Approaching quadratic equations from a right angle

KING-SHUN LEUNG

1. *Introduction*

The theory of quadratic equations (with real coefficients) is an important topic in the secondary school mathematics curriculum. Usually students are taught to solve a quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) *algebraically* (by factorisation, completing the square, quadratic formula), *graphically* (by plotting the graph of the quadratic polynomial $y = ax^2 + bx + c$ to find the x-intercepts, if any), and *numerically* (by the bisection method or Newton-Raphson method). Less well-known is that we can indeed solve a quadratic equation *geometrically* (by geometric construction tools such as *a ruler and compasses*, R&C for short). In this article we describe this approach. A more comprehensive discussion on geometric approaches to quadratic equations can be found in [1]. We have also gained much insight from [2] to develop our methods. The tool we use is a *set square* rather than the more common R&C. But the methods to be presented here can also be carried out with R&C. We choose a set square because it is more convenient (one tool is used instead of two). It is also powerful as any construction done with R&C can also be done with a set square, though the underlying geometric theorems may be different. Of course, we cannot draw a circle with a set square. But we can locate as many points as we wish on the circumference of a circle with given centre and radius (a circle whose circumference is not explicitly shown is called a *hypocircle*). Therefore, it is worthwhile to learn some constructions with a set square. The reader may refer to [3] and [4] for geometric constructions with different kinds of tools. The discussion here is somewhat informal. We hope that school teachers can incorporate some of the ideas in the design of learning activities for their students.

The organisation of this article is as follows. Section 2 following this introduction is on how a set square is used in our setting. All the basic constructions that will be used in our geometric method are described in this section. Then in Section 3, we find a number (more precisely, the y-coordinate of a certain point) that functions in exactly the same way as the *discriminant* in determining the number of real roots of a quadratic equation. In Section 4, we show how to find the real roots of a quadratic equation. We present the constructions leading to the complex solutions of a quadratic equation in Section 5 and Section 6 is devoted to the discussion about the choices of the points A, B and C used in our geometric method.

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2. *How a set square is used*

We use a set square in the following ways:

(C1) Draw a right angle with the right-angled tip. We never use the other two non-right-angled tips. So when we say the tip we always refer to the right-angled tip.

(C2) Use any of the three edges to draw a straight line, in particular, the line passing through two given points.

(C3) Draw the line perpendicular to a given line l and passing through a given point P . The case when P is not on l is shown in Figure 1.

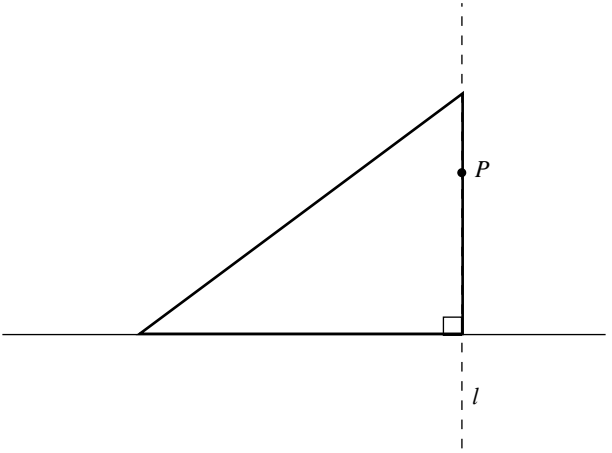


FIGURE 1

(C4) Given a line l and two points P and Q not on l, find the points of intersection of l and $\mathcal{H}(PQ)$, the hypocircle with PQ as a diameter, by placing one of the perpendicular edges on P and the other on Q in such a way that the tip is on l. There are two cases:

(a) l touches the tip (P and Q are on the same side of l) (see Figure 2), and

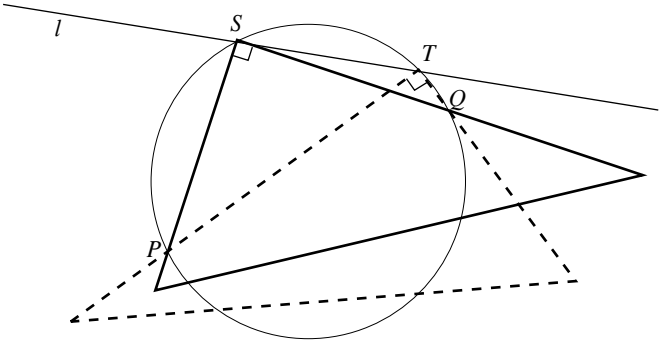


FIGURE 2

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(b) l cuts through the tip (P and Q are on different sides of l) (see Figure 3).

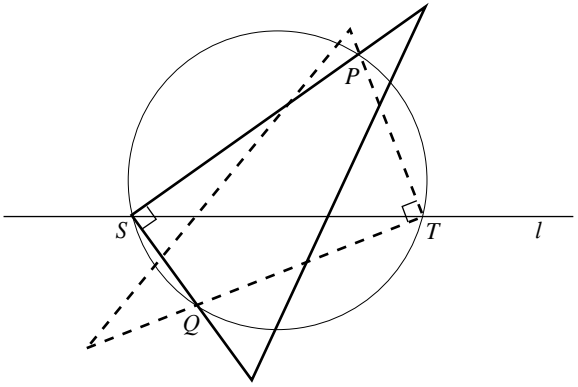


FIGURE 3

In practice, it is more accurate to find the points of intersection with (b) than with (a).

By the above fundamental constructions (C1) to (C4), we can solve a number of construction problems, for example

(C5) Find the midpoint of a line segment by drawing a rectangle with the given segment as one of its two diagonals (see Figure 4).

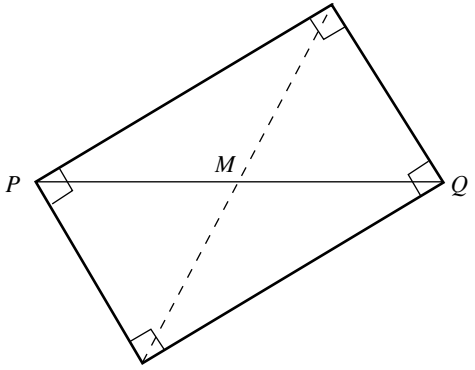


FIGURE 4

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3. *Discriminant*

Consider a quadratic equation

$$x^2 + bx + c = 0. \tag{1}$$

We assume $b > 0$ and $c > 0$. Locate the three points $A(0, -1)$, $B(-b, 0)$ and $C(-b, -c)$ on a graph paper. The line $x = -\frac{1}{2}b$ (called ℓ, which can be drawn by (C5) and then (C3)) intersects the hypocircle $\mathcal{H}(AC)$ at two points $D(-\frac{1}{2}b, u)$ and $D'(-\frac{1}{2}b, u')$ which can be found by using (C4)(b) twice. Notice that $|AO| = 1$, $|OB| = b$ and $|BC| = c$ are the coefficients of the polynomial $x^2 + bx + c$. Let $P(a, 0)$ be a point on the x-axis. It can be shown that the following are equivalent:

(i) α is a root of (1);

(ii) $\dfrac{|AO|}{|OP|} = \dfrac{|PB|}{|BC|}$;

(iii) $\angle APC$ is a right angle, (see Figure 5).

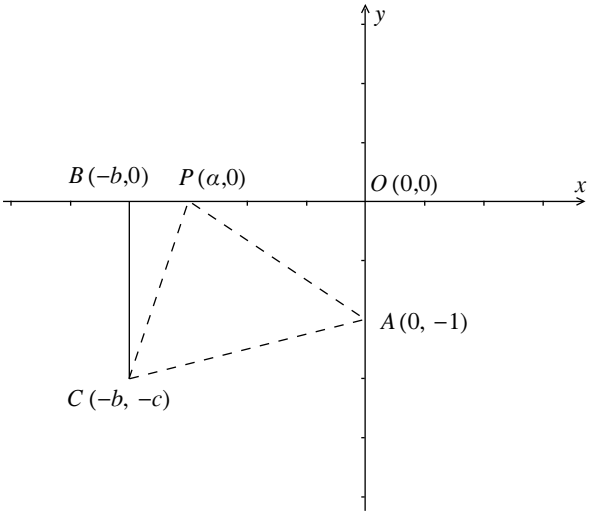


FIGURE 5

It follows from $AD \perp DC$ and $AD' \perp D'C$ that u and u' are the two roots of the following equation:

$$u^2 + (c + 1)u - \frac{b^2 - 4c}{4} = 0. \tag{2}$$

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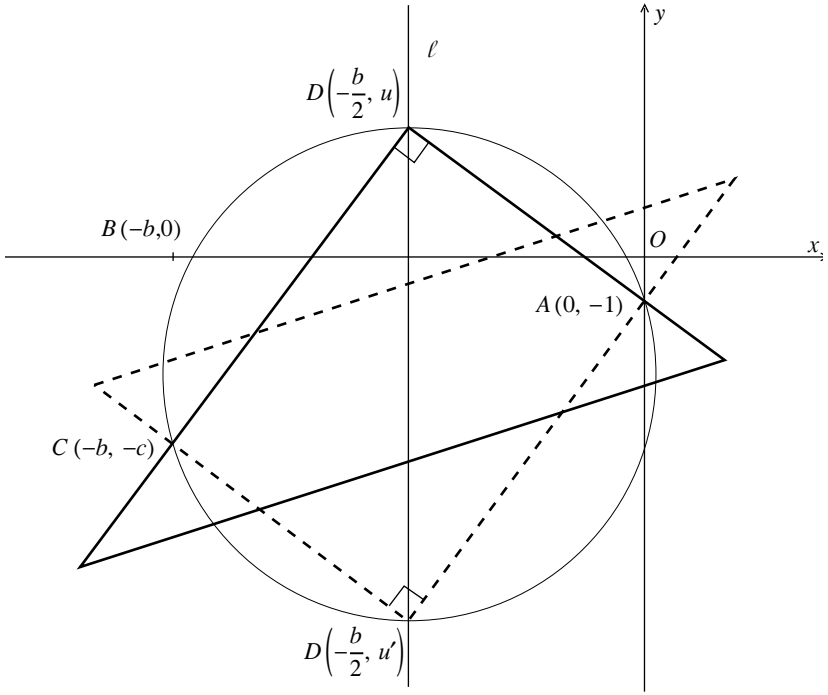


FIGURE 6

We see that the root u' is negative. The other root, u, is positive (see Figure 6), zero (see Figure 7) and negative (see Figure 8)) according as $\Delta = b^2 - 4c > 0$, $\Delta = b^2 - 4c = 0$ or $\Delta = b^2 - 4c < 0$. In other words, (1) has two distinct roots, one double real root or no real root according as $u > 0$, $u = 0$ or $u < 0$.

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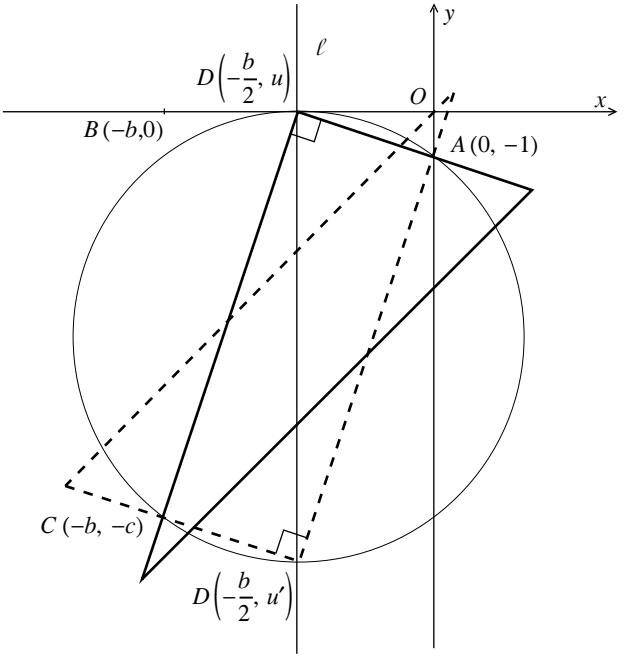


FIGURE 7

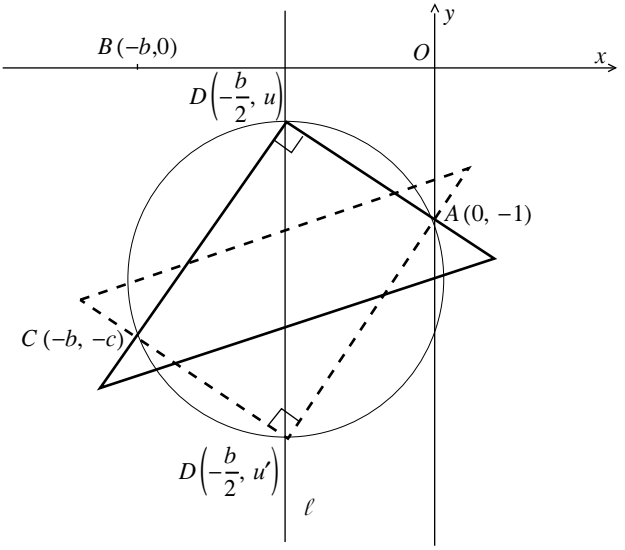


FIGURE 8

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Next we consider (1) with $b > 0$ and $c \leqslant 0$. We can see from Figures 9 and 10 that $u > 0$ always. That means (1) has two distinct real roots.

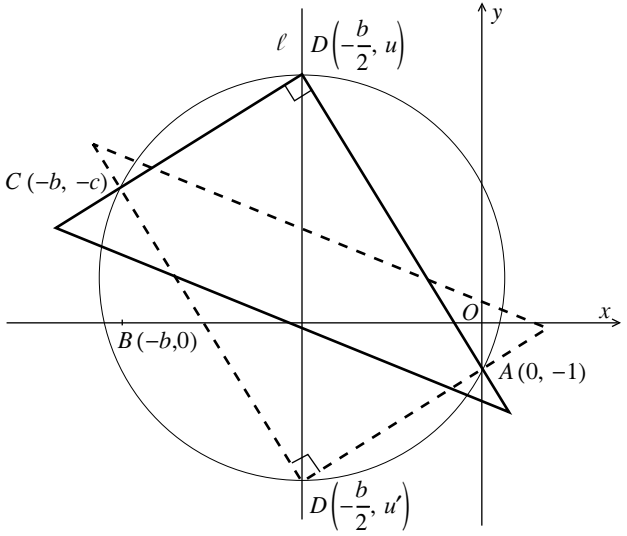


FIGURE 9: $c < 0$

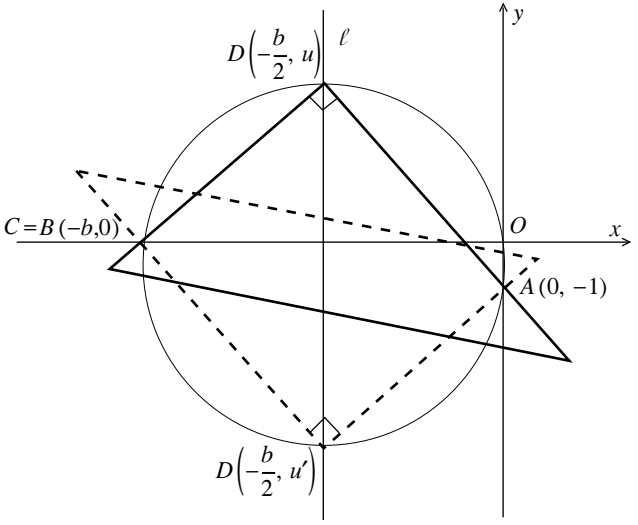


FIGURE 10: $c = 0$

The diagrams for the cases when $b < 0$ are just the reflections in the y-axis of the diagrams for $x^2 + (-b)x + c = 0$. Therefore, we need only show how to solve (1) with $b \geqslant 0$ in the following sections.

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4. *Finding the real roots of a quadratic equation*

(a) $c > 0, b > 0$

For the reason mentioned in the description of (C4) in Section 2, we present two methods here.

(i) Method 1 (simple)

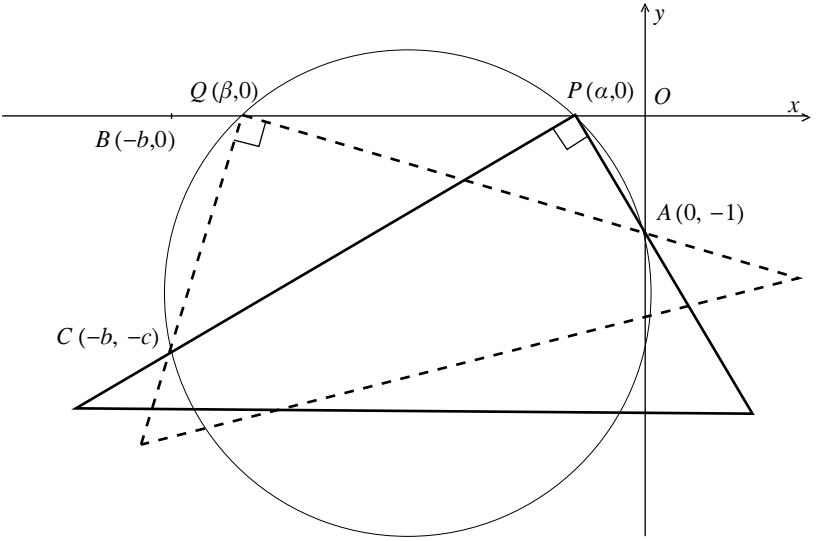


FIGURE 11

By applying (C4)(a) twice, we can find the points of intersection P and Q of the x-axis and the hypocircle $\mathcal{H}(AC)$ (see Figure 11). As $\angle APC$ is a right angle, we have $\frac{-1-0}{0-\alpha} \cdot \frac{-c-0}{-b-\alpha} = -1$. It follows that $\alpha^2 + b\alpha + c = 0$. By the same argument, we can see that $\beta^2 + b\beta + c = 0$.

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(ii) Method 2 (complicated)

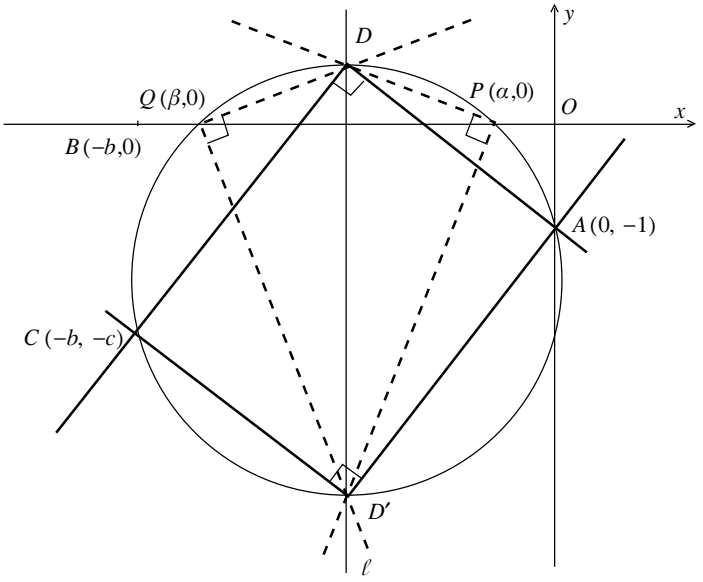


FIGURE 12

As has been mentioned in Section 3, we find the two points of intersection D and D' of the line ℓ and the hypocircle $\mathcal{H}(AC)$ (see Figure 12). By the same construction, we obtain the points P and Q, the points of intersection of the x-axis and the hypocircle with DD' as a diameter. But this hypocircle is just $\mathcal{H}(AC)$.

(b) $c = 0, b > 0$

(i) Method 1

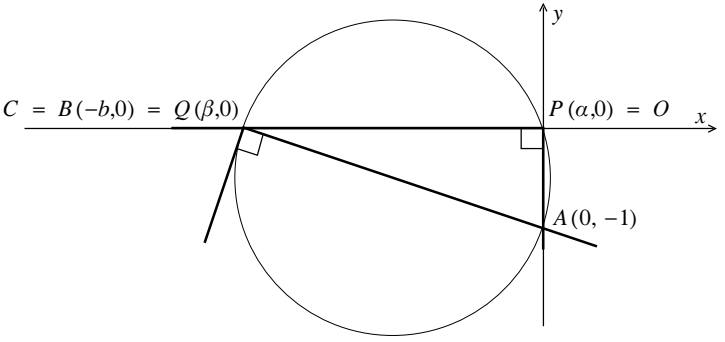


FIGURE 13

It is obvious that $\alpha = 0$ and $\beta = -b$ are the two roots of (1) (see Figure 13).

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(ii) Method 2

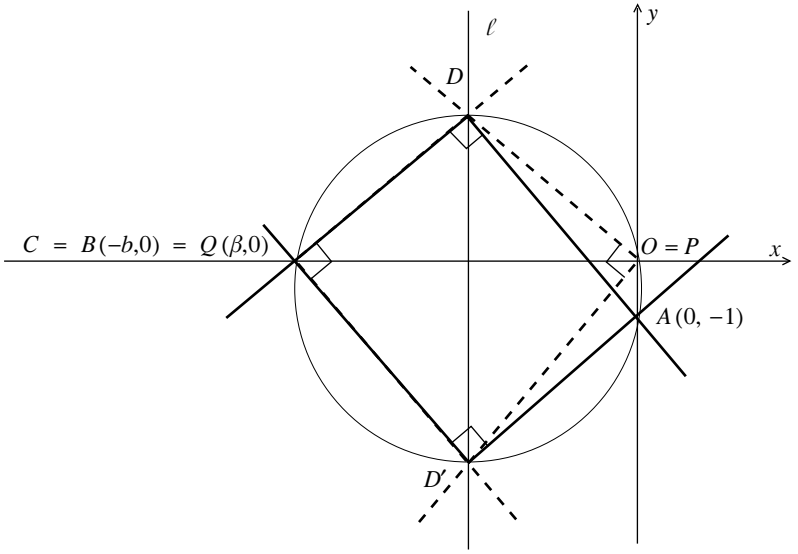


FIGURE 14

As in (a)(ii), the points P and Q can be located by applying (C4)(b) (see Figure 14).

(c) $c < 0, b = 0$

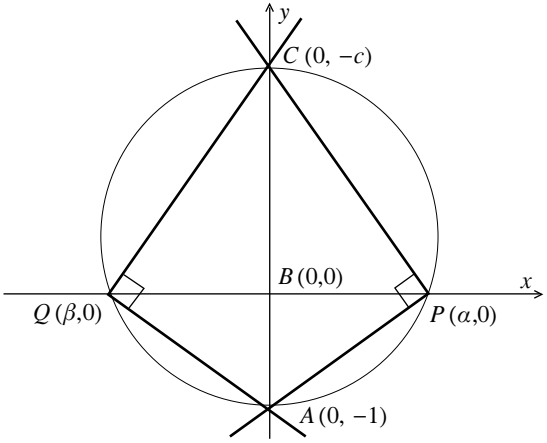


FIGURE 15

In this case, the line ℓ is just the y-axis, the points D and D' are C and A respectively (see Figure 15). This is a construction for finding the square roots of a positive real number.

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(d) $c < 0, b > 0$

(i) Method 1

As before, the points P and Q denote the points of intersection of the x-axis and $\mathcal{H}(AC)$ (see Figure 16). It can be shown in the same way that α and β are the two roots of (1).

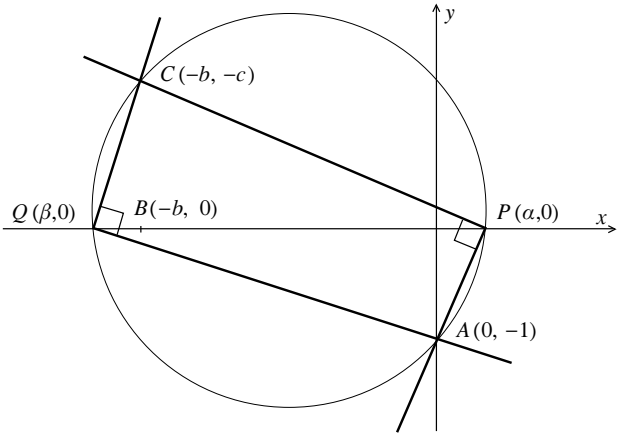


FIGURE 16

(ii) Method 2

We find the points of intersection D, D' of ℓ and $\mathcal{H}(AC)$ by applying (C4)(b) as before. By (C4)(b) again, we .nd the points of intersection P and Q of the x-axis and $\mathcal{H}(DD')\,(=\mathcal{H}(AC))$ (see Figure 17).

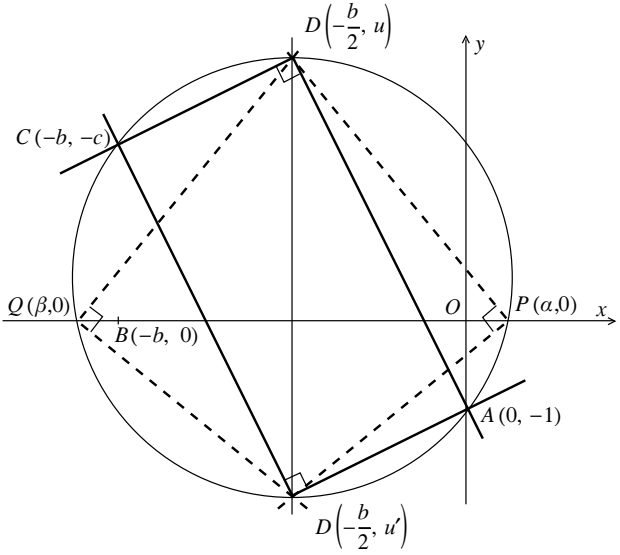


FIGURE 17

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(e) $c = 0, b = 0$

This case may be regarded as the limiting case when $c \to 0$ and $b \to 0$. The five points O, B, C, P and Q coincide.

5. *Finding the complex roots of a quadratic equation*

The xy-plane, the x-axis and the y-axis are identified with the Argand plane, the real axis and the imaginary axis respectively.

(a) $c > 0, b > 0$

Draw the line ℓ: $\mathrm{Re}(z) = -\frac{1}{2}b$ and locate the point $A'(-b, 1)$. The point A' will be used to find $\sqrt{c}$ as described below. By (C4)(b), we find the points of intersection S and T of the x-axis and the hypocircle $\mathcal{H}(A'C)$. Then by (C4)(b) again, we see that the x-axis intersects the hypocircle $\mathcal{H}(ST)$ at points P and Q (see Figure 18). Let $P = \left(-\frac{1}{2}b, p\right)$ and $Q = \left(-\frac{1}{2}b, -p\right)$. We can easily infer from (c) of Section 4 that $|SB| = |BT| = \sqrt{c}$. But since S, P, T and Q are concyclic, we have $|BP| = |BQ| = \sqrt{c}$. Then $c = \left(-\frac{1}{2}b - (-b)\right)^2 + p^2$. As a result, $p = \dfrac{\sqrt{4c - b^2}}{b}$. Hence the points P, identified with the complex number $-\dfrac{b}{2} + \dfrac{\sqrt{4c - b^2}}{2}\,i$ and Q, identified with $-\dfrac{b}{2} - \dfrac{\sqrt{4c - b^2}}{2}\,i$, are the two complex roots of (1).

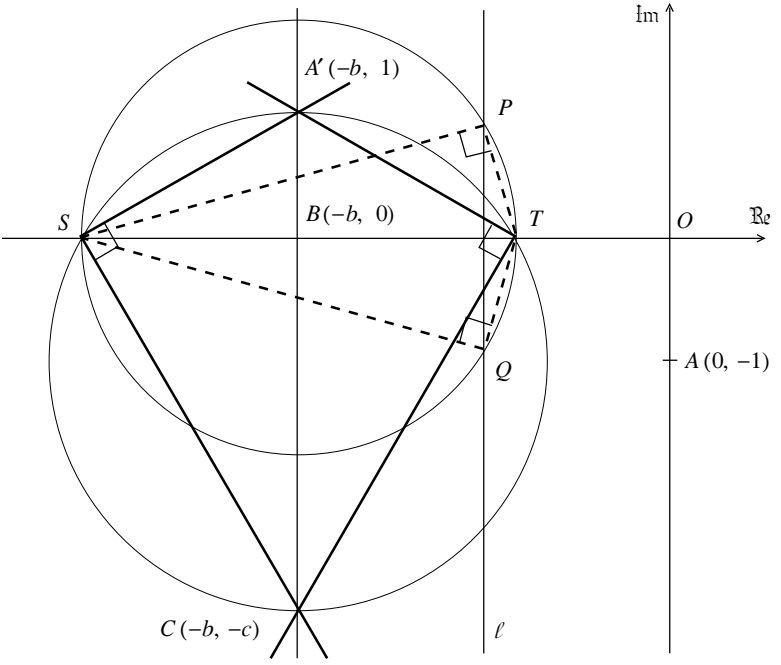


FIGURE 18

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(b) $c > 0, b = 0$

In this case the line ℓ' is the imaginary axis. It can be shown that P and Q are the points $(0, c)$ and $(0, -c)$ respectively (see Figure 19).

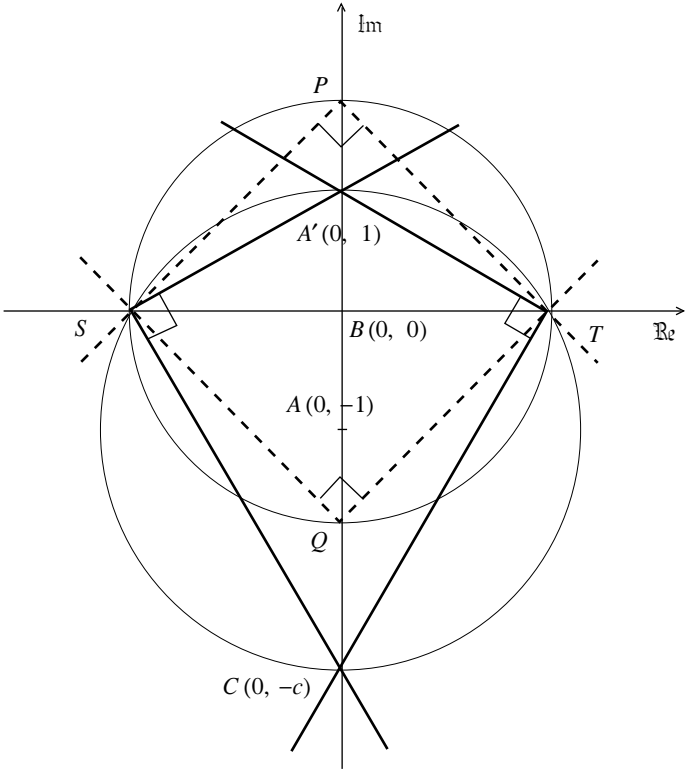


FIGURE 19

6. *Discussion*

Our aim is to construct the two roots of the equation $x^2 + bx + c = 0$ using the intersection of $f(x) = x^2 + bx + c$ with a circle whose diameter subtends right angles at the points of intersection (see Figure 20).

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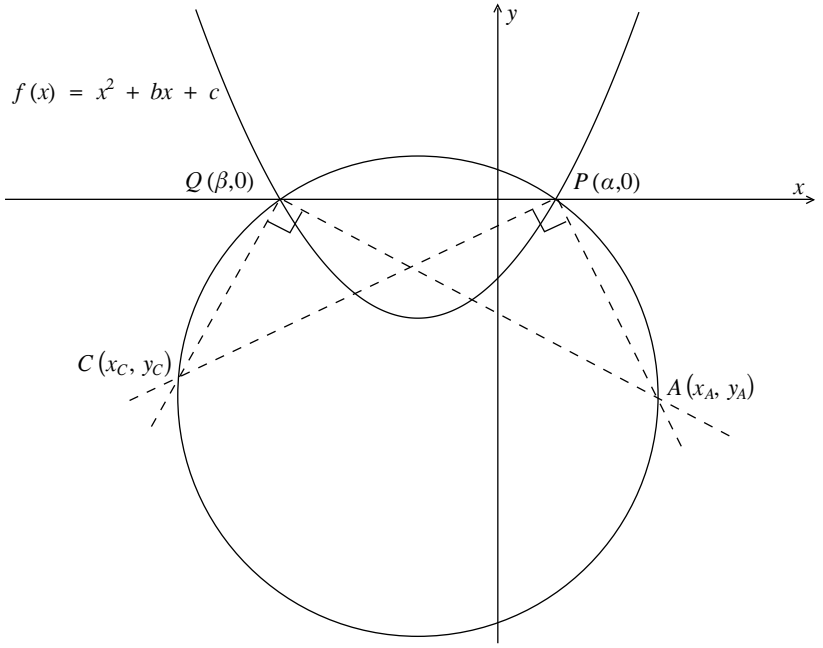


FIGURE 20

The choice of A and C needs to be such that they involve b and c in their coordinates as this is all the information we have to go on. Now since $CQ \perp AQ$, it follows that $\frac{0 - y_C}{\beta - x_C} \cdot \frac{0 - y_A}{\beta - x_A} = -1$ and so

$$y_A y_C = -\beta^2 + \beta(x_C + x_A) - x_A x_C. \tag{3}$$

Similarly, since $CP \perp AP$, it follows that $\frac{0 - y_C}{\alpha - x_C} \cdot \frac{0 - y_A}{\alpha - x_A} = -1$ and so

$$y_A y_C = -\alpha^2 + \alpha(x_C + x_A) - x_A x_C. \tag{4}$$

Therefore, it follows from (3) and (4) that

$$-\beta^2 + \beta(x_C + x_A) - x_A x_C = -\alpha^2 + \alpha(x_C + x_A) - x_A x_C$$

and thus $\beta^2 - \alpha^2 = (\beta - \alpha)(x_C + x_A)$, or $(\beta - \alpha)(\beta + \alpha) = (\beta - \alpha)(x_C + x_A)$, which implies that $\alpha + \beta = x_A + x_C = -b$ as $\alpha \neq \beta$. So if we fix $x_A = 0$, then $x_C = -b$ and $x_A x_C = 0$. Hence, it follows from (3) and (4) that $y_A y_C = -\beta^2 - \beta b = -\alpha^2 - \alpha b$. Since, by (1), $-\beta^2 - \beta b = -\alpha^2 - \alpha b = c$, we have $y_A y_C = c$. So by choosing $y_A = -1$ and $y_C = -c$, we have $A = (0, -1)$ and $C = (-b, -c)$. We can see from the above discussion that the choices of A and C are not unique. The same choices also apply when (1) has double roots.

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Nemo, continued from page 411.

5. Only I discern
   Infinite passion, and the pain
   Of finite hearts that yearn.

6. His wings were infinite, his youth eternal; the sun was a jewel on his finger as he passed it in benediction over the world.

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