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Epsilon-hypercyclic operators

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Abstract. Let X be a separable infinite-dimensional Banach space, and T a bounded linear operator on X; T is hypercyclic if there is a vector x in X with dense orbit under the action of T. For a fixed $\varepsilon \in (0, 1)$, we say that T is ε-hypercyclic if there exists a vector x in X such that for every non-zero vector $y \in X$ there exists an integer n with $\|T^n x - y\| \leq \varepsilon \|y\|$. The main result of this paper is a construction of a bounded linear operator T on the Banach space ℓ_1 which is ε-hypercyclic without being hypercyclic. This answers a question from V. Müller [Three problems, Mini-Workshop: Hypercyclicity and linear chaos, organized by T. Bermudez, G. Godefroy, K.-G. Grosse-Erdmann and A. Peris. *Oberwolfach Rep.* **3** (2006), 2227–2276].

1. *Introduction*

The aim of this paper is to study some (variations of) density properties of orbits of bounded linear operators acting on a real or complex separable Banach space X. Using a functional analysis terminology, an operator $T \in \mathcal{B}(X)$ is said to be *hypercyclic* if there exists a vector $x \in X$ such that the orbit $\mathcal{O}rb(x, T) = \{T^n x \mid n \geq 0\}$ of x under the action of T is dense in X. A vector x with dense orbit is called a *hypercyclic* vector for T.

The first examples of Banach and Hilbert space hypercyclic operators date back to the 1970s [**11**], and there is now an important literature on hypercyclicity properties and the dynamics of bounded linear operators. We refer the reader to the recent book [**2**] for more on this topic. It is natural in this context to investigate which properties of the orbit of a vector, weaker than denseness, imply either that the orbit itself is in fact dense, or that the operator is hypercyclic (i.e. some other orbit is dense in X). Let us mention here some of the results in this direction.

- If the orbit $\mathcal{O}rb(x, T)$ is somewhere dense in X, then it is dense in X [**4**]. This implies in particular that if the union of finitely many orbits $\mathcal{O}rb(x_1, T)$, $\mathcal{O}rb(x_2, T)$, $\ldots$, $\mathcal{O}rb(x_n, T)$ is dense in X, then one of these orbits must already be dense. This result was proved directly in [**6**] and [**10**].

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- Suppose that for some positive number d the orbit of $x \in X$ meets every open ball $B(y, d)$ of radius d. Then $\mathcal{O}rb(x, T)$ is not necessarily dense in X, but T must be hypercyclic [**7**].
- If x is a frequently hypercyclic vector for T, then $T \oplus T$ must be hypercyclic on $X \oplus X$ [**8**].

Recall (see [**1**] for details) that x is a frequently hypercyclic vector for T if for every non-empty open subset U of X, the set of positive integers n such that $T^n x \in U$ has positive lower density; in other words, for every non-empty open subset U of X, there exists a sequence $(n_k)_{k\geq 0}$ with $n_k = O(k)$ such that $T^{n_k} x \in U$.

On the other hand, some conditions on the orbit, which may look strong enough at first sight, do not imply that the operator is hypercyclic. For instance, we have the following.

- There exist operators which are weakly hypercyclic, i.e. for which there exists a vector x whose orbit is weakly dense in X, but still are not hypercyclic: examples of weighted shifts having this property are given in [**5**].
- For every $\varepsilon > 0$, there exists a bounded operator on the space $\ell_1(\mathbb{N})$ which has the following property: there exists a vector $x \in \ell_1(\mathbb{N})$ such that for every non-empty open subset U of $\ell_1(\mathbb{N})$, there exists a sequence $(n_k)_{k\geq 0}$ with $n_k = O(k^{1+\varepsilon})$ such that $T^{n_k} x \in U$, but $T \oplus T$ is not hypercyclic [**3**]. This shows that the result of [**8**] that every frequently hypercyclic operator on a Banach space satisfies the Hypercyclicity Criterion is in a sense optimal.

We investigate in this paper a weaker version of Feldman's result [**7**] already mentioned above; this result states that if given a positive ε there exists a vector x such that for every $y \in X$ one has $\|T^n x - y\| \leq \varepsilon$ for some integer n, then T is hypercyclic. Let us now consider the following notion.

Definition. Let ε be a number in $(0, 1)$. If x is a vector of X, we say that x is *ε-hypercyclic* for $T \in \mathcal{B}(X)$ if for every non-zero vector $y \in X$ there exists an integer n such that $\|T^n x - y\| \leq \varepsilon \|y\|$. The operator T is *ε-hypercyclic* if it admits an ε-hypercyclic vector.

This is in a sense a 'scaled' version of the ε-density considered in Feldman's work. It is obviously weaker than Feldman's condition, and in a sense more natural in this context. The following question was proposed in [**9**].

Question 1.1. Suppose that T is a bounded operator on X which admits for some $\varepsilon \in (0, 1)$ an ε-hypercyclic vector. Is it true that T is hypercyclic?

The restriction $\varepsilon \in (0, 1)$ comes from the fact the zero vector is trivially 1-hypercyclic for any operator T.

The main result of this paper gives a negative answer to Question 1.1.

THEOREM 1.2. *For every $\varepsilon \in (0, 1)$ there exists an ε-hypercyclic operator on the space $\ell_1(\mathbb{N})$ which is not hypercyclic.*

Nevertheless, we have the following theorem.

THEOREM 1.3. *If T is a bounded operator on X which is ε-hypercyclic for every $\varepsilon > 0$, then T must be hypercyclic.*

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Theorems 1.2 and 1.3 are proved in the next sections. The following question is still open.

Question 1.4. Let $\varepsilon \in (0, 1)$ and suppose that $T \in \mathcal{B}(H)$ is an ε-hypercyclic operator acting on a Hilbert space H. Must T be hypercyclic?

2. *Proof of Theorem 1.2*

2.1. *Outline of the proof of Theorem 1.2.* Fix $\varepsilon \in (0, 1)$ and a positive integer a such that $\varepsilon > 2^{-a+1}$. Let X be the space $\ell_1(\mathbb{N})$ endowed with the canonical basis $(e_n)_{n\geq 0}$. Our operator T will act on the ℓ_1-direct sum $Y = \bigoplus_{i=0}^{\infty} X$ of countably many copies of $\ell_1(\mathbb{N})$.

Let $(y^{(k)})_{k\geq 1}$ be a sequence of vectors of Y which has the following properties.

(i) The set $\{y^{(k)} \mid k \geq 1\}$ is dense in Y.

(ii) Each $y^{(k)} \in Y$, $k \geq 1$, can be written as a sequence $y^{(k)} = (y_0^{(k)}, y_1^{(k)}, \ldots, y_{k-1}^{(k)}, 0, \ldots)$, where each $y_j^{(k)}$ is a vector of $X = \ell_1(\mathbb{N})$ which is in the linear span of the vectors e_i, $0 \leq i \leq k-1$.

(iii) $2^{-k} \leq \|y_j^{(k)}\|$ for every $j = 0, \ldots, k-1$, and $\|y^{(k)}\| \leq (2^k/(1+2^{-a}))$.

For each $k \geq 1$ and each $j \leq k-1$, define $z_j^{(k)} = y_j^{(k)} + 2^{-a}\|y_j^{(k)}\| e_{k^2+j}$: it is a perturbation of the vector $y_j^{(k)}$ obtained by adding to it a (not too small) multiple of the basis vector e_{k^2+j}, which is far away from the support of $y_j^{(k)}$. We have $2^{-k} \leq \|z_j^{(k)}\|$ for each $j \leq k-1$. We then define $z^{(k)} \in Y$ for $k \geq 1$ by $z^{(k)} = (z_0^{(k)}, \ldots, z_{k-1}^{(k)}, 0, \ldots)$. Clearly $\|z^{(k)}\| \leq 2^k$.

Set $n_0 = n_0' = 0$. Our goal is to construct by induction a sequence $(S_j)_{j\geq 1}$ of bounded operators on X and two increasing sequences of positive integers $(n_k)_{k\geq 1}$ and $(n_k')_{k\geq 1}$ such that $n_{k-1}' \leq n_{k-1}' + n_{k-1} \leq n_k < n_k + k < n_k'$ for every $k \geq 1$ and the six following properties hold true.

(a) Each operator S_j is bounded and invertible with $\|S_j^{-1}\| \leq 2$.

(b) $S_j e_0 = e_0$ for every $j \geq 1$.

(c) $\|S_j S_{j-1} \ldots S_1\| \leq 2^{a+1}$ for every $j \geq 1$.

(d) $S_{n_k'} \ldots S_2 S_1 = I$ (the identity operator) for every $k \geq 1$.

(e) $S_j = I$ for every j such that $n_k - n_{k-1} < j \leq n_k + k$ for some $k \geq 1$.

(f) $\|S_{n_k} \ldots S_{j+1} z_j^{(k)}\| \leq 2^{-2k-a}$ for every $k \geq 1$ and every $j = 0, \ldots, k-1$.

Suppose that $(n_k)_{k\geq 1}$, $(n_k')_{k\geq 1}$ and $(S_j)_{j\geq 1}$ have been constructed so as to satisfy properties (a) to (f). Consider on Y the operator T which is the backward shift with operator-weights S_j^{-1}: for any sequence $(v_j)_{j\geq 0}$ of Y,

$$T(v_0, v_1, \ldots) = (S_1^{-1} v_1, S_2^{-1} v_2, \ldots).$$

Clearly T is bounded on Y with $\|T\| \leq 2$ by (a). For any $n \geq 1$ we have

$$T^n(v_0, v_1, \ldots) = (S_1^{-1} \ldots S_n^{-1} v_n, S_2^{-1} \ldots S_{n+1}^{-1} v_{n+1}, \ldots, S_{j+1}^{-1} \ldots S_{n+j}^{-1} v_{n+j}, \ldots).$$

For $k \geq 1$ define $x^{(k)} \in Y$ by

$$x^{(k)} = (\underbrace{0, \ldots, 0}_{n_k}, S_{n_k} \ldots S_1 z_0^{(k)}, S_{n_k} \ldots S_2 z_1^{(k)}, \ldots, S_{n_k} \ldots S_k z_{k-1}^{(k)}, 0, \ldots).$$

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By (f), we have $\|x^{(k)}\| \le k2^{-2k-a} \le 2^{-k-a}$, and thus the vector

$$x = \sum_{k=1}^{\infty} x^{(k)}$$

belongs to Y. In order to prove Theorem 1.2, we shall prove that x is ε-hypercyclic for T, but that T is not hypercyclic. In the forthcoming §§2.2– 2.9 we detail the steps of the proof of Theorem 1.2.

2.2. *The vector x is ε-hypercyclic for T.* Let $k \ge 1$. Observe that by (e) we can rewrite $x^{(k)}$ as

$$x^{(k)} = (\underbrace{0, \ldots, 0}_{n_k}, S_{n_k} \ldots S_1 z_0^{(k)}, S_{n_k+1} \ldots S_2 z_1^{(k)}, \ldots, S_{n_k+k-1} \ldots S_k z_{k-1}^{(k)}, 0, \ldots),$$

and so $T^{n_k} x^{(k)} = z^{(k)}$. Clearly $T^{n_k} x^{(m)} = 0$ for $m < k$, and for $m > k$ the vector $T^{n_k} x^{(m)}$ is obtained by shifting the vector $x^{(m)}$ n_k times backward. Thus $\|T^{n_k} x^{(m)}\| = \|x^{(m)}\|$ for $m > k$. Hence

$$\|T^{n_k} x - z^{(k)}\| = \sum_{m=k+1}^{\infty} \|x^{(m)}\| \le \sum_{m=k+1}^{\infty} 2^{-m-a} = 2^{-k-a}.$$

Let v be any non-zero vector of Y. Choose $k \ge 1$ such that $\|v - y^{(k)}\| < \varepsilon' \|v\|$, where $\varepsilon' > 0$ satisfies $(1 + \varepsilon')2^{1-a} + \varepsilon' < \varepsilon$. Then $\|y^{(k)}\| < \|v\|(1 + \varepsilon')$ and

$$\begin{aligned} \|T^{n_k} x - v\| &\le \|T^{n_k} x - z^{(k)}\| + \|z^{(k)} - y^{(k)}\| + \|y^{(k)} - v\| \\ &\le 2^{-k-a} + 2^{-a}\|y^{(k)}\| + \varepsilon' \|v\| \le \|y^{(k)}\| 2^{-a+1} + \varepsilon' \|v\| \\ &\le \|v\|((1 + \varepsilon')2^{-a+1} + \varepsilon') \le \varepsilon \|v\|. \end{aligned}$$

Hence x is an ε-hypercyclic vector for T.

2.3. *The operator T is not hypercyclic on Y.* Suppose on the contrary that there exists a vector $v = (v_0, v_1, \ldots)$ of Y which is hypercyclic for T. Then there exists an increasing sequence $(m_j)_{j \ge 0}$ of integers such that the quantity $\|T^{m_j} v - (e_0, 0, \ldots)\|$ tends to zero as j tends to infinity. In particular, reading this on the first coordinate yields that $\|S_1^{-1} S_2^{-1} \ldots S_{m_j}^{-1} v_{m_j} - e_0\|$ tends to zero. Here assumptions (b) and (c) come into play:

$$\begin{aligned} \|v_{m_j} - e_0\| &= \|v_{m_j} - S_{m_j} \ldots S_1 e_0\| \le \|S_{m_j} \ldots S_1\| \cdot \|S_1^{-1} \ldots S_{m_j}^{-1} v_{m_j} - e_0\| \\ &\le 2^{a+1} \|S_1^{-1} \ldots S_{m_j}^{-1} v_{m_j} - e_0\|. \end{aligned}$$

Hence $\|v_{m_j} - e_0\|$ tends to zero. Thus $\|v_{m_j}\|$ tends to 1, which is impossible since v belongs to Y, i.e. $\sum \|v_n\|$ is finite.

2.4. *Construction of the sequences $(n_k)_{k \ge 0}$, $(n'_k)_{k \ge 0}$ and $(S_j)_{j \ge 1}$.* Recall that we set formally $n_0 = n'_0 = 0$. Define the numbers n_k, n'_k inductively by setting

$$n_k = n'_{k-1} + 4k + 2a + 1 + n_{k-1}$$

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and

$$n'_k = n_k + 5k + 2a + 1.$$

We define the operators S_j by induction: at step k the operators S_j are constructed for $n'_{k-1} < j \le n'_k$. Therefore, let $k \ge 1$ and suppose that $S_j \in B(X)$ are already defined and invertible for $j \le n'_{k-1}$. For $0 \le i \le k-1$ write

$$w_i^{(k)} = S_1^{-1} \dots S_i^{-1} y_i^{(k)}$$

and

$$\alpha_i^{(k)} = 2^{-a} \|y_i^{(k)}\| \cdot \|S_1^{-1} \dots S_i^{-1} e_{k^2+i}\|.$$

Note that for $k = 1$ we have $w_0^{(1)} = y_0^{(1)}$ and $\alpha_0^{(1)} = 2^{-a}\|y_0^{(1)}\|$.

At step $k \ge 2$ we have already defined in particular the invertible operators $S_1, \dots, S_{k-1}$, since $k - 1 \le n'_{k-1}$.

We define the operators S_j, $n'_{k-1} < j \le n'_k$, by defining $S_j e_i$, depending on the values of i and j.

- For $i < k^2$, define:
  (1) $S_j e_i = e_i$ $(n'_{k-1} < j \le n'_k)$.
- For $k^2 \le i \le k^2 + k - 1$, define:
  (2a) $S_j e_i = 2e_i$ $(n'_{k-1} < j \le n'_{k-1} + a)$;
  (2b) $S_j e_i = -(w_{i-k^2}^{(k)}/2^a \alpha_{i-k^2}^{(k)}) + e_i$ $(j = n'_{k-1} + a + 1)$;
  (2c) $S_j e_i = (1/2)e_i$ $(n'_{k-1} + a + 1 < j < n'_{k-1} + 2a + 4k + 1 = n_k - n_{k-1})$;
  (2d) $S_j e_i = e_i$ $(n_k - n_{k-1} < j \le n_k + k)$;
  (2e) $S_j e_i = 2e_i$ $(n_k + k < j \le n_k + 5k + a)$;
  (2f) $S_j e_i = (w_{i-k^2}^{(k)}/2^a \alpha_{i-k^2}^{(k)}) + e_i$ $(j = n_k + 5k + a + 1)$;
  (2g) $S_j e_i = (1/2)e_i$ $(n_k + 5k + a + 1 < j \le n_k + 5k + 2a + 1 = n'_k)$.
- For $i > k^2 + k - 1$, define:
  (3a) $S_j e_i = (1/2)e_i$ $(n'_{k-1} < j \le n_k - n_{k-1})$;
  (3b) $S_j e_i = e_i$ $(n_k - n_{k-1} < j \le n_k + k)$;
  (3c) $S_j e_i = (1/2)e_i$ $(n_k + k < j \le n'_k - 1)$;
  (3d) $S_j e_i = 2^{n'_k - n'_{k-1} - n_{k-1} - k - 1} e_i$ $(j = n'_k)$.

For $k \ge 1$ let $M_k = \overline{\mathrm{sp}}[e_i \mid i = 0 \cdots k^2 + k - 1]$ and $L_k = \overline{\mathrm{sp}}[e_i \mid i > k^2 + k - 1]$.

2.5. *Boundedness and invertibility of the operators S_j.* We show first by induction on k that the operators S_j, $n'_{k-1} < j \le n'_k$, defined above are bounded, invertible and upper triangular, and that their inverses S_j^{-1} are also bounded and upper triangular.

As mentioned above, for $k = 1$ we have $w_0^{(1)} = y_0^{(1)} \in \mathbb{C} \cdot e_0$, so the operators S_j, $j \le n'_1$ are upper triangular. Moreover, for each $j \le n'_1$ we have $S_j(M_1) \subseteq M_1$ and $S_j(L_1) \subseteq L_1$. The operator $S_j|M_1$ is upper triangular with a positive main diagonal and $S_j|L_1$ is a non-zero scalar multiple of the identity operator. So S_j is bounded and invertible and its inverse S_j^{-1} is also bounded and upper triangular.

Suppose that $k \ge 2$ and that the operators S_j, S_j^{-1}, $j \le n'_{k-1}$, are bounded, invertible and upper triangular. For $0 \le i \le k-1$, $y_i^{(k)}$ belongs to the linear span of the vectors e_l, $l = 0 \cdots k - 1$, and so this is also the case for the vector $w_i^{(k)} = S_1^{-1} \dots S_{k-1}^{-1} y_i^{(k)}$. Hence

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the operators S_j, $n'_{k-1} < j \le n'_k$ defined by (1)–(3) are upper triangular. As above, we conclude that they are bounded and invertible, and that their inverses S_j^{-1} are also bounded and upper triangular.

We now have to show that the operators S_j satisfy conditions (a)–(f).

2.6. *Proof of properties (b), (e) and (d).* By definition, $S_j e_0 = e_0$ for all j and S_j is equal to the identity operator for $n_k - n_{k-1} < j \le n_k + k$. Hence conditions (b) and (e) are satisfied trivially. Then we have to prove by induction on k that $S_{n'_k} \dots S_1 = I$, i.e., that $S_{n'_k} \dots S_{n'_{k-1}+1} = I$.

• For $i < k^2$, clearly $S_{n'_k} \dots S_{n'_{k-1}+1} e_i = e_i$ since all the operators S_j, $n'_{k-1} + 1 \le j \le n'_k$, act on e_i as the identity operator by (1).

• For $i > k^2 + k - 1$, it is also easy to check using property (3) that $S_{n'_k} \dots S_{n'_{k-1}+1} e_i = e_i$ (just multiply all coefficients together).

• For $k^2 \le i \le k^2 + k - 1$, we have that $S_{n'_k} \dots S_{n'_{k-1}+1} e_i$ is equal to

$$S_{n'_k} \dots S_{n'_{k-1}+a+1}(2^a e_i) = S_{n'_k} \dots S_{n'_{k-1}+a+2}\left(-\frac{w^{(k)}_{i-k^2}}{\alpha^{(k)}_{i-k^2}} + 2^a e_i\right)$$

by (2b). Then since $w^{(k)}_{i-k^2}/\alpha^{(k)}_{i-k^2}$ is supported by the first k vectors e_l, $l = 0, \dots, k-1$, by (2c), (2d) and (2e) applied successively this quantity is equal to

$$S_{n'_k} \dots S_{n'_{k-1}+2a+4k+2}\left(-\frac{w^{(k)}_{i-k^2}}{\alpha^{(k)}_{i-k^2}} + 2^{-4k} e_i\right)$$

and so equal to

$$S_{n'_k} \dots S_{n_k+k+1}\left(-\frac{w^{(k)}_{i-k^2}}{\alpha^{(k)}_{i-k^2}} + 2^{-4k} e_i\right) = S_{n'_k} \dots S_{n_k+5k+a+1}\left(-\frac{w^{(k)}_{i-k^2}}{\alpha^{(k)}_{i-k^2}} + 2^a e_i\right).$$

Then the expression in (2f) destroys the quantity $w^{(k)}_{i-k^2}/\alpha^{(k)}_{i-k^2}$ in this expression, and we eventually get that

$$S_{n'_k} \dots S_{n'_{k-1}+2a+4k+2}\left(-\frac{w^{(k)}_{i-k^2}}{\alpha^{(k)}_{i-k^2}} + 2^{-4k} e_i\right) = S_{n'_k} \dots S_{n_k+5k+a+2}(2^a e_i) = e_i.$$

Hence $S_{n'_k} \dots S_{n'_{k-1}+1} = I$ and property (d) is proved.

2.7. *Proof of property (a).* We now have to prove by induction on k that $\|S_j^{-1}\| \le 2$ for every j with $n'_{k-1} < j \le n'_k$. Let $k \ge 1$ and suppose that $\|S_j^{-1}\| \le 2$ for every $j \le n'_{k-1}$. For $0 \le i \le k-1$ we have

$$S_1^{-1} \dots S_i^{-1} e_{k^2+i} = \|S_1^{-1} \dots S_i^{-1} e_{k^2+i}\| \cdot e_{k^2+i}$$

since the operators $S_1^{-1}, \dots, S_i^{-1}$ just multiply the vector e_{k^2+i} by some coefficient. Thus

$$S_1^{-1} \dots S_i^{-1} z_i^{(k)} = S_1^{-1} \dots S_i^{-1} y_i^{(k)} + 2^{-a} \|y_i^{(k)}\| S_1^{-1} \dots S_i^{-1} e_{k^2+i} = w_i^{(k)} + \alpha_i^{(k)} e_{k^2+i}.$$

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Let now $l \geq 0$ be such that $n'_l < i \leq n'_{l+1}$ and consider separately the two cases $i < n'_{l+1}$ and $i = n'_{l+1}$.

• If $i < n'_{l+1}$, set $r = \text{card}\,\{s \mid n'_l < s \leq i \text{ and } S_s \neq I\}$. Then

$$\|S_1^{-1} \dots S_i^{-1} y_i^{(k)}\| = \|S_{n'_l+1}^{-1} \dots S_i^{-1} y_i^{(k)}\| \leq 2^r \cdot \|y_i^{(k)}\|$$

by the induction assumption and (d). Since $l \leq k-1$, $k^2+1 > l^2+l-1$, and thus we compute $S_i^{-1} e_{k^2+i}$ by using the expressions of (3). The same is true for the computation of $S_{i-1}^{-1} S_i^{-1} e_{k^2+i}, \dots, S_{n'_l+1}^{-1} \dots S_i^{-1} e_{k^2+i}$, and we get that

$$S_1^{-1} \dots S_i^{-1} e_{k^2+i} = S_{n'_l+1}^{-1} \dots S_i^{-1} e_{k^2+i} = 2^r e_{k^2+i}.$$

Hence $\alpha_i^{(k)} \geq 2^{-a} \|w_i^{(k)}\|$.

• If $i = n'_{l+1}$, $S_1^{-1} \dots S_i^{-1} = I$, so we also have by the same computation $\alpha_i^{(k)} \geq 2^{-a} \|w_i^{(k)}\|$.

These inequalities imply that $\|S_{n'_{k-1}+a+1} e_i\| \leq 2$ and $\|S_{n_k+5k+a+1} e_i\| \leq 2$ for $k^2 \leq i \leq k^2+k-1$.

Let us now go back to the proof of property (a). Clearly $\|S_j^{-1}\| \leq 2$ for all j with $n'_{k-1} < j \leq n'_k$, $j \neq n'_{k-1}+a+1$ and $j \neq n_k+5k+a+1$. In order to prove that $\|S_j^{-1}\| \leq 2$ in these remaining two cases, we only have to check that $\|S_j^{-1} e_i\| \leq 2$ for every $i \geq 0$ (we use the ℓ_1-norm at this point).

• If $i < k^2$, then obviously, by (1), $\|S_{n'_{k-1}+a+1}^{-1} e_i\| \leq 2$ and $\|S_{n_k+5k+a+1}^{-1} e_i\| \leq 2$.

• Similarly, if $i > k^2+k-1$ then by (3) we have $\|S_{n'_{k-1}+a+1}^{-1} e_i\| \leq 2$ and $\|S_{n_k+5k+a+1}^{-1} e_i\| \leq 2$.

• Let $k^2 \leq i \leq k^2+k-1$. Then

$$S_{n'_{k-1}+a+1} S_{n_k+5k+a+1} e_i = S_{n_k+5k+a+1} S_{n'_{k-1}+a+1} e_i = e_i.$$

So we have $\|S_{n_k+5k+a+1}^{-1} e_i\| = \|S_{n'_{k-1}+a+1} e_i\| \leq 2$ by the considerations above, and $\|S_{n'_{k-1}+a+1}^{-1} e_i\| = \|S_{n_k+5k+a+1} e_i\| \leq 2$ for the same reason. This finishes the proof of (a).

2.8. *Proof of property (f).* Let $k \geq 1$ and $0 \leq i \leq k-1$. Then

$$\begin{aligned}
\|S_{n_k} \dots S_{i+1} z_i^{(k)}\| &= \|S_{n_k} \dots S_1 (S_1^{-1} \cdots S_i^{-1}) z_i^{(k)}\| \\
&= \|S_{n_k} \dots S_1 (w_i^{(k)} + \alpha_i^{(k)} e_{k^2+i})\| \\
&= \|S_{n_k} \dots S_{n'_{k-1}+1} (w_i^{(k)} + \alpha_i^{(k)} e_{k^2+i})\| \\
&= \|S_{n_k} \dots S_{n'_{k-1}+a+1} (w_i^{(k)} + 2^a \alpha_i^{(k)} e_{k^2+i})\| \\
&= \|S_{n_k} \dots S_{n'_{k-1}+a+2} (2^a \alpha_i^{(k)} e_{k^2+i})\| \\
&= \|S_{n_k} \dots S_{n'_{k-1}+2a+4k+2} (2^{-4k} \alpha_i^{(k)} e_{k^2+i})\| \\
&= \|2^{-4k} \alpha_i^{(k)} e_{k^2+i}\| \\
&= 2^{-4k} \alpha_i^{(k)}.
\end{aligned}$$

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Then (f) is proved by observing that

$$2^{-4k}\alpha_i^{(k)} = 2^{-4k}2^{-a}\|y_i^{(k)}\| \cdot \|S_1^{-1} \dots S_i^{-1} e_{k^2+i}\| \le 2^{-4k}2^{-a}2^k \cdot 2^k = 2^{-2k-a}.$$

2.9. *Proof of property (c).* It remains to show that $\|S_j \dots S_1\| \le 2^{a+1}$ for all $j \ge 1$. By (d), it is sufficient to show that $\|S_j \dots S_{n'_{k-1}}\| \le 2^{a+1}$ for all $k \ge 1$ and all $n'_{k-1} < j \le n'_k$. Equivalently, using again the ℓ_1-norm, it must be proved that $\|S_j \dots S_{n'_{k-1}+1} e_i\| \le 2^{a+1}$ for every $i \ge 0$ and $n'_{k-1} < j \le n'_k$.

• For $i < k^2$, this is clear since the operators S_j, $n'_{k-1} < j \le n'_k$, act on e_i as the identity operator.

• For $i > k^2 + k - 1$, this is also clear: $\|S_j \dots S_{n'_{k-1}+1} e_i\| \le 1$ for all j with $n'_{k-1} + 1 \le j \le n'_k$ (just multiply the coefficients, the worst case being when $j = n'_k$).

• For $k^2 \le i \le k^2 + k - 1$, the sequence

$$S_{n'_{k-1}+1} e_i,\ S_{n'_{k-1}+2} S_{n'_{k-1}+1} e_i,\ \dots,\ S_{n'_k} \dots S_{n'_{k-1}+1} e_i$$

is equal to $2e_i, \dots, 2^a e_i, -(w_{i-k^2}^{(k)}/\alpha_{i-k^2}^{(k)}) + 2^a e_i, \dots, -(w_{i-k^2}^{(k)}/\alpha_{i-k^2}^{(k)}) + 2^{-4k} e_i, \dots, -(w_{i-k^2}^{(k)}/\alpha_{i-k^2}^{(k)}) + 2^{-4k} e_i, \dots, -(w_{i-k^2}^{(k)}/\alpha_{i-k^2}^{(k)}) + 2^a e_i, 2^a e_i, \dots, e_i$. Hence

$$\max_{n'_{k-1}+1 \le j \le n'_k} \|S_j \dots S_{n'_{k-1}+1} e_i\| = \left\| -\frac{w_{i-k^2}^{(k)}}{\alpha_{i-k^2}^{(k)}} + 2^a e_i \right\| = \left| \frac{w_{i-k^2}^{(k)}}{\alpha_{i-k^2}^{(k)}} \right| + 2^a \le 2^{a+1}.$$

This proves (c).

Thus the operators S_j, $j \ge 1$, satisfy all the properties (a) to (f), and consequently the operator T defined here is ε-hypercyclic but not hypercyclic on $Y = \bigoplus_{\ell_1} \ell_1(\mathbb{N})$. This finishes the proof of Theorem 1.2. □

2.10. *Possible extensions of the method.* In the same way it is possible to construct a non-hypercyclic operator T on $\ell_1(\mathbb{N})$ such that $T \oplus T$ is ε-hypercyclic. Indeed, consider the space Y as in Theorem 1.2 and a sequence of pairs of vectors $(y^{(k)}, \widetilde{y}^{(k)})$ which is dense in $Y \oplus Y$. In the same way one can construct a vector $x \oplus \widetilde{x} \in Y \oplus Y$ which is ε-hypercyclic for $T \oplus T$. One can even have that $T_n = \underbrace{T \oplus \dots \oplus T}_{n}$ is ε-hypercyclic for each $n \ge 1$. Details are left to the reader.

3. *Proof of Theorem 1.3*

3.1. *Point spectrum of the adjoint of an ε-hypercyclic operator.* We need the following auxiliary result.

LEMMA 3.1. *Let $0 < \varepsilon < 1$ and let $T \in B(X)$ be an ε-hypercyclic operator. Then the point spectrum $\sigma_p(T^*)$ of the adjoint of T is empty.*

Proof. Suppose on the contrary that α belongs to $\sigma_p(T^*)$. Let $y^* \in X^*$ satisfy $\|y^*\| = 1$ and $T^* y^* = \alpha y^*$, and let $x \in X$ be an ε-hypercyclic vector for T. We distinguish two cases.

• *First case.* We have either $\langle x, y^* \rangle = 0$, or $|\alpha| \le 1$. Choose $t > (\|x\| + 1)/(1 - \varepsilon)$ and $y \in X$ with $\|y\| = 1$ and $\langle y, y^* \rangle > 1 - \varepsilon/t$. Since x is an ε-hypercyclic vector for T,

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there exists an $n \geq 0$ such that $\|T^n x - ty\| \leq \varepsilon \|ty\| = t\varepsilon$. So $|\langle T^n x - y, y^* \rangle| \leq t\varepsilon$. On the other hand,

$$|\langle T^n x - ty, y^* \rangle| \geq |\langle ty, y^* \rangle| - |\langle T^n x, y^* \rangle| \geq t - \varepsilon - |\alpha|^n |\langle x, y^* \rangle| \geq t - 1 - \|x\|.$$

Thus $t - 1 - \|x\| \leq t\varepsilon$ and so $t \leq (1 + \|x\|)/(1 - \varepsilon)$, which contradicts the assumption.

• *Second case.* We have $\langle x, y^* \rangle \neq 0$ and $|\alpha| > 1$. Choose $y \in X$ such that $0 \neq \|y\| < |\langle x, y^* \rangle|/(1 + \varepsilon)$. There exists $n \geq 0$ such that $\|T^n x - y\| \leq \varepsilon \|y\|$, and thus $|\langle T^n x - y, y^* \rangle| \leq \varepsilon \|y\|$. On the other hand,

$$|\langle T^n x - y, y^* \rangle| \geq |\langle T^n x, y^* \rangle| - |\langle y, y^* \rangle| \geq |\alpha^n| \cdot |\langle x, y^* \rangle| - \|y\| > |\langle x, y^* \rangle| - \|y\|.$$

Thus $|\langle x, y^* \rangle| - \|y\| < \varepsilon \|y\|$, and so $\|y\| > (|\langle x, y^* \rangle|)(1 + \varepsilon)$, which contradicts the assumption again. □

3.2. *Completing the proof of Theorem 1.3.* Lemma 3.1 shows that we can assume that X is infinite dimensional. We are going to prove that T is topologically transitive, i.e. that for all non-empty open subsets U and V of X there exists an integer $n \geq 1$ such that $T^n(U) \cap V$ is non-empty. Let $u \in U$ and $v \in V$ be two non-zero vectors of U and V respectively, and let $\delta > 0$ be so small that $B(u, \delta) \subseteq U$, $B(v, \delta) \subseteq V$ and $\delta < \min(\|u\|, \|v\|)$. Let $x \in X$ be an ε-hypercyclic vector for T, where $\varepsilon < \delta/(6 \max(\|u\|, \|v\|))$. There exists $n_0 \geq 0$ such that $\|T^{n_0} x - u\| \leq \varepsilon \|u\| < \delta$, and so $T^{n_0} x$ belongs to U. Let us now show that there exist infinitely many n such that $T^n x$ belongs to V. Suppose on the contrary that there are only finitely many such integers $n_1, \ldots, n_k$. As above, for each $v' \in X$ with $\|v' - v\| < (2\delta/3)$ there exists an integer $n(v')$ which satisfies $\|T^{n(v')} x - v'\| \leq \varepsilon \|v'\| \leq 2\varepsilon \|v\| < \delta/3$. Since $\|T^{n(v')} x - v\| \leq \|T^{n(v')} x - v'\| + \|v' - v\| < \delta$, we have $n(v') \in \{n_1, \ldots, n_k\}$ and the ball $B(v, (2\delta)/3)$ is covered by a finite number of balls $B(T^{n_1} x, \delta/3), \ldots, B(T^{n_k} x, \delta/3)$. However, in an infinite dimensional space this is not possible. Hence there are infinitely many n with $\|T^n x - v\| < \delta$, and in particular, there exists $n_1 > n_0$ such that $T^{n_1} x$ is in V. Hence $T^{n_1 - n_0} T^{n_0} x = T^{n_1} x \in V \cap T^{n_1 - n_0}(U)$, and consequently T is hypercyclic. □

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