$ such that p_i and p_j are compatible whenever $f(i) = f(j)$.
   - [2, Theorem 2.26]: Assume that P is a $<\lambda$-support iteration of iterands that all are: stationary λ^+-Knaster, strategically $<\lambda$-closed, and any two compatible conditions have a greatest lower bound, as do decreasing ω-sequences. Then P is stationary λ^+-Knaster.

   Note that our Q satisfies the requirements, and that our proof that P is λ^+-cc actually shows that P is stationary λ^+-Knaster.

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