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A geometric characterization of linear hyperbolic flows on $\mathbb{C}^n$

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Abstract. We prove that a polynomial vector field on $\mathbb{C}^n$, $n \geq 2$, whose corresponding projective foliation has only singularities of hyperbolic type is linear in some affine chart provided that it is transverse to a sequence of spheres bounding balls exhausting $\mathbb{C}^n$.

1. *Introduction and main result*

A very basic example of non-discrete holomorphic dynamics on the complex projective space $\mathbb{C}P(n)$ is given by a linear vector field X on $\mathbb{C}^n$. Indeed, once regarded as a holomorphic foliation with singularities $\mathcal{F}_0(X)$ on $\mathbb{C}^n$, having as leaves the non-singular orbits of X, its dynamics extend to $\mathbb{C}P(n)$ in a natural way as the dynamics of the corresponding singular holomorphic foliation $\mathcal{F}(X)$ on $\mathbb{C}P(n)$. The foliation $\mathcal{F}(X)$ is the extension of $\mathcal{F}_0(X)$ to $\mathbb{C}P(n)$. The topological behavior and classification of linear (holomorphic) flows on $\mathbb{C}P(n)$ was given by Camacho *et al.* in [**3**] where the classical notions of *Poincaré domain* and *Siegel domain* play a fundamental role, as well as the variety $M(X)$ of tangency points of X with the foliation by real codimension one spheres centered at the singularity. Thus, one natural question is raised: *How to characterize linear flows on* $\mathbb{C}P(n)$. A very first remark is that a holomorphic vector field on $\mathbb{C}P(n)$ is necessarily linear in an affine chart and, conversely, any linear vector field on $\mathbb{C}^n$ is the restriction of a holomorphic vector field on $\mathbb{C}P(n)$. In this paper, we pursue further characterizations of linear flows on $\mathbb{C}P(n)$ and, in connection with the basic spirit in [**3**], we look for such a characterization in terms of the tangency variety $M(X)$. Since any one-dimensional holomorphic foliation with singularities on $\mathbb{C}P(n)$ is given in an affine chart by a polynomial vector field with isolated singularities, we consider the following problem: *Given a polynomial vector field X with isolated singularities on $\mathbb{C}^n$, $n \geq 2$, decide whether it corresponds to a linear flow, in terms of its tangency variety $M(X)$ with spheres on $\mathbb{C}^n$.*

Our result deals with the simplest case, i.e. the case where the tangency variety reduces to a singular point in $\mathbb{C}^n$. This property is well known as valid for linear vector fields in the

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Poincaré domain and our main result is a converse of this fact, taking into consideration the singular set of the corresponding foliation on the projective space.

THEOREM 1. *Let X be a polynomial vector field with isolated singularities on $\mathbb{C}^n$, $n \geq 2$ and assume that the corresponding foliation $\mathcal{F}(X)$ on $\mathbb{C}P(n)$ has singularities of hyperbolic type. Then the following conditions are equivalent.*

(i) *$\mathcal{F}(X)$ has (at least) n separatrices on $\mathbb{C}^n$ and is transverse to a sequence of spheres $S^{2n-1}(p_j, R_j) \subset \mathbb{C}^n$ where $\lim_{j\to\infty} R_j = +\infty$.*

(ii) *X is linear (of Poincaré hyperbolic type) in some affine chart on $\mathbb{C}^n$.*

Let us recall some of the notions involved in the above statement. Given a linear vector field

$$X = \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j}$$

on $\mathbb{C}^n$ we say that X is in the *Poincaré domain* if the convex hull of the set $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}^2$ does not contain the origin and also that X is *hyperbolic* if $\lambda_i/\lambda_j \in \mathbb{C}\backslash\mathbb{R}$, for all $i \neq j$. By a *separatrix* of X we mean a non-singular orbit of X on $\mathbb{C}^n$ that is closed outside the singular set of X and accumulates some singularity of X. Applying a classical result of Remmert and Stein (see [**5**]) on extension of analytic subsets, we conclude that a separatrix of X is an irreducible analytic curve $C \subset \mathbb{C}^n$ that is invariant by X and such that $C \cap \mathrm{sing}(X) \neq \emptyset$.

Remark 1. According to a result of Brunella [**2**], given a relatively compact open subset $\Omega \subset \mathbb{C}^n$, $n \geq 2$, with smooth and strongly convex boundary, and given a holomorphic vector field X in a neighborhood of $\bar{\Omega}$, transverse to the boundary $\partial\Omega$, there is a diffeomorphism $\Phi\colon \bar{\Omega} \to B[0, 1] \subset \mathbb{C}^n$ sending the foliation $\mathcal{F}(X)$ onto a holomorphic foliation $\mathcal{G}$ transverse to the spheres $S^{2n-1}(0; R)$, $0 < R \leq 1$. Using this, one may state a version of Theorem 1 for polynomial vector fields X on $\mathbb{C}^n$, $n \geq 3$, that are transverse to the boundaries $\partial\Omega_j$ of a sequence of bounded domains $\Omega_j \subset\subset \mathbb{C}^n$ with smooth strongly convex boundary and such that $\lim_{j\to\infty} \Omega_j = \mathbb{C}^n$.

In what follows we give an example showing that our hypothesis on the existence of n separatrices and on $\mathrm{sing}(\mathcal{F}(X))$ cannot be dropped.

Example 1. Let X be the polynomial vector field on $\mathbb{C}^3$ given by

$$X(x, y, z) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + (x + z)\frac{\partial}{\partial z}.$$

We claim that *X is transverse to all spheres $S^5(0; R)$, $R > 0$*. Indeed, the equations of the set of tangency points of X and the sphere $S^5(0, R)$ are $|x|^2 + |y|^2 + |z|^2 = R^2$ and $x \cdot \bar{x} + y \cdot \bar{y} + (x + z)\bar{z} = 0$. This system of equations is equivalent to $|x|^2 + |y|^2 + |z|^2 = R^2$ and $x\bar{z} = -R^2$. Now, $x\bar{z} = -R^2 \Rightarrow \bar{x}z = -R^2$. Also for any complex number $\alpha \in \mathbb{C}$ we have $\alpha^2 + \bar{\alpha}^2 = 2[\mathrm{Re}(\alpha)]^2 \geq 0$. Therefore, summing up the equations $(x+\bar{z})^2 = x^2 + 2x\bar{z} + (\bar{z})^2$ and $(\bar{x} + z)^2 = (\bar{x})^2 + 2\bar{x}z + z^2$ we obtain $2[\mathrm{Re}(x + \bar{z})]^2 = 2[\mathrm{Re}(x)]^2 + 4x\bar{z} + 2[\mathrm{Re}(z)]^2$ $\Rightarrow 0 \leq [\mathrm{Re}(x)]^2 = [\mathrm{Re}(x)]^2 + [\mathrm{Re}(z)]^2 - 2R^2 \Rightarrow [\mathrm{Re}(x)]^2 + [\mathrm{Re}(z)]^2 \geq 2R^2 > R^2$,

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which is absurd because $|x|^2 + |y|^2 + |z|^2 = R^2$. Thus, X is an example of a vector field, polynomial on $\mathbb{C}^3$, transverse to all spheres $S^5(R; 0)$ but not linear in the sense of Theorem 1. Note that the origin is a singularity having only one separatrix, the z-axis.

2. *Proof of Theorem 1*

From now on we shall consider the following framework of Theorem 1. We have X a polynomial vector field with isolated singularities on $\mathbb{C}^n$, $n \geq 2$ and $\mathcal{F} = \mathcal{F}(X)$ is the foliation induced by X on the complex projective space $\mathbb{C}P(n)$. We denote by $\text{sing}(\mathcal{F})$ the singular set of $\mathcal{F}$ on $\mathbb{C}P(n)$ and by $B[p; R] \subset \mathbb{C}^n$ the closed ball of center $p \in \mathbb{C}^n$ and radius $R > 0$, $B(p; R)$ is the corresponding open ball and $S^{2n-1}(p; R) = \partial B[p; R]$ is the boundary sphere. We will assume that the singularities of $\mathcal{F}$ in $\mathbb{C}P(n)$ are of *hyperbolic type* in the following sense: given $p \in \text{sing}(\mathcal{F})$ the foliation $\mathcal{F}$ is represented in a neighborhood U of p on $\mathbb{C}P(n)$, by a holomorphic vector field X_p with $\text{sing}(X_p) = \{p\}$ and linear part $DX_p(p)$ having eigenvalues $\lambda_1, \ldots, \lambda_n$ such that the quotient $\lambda_i/\lambda_j \in \mathbb{C}\backslash\mathbb{R}$, for all $i \neq j$.

Remark 2. Suppose that X_p is analytically linearizable in a neighborhood of p, in this case we may write

$$X_p(x_1, \ldots, x_n) = \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j}$$

in suitable local coordinates $(x_1, \ldots, x_n) \in U$ with $x_j(p) = 0$, for all j and, in particular, $\mathcal{F}$ has exactly n separatrices passing through p corresponding to the coordinate axes. Also from the linear model we obtain that $\mathcal{F}$ has exactly n invariant hypersurfaces passing through p corresponding to the coordinate hyperplanes. These hypersurfaces intersect transversely along invariant submanifolds and each $n - 1$ of them meet along a separatrix.

Given a singularity $p \in \text{sing}(X)$ we denote by $\text{Sep}(\mathcal{F}; p)$ the germ of analytic set obtained by the union of separatrices through p and by $\Lambda(\mathcal{F}; p)$ the germ of codimension one analytic set obtained by the union of the invariant hypersurfaces through p.

PROPOSITION 1. *Suppose that $\mathcal{F}$ is transverse to the spheres $S^{2n-1}(p_j; R_j)$ for some sequence $p_j \in \mathbb{C}^n$ and $R_j > 0$ with $\lim R_j = +\infty$. Then:*

(i) *$\mathcal{F}|_{\mathbb{C}^n}$ has a single singularity on $\mathbb{C}^n$ say $o \in \mathbb{C}^n$; moreover, the vector field X is analytically linearizable in a neighborhood of this singularity;*

(ii) *the sets $\text{Sep}(\mathcal{F}, o)$ and $\Lambda(\mathcal{F}; o)$ are germs of algebraic invariant sets.*

Proof. According to [**6**], if $\mathcal{F}$ is transverse to a sphere $S^{2n-1}(p; R)$, then $\mathcal{F}$ has a unique singularity in the ball $B(p; R)$. After a Möbius transformation we can assume that this singular point is the origin and, also by [**6**], we have that $\mathcal{F}$ is transverse to the spheres $S^{2n-1}(p; R')$ with $0 < R' \leq R$. This also implies that the singularity $0 \in \text{sing}(X)$ is in the Poincaré domain. Now we have two possibilities. If there are no resonances for the linear part $DX(0)$ then by the Poincaré–Dulac linearization theorem, X is analytically conjugate to its linear part in a neighborhood of the origin [**1, 4**]. Assume now that the linear part has some resonance. Then an immediate analysis on the Poincaré–Dulac normal form for X in a neighborhood of 0 shows that since it has n separatrices through the origin (note that

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the origin is the only singularity in $\mathbb{C}^n$) this singularity is still analytically linearizable. This proves (i).

Let us prove (ii). First we remark that we can assume that the spheres $S^{2n-1}(p_j; R_j)$ have a common point and then that $p_j = 0$, for all j with $R_j \nearrow +\infty$ and $\mathcal{F}$ is singular at $0 \in \mathbb{C}^n$. Thus, we have $\mathcal{F}$ transverse to $S^{2n-1}(0; R_j)$ with $\lim R_j = +\infty$. By the remark above, this implies that $\mathbb{C}^n \cap \mathrm{sing}(\mathcal{F}) = \{0\}$. We denote by $\Lambda(\mathcal{F}) = \mathrm{Sat}_{\mathcal{F}}(\Lambda(\mathcal{F}; 0))$ the saturation of $\Lambda(\mathcal{F}; 0)$ by $\mathcal{F}$ on $\mathbb{C}^n$.

CLAIM 1. *$\Lambda(\mathcal{F})$ is an analytic subset of codimension one of $\mathbb{C}^n$.*

Proof. Owing to [**6**], if $\mathcal{F}$ is transverse to $S^{2n-1}(0; R)$, then the restriction $\mathcal{F}|_{B[0;R]-\{o\}}$ is C^w-conjugate to the foliation $\mathcal{F}|_{S^{2n-1}(0;R)} \times (0, 1]$. This, together with the above remark, gives that $\mathcal{F}|_{B[0;R]-\{0\}}$ and $\mathcal{F}|_{B[0;R']-\{0\}}$ are C^w-conjugate for all $0 < R' \leq R$. Thus, $\mathcal{F}|_{B[0;R]-\{0\}}$ is C^w-conjugate to the restriction $\mathcal{F}_{X_0}|_{B[0;\varepsilon]-\{0\}}$, of the foliation defined by the linear model

$$X_0 = \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j},$$

to any punctured small ball $B[0; \varepsilon] - \{0\}, \varepsilon > 0, \varepsilon \approx 0$. This implies that $\Lambda(\mathcal{F}) \cap B[0; R_j]$ is closed and therefore, by the Remmert–Stein theorem, analytic of codimension one on $B[0; R_j]$. Since $\lim R_j = +\infty$ the claim follows. □

Now we analyze the behavior of $\Lambda(\mathcal{F})$ 'at infinity', i.e. in a neighborhood of the hyperplane $\mathbb{C}P(n-1)_\infty = \mathbb{C}P(n) \backslash \mathbb{C}^n$. We have two cases.

Case 1. $\mathbb{C}P(n-1)_\infty$ is invariant by $\mathcal{F}$. In this case, by the index theorem, there necessarily exists some singularity $p \in \mathrm{sing}(\mathcal{F}) \cap \mathbb{C}P(n-1)_\infty$ (see [**8, 10**]). We consider the local linear or Poincaré–Dulac normal form model X_p for $\mathcal{F}$ at p and the germs of analytic sets $\mathrm{Sep}(\mathcal{F}, p)$ and $\Lambda(\mathcal{F}, p)$. Since $\mathbb{C}P(n-1)_\infty$ is invariant of codimension one, we must have $\mathbb{C}P(n-1)_\infty \cap U \subset \Lambda(\mathcal{F}, p)$ for a small neighborhood U of p where $\mathcal{F}$ is conjugate to its local model. In particular, $\Lambda(\mathcal{F}, p)$ has $\ell \leq n-1$-irreducible components $\Lambda_1(\mathcal{F}, p), \ldots, \Lambda_\ell(\mathcal{F}, p)$ that are transverse to $\mathbb{C}P(n-1)_\infty$ and, therefore, $\Lambda_j(\mathcal{F}; p) \backslash \{p\} \subset \mathbb{C}^n$ for all $j \in \{1, \ldots, \ell\}$. For an index j sufficiently large, we must have $S^{2n-1}(p, R_j) \cap \Lambda_k(\mathcal{F}, p) \neq \emptyset$, for all $k \in \{1, \ldots, \ell\}$ and, therefore, $\Lambda_k(\mathcal{F}; p) \cap S^{2n-1}(p; R_j)$ is a closed (compact) leaf of the induced foliation $\mathcal{F}|_{S^{2n-1}(p;R_j)}$. By uniqueness of these compact leaves, we must have $\ell = n-1$ and $\Lambda_k(\mathcal{F}; p) \cap S^{2n-1}(p; R_j) \subset \Lambda(\mathcal{F}) \cap S^{2n-1}(p; R_j)$, and by invariance and transversality we have $\Lambda_k(\mathcal{F}; p) \cap \mathbb{C}^n \subset \Lambda(\mathcal{F})$, for all $k \in \{1, \ldots, n-1\}$. This proves that $\Lambda(\mathcal{F})$ has $n-1$ irreducible components that extend analytically to $\mathbb{C}P(n)$ as $\Lambda_1(\mathcal{F}; p), \ldots, \Lambda_{n-1}(\mathcal{F}; p)$ in a neighborhood of $\mathbb{C}P(n-1)_\infty$.

Remark 3. Alternatively one can prove that the irreducible components of $\Lambda(\mathcal{F})$ do not accumulate on regular points $q \in \mathbb{C}P(n-1)_\infty \backslash \mathrm{sing}(\mathcal{F})$. Indeed, if $q \in \mathbb{C}P(n-1)_\infty \backslash \mathrm{sing}(\mathcal{F})$ is accumulated by $\Lambda(\mathcal{F})$, then we choose a tubular flow box for $\mathcal{F}$ around q, say a polydisc $D \times \Delta^{n-1}$ with coordinates $(x, y) = (x, y_2, \ldots, y_n) \in D \times \Delta^{n-1}$ such that $\mathbb{C}P(n-1)_\infty \cap (D \times \Delta^{n-1})$ is given by $\{y_n = 0\}$ and the restriction $\mathcal{F}|_{D \times \Delta^{n-1}}$ is given by the

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vector field $\partial/\partial x$ and q is given by $x = y = 0$. The intersection $\Lambda(\mathcal{F}) \cap (D \times \Delta^{n-1})$ is an analytic subset of $(D \times \Delta^{n-1}) \backslash \mathbb{C}P(n-1)_\infty = (D \times \Delta^{n-1}) \backslash \{y_n = 0\} \subset \mathbb{C}^n$. Thus we can choose a (reduced) holomorphic function $f = f(x, y)$ on $(D \times \Delta^{n-1}) \backslash \{y_n = 0\}$ such that $\{f(x, y) = 0\} = \Lambda(\mathcal{F}) \cap (D \times \Delta^{n-1})$. Since $\Lambda(\mathcal{F})$ is $\mathcal{F}$-invariant we must have that $f(x, y) = 0 \Rightarrow f(tx_1, x_2, \ldots, x_{n-1}, y) = 0$, for all $t \approx 0$, $t \in \mathbb{C}$ and therefore $(\partial f/\partial x_1)(x, y) = 0$, thus we conclude that $\Lambda(\mathcal{F}) \cap (D \times \Delta^{n-1})$ must be of the form $D \times S$ (note that D is a complex disc representing any plaque of $\mathcal{F}$ in the flow box) and S is a codimension one analytic submanifold of the transverse section Δ^{n-1}. Since $\Lambda(\mathcal{F})$ is closed on $\mathbb{C}^n$ it follows that S is closed in $\Delta^{n-1} \backslash \{y_n = 0\}$. Either $\bar{S}$ is analytic or S accumulates entirely on $\{y_n = 0\}$. Suppose that $\bar{S}$ is not analytic, then $\Lambda(\mathcal{F})$ accumulates on all points of $\mathbb{C}P(n-1)_\infty$. This implies that $\Lambda(\mathcal{F})$ accumulates on some singularity $p \in \operatorname{sing}(\mathcal{F}) \cap \mathbb{C}P(n-1)_\infty$ and gives a contradiction because in a small neighborhood U_p of p on $\mathbb{C}P(n)$ we must have $\Lambda(\mathcal{F}) \cap U_p \subseteq \Lambda(\mathcal{F}; p) \backslash \mathbb{C}P(n-1)_\infty$, which does not accumulate on $\mathbb{C}P(n-1)_\infty \cap U_p$.

Case 2. $\mathbb{C}P(n-1)_\infty$ *is not invariant by* $\mathcal{F}$. In this case we investigate whether $\Lambda(\mathcal{F})$ accumulates only on singular points in $\mathbb{C}P(n-1)_\infty$. Indeed, suppose then that there is $q \in \mathbb{C}P(n-1)_\infty \backslash \operatorname{sing}(\mathcal{F})$ accumulated on by $\Lambda(\mathcal{F})$. We can choose a tubular flow box neighborhood T of q on $\mathbb{C}P(n)$ such that $T \cap \mathbb{C}P(n-1)_\infty$ is a disc D centered at q and $\mathcal{F}|_T$ is holomorphically conjugate to the trivial foliation by discs $z \times \Sigma$, $z \in D$ on $T \simeq D \times \Sigma$. Any analytic codimension one invariant subset Γ of $(D \times \Sigma) \backslash D$ must be of the form $\Gamma = \{(x, y) \in D \times \Sigma : y \neq 0, f(x, y) = 0\}$ where f is a holomorphic function on $D \times \Sigma \backslash \{y = 0\}$ such that if $f(x, y) = 0$ then $f(x, ty) = 0$ for all $t \in \mathbb{C}$ with $0 < |t| \leq 1$. Therefore $(\partial f/\partial y)(x, y) = 0$, for all $(x, y) \in \Gamma$ and, therefore, Γ is of the form $\Gamma = (C \times \Sigma) \backslash D$ for some analytic subset $C \subset D$ with $\operatorname{codim}_D C = 1$. This implies that Γ extends analytically to $D \times \Sigma$ as $C \times \Sigma$. Thus, we can assume that q is a singular point of $\mathcal{F}$ in $\mathbb{C}P(n-1)_\infty$. In this case an argumentation similar to that in Case 1 shows that $\Lambda(\mathcal{F})$ must be locally contained in the subset $\Lambda(\mathcal{F}, q) \backslash \mathbb{C}P(n-1)_\infty$ and, therefore, $\Lambda(\mathcal{F})$ extends analytically to a neighborhood of q in $\mathbb{C}P(n)$.

This shows that $\Lambda(\mathcal{F})$ extends as an analytic subset of codimension one of $\mathbb{C}P(n)$. Using Chow's theorem [**5**] we conclude that $\Lambda(\mathcal{F})$ is algebraic proving the lemma. □

Denote by $\operatorname{Sep}(\mathcal{F})$ the algebraic subset of $\mathbb{C}P(2)$ that satisfies $\operatorname{Sep}(\mathcal{F}) \cap U_p = \operatorname{Sep}(\mathcal{F}; p) \cap U_p$ for every singularity $p \in \operatorname{sing}(\mathcal{F})$. Fix an irreducible component C of $\operatorname{Sep}(\mathcal{F})$ on $\mathbb{C}P(n)$.

PROPOSITION 2. *There exists a system* $\{\eta_j\}_{j=2}^n$ *of closed meromorphic one forms defined in a neighborhood* V *of the set* $C \subseteq \operatorname{Sep}(\mathcal{F})$ *such that* $\mathcal{F}|_V$ *is given by* $\eta_2 = \cdots = \eta_n = 0$ *outside the union of polar divisors. Also we have that* $(\eta_j)_\infty$ *is an irreducible component of* $V \cap \Lambda(\mathcal{F})$ *for every* $j \in \{2, \ldots, n\}$, $(\eta_j)_\infty$ *has order one, for all* $j \in \{2, \ldots, n\}$ *and* $(\eta_2)_\infty \cup \cdots \cup (\eta_n)_\infty = \Lambda(\mathcal{F}) \cap V$.

Proof. In the proof of Proposition 2 we shall need the following lemma.

LEMMA 1. *Denote by* L_0 *the leaf* $C \backslash \operatorname{sing}(\mathcal{F})$. *Then the holonomy group* $\operatorname{Hol}(\mathcal{F}, L_0)$ *of* L_0 *is abelian analytically linearizable in a diagonal way: given any transverse disc* Σ *to*

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L_0 there exist local coordinates $z = (z_1, \ldots, z_{n-1}) \in \Sigma$ centered at $\Sigma \cap L_0$ such that any element $f \in \mathrm{Hol}(\mathcal{F}, L_0, \Sigma)$ writes in these coordinates as $f(z) = (\mu_1^f z_1, \ldots, \mu_{n-1}^f z_{n-1})$ for some $\mu_j^f \in \mathbb{C}^*$.

Proof. Fix a point $p_0 \in L_0 \cap \mathbb{C}^n$ and a transverse $(n-1)$-disc $\Sigma_0 \simeq \{z \in \mathbb{C}^{n-1} : |z| < 1\}$, $\Sigma_0 \pitchfork L_0$ with $\Sigma_0 \cap L_0 = \{p_0\}$. By the remark in the beginning of the proof of Proposition 1, the foliation $\mathcal{F}|_{B[0,R]-\{0\}}$ is C^w-conjugate to $\mathcal{F}|_{B[0,\varepsilon]-\{0\}}$. Regarded as real foliations, this implies that the corresponding leaves have conjugate holonomy groups. Now, clearly any C^w-conjugation must preserve the leaves in $\mathrm{Sep}(\mathcal{F})$ therefore, assuming $L_0 \subset \{z_n = 0\}$ around the origin and using the local model

$$X_0 = \sum_{j=1}^{n} \lambda_j z_j \frac{\partial}{\partial z_j} \quad \text{for } \mathcal{F}|_{B[0;\epsilon]},$$

we can conclude that for any $R > 0$ the holonomy of the leaf corresponding to $L_0 \cap B[0; R] = C \cap B[0; R] \setminus \{0\}$ of the restriction $\mathcal{F}|_{B[0;R]-\{0\}}$ is cyclic generated by a hyperbolic element with diagonal normal form

$$f(z_1, \ldots, z_{n-1}) = (e^{2\pi i(\lambda_1/\lambda_n)} z_1, \ldots, e^{2\pi i(\lambda_{n-1}/\lambda_n)} z_{n-1})$$

where $\lambda_i/\lambda_j \notin \mathbb{R}$, for all $i \neq j$. This is a hyperbolic linear diagonal map. Let us compute the contribution of the points at the infinity $q \in C \cap \mathbb{C}P(n-1)_\infty$ for the holonomy of L_0. Clearly we are interested only in those points $q \in C \cap \mathbb{C}P(n-1)_\infty \cap \mathrm{sing}(\mathcal{F})$. Given such a point q we can choose a disc $D \subset C$ transverse to $\mathbb{C}P(n-1)_\infty$ such that $D \cap \mathbb{C}P(n-1)_\infty = \{q\}$ and, therefore, $D \setminus \{q\} \subset \mathbb{C}^n$. The local holonomy of the singularity $q \in C$ can be calculated by a map $f_{\gamma_q} : \Sigma_0, p_0 \to \Sigma_0, p_0$ where $\gamma_q \subset D \setminus \{q\}$ is a simple loop around q and p_0 is chosen in L_0 such that $p_0 \in \gamma_q$. Thus, the contribution of the singularity $q \in C \cap \mathrm{sing}(\mathcal{F}) \cap \mathbb{C}P(n-1)_\infty$ to the holonomy group of the leaf L_0 is already in the holonomy group of $\mathcal{F}|_{B[0;R]-\{0\}}$ for $R > 0$ large enough. This proves Lemma 1. □

Remark 4. We point out that, according to the above proof, the local holonomy of a singularity $q \in C \cap \mathrm{sing}(\mathcal{F}) \cap \mathbb{C}P(n-1)_\infty$ is always analytically linearizable in a diagonal way, therefore *such a singularity is also analytically linearizable*.

Now we proceed. For p_0 and $\Sigma_0 \ni p_0$ as above we choose local holomorphic coordinates $z_0 = (z_1^0, \ldots, z_{n-1}^0) \in \Sigma_0$ centered at p_0 and such that the (local) holonomy f of the (local) branch of C through $0 \in \mathbb{C}^n$ is linear in this chart

$$f(z_0) = (e^{2\pi i(\lambda_1/\lambda_n)} z_1^0, \ldots, e^{2\pi i(\lambda_{n-1}/\lambda_n)} z_{n-1}^0)$$

with $|e^{e\pi i(\lambda_j/\lambda_n)}| \neq 1$, for all $j \in \{1, \ldots, n-1\}$. Then, by commutativity, every element $g \in \mathrm{Hol}(\mathcal{F}, L_0, \Sigma_0)$ is also linear diagonal in this chart say $g(z_0) = (\mu_g^1 z_1^0, \ldots, \mu_g^{n-1} z_{n-1}^0)$ with $\mu_g^j \in \mathbb{C}^*$. We define $\eta_2, \ldots, \eta_n$ on Σ_0 by $\eta_{j+1} = dz_j^0/z_j^0$, $j = 1, \ldots, n-1$.

CLAIM 2. *η_j is invariant by the holonomy group $\mathrm{Hol}(\mathcal{F}, L_0, \Sigma_0)$, i.e. $g^*(\eta_j) = \eta_j$, for all $g \in \mathrm{Hol}(\mathcal{F}, L_0, \Sigma_0)$.*

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Proof. Indeed,

$$g^*(\eta_j) = \frac{d(\mu_g^j z_j^0)}{\mu_g^j z_j^0} = \frac{dz_j^0}{z_j^0} = \eta_j, \quad \text{for all } j, \text{ for all } g \in \mathrm{Hol}(\mathcal{F}, L_0, \Sigma_0). \qquad \square$$

Therefore, by holonomy extension, we can define η_j in a neighborhood V^* of L_0 in $\mathbb{C}P(n)$. Let us show how to extend η_j to a neighborhood of a given singularity $p \in \mathrm{sing}(\mathcal{F}) \cap C$. Take local coordinates $(x_1, \ldots, x_n) \in U$ centered at p such that $\mathcal{F}|_U$ is given by a holomorphic vector field

$$X_p = \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j} \quad \text{with } \lambda_i/\lambda_j \in \mathbb{C}\backslash\mathbb{R}, \text{ for all } i \neq j.$$

We may assume that $C \cap U = \{x_2 = \cdots = x_n = 0\}$ is the x_1-axis. Dividing by λ_1 we may rewrite

$$X_p = x_1 \frac{\partial}{\partial x_1} + \sum_{j=2}^{n} \frac{\lambda_j}{\lambda_1} x_j \frac{\partial}{\partial x_j}$$

and define local one-forms η_j^p in U for $j = 1, \ldots, n$ by

$$\eta_j^p := \frac{d(x_j x_1^{-\lambda_j/\lambda_1})}{(x_j x_1^{-\lambda_j/\lambda_1})} = \frac{dx_j}{x_j} - \frac{\lambda_j}{\lambda_1}\frac{dx_1}{x_1}, \quad j = 2, \ldots, n.$$

Now we fix a point $p_1 \in C \cap U$ say $p_1 = (x_1^1, 0, \ldots, 0)$ and a small disc $D = \{(x_1, 0, \ldots, 0) : |x_1 - x_1^1| < \varepsilon\} \subset C \cap U$ not containing the point $p = (0, 0, \ldots, 0) \in C$. Take also a small $(n-1)$-disc Σ centered at $0 \in \mathbb{C}^{n-1}$ such that the polydisc $D \times \Sigma$ is contained in U. Take a loop $\gamma_{x_1^1}(t) = (e^{it}x_1^1, 0, \ldots, 0) \subset D \times 0 \subset C\backslash\{p\}$ around the singularity $p \in C$ and denote by $f_\gamma : (x_1^1 \times \Sigma, x_1^1) \hookleftarrow$ the corresponding local holonomy at the transverse $(n-1)$-disc. We have

$$f_\gamma(x_1^1, x_2, \ldots, x_n) = (x_1^1, e^{2\pi i(\lambda_2/\lambda_1)}x_2, \ldots, e^{2\pi i(\lambda_2/\lambda_1)}x_n)$$

by straightforward computation using the linear model X_p. Now let $\hat{x}_j := x_j x_1^{-\lambda_j/\lambda_1}$, $j = 2, \ldots, n$. Then we can choose uniform branches of the functions $\hat{x}_j$ in the polydisc $D \times \Sigma$. Note that the $(n-1)$-cycle $(\hat{x}_2, \ldots, \hat{x}_n) : D \times \Sigma \to \mathbb{C}^{n-1}$ gives a flow-box trivialization for $\mathcal{F}$ in $D \times \Sigma$. Therefore, the holonomy extension of the linearizing coordinates for f_γ from $x_1^1 \times \Sigma$ to $D \times \Sigma$ is given exactly by $(\hat{x}_2, \ldots, \hat{x}_n) \in D \times \Sigma$. On the other hand, on $x_1 \times \Sigma$ we can take holomorphic coordinates $z = (z_2, \ldots, z_n) \in x_1 \times \Sigma$ such that z linearizes the holonomy group $\mathrm{Hol}(\mathcal{F}, L_0, x_1^1 \times \Sigma)$. By definition, up to reordering we can assume that $\eta_j|_{x_1^1 \times \Sigma} = dz_j/z_j$, $j = 2, \ldots, n$. The linearizing coordinates for a hyperbolic diffeomorphism are unique up to diagonal linear changes, therefore we must have, up to reordering, $z_j = a_j \hat{x}_j$ in $D \times \Sigma$ for all $j \in \{2, \ldots, n\}$ for some constant $a_j \in \mathbb{C}^*$. This implies $dz_j/z_j = d(\hat{x}_j)/\hat{x}_j$ in $D \times \Sigma$, that is, $\eta_j = \eta_j^p$ in $D \times \Sigma$, for all $j \in \{2, \ldots, n\}$. This shows that the forms η_j extend meromorphically with simple poles to the neighborhood U of p in $\mathbb{C}P(n)$. This proves Proposition 2. $\square$

PROPOSITION 3. *There exists a system* $\{\eta_j\}_{j=2}^n$ *of closed meromorphic one-forms* η_j *defined in a neighborhood* W *of* $\mathrm{Sep}(\mathcal{F})$ *on* $\mathbb{C}P(n)$ *such that:*

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(i) $\mathcal{F}|_W$ *is defined by* $\eta_2 = \cdots = \eta_n = 0$ *off the union of polar sets;*
(ii) $(\eta_j)_\infty \subset \Lambda(\mathcal{F}) \cap W$ *has order one, for all* j *and* $\Lambda(\mathcal{F}) \cap W = \bigcup_{j=2}^n (\eta_j)_\infty$*;*
(iii) $\bigcap_{j=2}^n (\eta_j)_\infty = \mathrm{Sep}(\mathcal{F})$.

Proof. For any two irreducible components $C, \tilde{C}$ of $\mathrm{Sep}(\mathcal{F})$ we have constructed $(n-1)$-uples of forms $(\eta_2, \ldots, \eta_n)$ and $(\tilde{\eta}_2, \ldots, \tilde{\eta}_n)$, respectively, in neighborhoods V and V of C and C, respectively. Take any point $p \in C \cap \tilde{C}$. Clearly $p \in \mathrm{sing}(\mathcal{F})$ and we can consider the $(n-1)$-uples $(\eta_2, \ldots, \eta_n)$ and $(\tilde{\eta}_2, \ldots, \tilde{\eta}_n)$ in a common neighborhood U of p in $\mathbb{C}P(n)$, where $\mathcal{F}$ is given by a linear model X_p as in the proof of Proposition 2. Up to reordering the local coordinates $(x_1, \ldots, x_n) \in U$, we can assume that $C \cap U = \{x_2 = \cdots = x_n = 0\} = x_1$-axis and $\tilde{C} \cap U = \{x_1 = x_3 = \cdots = x_n = 0\} = x_2$-axis. By the proof of Proposition 2 we then have (up to reordering the $(n-1)$-uples):

$$\eta_j|_U = \frac{dx_j}{x_j} - \frac{\lambda_j}{\lambda_1}\frac{dx_1}{x_1}, \quad \text{for all } k \in \{2, \ldots, n\} \quad \text{and}$$
$$\tilde{\eta}_k|_U = \frac{dx_k}{x_k} - \frac{\lambda_k}{\lambda_2}\frac{dx_2}{x_2}, \quad \text{for all } k \in \{1, \ldots, n\}\backslash\{2\}.$$

For simplicity let us consider the case $n = 3$, the general case is similar as it will be clear from what follows. Then we have the pairs

$$\begin{cases} \eta_2 = \dfrac{dx_2}{x_2} - \dfrac{\lambda_2}{\lambda_1}\dfrac{dx_1}{x_1} \\ \eta_2 = \dfrac{dx_3}{x_3} - \dfrac{\lambda_3}{\lambda_1}\dfrac{dx_1}{x_1} \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\eta}_1 = \dfrac{dx_1}{x_1} - \dfrac{\lambda_1}{\lambda_2}\dfrac{dx_2}{x_2} \\ \tilde{\eta}_3 = \dfrac{dx_3}{x_3} - \dfrac{\lambda_3}{\lambda_2}\dfrac{dx_2}{x_2} \end{cases}$$

Thus, we have in U

$$\eta_2 = -\frac{\lambda_1}{\lambda_2} \cdot \tilde{\eta}_1 \quad \text{and} \quad \eta_3 = \tilde{\eta}_3 + \frac{\lambda_2}{\lambda_3} \cdot \eta_2 = \tilde{\eta}_3 - \frac{\lambda_1}{\lambda_3} \cdot \tilde{\eta}_1.$$

The expressions

$$\eta_2 = -\frac{\lambda_1}{\lambda_2} \cdot \tilde{\eta}_1 \quad \text{and} \quad \eta_3 = \tilde{\eta}_3 - \frac{\lambda_1}{\lambda_3} \cdot \tilde{\eta}_1$$

show how to glue the pairs (η_2, η_3) and $(\tilde{\eta}_1, \tilde{\eta}_3)$ to obtain globally defined pairs of one-forms in the neighborhood $V \cup \tilde{V}$ of $C \cup \tilde{C}$ in $\mathbb{C}P(3)$. This shows the existence of the $(n-1)$-uple $\{\eta_j\}_{j=2}^n$ and of the neighborhood W of $\mathrm{Sep}(\mathcal{F})$ in $\mathbb{C}P(n)$. □

PROPOSITION 4. *The one-forms* $\{\eta_j\}_{j=2}^n$ *on Proposition 3 extend to rational one-forms on* $\mathbb{C}P(n)$.

Proof. Let Λ_j, $j = 1, \ldots, r$ the irreducible components of $\Lambda(\mathcal{F})$ each of them of codimension one on $\mathbb{C}P(n)$. Then $\bigcap_{j=1}^r \Lambda_j = \mathrm{Sep}(\mathcal{F})$ has dimension one and the intersection is transverse at the points $p \in \mathrm{Sep}(\mathcal{F})\backslash\,\mathrm{sing}(\mathcal{F})$. Thus, $r = n-1$ and $\mathrm{Sep}(\mathcal{F})$ is an algebraic complete intersection on $\mathbb{C}P(n)$. This implies that the complement $M = \mathbb{C}P(n)\backslash\,\mathrm{Sep}(\mathcal{F})$ is a $n - (n-1) + 1 = 2$-complete manifold in the sense of Levi (see [**9**]). According to Levi's extension theorem if $K \subset M$ is a compact subset such that $M\backslash K$ is connected, then any meromorphic one-form defined on $M\backslash K$ admits a unique meromorphic extension to M. Take then W a neighborhood of $\mathrm{Sep}(\mathcal{F})$ as in

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Proposition 3 and small enough such that W is connected. Let $M = \mathbb{C}P(n)\setminus \mathrm{Sep}(\mathcal{F})$ and $K = M\setminus W = \mathbb{C}P(n)\setminus W$. Then K is a compact subset of M such that $M\setminus K = W$. By Levi's extension theorem, the 1-forms $\{\eta_j\}_{j=2}^n$ admit unique meromorphic extensions still denoted $\{\eta_j\}_{j=2}^n$ to $M = \mathbb{C}P(n)\setminus \mathrm{Sep}(\mathcal{F})$. Clearly these extensions are then defined in $\mathbb{C}P(n)$ and must be closed rational one forms with simple poles on $\mathbb{C}P(n)$. □

Now, let $\Lambda_1^0, \ldots, \Lambda_n^0$ be the irreducible components of $\Lambda(\mathcal{F}) \cap \mathbb{C}^n$ and choose irreducible polynomials $P_1, \ldots, P_n \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\Lambda_j^0 = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : P_j(z_1, \ldots, z_n) = 0\}$, $j = 1, \ldots, n$. The singular set $\mathrm{sing}(X) = \mathrm{sing}(\mathcal{F}) \cap \mathbb{C}^n$ is given by $\mathrm{sing}(X) = \bigcap_{j=1}^n \Lambda_j^0$ so that $\mathrm{sing}(X) = \{P_1 = \cdots = P_n = 0\}$. Let $\Lambda_j \subset \mathbb{C}P(n)$ be the irreducible component of $\Lambda(\mathcal{F})$ containing Λ_j^0. Since the intersection of the Λ_j is everywhere transverse, it follows that $\#\,\mathrm{sing}(X) = \prod_{j=1}^n \deg(P_j)$ and, therefore, $\deg(P_j) = 1$, for all $j \in \{1, \ldots, n\}$. The polynomial P_j vanishes at the origin $0 = \mathrm{sing}(X)$ of $\mathbb{C}^n$ and, therefore, since they intersect transversely at every point, we can find an affine change of coordinates $T(z_1, \ldots, z_n) = (y_1, \ldots, y_n)$ on $\mathbb{C}^n$ such that $P_j(y_1, \ldots, y_n) = y_j$, $j = 1, \ldots, n$. The one-forms $\eta_j (j = 2, \ldots, n)$ write as

$$\eta_j = \sum_{k=1}^n \alpha_k^j \cdot \frac{dy_k}{y_k}$$

for some $\alpha_k^j \in \mathbb{C}$ and

$$X = \sum_{j=1}^n A_j(y_1, \ldots, y_n)\frac{\partial}{\partial y_j}$$

is polynomial such that $\eta_j(X) \equiv 0$, for all $j = 2, \ldots, n$. Since the hyperplanes $\{y_j = 0\} = \Lambda_j^0$ are X-invariant we must have $A_j(y_1, \ldots, y_n) = y_j \cdot B_j(y_1, \ldots, y_n)$ for some polynomials $B_j \in \mathbb{C}[y_1, \ldots, y_n]$. Thus,

$$X = \sum_{j=1}^n y_j \cdot B_j(y_1, \ldots, y_n)\frac{\partial}{\partial y_j}.$$

Hence,

$$0 = \eta_j(X) = \sum_{k=1}^n \alpha_k^j \cdot B_k(y_1, \ldots, y_n) \quad \text{for all } j \in \{2, \ldots, n\}.$$

Therefore,

$$\sum_{k=1}^n \alpha_k^j \cdot B_k(0, \ldots, 0) = 0 \quad \text{for all } j \in \{2, \ldots, n\}.$$

Let then

$$Z := \sum_{k=1}^n B_k(0, \ldots, 0) y_k \frac{\partial}{\partial y_k}$$

be a linear vector field. By construction we have $\eta_j(Z) = 0$, for all $j \in \{2, \ldots, n\}$. Therefore, the foliations $\mathcal{F}(X)$ defined by X and $\mathcal{F}(Z)$ defined by Z coincide on $\mathbb{C}^n$ and, thus, $X = H \cdot Z$ for some polynomial H on $\mathbb{C}^n$. Since $\mathrm{sing}(X) = \{0\} = \mathrm{sing}(Z)$ the

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polynomial H has no zeros on $\mathbb{C}^n$ and it is therefore constant, say $H = \lambda \in \mathbb{C}^*$. Thus, we have proved that X is linear in the variables $(y_1, \ldots, y_n) \in \mathbb{C}^n$ of the form

$$X(y_1, \ldots, y_n) = \lambda \cdot \sum_{k=1}^{n} B_k(0, \ldots, 0) y_k \frac{\partial}{\partial y_k}.$$

This ends the proof of Theorem 1.

As an immediate consequence of Theorem 1 and Remark 1, we obtain the following.

COROLLARY 1. *Let X be a polynomial vector field with isolated singularities on $\mathbb{C}^n$. Suppose that there is a C^1 (real) diffeomorphism $F : \mathbb{C}P(n) \to \mathbb{C}P(n)$ that is a convex diffeomorphism on $\mathbb{C}^n$ and defines an equivalence on $\mathbb{C}P(n)$ between the foliations $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ for some Poincaré hyperbolic linear vector field Y on $\mathbb{C}^n$. Then X is also linear of Poincaré hyperbolic type.*

We also have the following.

COROLLARY 2. *Let X be a polynomial vector field with isolated singularities on $\mathbb{C}^n$ whose corresponding foliation $\mathcal{F}(X)$ has only hyperbolic linearizable singularities on $\mathbb{C}P(n)$. Suppose that there is a C^1 (real) convex diffeomorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ that defines an equivalence on $\mathbb{C}^n$ between the foliations $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ for some Poincaré hyperbolic linear vector field Y on $\mathbb{C}^n$. Then X is also linear of Poincaré hyperbolic type.*

For more on transversality of (codimension one) holomorphic foliations and real manifolds, see [**7**].

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