ON THE COMPUTABILITY OF OPTIMAL SCOTT SENTENCES

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Abstract. Given a countable mathematical structure, its *Scott sentence* is a sentence of the infinitary logic $\mathcal{L}_{\omega_1\omega}$ that characterizes it among all countable structures. We can measure the complexity of a structure by the least complexity of a Scott sentence for that structure. It is known that there can be a difference between the least complexity of a Scott sentence and the least complexity of a computable Scott sentence; for example, Alvir, Knight, and McCoy showed that there is a computable structure with a Π_2 Scott sentence but no computable Π_2 Scott sentence. It is well known that a structure with a Π_2 Scott sentence must have a computable Π_4 Scott sentence. We show that this is best possible: there is a computable structure with a Π_2 Scott sentence but no computable Σ_4 Scott sentence. We also show that there is no reasonable characterization of the computable structures with a computable Π_n Scott sentence by showing that the index set of such structures is Π^1_1-*m*-complete.

§1. Introduction. Let $\mathcal{A}$ be a countable mathematical structure, such a countable graph, group, or ring. Suppose that we want to characterize $\mathcal{A}$ by writing down a sentence, or theory, which characterizes $\mathcal{A}$ up to isomorphism. If we work in elementary first-order logic then, as a consequence of compactness, we cannot do this for most countable structures $\mathcal{A}$. However, suppose we strengthen our logic to the logic $\mathcal{L}_{\omega_1\omega}$ which allows countably infinite conjunctions and disjunctions. Scott [24] showed that for any countable structure $\mathcal{A}$ there is a sentence φ of $\mathcal{L}_{\omega_1\omega}$ that characterizes $\mathcal{A}$ up to isomorphism among countable structures, i.e., for all countable $\mathcal{B}$,

$$\mathcal{B} \models \varphi \iff \mathcal{A} \cong \mathcal{B}.$$

We call such a sentence a *Scott sentence* for $\mathcal{A}$. This fact implies, for example, that while isomorphism is analytic-complete if we fix a particular $\mathcal{L}$-structure $\mathcal{A}$ then the set $\{\mathcal{B} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{A} \cong \mathcal{B}\}$ is actually Borel.

The standard proof that every countable structure has a Scott sentence uses the back-and-forth relations. First, one shows that they must stabilize at some countable ordinal, and then from this one can extract a Scott sentence, called the *canonical Scott sentence*. This Scott analysis of a structure has played an important role in the study of Vaught's conjecture, e.g., in Morley's Theorem [21] that the number of non-isomorphic countable models of a theory is either at most $\aleph_1$ or is exactly $2^{\aleph}_0$.

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For each particular structure $\mathcal{A}$, there is some ordinal α at which the back-and-forth relations stabilize. This gives a way of assigning an ordinal rank to each countable structure giving a measure of that structure's complexity. There are numerous non-equivalent definitions of Scott rank depending on the particular back-and-forth relations one chooses or exactly what one means by stabilizing. Within the last ten years the most commonly used has been the *(unparametrized) Scott rank* due to Montalbán [16] which measures the least complexity of a Scott sentence for $\mathcal{A}$, namely, the Scott rank is the least α such that $\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence. The following theorem shows that this definition is particularly robust.

THEOREM 1.1 (Montalbán [16]). *Let $\mathcal{A}$ be a countable structure, and α a countable ordinal. The following are equivalent*:

(1) *$\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.*
(2) *Every finite tuple has an automorphism orbit in $\mathcal{A}$ which is Σ_α-definable, i.e., $\mathcal{A}$ has a Scott family of Σ_α formulas (without parameters.)*
(3) *The set $\{\mathcal{B} \in \mathrm{Mod}(\mathcal{L}) : \mathcal{B} \cong \mathcal{A}\}$ of isomorphic copies of $\mathcal{A}$ is $\mathbf{\Pi}^0_{\alpha+1}$.*
(4) *$\mathcal{A}$ is (boldface) $\mathbf{\Delta}^0_\alpha$-categorical.*

In computable structure theory, there has been a long history of studying the Scott analysis of computable structures, or more generally how the Scott analysis of a structure relates computationally to the structure. Nadel [22] showed that every computable structure $\mathcal{A}$ has Scott rank at most $\omega_1^{\mathcal{A}} + 1$. Harrison [8], Millar and Knight [13] (building on work of Makkai [15]), Harrison-Trainor, Igusa, and Knight [11], Harrison-Trainor [9], and Alvir, Greenberg, Harrison-Trainor, and Turetsky [1] constructed various examples of computable structures with non-computable Scott ranks.

In particular, if $\mathcal{A}$ is computable and has computable Scott rank $< \omega_1^{CK}$, then the canonical Scott sentence constructed by Scott will also be a computable sentence. However this construction is not optimal: Alvir, Knight, and McCoy [2] showed that there is a computable structure with a Π_2 Scott sentence but with no computable Π_2 Scott sentence. It is well-known (see, e.g., [19, Lemma VI.14]) that if a computable structure $\mathcal{A}$ has a Π_α Scott sentence, with α computable, then it has a computable $\Pi_{2\alpha}$ Scott sentence. This is obtained by noting that if $\mathcal{A}$ has a Π_α Scott sentence φ, then $\mathcal{A}$ has the property that for countable structure $\mathcal{B}$,

$$\mathcal{A} \leq_\alpha \mathcal{B} \iff \mathcal{A} \cong \mathcal{B}.$$

Writing out the definition of the back-and-forth relations, we can see that there is a computable $\Pi_{2\alpha}$ sentence ψ such that

$$\mathcal{B} \models \psi \iff \mathcal{A} \leq_\alpha \mathcal{B}.$$

This ψ is a computable $\Pi_{2\alpha}$ Scott sentence.

In this article, we restrict to the case of structures with Π_2 Scott sentences, which are called by Montalbán [18] the $\exists$-atomic structures because every automorphism orbit is isolated by a (finitary) existential formula. Even in this seemingly simple class of structures we find that there is significant complexity. As described above, such structures have a computable Π_4 Scott sentence; we improve the result of [2] to show that this upper bound is optimal.

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THEOREM 1.2. *There is a computable structure with a* Π_2 *Scott sentence but no computable* Σ_4 *Scott sentence.*

By taking Marker extensions / jump inversions, one can also obtain that for any n, there is a computable structure with a Π_n Scott sentence but no computable Σ_{n+2} Scott sentence. (One can also extend this to non-limit ordinals though we will remain at finite levels since all of the complexity appears there; see Corollary 5.8.) We leave open the question of improving this to an example with no Σ_{2n} Scott sentence.

In the second part of this article, we consider the *effective Scott rank*, i.e., the least α such that a given structure $\mathcal{A}$ has a computable $\Pi_{\alpha+1}$ Scott sentence (when a computable Scott sentence exists). Whether or not this effective Scott rank is as robust as the (non-effective) Scott rank was an open problem from [2]. It is known that there are a few conditions equivalent to having a formally Σ_α Scott family; a "boldface" version of this result led to many of the equivalences of Theorem 1.1, so it was not unreasonable to expect similar robustness in the effective setting. In fact, it was known that for a computable structure, having a formally Σ_α Scott family implies that the structure has a computable $\Pi_{\alpha+1}$ Scott sentence. Whether the reverse implication held was left open.

In Section 4, we show that the effective Scott rank is not robust and that there is no good way to characterize when a structure has a computable Π_n Scott sentence, even when the structure is computable.

THEOREM 1.3. *With* $(\mathcal{A}_i)_{i\in\omega}$ *an effective list of (possibly partial) structures in a sufficiently rich language, the set*

$$\{i \mid \mathcal{A}_i \text{ has a computable } \Pi_2 \text{ Scott sentence}\}$$

is Π^1_1*-m-complete.*

By a sufficiently rich language we mean any language which is universal, e.g., those including at least a binary relation symbol. This result was also obtained independently and around the same time by Knight, Lange, and McCoy [14].

Again by taking Marker extensions / jump inversions, one can replace Π_2 with Π_n, or through the hyperarithmetic hierarchy, as in Corollary 5.9. One can obtain from this theorem several interesting corollaries which we give in Section 5, for example, we have the following.

COROLLARY 1.4. *There is a computable structure* $\mathcal{A}$ *with a* Π_2 *Scott sentence but no computable* Π_2 *Scott sentence, but with a computable* Π_2 *sentence* φ *such that, for all hyperarithmetic* $\mathcal{B}$*,*

$$\mathcal{B} \models \varphi \iff \mathcal{A} \cong \mathcal{B}.$$

We leave open several questions. Though there are further generalizations that one can ask (e.g., to infinite ordinals), we see the two mains questions to be the following.

QUESTION 1.5. *Is there, for every* n*, a computable structure* $\mathcal{A}$ *with a* Π_n *Scott sentence but with no computable* Σ_{2n} *Scott sentence*?

QUESTION 1.6. *Is the set of computable structures with a* Π_2 *Scott sentence and a computable* Π_3 *Scott sentence* Π^1_1*-m-complete*?

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For Question 1.5, recall that the upper bound of Π_{2n} was obtained as follows. Given $\mathcal{A}$ with a Π_n Scott sentence, $\mathcal{A} \leq_n \mathcal{B} \Longrightarrow \mathcal{A} \cong \mathcal{B}$. We find a Π_{2n} sentence ψ such that $\mathcal{B} \models \psi \iff \mathcal{A} \leq_n \mathcal{B}$. Chen, Gonzalez, and Harrison-Trainor [5] have recently showed that the set of pairs $\{(\mathcal{A}, \mathcal{B}) : \mathcal{A} \leq_n \mathcal{B}\}$ is $\mathbf{\Pi}^0_{2n}$-complete. This means that there is no way of defining the set $\{\mathcal{B} : \mathcal{A} \leq_n \mathcal{B}\}$ in a way that is both better than Π_{2n} and which is "schematic" in $\mathcal{A}$. On the other hand, they showed that the set $\{\mathcal{B} : \mathcal{A} \leq_n \mathcal{B}\}$ is $\mathbf{\Pi}^0_{n+2}$. This relies on the fact that every Π_n-type in $\mathcal{A}$ is Π_n-definable, and so is a non-effective argument. The case of Question 1.5 solved in this article, $n = 2$, satisfies $n + 2 = 2n = 4$ and the distinction does not yet show up at this level. Thus one might expect Question 1.5 for $n \geq 3$ to involve some new insight.

§2. A simplifying remark. Given a structure $\mathcal{A}$, we can consider the structure $\mathcal{A} \cdot \omega$ which consists of an equivalence relation E with infinitely many equivalence classes, on each of which is a copy of $\mathcal{A}$. Then:

(1) if $\mathcal{A}$ has a (computable) Π_α Scott sentence, then $\mathcal{A} \cdot \omega$ will have a (computable) Π_α Scott sentence; and
(2) if $\mathcal{A}$ has no (computable) Π_α Scott sentence, then $\mathcal{A} \cdot \omega$ will have no (computable) $\Sigma_{\alpha+1}$ Scott sentence.

(1) is straightforward, and (2) uses the fact that $\mathcal{B}$ has a $\Sigma_{\alpha+1}$ Scott Sentence if and only if there is $\bar{b} \in \mathcal{B}$ such that $(\mathcal{B}, \bar{b})$ has a Π_α Scott sentence. (This fact was first stated by Montalbán in [16] and proved in [1, 19].)

Given the above facts, we do not need to consider Σ Scott sentences, as e.g., the fact that there is a computable structure with a Π_2 Scott sentence but no computable Π_2 Scott sentence yields that there is a structure with a Π_2 Scott sentence but no computable Σ_3 Scott sentence. When we prove Theorem 1.2 we will prove that there is a computable structure with a Π_2 Scott sentence but no computable Π_3 Scott sentence; it will follow that there is a computable structure with a Π_2 Scott sentence but no computable Σ_4 Scott sentence.

§3. A computable structure with a Π_2 Scott sentence but no computable Π_3 Scott sentence. As a warmup we sketch a new construction of a computable structure with a Π_2 Scott sentence but no computable Π_2 Scott sentence. Theorems 1.2 and 1.3 will build on this technique and so we begin by presenting it in its simplest form.

THEOREM 3.1 (Alvir, Knight, and McCoy [2]). *There is a computable structure with a Π_2 Scott sentence but with no computable Π_2 Scott sentence.*

PROOF SKETCH. We list all of the computable Π_2 sentences as $(\theta_e)_{e\in\omega}$, where

$$\theta_e = \bigwedge\!\!\!\bigwedge_{i\in\omega} \forall \bar{x}_{e,i} \;\; \varphi_{e,i}(\bar{x}_{e,i}),$$

where the $\varphi_{e,i}(\bar{x}_{e,i})$ are computable Σ_1 formulas uniformly in e, i and the arities of each of the $\bar{x}_{e,i}$ are also computable in the indices.

We can take $\mathcal{A}$ to be the "bouquet graph" $\mathcal{G}_1(\mathcal{F})$ of a collection $\mathcal{F}$ of subsets of ω. $\mathcal{A}$ will be $\exists$-atomic and hence have a Π_2 Scott sentence. Recall that by Lemma 8.17 of [20] the structure $\mathcal{G}_1(\mathcal{F})$ is $\exists$-atomic exactly when $\mathcal{F}$ is discrete.

The structure $\mathcal{A}$ will consist of a number of elements each of which is given various c.e. labels. We can think of $\mathcal{A}$ as consisting of the elements which are the centers of

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the flowers, labeled by labels ℓ_n; putting a label ℓ_n on a flower means to add a loop of length $n+3$ to the flower.

We divide our label into *sort labels* $(u_e)_{e\in\omega}$ such that only exactly one label holds of each element, dividing the domain into the disjoint sets $U_e = \{x \in A : u_e(x)\}$; we think of these as different sorts of the structure, and call the elements of U_e the *eth sort*. Then we have two other sets of labels $(\ell_i)_{i\in\omega}$ and $(\ell_i^\dagger)_{i\in\omega}$ which we call simply *labels*. For example, we may use loops of length $2e+3$ for the labels u_e, and of length $4i+4$ for the ℓ_i, and $4i+6$ for the labels $\ell_i^\dagger$.

We use the eth sort to diagonalize against θ_e. We do this by simultaneously building a structure $\mathcal{B}_e$ such that if $\mathcal{A} \models \theta_e$ then $\mathcal{B}_e \models \theta_e$ and $\mathcal{A} \ncong \mathcal{B}$. $\mathcal{B}_e$ will differ from $\mathcal{A}$ only on the eth sort U_e. We build $\mathcal{A}$ by finite approximations $\mathcal{A} = \bigcup_s \mathcal{A}_s$. $\mathcal{B}_e$ will also be built by approximations $\mathcal{B}_e = \bigcup_s \mathcal{B}_{e,s}$ with each $\mathcal{B}_{e,s} \cong \mathcal{A}_s$.

When we describe the construction of $\mathcal{A} = \bigcup \mathcal{A}_s$, we will describe the construction of the eth sort. The constructions for the different sorts should be thought of as happening simultaneously.

During the construction certain stages will be *e-expansionary stages* where we get evidence that $\mathcal{A}_s \models \theta_e$. We use $k = k[s]$ to keep track of the number of expansionary stages and $n = n[s] = k[s]+1$ to keep track of the number of flowers in $\mathcal{A}_s$. At stage s, $\mathcal{A}_s$ will have elements $a_1, \dots, a_n$ and $\mathcal{B}_{e,s}$ will have elements $b_1, \dots, b_{n-1}, c$.

Construction.

Stage 0. We begin with $n = n[0] = 1$ and $k = k[0] = 0$. $\mathcal{A}_0$ consists of a single element a_1, and $\mathcal{B}_{e,0}$ of a single element c. We give a_1 and c the same label ℓ_0.

Stage $s+1$. Suppose that we have constructed $\mathcal{A}_s$ and $\mathcal{B}_{e,s}$ with $k = k[s]$ and $n = n[s] = k+1$. The elements of $\mathcal{A}_s$ will be $a_1, \dots, a_n$. For each $i \le n$, a_i will have the labels ℓ_j for $j \le i$; and for $i < n$, a_i will have the label $\ell_i^\dagger$. The elements of $\mathcal{B}_{e,s}$ will be $b_1, \dots, b_{n-1}, c$. For each $i \le n-1$, b_i will have the labels ℓ_j for $j \le i$ and $\ell_i^\dagger$. The element c will have the labels ℓ_j for $j \le n$.

Let s_k be the last expansionary stage. Check whether

$$\mathcal{A}_s \models \bigwedge_{i\le k} \forall \bar{x}_{e,i} \in \mathcal{A}_{s_k} \;\; \varphi_{e,i}(\bar{x}_{e,i}).$$

If not, then this is not an expansionary stage. Make no changes to $\mathcal{A}$, $\mathcal{B}_e$, $k[s+1] = k[s]$, or $n[s+1] = n[s]$. If so then this stage is an expansionary stage. In $\mathcal{A}_{s+1}$ put the label $\ell_i^\dagger$ on a_n, and add a new element a_{n+1} to with labels ℓ_i for $i \le n+1$. In $\mathcal{B}_{e,s+1}$ add a new element b_n with labels ℓ_i for $i \le n$ and $\ell_n^\dagger$. Put the label ℓ_{n+1} on c. Set $k[s+1] = k[s]+1$ and $n[s+1] = n[s]+1$.

End construction.

We will not give the full details of the verification. However, the general idea of the argument is as follows. We argue that there are infinitely many expansionary stages if and only if $\mathcal{A} \models \theta_e$. On the one hand, if there are infinitely many expansionary stages s then for every k there is s such that

$$\mathcal{A}_s \models \bigwedge_{i\le k} \forall \bar{x}_{e,i} \in \mathcal{A}_{s_k} \;\; \varphi_{e,i}(\bar{x}_{e,i}),$$

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and so as $\varphi_{e,i}$ is Σ_1 we have $\mathcal{A} \models \theta_e$. On the other hand, if there are only finitely many expansionary stages, then there is i and $\bar{x}$ such that for all s we have

$$\mathcal{A}_s \models \neg\varphi_{e,i}(\bar{x}).$$

Then $\mathcal{A} \models \neg\theta_e$.

If there are infinitely many expansionary stages, then $\mathcal{B}_e \not\cong \mathcal{A}$ as the element $c \in \mathcal{B}_e$ has infinitely many labels ℓ_i for each i, but $\mathcal{A}$ has no such element. We can also argue that $\mathcal{B}_{e,s} \models \theta_n$; the tricky part is to check tuples $\bar{x}_{e,i}$ containing c. Here we use the fact that at the $k+1$st expansionary stage $s > s_k$ we have

$$\mathcal{A}_s \models \bigwedge_{i \leq k} \forall \bar{x}_{e,i} \in \mathcal{A}_{s_k} \;\; \varphi_{e,i}(\bar{x}_{e,i}),$$

and that that the isomorphism $\mathcal{A}_s \cong \mathcal{B}_{e,s}$ extends the isomorphism $\mathcal{A}_{s_k} \cong \mathcal{B}_{e,s_k}$.

Finally, we need to check that $\mathcal{A}$ is always $\exists$-atomic and hence always has a Π_2 Scott sentence. Any element with a label $\ell_i^{\dagger}$ is isolated by that label in its sort; and in each sort, there is at most one element without such a label (and only if there were finitely many expansionary stages in that sort), and that element is isolated by a label ℓ_i which no other element of the sort has. ⊣

The construction of a computable structure with a Π_2 Scott sentence but no computable Π_3 Scott sentence is similar, but the guessing as to whether a Π_3 sentence is true in $\mathcal{A}$ is more involved. Otherwise, the general ideas in the argument remain the same. Recall that by proving that such a structure exists, we have also shown that there is a computable structure with a Π_2 Scott sentence but no computable Σ_4 Scott sentence.

THEOREM 1.2. *There is a computable structure with a Π_2 Scott sentence but no computable Σ_4 Scott sentence.*

PROOF. We list all of the computable Π_3 sentences as $(\theta_e)_{e\in\omega}$, where

$$\theta_e = \bigwedge\!\!\!\bigwedge_{i\in\omega} \forall \bar{x}_{e,i} \bigvee\!\!\!\bigvee_{j\in\omega} \exists \bar{y}_{e,i,j} \;\; \varphi_{e,i,j}(\bar{x}_{e,i}, \bar{y}_{e,i,j}),$$

where the $\varphi_{e,i,j}(\bar{x}_{e,i}, \bar{y}_{e,i,j})$ are computable Π_1 formulas uniformly in e, i, j and the arities of each of the $\bar{x}_{e,i}$ and $\bar{y}_{e,i,j}$ are also computable in the indices.

We will build a structure $\mathcal{A}$ which is $\exists$-atomic. The structure $\mathcal{A}$ consist of a number of elements each of which is given various c.e. labels. Formally, $\mathcal{A}$ can be taken to be the "bouquet graph" $\mathcal{G}_1(\mathcal{F})$ of a collection $\mathcal{F}$ of subsets of ω—see Section VIII of [20]. As we construct $\mathcal{A}$, we think of ourselves as giving the centers of various "flower graphs" and when we add labels this corresponds to adding loops to the flower graphs. Thus in a computable construction of $\mathcal{A}$ we can add labels to elements in a c.e. way.

We introduce two sets of distinguished labels. First, we have *sort labels* $(u_e)_{e\in\omega}$ such that only exactly one label holds of each element, dividing the domain into the disjoint sets $U_e = \{x \in A : u_e(x)\}$; we think of these as different sorts of the structure, and call the elements of U_e the *eth sort*. The second set of labels $(\ell_i)_{i\in\omega}$ we will just call *labels*, and they will be the most important labels for the construction.

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Within U_e we will diagonalize against θ_e. We will do this by constructing another countable structure $\mathcal{B}_e$ such that if $\mathcal{A} \models \theta_e$ then $\mathcal{B}_e \models \theta_e$ and $\mathcal{A} \not\cong \mathcal{B}_e$. $\mathcal{B}_e$ will be the same as $\mathcal{A}$ except for the eth sort U_e on which they will differ. At each stage s we will have a finite approximation $\mathcal{A}_s$ to $\mathcal{A}$, with $\mathcal{A} = \bigcup_s \mathcal{A}_s$. $\mathcal{B}_e$ will also be built by finite approximations $\mathcal{B}_e = \bigcup_s \mathcal{B}_{e,s}$ with each $\mathcal{B}_{e,s} \cong \mathcal{A}_s$. Thus one can think of $\mathcal{A}$ and $\mathcal{B}_e$ as direct limits of direct systems of the same finite structures but with different embeddings.

$$\mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \cdots .$$

At each stage s, each element of $\mathcal{A}_s$ will satisfy some finite set of labels (in fact two, a domain label and another label) that is satisfied by no other elements except duplicates of itself which satisfy exactly the same labels. Thus the isomorphism $\mathcal{A}_s \to \mathcal{B}_{e,s}$ will be easily and uniquely determined up to mapping duplicates.

For each e, we have requirements $\mathcal{R}^e_{i,\bar{b}}$ for each $i \in \omega$ and $\bar{b} \in \mathcal{B}_e$ of the right arity; to make sense of this, we must fix ahead of time the domain of $\mathcal{B}_e$ even though these elements will only be added to $\mathcal{B}_e$ slowly over time. The elements $\bar{b}$ may be in any sort of $\mathcal{B}_e$, not just the eth. Each requirement will have a stage $t = t(\mathcal{R}^e_{i,\bar{b}})$ when it began working; it will only start working at some stage when $\bar{b} \in \mathcal{B}_{e,t}$. At this stage, there will be an element $\bar{a}$ of $\mathcal{A}_t$ which corresponds to $\bar{b} \in \mathcal{B}_{e,t}$ at this stage t via the isomorphism $\mathcal{A}_t \cong \mathcal{B}_{e,t}$. The requirement will be met if whenever $\mathcal{A} \models \bigvee_j \exists \bar{y}_{e,i,j}\ \varphi_{e,i,j}(\bar{a}, \bar{y}_{e,i,j})$ then $\mathcal{B}_e \models \bigvee_j \exists \bar{y}_{e,i,j}\ \varphi_{e,i,j}(\bar{b}, \bar{y}_{e,i,j})$; equivalently, if $\mathcal{B}_e \models \bigwedge_j \forall \bar{y}_{e,i,j}\ \neg\varphi_{e,i,j}(\bar{b}, \bar{y}_{e,i,j})$ then $\mathcal{A} \models \bigwedge_j \forall \bar{y}_{e,i,j}\ \neg\varphi_{e,i,j}(\bar{a}, \bar{y}_{e,i,j})$. One can think of the requirements for a given e as splitting up, into the different possible outermost witnesses, the supreme requirement that if $\mathcal{A} \models \theta_e$ then $\mathcal{B}_e \models \theta_e$ (or equivalently if $\mathcal{B} \models \neg\theta_e$ then $\mathcal{A} \models \neg\theta_e$). The requirement will require attention when we believe that $\mathcal{B}_e \models \forall \bar{y}_{e,i,j}\ \neg\varphi_{e,i,j}(\bar{b}, \bar{y}_{e,i,j})$. This is Π^0_2 and so if this is the case, this requirement will require attention infinitely often. Meeting a single requirement in this infinitary manner will be enough, as for $\bar{b} \in \mathcal{B}_e$ we have that if $\mathcal{B}_e \models \forall \bar{y}_{e,i,j}\ \neg\varphi_{e,i,j}(\bar{b}, \bar{y}_{e,i,j})$ then $\mathcal{B}_e \not\models \theta_e$ and the requirement will have the outcome that $\mathcal{B}_e \cong \mathcal{A}$ so that $\mathcal{A} \not\models \theta_e$. Thus even though the injury may be infinite, any guessing is simply finitary since if a requirement is injured infinitely often, then it does not need to be satisfied as a higher priority requirement will be satisfied. In addition to the value $t = t(\mathcal{R}^e_{i,\bar{b}})$, each requirement will also have a value $k = k(\mathcal{R}^e_{i,\bar{b}})$ which denotes the number of times that it has previously received attention.

The general idea of the construction is that, over time, we work to build $\mathcal{B}_e$ taking, at each stage, a step towards making $\mathcal{B}_e \not\cong \mathcal{A}$. In particular, there will be some special element of the eth sort of $\mathcal{B}_e$ which we will work to make different from each element of $\mathcal{A}$. However, whenever we see evidence that $\mathcal{B}_e \models \neg\theta_e$, we will undo our previous work to make $\mathcal{B}_e$ different from $\mathcal{A}$; since this makes $\mathcal{B}_e$ look similar to $\mathcal{A}$, we can at the same time transfer our evidence that $\mathcal{B}_e \models \neg\theta_e$ to evidence that $\mathcal{A} \models \neg\theta_e$. If in fact $\mathcal{B}_e \models \neg\theta_e$, then we will continually find evidence of this, continually rolling back our construction, so that actually $\mathcal{B}_e$ never becomes different from $\mathcal{A}$. Since $\mathcal{A} \cong \mathcal{B}_e$, $\mathcal{A} \models \neg\theta_e$. On the other hand, if $\mathcal{B}_e \models \theta_e$, then we will have $\mathcal{A} \not\cong \mathcal{B}_e$ so that θ_e cannot be a Scott sentence for $\mathcal{A}$.

We describe the construction of $\mathcal{A}$ and the $\mathcal{B}_e$ stage-by-stage. We will describe for a fixed e the construction of the eth sort of $\mathcal{A}$ together with the eth sort of $\mathcal{B}_e$.

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However all of these constructions for the different sorts should be thought of as happening simultaneously. All of the other sorts of $\mathcal{B}_e$ will be exactly the same as $\mathcal{A}$, so when constructing the eth sort of $\mathcal{A}$ we need only discuss $\mathcal{B}_e$ and not any other $\mathcal{B}_m$. In the construction that follows, without explicitly saying so we put the label u_e on all elements that we add to $\mathcal{A}$ and $\mathcal{B}_e$ so that they are in U_e. We sometimes suppress e and the label u_e in what follows, e.g., we write for $\theta = \theta_e$

$$\theta = \bigwedge\!\!\bigwedge_{i\in\omega} \forall \bar{x}_i \bigvee\!\!\bigvee_{j\in\omega} \exists \bar{y}_{i,j} \;\; \varphi_{i,j}(\bar{x}_i, \bar{y}_{i,j}).$$

However, when we are guessing at whether some formula is true in $\mathcal{A}$ or $\mathcal{B}_e$, we must allow the quantifiers to quantify over all of the sorts; we will remind the reader of this later.

We begin at stage 0 with a single element a_0 with no labels (other than u_e, which we no longer mention). At each stage s we will have a number $n = n[s]$. There will be *n active elements* designated $a_1[s], \ldots, a_n[s]$. Each active element will have a label that no other active element has. The rest of the elements of $\mathcal{A}_s$ will be *duplicate elements* and will have exactly the same labels as one of the active elements. When we declare a (formerly active) element to be a duplicate of a_i, we will give it exactly the same labels as a_i, and whenever a_i gets a new label, the element will as well; thus if later a_i becomes a duplicate of some other a_j, then any duplicates of a_i will become duplicates of a_j. When an active element a_i is declared to be a duplicate of another, becoming inactive, it gives up the designation a_i. Recall that $\mathcal{B}_{e,s}$ will be isomorphic to $\mathcal{A}_s$; however, $\mathcal{B}_e$ will follow different embeddings from one stage to the next. The active elements of $\mathcal{B}_{e,s}$ will be $b_1[s], \ldots, b_{n-1}[s], c$. The isomorphism $\mathcal{A}_s \to \mathcal{B}_{e,s}$ will map $a_i \mapsto b_i$ and $a_n \mapsto c$. (The duplicate elements will also be mapped correspondingly.)

To illustrate the idea of the construction, we will describe a generic stage of the construction when no requirement requires attention. Suppose that we have defined $\mathcal{A}_s$ and $\mathcal{B}_{e,s}$ with $n = n[s]$. Define $\mathcal{A}_{s+1}$ as follows. Add a new element a_{n+1}, and give a_{n+1} all of the labels that a_n had in $\mathcal{A}_s$. Give a_n and a_{n+1} each a new label unique only to it. For $\mathcal{B}_{e,s+1}$, add a new element b_n and let b_n have the same labels as a_n, and let c have the same labels as a_{n+1}. If we do this at every stage, then $\mathcal{A}$ will have elements $a_1, a_2, \ldots$ and $\mathcal{B}_e$ will have corresponding elements $b_1, b_2, \ldots$ with the same labels, but also an additional element c which does not correspond to any element of $\mathcal{A}$. Thus $\mathcal{A} \not\cong \mathcal{B}_e$. However we have not ensured that if $\mathcal{A} \models \theta_e$ then $\mathcal{B}_e \models \theta_e$.

To ensure that if $\mathcal{A} \models \theta_e$ then $\mathcal{B}_e \models \theta_e$, our strategy (for some i and $\bar{b}$) is to wait for stages at which we think that $\mathcal{B}_{e,s} \models \bigwedge\!\!\bigwedge_j \forall \bar{y}_{i,j}\, \neg\varphi^e_{i,j}(\bar{b}, \bar{y}_{i,j})$. This is evidence that $\mathcal{B}_e \not\models \theta_e$. At stage $s+1$ instead of taking our usual action towards making $\mathcal{B}_e$ different from $\mathcal{A}$, we instead revert some of our previous actions to make $\mathcal{B}_e$ look like $\mathcal{A}$. If $\mathcal{B}_e \not\models \theta_e$ then we will get more and more evidence of this, and we will instead have $\mathcal{B}_e \cong \mathcal{A}$ and thus $\mathcal{A} \not\models \theta_e$. How we do this is the heart of the construction.

Construction.

Stage 0. We begin with $n = n[0] = 1$ and $\mathcal{A}_0$ consisting of a single element a_1, and $\mathcal{B}_{e,0}$ of a single element c. We give a_1 a label and give c the same label.

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Stage $s+1$. Suppose that we have constructed $\mathcal{A}_s$ and $\mathcal{B}_{e,s}$ with $n = n[s]$ and active elements $a_1, \ldots, a_n$. We first check whether any requirement $\mathcal{R}^e_{i,\bar{b}}$ requires attention.

Given $\mathcal{R}^e_{i,\bar{b}}$, let $t = t(\mathcal{R}^e_{i,\bar{b}})$ be the corresponding stage and let $k = k(\mathcal{R}^e_{i,\bar{b}})$. Then we say that $\mathcal{R}^e_{i,\bar{b}}$ *requires attention* if

$$\mathcal{B}_{e,s} \models \bigwedge_{j \leq k} \forall \bar{y}_{i,j} \in \mathcal{B}_{e,t+k} \ \ \neg\varphi^e_{i,j}(\bar{b}, \bar{y}_{i,j}).$$

Note that this quantifier is not restricted to the eth sort of $\mathcal{B}_{e,t+k}$ but is over the elements of all sorts.

If no requirement requires attention, then define $\mathcal{A}_{s+1}$ as follows. Add a new element a_{n+1}, and give a_{n+1} all of the labels that a_n had in $\mathcal{A}_s$. Give each of a_n and a_{n+1} a new label unique only to it. Any duplicates of a_n receive the same labels that a_n did. For $\mathcal{B}_{e,s+1}$, add a new element b_n and let b_n have the same labels as a_n, and let c have the same labels as a_{n+1}. There may be elements of $\mathcal{B}_{e,s}$ which look like duplicates of c; these are the elements which correspond to the duplicates of a_n. We call these *pending duplicates* and we now give them the same labels as b_n, making them duplicates of b_n. Let $\mathcal{R}^e_{i,\bar{b}}$ be the next highest priority requirement; we may assume, by appropriately listing the requirements, that $\bar{b} \in \mathcal{B}_{e,s+1}$. We initialize $\mathcal{R}^e_{i,\bar{b}}$ by setting $t(\mathcal{R}^e_{i,\bar{b}}) = s+1$ and $k(\mathcal{R}^e_{i,\bar{b}}) = 0$.

Otherwise, let $\mathcal{R}^e_{i,\bar{b}}$ be the highest priority requirement which requires attention, and we will act on the requirement as follows. Let $t = t(\mathcal{R}^e_{i,\bar{b}})$ be the corresponding stage, and $n^* = n[t]$ the corresponding value of n. Let $n[s+1] = n[t] = n^*$. In $\mathcal{A}_{s+1}$, declare $a_{n^*+1}[s], \ldots, a_n[s]$ to be duplicates of a_{n^*}; they give up these designations $a_{n^*+1}, \ldots, a_n$. Moreover, any duplicates of these elements also become duplicates of a_{n^*}. Add labels to these elements and a_{n^*} so that they have the same labels, i.e., for each label of one of these elements, give it to each other element. So in $\mathcal{A}_{s+1}$ we have active elements $a_1[s+1], \ldots, a_{n^*}[s+1]$ together with duplicates of these elements. In $\mathcal{B}_{e,s+1}$ add each label of any of $b_{n^*}, \ldots, b_{n-1}, c$ to each of the others. In $\mathcal{B}_{e,s+1}$ there are elements which correspond to the duplicates of $a_n[s+1]$. One should think of these not as duplicates of c, but as duplicates of an element b_n which does not yet exist. We call these elements *pending duplicates*. Then put $b_i[s+1] := b_i[s]$ for $i < n^*$. Thus, at stage $s+1$, the correspondence between $\mathcal{A}_{s+1}$ and $\mathcal{B}_{e,s+1}$ is $a_i \mapsto b_i$ for $i < n^*$ and $a_{n^*} \mapsto c$ (together with the duplicates mapping to corresponding duplicates, with duplicates of a_n mapping to pending duplicates). Each lower priority e-requirement is injured, and increment $k(\mathcal{R}^e_{i,\bar{a}})$ by 1.

End construction.

We must now verify that the construction works. Standard arguments show that for each e, there are two possibilities. First, it might be that some requirement $\mathcal{R}^e_{i,\bar{b}}$ is, after some point, never injured and acts infinitely often. In this case, let $t = t(\mathcal{R}^e_{i,\bar{b}})$ and let $n = n[t]$. Since no higher priority requirement acts, the elements $a_1, \ldots, a_n$ never become duplicates of any other elements, and every time $\mathcal{R}^e_{i,\bar{b}}$ acts every other element which is not already becomes a duplicate of one of these elements. Thus the

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eth sort of $\mathcal{A}$ consists of n elements $a_1, \ldots, a_n$ together with duplicates of them. We write $a_1[\infty], \ldots, a_n[\infty]$ for these elements.

The other possibility is that each requirement acts only finitely many times. If, after some stage s with $n = n[s]$, no requirement with $t(\mathcal{R}^e_{i,\bar{b}}) < s$ ever acts, then after this stage $a_1, \ldots, a_n$ never become duplicates. In the limit, we build infinitely many elements $a_1[\infty], a_2[\infty], a_3[\infty], \ldots$. (Note that the first time we introduce some a_i, this element may become a duplicate; but there will be some point after which this stops happening.) All of the other elements will be duplicates of these.

LEMMA 3.2. *$\mathcal{A}$ has a Π_2 Scott sentence.*

PROOF. Recall that by Lemma 8.17 of [20] the structure $\mathcal{G}_1(\mathcal{F})$ is $\exists$-atomic exactly when $\mathcal{F}$ is discrete. Therefore it suffices to show that for each element of $\mathcal{A}$, there is a finite set of labels on that element such that no other element has the same labels. Each element is in only one sort U_e, so it suffices to only compare elements within the same sort. Within each sort we will argue that each element has a label, that we call the distinguishing label, which it has and no other element has other than duplicates of that element.

Each element $a_i[\infty]$, when it is created, is given a label which is never given to $a_1[\infty], \ldots, a_{i-1}[\infty]$. If $a_{i+1}[\infty]$ exists then, when it is created, $a_i[\infty]$ is given a label that is never given to $a_{i+1}[\infty]$; and no $a_j[\infty]$, $j > i+1$, is ever given that label. The other elements of $\mathcal{A}$ are duplicates of the $a_i[\infty]$ and hence share the same distinguishing label as the element they duplicate. ⊣

LEMMA 3.3. *Fix i and $\bar{b}$. Suppose that there is a stage s after which $\mathcal{R}^e_{i,\bar{b}}$ is never injured. Let $t = t(\mathcal{R}^e_{i,\bar{b}})$. Suppose that*

$$\mathcal{B}_e \models \bigwedge\!\!\!\bigwedge_j \forall \bar{y}_{i,j} \ \neg\varphi_{i,j}(\bar{b}, \bar{y}_{i,j}).$$

Then there are infinitely many stages at which $\mathcal{R}^e_{i,\bar{b}}$ requires attention.

PROOF. Given $s_0 = s$, we will argue that there are infinitely many stages s_u at which $\mathcal{R}^e_{i,\bar{b}}$ requires attention. Given s_u, we have that

$$\mathcal{B}_e \models \bigwedge\!\!\!\bigwedge_j \forall \bar{y}_{i,j} \ \neg\varphi_{i,j}(\bar{b}, \bar{y}_{i,j}),$$

and so there must be some stage $s_{u+1} > s_u$ such that

$$\mathcal{B}_{e,s_{u+1}} \models \bigwedge_{j \leq u+1} \forall \bar{y}_{i,j} \in B_{e,s_u} \ \neg\varphi_{i,j}(\bar{b}, \bar{y}_{i,j}).$$

At this stage s_{u+1}, $\mathcal{R}^e_{i,\bar{b}}$ requires attention. (Since $\mathcal{R}^e_{i,\bar{b}}$ is not injured, no higher priority requirement can require attention at this stage.) ⊣

LEMMA 3.4. *Suppose that $\mathcal{B}_e \models \neg\theta_e$. Then $\mathcal{A} \cong \mathcal{B}_e$ and $\mathcal{A} \models \neg\theta_e$*

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PROOF. Then there is i and $\bar{b} \in \mathcal{B}_e$ such that

$$\mathcal{B}_e \models \bigwedge\!\!\!\bigwedge_j \forall \bar{y}_{i,j} \ \neg\varphi_{i,j}(\bar{b}, \bar{y}_{i,j}).$$

By the previous lemma, for any such i and $\bar{b}$, if there is some stage after which it is not injured, the requirement $\mathcal{R}^e_{i,\bar{b}}$ will require attention infinitely many times. Thus in particular there is some highest priority requirement $\mathcal{R}^e_{i,\bar{b}}$ which requires attention infinitely often.

Let $t = t(\mathcal{R}^e_{i,\bar{b}})$. Let $n = n[t]$. Let s_0 be the stage at which $\mathcal{R}^e_{i,\bar{b}}$ is initiated, and let $s_1, s_2, \ldots$ be the stages at which it requires attention. At each of these stages, $\mathcal{A}_{s_i}$ consists of $a_1, \ldots, a_e$ and duplicates, and $\mathcal{B}_{e,s_i}$ consists of $b_1, \ldots, b_{n-1}, c$ and duplicates; thus the isomorphisms $a_1, \ldots, a_e \mapsto b_1, \ldots, b_{n-1}, c$ at these stages extend to an isomorphism $\mathcal{A} \cong \mathcal{B}_e$. (For this, we also need to know that corresponding elements have the same number of duplicates; this is maintained at each stage of the construction.) Thus $\mathcal{A} \cong \mathcal{B}_e$, and since $\mathcal{B}_e \models \neg\theta_e$ we have $\mathcal{A} \models \neg\theta_e$. ⊣

As a result, if $\mathcal{A} \models \theta_e$ then $\mathcal{B}_e \models \theta_e$. Now we show that if $\mathcal{B}_e \models \theta_e$ then $\mathcal{A} \ncong \mathcal{B}_e$.

LEMMA 3.5. *Suppose that* $\mathcal{B}_e \models \theta_e$. *Then* $\mathcal{A} \ncong \mathcal{B}_e$.

PROOF. We will argue that each requirement, after it is initialized and unless it is injured, requires attention only finitely many times. We argue by induction; given a requirement $\mathcal{R}^e_{i,\bar{b}}$, suppose that the higher priority requirements only require attention finitely many times. Then there is a stage s_0 after which $\mathcal{R}^e_{i,\bar{b}}$ is never injured. Let $t = t(\mathcal{R}^e_{i,\bar{b}})$. Whenever $\mathcal{R}^e_{i,\bar{b}}$ acts for the kth time at stage $s+1$, it is because

$$\mathcal{B}_{e,s} \models \bigwedge_{j \le k} \forall \bar{y}_{i,j} \in \mathcal{B}_{e,t+k} \ \neg\varphi_{i,j}(\bar{b}, \bar{y}_{i,j}).$$

Then if $\mathcal{R}^e_{i,\bar{b}}$ requires attention infinitely many times, we see that $\mathcal{B}_e \models \bigwedge\!\!\!\bigwedge_j \forall \bar{y}_{i,j} \ \neg\varphi_{i,j}(\bar{b}, \bar{y}_{i,j})$ and so $\mathcal{B}_e \models \neg\theta_e$, contradicting the hypotheses of this lemma. Thus each requirement requires attention only finitely many times.

In particular, there are infinitely many "true expansionary stages" and $\mathcal{A} \ncong \mathcal{B}_e$ because the element c in $\mathcal{B}_e$ does not correspond to any element of $\mathcal{A}$. ⊣

We have proved that if $\mathcal{A} \models \theta_e$ then $\mathcal{B}_e \models \theta_e$ and $\mathcal{A} \ncong \mathcal{B}_e$. Thus θ_e cannot be a Scott sentence for $\mathcal{A}$. It follows that, since the θ_e list all computable Π_3 sentences, $\mathcal{A}$ cannot have a computable Π_3 Scott sentence. ⊣

§4. There is no characterization of the structures with a computable Π_2 Scott sentence. Given that we know that some computable structures with Π_2 Scott sentences have no computable Π_2 Scott sentences (or even computable Π_3 Scott sentences), we might like to know how to decide whether or not some particular structure with a computable Π_2 Scott sentence has a computable Π_2 Scott sentence.

Having a Π_2 Scott sentence is equivalent to being $\exists$-atomic, which means that it has a Scott family of existential formulas. This is equivalent (see [18]) to the following statement: for each tuple $\bar{a} \in \mathcal{A}$ there is an existential sentence $\varphi(\bar{x})$ such that for any universal sentence $\psi(\bar{x})$ with $\mathcal{A} \models \psi(\bar{x})$, $\mathcal{A} \models \forall \bar{x} \ \varphi(\bar{x}) \to \psi(\bar{x})$. Thus

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it is not too difficult to tell whether a structure has a Π_2 Scott sentence: Π^0_4 if one counts the quantifiers (though we leave it as an open question to determine if this is best possible).

Since it is not too hard to decide whether a computable structure $\mathcal{A}$ has a Π_2 Scott sentence, we might as well assume that our given structure $\mathcal{A}$ does. One possible attempt to characterize whether $\mathcal{A}$ has a computable Π_2 Scott sentence is to ask whether it is effectively $\exists$-atomic, that is, whether it has a c.e. Scott family of existential formulas. Alvir, Knight, and McCoy [2] proved that if a computable structure $\mathcal{A}$ has a c.e. Scott family of existential formulas, then it has a computable Π_2 Scott sentence. However the converse fails (and while our Theorem 1.3 gives a counterexample it is not too hard to construct a one directly).

In this section we will prove that there is no characterization of the structures with a computable Π_2 Scott sentence. We will use the technique of index set complexity which has been used to great effect in computable structure theory, as suggested in [7], e.g., to show that there is no characterization of when a computable structure has a decidable or automatic presentation [4, 10]. Given a listing of all (partial) computable structures in a given langauge, we consider the set of (indices for) structures with the particular property we are interested in. In this case, we consider the set

$$\{i \mid \mathcal{A}_i \text{ has a computable } \Pi_2 \text{ Scott sentence}\}.$$

This set is naively Π^1_1, because $\mathcal{A}i$ has a computable Π_2 Scott sentence if and only if (a) $\mathcal{A}$ has a Π_2 Scott sentence and (b) there is a computable Π_2 sentence φ such that for all countable structures $\mathcal{B}$, $\mathcal{B} \models \varphi$ if and only if $\mathcal{A} \equiv_2 \mathcal{B}$. Note that given (a), if $\mathcal{A} \equiv_2 \mathcal{B}$ then $\mathcal{A} \cong \mathcal{B}$. While (a) is Π^0_4, (b) is Π^1_1. If there was a better characterization of when a structure had a computable Scott sentence, then this index set would be simpler than Π^1_1, e.g., hyperarithmetic. However we show that the set is Π^1_1-*m*-complete, and hence there is no simpler characterization.

THEOREM 1.3. *With $(\mathcal{A}_i)_{i\in\omega}$ an effective list of (possibly partial) structures in a sufficiently rich language, the set*

$$\{i \mid \mathcal{A}_i \textit{ has a computable } \Pi_2 \textit{ Scott sentence}\}$$

is Π^1_1-m-complete.

PROOF. By general results on the universality of certain languages up to effective bi-interpretability, we can construct examples in whatever language we wish to. We explained above why the class is Π^1_1, and it remains to show that it is Π^1_1-*m*-complete. Let $T \subseteq \omega^{<\omega}$ be a tree. We will define a computable structure $\mathcal{A} = \mathcal{A}_T$ which is $\exists$-atomic and hence has a Π_2 Scott sentence, and such that $\mathcal{A}$ has a computable Π_2 Scott sentence if and only if T is well-founded.

We list all of the computable Π_2 sentences as $(\theta_e)_{e\in\omega}$, where

$$\theta_e = \bigwedge_{i\in\omega} \forall \bar{x}_{e,i} \varphi_{e,i}(\bar{x}_{e,i}),$$

where the $\varphi_{e,i}(\bar{x}_{e,i})$ are computable Σ_1 formulas uniformly in e, i and the arities of each of the $\bar{x}_{e,i}$ are also computable in the indices.

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$\mathcal{A}$ will be $\exists$-atomic and hence have a Π_2 Scott sentence. It will consist of a number of elements each of which is given various c.e. labels. As before, we can take $\mathcal{A}$ to be the "bouquet graph" $\mathcal{G}_1(\mathcal{F})$ of a collection $\mathcal{F}$ of subsets of ω. We introduce three sets of labels. First, we have *sort labels* $(u_e)_{e\in\omega}$ such that only exactly one label holds of each element, dividing the domain into the disjoint sets $U_e = \{x \in A : u_e(x)\}$; we think of these as different sorts of the structure, and call the elements of U_e the *eth sort*. Then we will have two set of labels $(\ell_\sigma)_{\sigma\in\omega^{<\omega}}$ and $(\ell^\dagger_\sigma)_{\sigma\in\omega^{<\omega}}$ we will just call *labels*.

Within U_e we will diagonalize against θ_e, though this diagonalization may only be successful if T has an infinite path. Given a path π through T, we will diagonalize by constructing another countable structure $\mathcal{B}_e = \mathcal{B}_{e,\pi}$ such that if $\mathcal{A} \models \theta_e$ then $\mathcal{B}_e \models \theta_e$ and $\mathcal{A} \not\cong \mathcal{B}_e$. At each stage s we will have an approximation $\mathcal{A}_s$ to $\mathcal{A}$, with $\mathcal{A} = \bigcup_s \mathcal{A}_s$. $\mathcal{B}_e$ will differ from $\mathcal{A}$ only on the eth sort U_e. $\mathcal{B}_e$ will also be built by approximations $\mathcal{B}_e = \bigcup_s \mathcal{B}_{e,s}$ with each $\mathcal{B}_{e,s} \cong \mathcal{A}_s$, though the construction of $\mathcal{B}_e$ is non-computable as it requires knowing a path through T.

When we describe the construction of $\mathcal{A} = \bigcup \mathcal{A}_s$, we will describe the construction of the eth sort. The constructions for the different sorts should be thought of as happening simultaneously.

During the construction certain stages will be *e-expansionary stages* where we get evidence that $\mathcal{A} \models \theta_e$. We use the variable $k[s]$ to keep track of the number of expansionary stages. The elements of $\mathcal{A}$ will all be of the form a_σ for some $\sigma \in T$. We write T_s for the set of σ such that $a_\sigma \in \mathcal{A}_s$, i.e., for the set of σ which correspond to elements of $\mathcal{A}_s$ at stage s. T_s will be a subtree of T, with the property that if one child of $\sigma \in T_s$ is in T_s, then all children of σ are in T_s.

Construction of the eth sort of $\mathcal{A}$.

Stage 0. We begin with $\mathcal{A}_0$ consisting of a single element $a_\varnothing$ with the single label $\ell_\varnothing$. Thus $T_0 = \{\varnothing\}$. Begin with $k = 0$ as we have not yet had any expansionary stages.

Stage $s + 1$. Suppose that we have constructed $\mathcal{A}_s$ with $k = k[s]$. We say that a tuple $\bar{x}$ is k-small it consists of elements $a_\sigma \in \mathcal{A}$ of the eth sort with $\sigma \in \{0, \dots, k\}^{\leq k}$ and elements not of the eth sort but among the first k elements of $\mathcal{A}$. First, we check whether this is an expansionary stage[1]: Check whether, for each $i \leq k$ and each k-small tuple $\bar{x}_i$ we have

$$\mathcal{A}_s \models \varphi_i(\bar{x}_i),$$

where the existential quantifier in $\varphi_i(\bar{x}_i)$ is witnessed by one of the first s witnesses (being careful to use an appropriately dovetailed listing of possible witnesses). If this is the case, then stage $s + 1$ is an expansionary stage, and set $k[s + 1] = k[s] + 1$. Otherwise, set $k[s + 1] = k[s] + 1$.

If stage $s + 1$ is not an expansionary stage, set $\mathcal{A}_{s+1} = \mathcal{A}_s$. If stage $s + 1$ is an expansionary stage:

[1]The conditions to have an expansionary stage, and exactly which elements we should add to $\mathcal{A}_{s+1}$ at an expansionary stage, are quite subtle. We must ensure that we can perform the construction of $\mathcal{B}_e$ given below and prove Lemmas 4.4 and 4.5.

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(1) for each $\sigma \in T_s \cap \{0, \dots, k\}^{\leq k}$ with a child τ in $T \cap \{0, \dots, k\}^{\leq k}$, put the label $\ell_\sigma^\dagger$ on a_σ;
(2) for each $\sigma \in T_s \cap \{0, \dots, k\}^{\leq k}$ which is a leaf of T_s and has a child τ in $T \cap \{0, \dots, k\}^{\leq k}$, and each child $\tau \in T$ of σ, add a new element a_τ with the labels ℓ_ρ for $\rho \preceq \tau$. Let T_{s+1} be T_s together with all of these new τ.

End construction.

What $\mathcal{A}$ looks like depends on whether there are finitely many or infinitely many expansionary stages.

(1) If there are finitely many expansionary stages, say k, then $\mathcal{A}$ has finitely many elements a_σ with each $|\sigma| \leq k$. Write T_∞ for the set of such σ; $T_\infty = \bigcup T_s$ is a subtree of T. Each a_σ has the labels ℓ_ρ for $\rho \preceq \sigma$; and if σ is not a leaf of T_∞, then a_σ also has the label $\ell_\sigma^\dagger$.
(2) If there are infinitely many expansionary stages, then $\mathcal{A}$ has elements a_σ for $\sigma \in T$, and each a_σ has the labels ℓ_ρ for $\rho \preceq \sigma$ and the label $\ell_\sigma^\dagger$. (We have $T_\infty = \bigcup T_s = T$.)

Thus it is easy to see that $\mathcal{A}$ has a Π_2 Scott sentence.

LEMMA 4.1. *$\mathcal{A}$ has a Π_2 Scott sentence.*

PROOF. For each a_σ, either a_σ has the label $\ell_\sigma^\dagger$ and no other element has this label, or σ is a leaf of T_∞ and a_σ is the unique element with the label ℓ_σ. ⊣

LEMMA 4.2. *Suppose that T is well-founded. Then $\mathcal{A}$ has a computable Π_2 Scott sentence.*

PROOF. Note that $T_\infty = \bigcup T_s$ is a c.e. set. Moreover, the set of non-leaves of T_∞ is also a c.e. set. Consider the computable Π_2 sentence which says that:

(1) if x has the label ℓ_σ then it also has all of the labels ℓ_ρ for $\rho \prec \sigma$;
(2) if x has the label $\ell_\sigma^\dagger$ then it has the label ℓ_σ and does not have any label ℓ_τ for $\tau \succ \sigma$;
(3) no x has labels ℓ_{σ_1} and ℓ_{σ_2} for incompatible σ_1 and σ_2;
(4) if x has the label ℓ_σ then $\sigma \in T_\infty$;
(5) every x has the label $\ell_\varnothing$ and if x has the label ℓ_σ and σ is not a leaf of T_∞ then x has either the label $\ell_\sigma^\dagger$ or some label ℓ_τ where τ is a child of σ on T;
(6) if x has the label $\ell_\sigma^\dagger$, then σ is a non-leaf of T_∞;
(7) for each $\sigma \in T_\infty$, there is some x with the label ℓ_σ;
(8) for each $\sigma \in T_\infty$ which is not a leaf, there is some x with the label $\ell_\sigma^\dagger$;
(9) for every distinct x and y, either there are some incompatible σ, τ such that one element has the label ℓ_σ and the other has the label ℓ_τ, or there is some σ and a child σ^* of σ such that one element has the label $\ell_\sigma^\dagger$ and the other has the label ℓ_{σ^*}.

Since T is well-founded, T_∞ must also be well-founded. We can check that this sentence is true of $\mathcal{A}$.

Suppose that a structure $\mathcal{C}$ satisfies this sentence. For each non-leaf σ of T_∞, by (8) there is an element c_σ with the label $\ell_\sigma^\dagger$ (and thus, by (1) and (2) also the labels ℓ_ρ for $\rho \preceq \sigma$, and by (2) and (3) these are the only labels). By (9) and (2) no other element of $\mathcal{C}$ has the label $\ell_\sigma^\dagger$.

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For each leaf σ of T_∞ by (7) there is an element c_σ with the label ℓ_σ. By (1) it also has all of the labels ℓ_ρ for $\rho \prec \sigma$, and by (2), (3), and (6) it has no other labels. Thus we have shown that $\mathcal{A}$ embeds as a substructure of $\mathcal{C}$.

Suppose that d is some other element of $\mathcal{C}$, not included among the c_σ above. By (5) d has the label $\ell_\varnothing$. Let $\sigma_0 = \varnothing$. We argue inductively as follows. Given σ_i such that d has the label ℓ_{σ_i}, by (5) either (a) σ_i is a leaf of T_∞, (b) d has the label k_{σ_i}, or (c) d has some label $\ell_{\sigma_{i+1}}$ for some child $\sigma_{i+1} \in T_\infty$ of σ_i. Since T_∞ is well-founded, we can repeat this process to build a sequence of children $\sigma_0 = \varnothing, \sigma_1, \ldots, \sigma_m$ in T_∞ with either (a) σ_m is a leaf of T_∞, or (b) d has the label k_{σ_m}. Recall that there is already an element c_{σ_m} with the label k_{σ_m}, and by (2) and (9) there are no other such elements. Thus σ_m is a leaf of T_∞ and d has the label ℓ_{σ_m}. But we already have an element c_{σ_m} with the label ℓ_{σ_m}, and by (9) we cannot have two such elements. So no such d exists, that is, $\mathcal{C} \cong \mathcal{A}$. ⊣

LEMMA 4.3. *Suppose that $\mathcal{A} \models \theta_e$. Then there are infinitely many expansionary stages.*

PROOF. Let $s_1 < s_2 < s_3 < \cdots < s_k$ be expansionary stages. We must argue that there will later be another expansionary stage, the $k+1$st. For each $i \leq k$ and k-small tuple $\bar{x}_i$ we have

$$\mathcal{A} \models \varphi_i(\bar{x}_i).$$

Since φ_i is existential, and $\mathcal{A} = \bigcup_s \mathcal{A}_s$, there must be some stage $s_{k+1} > s_k$ such that for each such i and $\bar{x}_i$ we have

$$\mathcal{A}_{s_{k+1}} \models \varphi_i(\bar{x}_i)$$

with the existential quantifier in φ_i witnessed at this stage. This stage s_{k+1} is expansionary. ⊣

Suppose that T has an infinite path π. Let $\mathcal{B}_e$ be $\mathcal{A}$ together with another element c of the eth sort satisfying ℓ_ρ for each $\rho \prec \pi$. Clearly $\mathcal{B}_e \ncong \mathcal{A}$. However we also want to know that if $\mathcal{A} \models \theta_e$, then $\mathcal{B}_e \models \theta_e$. To see this, we will show that $\mathcal{B}_e$ is also a union $\mathcal{B}_e = \bigcup \mathcal{B}_{e,s}$ and that $\mathcal{B}_{e,s} \cong \mathcal{A}_s$. Thus one can think of $\mathcal{A}$ and $\mathcal{B}_e$ as direct limits of direct systems of the same structures but with different embeddings.

$$\mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \cdots .$$

However the definition of $\mathcal{B}_{e,s}$ depends on the path π and thus is non-computable.

At stage s, let π_s be the longest initial segment of π which on T_s. Then π_s is a leaf of T_s—this is because T_s has the property that if there is any child of π_s in T_s, then all children of π_s from T are in T_s. $\mathcal{B}_{e,s}$ will have domain consisting of elements $\{b_\sigma \mid \sigma \in T_s, \sigma \neq \pi_s\} \cup \{c\}$. We put the same labels on b_σ as on a_σ, and the same labels on c as on a_{π_s}. Note that since π_s is on the infinite path π, there cannot be a label k_{π_s} on c and a_{π_s}. We must check that $\mathcal{B}_e$ is in fact the union of these $\mathcal{B}_{e,s}$.

LEMMA 4.4. *If there are infinitely many expansionary stages, then* $\lim_s \pi_s = \pi$, *and so* $\mathcal{B}_e = \bigcup_s \mathcal{B}_{e,s}$.

PROOF. Suppose not, so that for some t for all $s \geq t$ we have $\pi_s = \pi_t$. But then there is k such that $\pi \upharpoonright_{|\pi_t|+1} \in \{0, \ldots, k\}^{\leq k}$, and at some stage $s \geq t$ there is a k'th

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expansionary stage for some $k' \geq k$. At this stage, we add the children of π_t on T to T_{s+1}, so that π_{t+1} is one of these children and strictly extends π_t. ⊣

Now we can show that $\mathcal{B}_e \models \theta_e$.

LEMMA 4.5. *Suppose that T has an infinite path. If there are infinitely many expansionary stages, then $\mathcal{B}_e \models \theta_e$.*

PROOF. Fix i and $\bar{y}_i \in \mathcal{B}_e$. Fix some stage s sufficiently large such that (a) by stage s there have been k expansionary stages, (b) for each b_σ in $\bar{y}_i$ of the eth sort, $\sigma \in \{0, \dots, k\}^{\leq k}$ and $|\sigma| < |\pi_s|$, and (c) each element of $\bar{y}_i$ not in the eth sort is among the first k elements of $\mathcal{A}$ (recalling that $\mathcal{B}_e$ is the same as $\mathcal{A}$ on these other sorts). Let $\bar{x}_i$ be $\bar{y}_i$ except that each b_σ in $\bar{y}_i$ is replaced by a_σ, and c is replaced by a_{π_s}, and elements not of the e sort are kept the same. Then at some stage $s' \geq s$ there is for the first time a k'th expansionary stage with $\pi_s \in \{0, \dots, k'\}^{\leq k'}$. Since $\bar{x}_i$ is k-small we have $\mathcal{A}_s \models \varphi_i(\bar{x}_i)$. Since $\mathcal{B}_{e,s} \cong \mathcal{A}_s$ via an isomorphism mapping $b_\sigma \mapsto a_\sigma$ and $c \mapsto a_{\pi_s}$ (and hence $\bar{y}_i \mapsto \bar{x}_i$), we have $\mathcal{B}_{e,s} \models \varphi_i(\bar{y}_i)$. Since φ_i is Σ_1, $\mathcal{B}_e \models \varphi_i(\bar{y}_i)$. Thus, since this is true for all i and $\bar{y}_i \in \mathcal{B}_e$, $\mathcal{B}_e \models \theta_e$. ⊣

We have shown in Lemma 4.1 that $\mathcal{A}$ has a Π_2 Scott sentence. If T is well-founded, then Lemma 4.2 says that $\mathcal{A}$ has a computable Π_2 Scott sentence. Otherwise, if T has an infinite path, we argue that $\mathcal{A}$ has no computable Π_2 Scott sentence. If it did, say θ_e, then we have $\mathcal{A} \models \theta_e$. Then by Lemma 4.3 there are infinitely many e-expansionary stages. We construct the structure $\mathcal{B}_{e,s} \not\cong \mathcal{A}$ using a path through T, and by Lemma 4.5 we have that $\mathcal{B}_e \models \theta_e$. Thus in fact θ_e cannot have been a Scott sentence for $\mathcal{A}$. Thus if T has an infinite path, we argue that $\mathcal{A}$ has no computable Π_2 Scott sentence. ⊣

§5. Corollaries and other commentary.

5.1. Complexity of Scott families. In [2] it was shown that if a computable structure has a c.e. Scott family of computable Σ_α formulas then the structure has a computable $\Pi_{\alpha+1}$ Scott sentence. (Note that it is easy to construct uncountably many structures with a c.e. Scott family of Σ_1 formulas, so it is necessary here that the structure is computable.) On the other hand, in the same paper it was shown that if a structure has a computable $\Pi_{\alpha+1}$ Scott sentence then it has a Scott family of computable Σ_α formulas, but as the following theorem shows the Scott family is not necessarily c.e.

COROLLARY 5.1. *There is a computable structure with a computable Π_2 Scott sentence but with no c.e. Scott family of computable Σ_1 formulas.*

PROOF. Any computable structure with a c.e. Scott family of computable Σ_1 formulas has a computable Π_2 Scott sentence. However the index set of computable structures with a c.e. Scott family of computable Σ_1 formulas is Σ^0_5 while the index set of computable structures with a computable Π_2 Scott sentence is Π^1_1-m-complete. Thus there must be a computable structure with a computable Π_2 Scott sentence but with no c.e. Scott family of computable Σ_1 formulas. ⊣

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Despite this, one can ask how bad the Scott family for such a structure must be. Note that by Theorem 1.2 these give necessary but not sufficient conditions to have a computable Π_2 Scott sentence.

PROPOSITION 5.2. *Suppose A is computable and has a computable $\Pi_{\alpha+1}$ Scott sentence. Then A has a c.e. Scott family of computable $\Pi_{\alpha+1}$ formulas.*

PROOF. We will describe a uniform procedure that given $\bar{a} \in A$ produces a computable $\Pi_{\alpha+1}$ formula defining the automorphism orbit of $\bar{a}$. We know that each of these automorphism orbits is definable by a computable Σ_α formula, and so it is enough to find a formula that supports the computable Π_α type of $\bar{a}$. It is Π_α to list all the computable Π_α formulas true of $\bar{a}$. Taking the conjunction of all such formulas we get a formula that supports the type and thus defines the automorphism orbit of $\bar{a}$. This is a conjunction of a Π_α set of Π_α formulas, which is also a conjunction of a $\Sigma_{\alpha+1}$ set of $\Pi_{\alpha+1}$ formulas, and so is equivalent to a computable $\Pi_{\alpha+1}$ formula (see [3, Proposition 7.12]). Given $\bar{a}$, we can find (an index for) the Π_α set of indices for the Π_α formulas we want to take the conjunction of, and so checking that Proposition 7.12 of [3] is uniform we can uniformly in $\bar{a}$ find an index for the computable $\Pi_{\alpha+1}$ formula equivalent to the conjunction. Thus we get a c.e. Scott family of computable $\Pi_{\alpha+1}$ formulas. ⊣

PROPOSITION 5.3. *Suppose A is a computable structure with a computable $\Pi_{\alpha+1}$ Scott sentence. Then A has a $\Sigma^0_{\alpha+2}$ Scott family of computable Σ_α formulas.*

PROOF. Since $\mathcal{A}$ has a computable $\Pi_{\alpha+1}$ Scott sentence, it has a Scott family of computable Σ_α formulas, and so for each tuple $\bar{a}$ there is a computable Σ_α formula defining the automorphism orbit of $\bar{a}$. Now consider the set of all pairs $(\bar{a}, \psi)$ where $\bar{a}$ is a tuple from $\mathcal{A}$ and ψ is a Σ_α formula such that

(1) $\mathcal{A} \models \psi(\bar{a})$;
(2) for all $\bar{b}$, if $\mathcal{A} \models \psi(\bar{b})$ then for all computable Σ_α formulas $\theta(\bar{x})$, if $\mathcal{A} \models \theta(\bar{a})$ then $\mathcal{A} \models \theta(\bar{b})$.

(1) is Σ^0_α while (2) is $\Pi^0_{\alpha+1}$. Thus this set is $\Pi^0_{\alpha+1}$. The projection of this set onto the second coordinate is a $\Sigma^0_{\alpha+2}$ Scott family. ⊣

QUESTION 5.4. *Let A be a computable structure with a computable $\Pi_{\alpha+1}$ Scott sentence. Are the bounds on the Scott family obtained in Propositions 5.2 and 5.3 best possible*?

5.2. Pseudo-Scott sentences. Recall that a pseudo-Scott sentence for a computable structure $\mathcal{A}$ is a computable sentence φ such that for all *computable* structure $\mathcal{B}$,

$$\mathcal{B} \models \varphi \iff \mathcal{A} \cong \mathcal{B}.$$

There are several examples of structures with a computable pseudo-Scott sentence of a certain complexity but no computable Scott sentence of that complexity [12, 23]. As a corollary we get a much stronger example.

COROLLARY 5.5. *There is a computable structure $\mathcal{A}$ with a Π_2 Scott sentence but no computable Π_2 Scott sentence, but with a computable Π_2 sentence φ such that, for*

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all hyperarithmetic $\mathcal{B}$,

$$\mathcal{B} \models \varphi \iff \mathcal{A} \cong \mathcal{B}.$$

PROOF. Consider the index set of computable structures $\mathcal{A}$ with a Π_2 Scott sentence and such that there is a computable Π^0_2 sentence φ such that for all hyperarithmetic structures $\mathcal{B}$, $\mathcal{B} \models \varphi$ if and only if $\mathcal{A} \equiv_2 \mathcal{B}$ (so that $\mathcal{A} \cong \mathcal{B}$). Since we are quantifying universally over hyperarthimetic structures, this index set is Σ^1_1. Moreover, it is a superset of the Π^1_1-*m*-complete set of computable structures with a computable Π_2 sentence. Thus it must be a proper superset, proving the corollary. ⊣

5.3. Working in $\mathrm{Mod}(\mathcal{L})$**.** Sometimes by working in $Mod(\mathcal{L})$ rather than with index sets we get a stronger theorem. In this case, it depends on how we relativize the statement and whether we work with boldface or lightface Borel classes.

PROPOSITION 5.6. *The set of structures with a computable* Π_2 *Scott sentence is a Borel—in fact (boldface)* $\mathbf{\Sigma}^0_3$*—set in* $Mod(\mathcal{L})$*. It is (lightface)* Π^1_1 *but, in a sufficiently rich language, not (lightface)* Σ^1_1*.*

PROOF. A similar argument as above (in the first paragraph of Theorem 1.3) shows that it is (lightface) Π^1_1. It is also (boldface) $\mathbf{\Sigma}^1_1$: listing out all of the countably many structures with a computable Π_2 Scott sentence, together with their Scott sentences, a structure has a computable Π_2 Scott sentence if and only if it is isomorphic to one of these. This is (boldface) $\mathbf{\Sigma}^0_3$: given $\mathcal{A}$, we ask whether there exists a structure in this list such that $\mathcal{A}$ satisfies the corresponding Π^0_2 Scott sentence. If this set was (lightface) Σ^1_1, then the index set of computable such structures would be Σ^1_1; but we know that this index set is Π^1_1-*m*-complete. ⊣

COROLLARY 5.7. *The set of structures* $\mathcal{A}$ *with an* $\mathcal{A}$*-computable* Π_2 *Scott sentence is* Π^1_1*-Wadge-complete set in* $Mod(\mathcal{L})$*.*

PROOF. The proof is the same as that of Theorem 1.3, working relative to the tree T. ⊣

5.4. Jump inversion. We can use the method of jump inversion or Marker extensions to generalize our results from Π_2 sentences to Π_n sentences for arbitrary n. We get the following corollaries of our main theorem.

COROLLARY 5.8. *For each computable ordinal* α *there is a computable structure with a* $\Pi_{\alpha+2}$ *Scott sentence but with no computable* $\Sigma_{\alpha+4}$ *Scott sentence.*

COROLLARY 5.9. *With* $(\mathcal{A}_i)_{i\in\omega}$ *an effective list of (possibly partial) structures in a sufficiently rich language, for each computable ordinal* α*, the set*

$$\{i \mid \mathcal{A}_i \text{ has a computable } \Pi_{\alpha+2} \text{ Scott sentence}\}$$

is Π^1_1*-m-complete.*

Note that Marker extensions are only additive, so that we do not answer whether every computable structure with a Π_n Scott sentence has a computable Σ_{2n} Scott sentence for $n \geq 3$.

The particular flavour of jump inversion that we use is due to Goncharov, Harizanov, Knight, McCoy, Miller, and Solomon [6] and also given in Chapter X.3

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of [16]. In Chapter X.3 of [16] Montalbán shows that given a computable ordinal α and a structure $\mathcal{A}$ there is a structure $\Phi_\alpha(\mathcal{A})$ such that $\mathcal{A}$ is effectively bi-interpretable with the α-canonical structural jump of the image. There is a uniform effective construction from $\mathcal{A}$ of $\Phi_\alpha(\mathcal{A})$ as discussed in [5]. We leave to [19] the definitions and properties of effective bi-interpretations and jumps of structures (see also [17]).

If two structures are effectively bi-interpretable, then one has a Π_β Scott sentence if and only if the other does. Thus $\mathcal{A}$ has a Π_β Scott sentence if and only if the α-canonical structural jump of $\Phi_\alpha(\mathcal{A})$ does. And the α-canonical structural jump of a structure has a Π_β Scott sentence if and only if the original structure has a $\Pi_{\alpha+\beta}$ Scott sentence. The same is true for computable Scott sentences. Thus we have the following.

PROPOSITION 5.10. *Let $\mathcal{A}$ be a computable structure and let α and β be computable ordinals.*

- *$\mathcal{A}$ has a Π_β Scott sentence if and only if $\Phi_\alpha(\mathcal{A})$ has a $\Pi_{\alpha+\beta}$ Scott sentence.*
- *$\mathcal{A}$ has a computable Π_β Scott sentence if and only if $\Phi_\alpha(\mathcal{A})$ has a computable $\Pi_{\alpha+\beta}$ Scott sentence.*

The same is true for Σ Scott sentences.

From this we can prove the corollaries above.

PROOF OF COROLLARY 5.8. Let $\mathcal{A}$ be a computable structure with a Π_2 Scott sentence but no computable Σ_4 Scott sentence. Then $\Phi_\alpha(\mathcal{A})$ is a computable structure with a $\Pi_{\alpha+2}$ Scott sentence but no computable $\Sigma_{\alpha+4}$ Scott sentence. ⊣

PROOF OF COROLLARY 5.9. Given a Π^1_1 set X, we can construct for $i \in X$ a computable structure $\mathcal{A}_i$ such that $i \in X$ if and only if $\mathcal{A}_i$ has a computable Π_2 Scott sentence. Then $i \in X$ if and only if $\Phi_\alpha(\mathcal{A}_i)$ has a computable $\Pi_{\alpha+2}$ Scott sentence. ⊣

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REFERENCES

[1] R. ALVIR, N. GREENBERG, M. HARRISON-TRAINOR, and D. TURETSKY, *Scott complexity of countable structures*. ***Journal of Symbolic Logic***, vol. 86 (2021), no. 4, pp. 1706–1720.

[2] R. ALVIR, J. F. KNIGHT, and C. MCCOY, *Complexity of Scott sentences*. ***Fundamenta Mathematicae***, vol. 251 (2020), no. 2, pp. 109–129.

[3] C. J. ASH and J. KNIGHT, ***Computable Structures and the Hyperarithmetical Hierarchy***, volume 144 of Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 2000.

[4] N. BAZHENOV, M. HARRISON-TRAINOR, I. KALIMULLIN, A. MELNIKOV, and K. M. NG, *Automatic and polynomial-time algebraic structures*. ***Journal of Symbolic Logic***, vol. 84 (2019), no. 4, pp. 1630–1669.

[5] R. CHEN, D. GONZALEZ, and M. HARRISON-TRAINOR, *Optimal syntactic definitions of back-and-forth types*, preprint.

[6] S. GONCHAROV, V. HARIZANOV, J. KNIGHT, C. MCCOY, R. MILLER, and R. SOLOMON, *Enumerations in computable structure theory*. ***Annals of Pure and Applied Logic***, vol. 136 (2005), no. 3, pp. 219–246.

[7] S. GONCHAROV and J. A. KNIGHT, *Computable structure and non-structure theorems*. ***Algebra and Logic***, vol. 41 (2002), pp. 351–373.

https://doi.org/10.1017/jsl.2025.10143 Published online by Cambridge University Press

[8] J. Harrison, *Recursive pseudo-well-orderings*. ***Transactions of the American Mathematical Society***, vol. 131 (1968), pp. 526–543.

[9] M. Harrison-Trainor, *Scott ranks of models of a theory*. ***Advances in Mathematics***, vol. 330 (2018), pp. 109–147.

[10] ———, *There is no classification of the decidably presentable structures*. ***Journal of Mathematical Logic***, vol. 18 (2018), no. 2, p. 1850010.

[11] M. Harrison-Trainor, G. Igusa, and J. F. Knight, *Some new computable structures of high rank*. ***Proceedings of the American Mathematical Society***, vol. 146 (2018), no. 7, pp. 3097–3109.

[12] M.-C. Ho, *Describing groups*. ***Proceedings of the American Mathematical Society***, vol. 145 (2017), no. 5, pp. 2223–2239.

[13] J. F. Knight and J. Millar, *Computable structures of rank ω_1*. ***Journal of Mathematical Logic***, vol. 10 (2010), nos. 1–2, pp. 31–43.

[14] J. F. Knight, K. Lange, and C. McCoy, *Computable $\prod_2$ scott sentences*, preprint.

[15] M. Makkai, *An example concerning Scott heights*. ***The Journal of Symbolic Logic***, vol. 46 (1981), no. 2, pp. 301–318.

[16] A. Montalbán, *A robuster Scott rank*. ***Proceedings of the American Mathematical Society***, vol. 143 (2015), no. 12, pp. 5427–5436.

[17] ———, *Notes on the jump of a structure*, ***Mathematical Theory and Computational Practice***, (K. Ambos-Spies, B. Löwe, and W. Merkle, editors), Springer, Berlin, 2009, pp. 372–378.

[18] ———, *Effectively existentially-atomic structures*, ***Computability and Complexity*** (A. Day, M. Fellows, N. Greenberg, B. Khoussainov, A. Melnikov, and F. Rosamond, editors), volume 10010 of Lecture Notes in Computer Science, Springer, Cham, 2017, pp. 221–237.

[19] ———, *Computable structure theory: Beyond the arithmetic*. In preparation.

[20] ———, ***Computable Structure Theory: Within the Arithmetic***, Perspectives in Logic, Cambridge University Press, Cambridge, 2021.

[21] M. Morley, *The number of countable models*. ***The Journal of Symbolic Logic***, vol. 35 (1970), pp. 14–18.

[22] M. Nadel, *Scott sentences and admissible sets*. ***Annals of Mathematical Logic***, vol. 7 (1974), pp. 267–294.

[23] S. B. Quinn, *Scott sentences for equivalence structures*. ***Archive for Mathematical Logic***, vol. 59 (2020), nos. 3–4, pp. 453–460.

[24] D. Scott, *Logic with denumerably long formulas and finite strings of quantifiers*, ***Theory of Models Proceedings of the 1963 International Symposium at Berkeley)*** (J. W. Addison, L. Henkin, and A. Tarski, editors), North-Holland, Amsterdam, 1965, pp. 329–341.

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