Proc. Camb. Phil. Soc. (1965), **61**, 7
With 5 text-figures
Printed in Great Britain

Mrs Perkins's quilt

By G. B. TRUSTRUM
University of Sussex

(*Received* 20 *August* 1964)

In a recent paper of the same title (2), Conway considered the problem of dividing a square of side N, into squares of integral side. The case when we demand that all these squares are of unequal size has been considered in some detail by various authors ((1), (4), (6), (7)). Conway considered the case with the condition relaxed. Dudeney ((3), Problem 173) has considered the same problem for the special case $N = 13$; hence the title of this paper.

Definition 1. *A dissection of order n of a rectangle is a set of n squares of integral side which cover the rectangle and are such that no two have a common area. A dissection is* proper *if $n > 1$: it is* prime *if the sides of the squares involved have no common factor. We define the function $f(N)$ as the least possible order of a prime dissection of the square side N.*

Obviously $f(N) \leqslant N^2$. Conway showed that

$$\log_2 N \leqslant f(N) \leqslant 6\sqrt[3]{N} + 1.$$

In this paper we show that $f(N) = O(\log N)$. This had been conjectured by Moser ((5), pages 79–80 and 87–88). We show in fact that, for $N > 2$,

$$f(N) \leqslant 6\log_2(3N-1) - 10 < 6\log_2 N.$$

Definition 2. *$g(n, i)$ is the least possible order of any dissection (not necessarily prime) of a rectangle sides n and $n+i$.*

Lemma 1. *For $N > 2$, $f(N) \leqslant 2g(\frac{1}{2}\{N-i\}, i) + 2$ where $i = 1, 2, 4$ according as N is of the form $2m+1$, $4m$, $4m+2$.*

Proof. It is sufficient to show that there exists a prime dissection of order

$$2g(\tfrac{1}{2}\{N-i\}, i) + 2.$$

We first divide the square of side N as follows:

(*a*) If N is odd, divide into squares sides $\frac{1}{2}(N-1)$ and $\frac{1}{2}(N+1)$ and two rectangles $\frac{1}{2}(N-1) \times \frac{1}{2}(N+1)$.

(*b*) If N is divisible by 4, divide into squares sides $\frac{1}{2}(N-2)$ and $\frac{1}{2}(N+2)$ and two rectangles $\frac{1}{2}(N-2) \times \frac{1}{2}(N+2)$.

(*c*) If N is even but not divisible by 4, divide into squares sides $\frac{1}{2}(N-4)$ and $\frac{1}{2}(N+4)$ and two rectangles $\frac{1}{2}(N-4) \times \frac{1}{2}(N+4)$.

Figure 1 illustrates the method. Notice that in each case the sides of the squares are coprime. The dissection is completed by dissections of the rectangles. If these are of least possible order the original square will be dissected into $2g(\frac{1}{2}(N-i), i) + 2$ squares. Since it contains two squares whose sides are coprime, the dissection will be prime.

https://doi.org/10.1017/S0305004100038573 Published online by Cambridge University Press

LEMMA 2. $g(kn, ki) \leqslant g(n, i)$.

Proof. The proof is immediate—if an $n \times (n+i)$ rectangle can be dissected into m squares then so can a $kn \times k(n+i)$ rectangle.

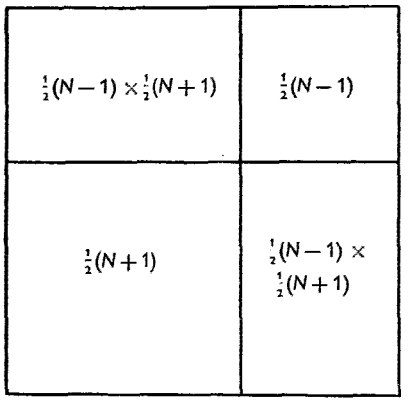


Fig. 1

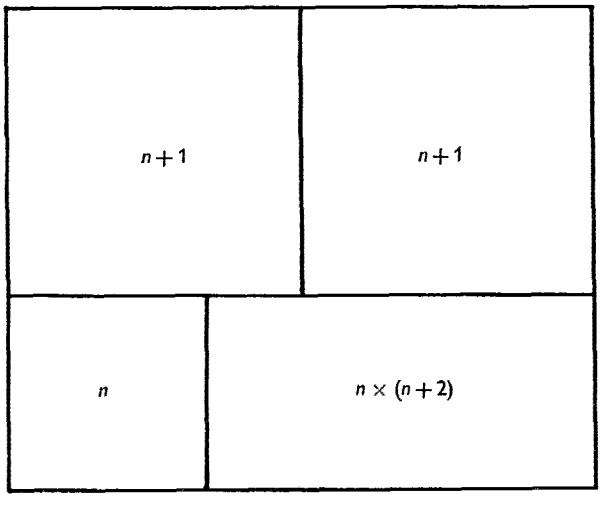


Fig. 2

We now list three rules for dissecting rectangles, which will enable us to state further inequalities concerning $g(n, i)$ for $i = 1, 2, 4$.

Rule 1. ($i = 1$, i.e. a rectangle of the form $m \times (m+1)$.)

(*a*) If m (> 1) is odd ($= 2n+1$ say) divide into squares of sides $n+1$, $n+1$ and n and a rectangle $n \times (n+2)$. For $m = 1$ divide into two squares.

(*b*) If m (> 2) is even ($= 2n+2$ say) divide into squares of sides $n+1$, $n+1$ and $n+2$ and a rectangle $n \times (n+2)$. For $m = 2$ divide into three squares. Figure 2 illustrates the method.

Rule 2. ($i = 2$, i.e. a rectangle of the form $m \times (m+2)$.)

(*a*) If m is even rectangle is of the form $2n \times 2(n+1)$ and Lemma 2 is used to obtain the required inequality.

https://doi.org/10.1017/S0305004100038573 Published online by Cambridge University Press

(*b*) If $m = 1$ divide into three squares. If $m = 3$ divide into four squares. If m is of the form $2n+3$ ($n \geqslant 1$), i.e. a $(2n+3) \times (2n+5)$ rectangle divide into four squares sides 1, $n+1$, $n+2$ and $n+3$ and a rectangle $n \times (n+4)$. Figure 3 illustrates the method.

Rule 3. ($i = 4$, i.e. a rectangle of the form $m \times (m+4)$.)

(*a*) If m is even, Lemma 2 is used to obtain the required inequality.

(*b*) If $m = 1$ or 3 divide into five squares. If $m = 5$ or 7 divide into six squares.

(*c*) If $m = 4n+5$ ($n \geqslant 1$), i.e. a $(4n+5) \times (4n+9)$ rectangle divide into six squares sides 1, 1, 1, $2n+1$, $2n+4$ and $2n+5$ and a rectangle $2n \times 2(n+4)$.

(*d*) If $m = 4n+7$ ($n \geqslant 1$), i.e. a $(4n+7) \times (4n+11)$ rectangle divide into six squares sides 1, 1, 1, $2n+3$, $2n+4$, and $2n+7$ and a rectangle $2n \times 2(n+4)$. Figures 4 and 5 illustrate the method.

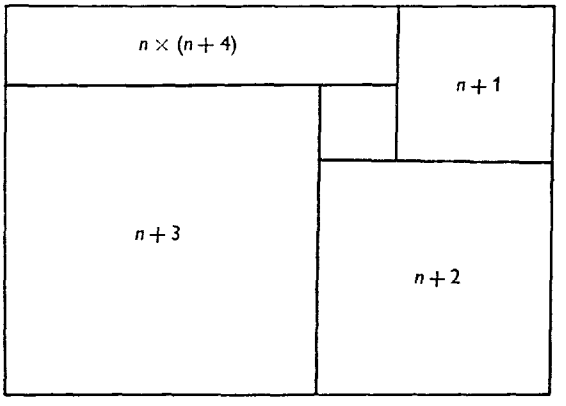


Fig. 3

The following inequalities are a consequence of the above rules and of Lemma 2:

$$\left.\begin{array}{lll} g(1,1) \leqslant 2, & g(1,2) \leqslant 3, & g(1,4) \leqslant 5, \\ & & g(3,4) \leqslant 5, \\ g(2,1) \leqslant 3, & g(3,2) \leqslant 4, & g(5,4) \leqslant 6, \\ & & g(7,4) \leqslant 6; \end{array}\right\} \quad \text{(A)}$$

$$\left.\begin{array}{lll} g(2n+1,1) \leqslant g(n,2)+3, & g(2n,2) \leqslant g(n,1), & g(2n,4) \leqslant g(n,2), \\ & & g(4n+5,4) \leqslant g(n,4)+6, \\ g(2n+2,1) \leqslant g(n,2)+3, & g(2n+3,2) \leqslant g(n,4)+4, & g(4n+7,4) \leqslant g(n,4)+6, \end{array}\right\} \quad \text{(B)}$$

all of (B) being for $n > 0$.

Definition 3. $h(n,1) = h(n,2) = 3\log_2(3n+1) - 3$,

$$h(n,4) = 3\log_2(3n+5) - 4.$$

It is easily seen that the functions $h(n,i)$ satisfy all the inequalities of (A) and (B) with the inequality signs reversed.

https://doi.org/10.1017/S0305004100038573 Published online by Cambridge University Press

THEOREM 1. *The $h(n,i)$ $(i = 1,2,4)$ are upper bounds for the corresponding $g(n,i)$.*

Proof. Let n^* be the least value of n for which the result is untrue. Then we have

$$g(n^*,i) > h(n^*,i), \quad \text{for some } i. \tag{1}$$

By inspection $g(n^*,i)$ is not one of those listed in (A), so it must occur on the left-hand side of one of the inequalities in (B). This will be of the form

$$g(n^*,i) \leqslant g(m,j)+c, \quad \text{for some } c,j, \quad \text{and some} \quad m < n^*. \tag{2}$$

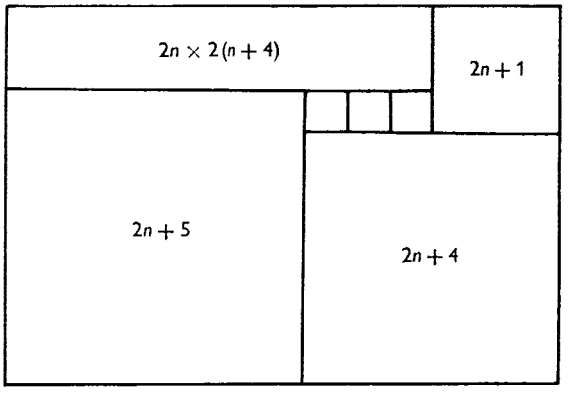


Fig. 4

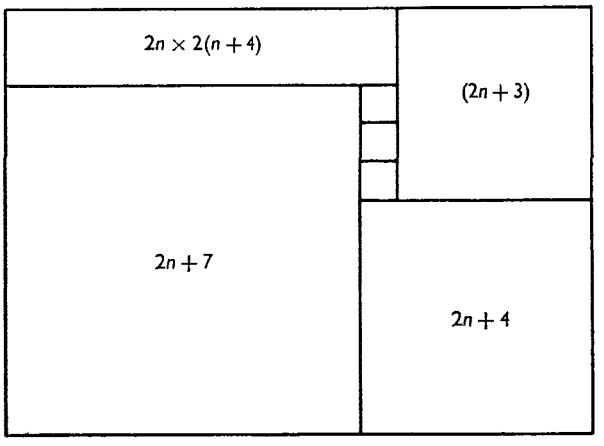


Fig. 5

There will be a corresponding inequality

$$h(n^*,i) \geqslant h(m,j)+c. \tag{3}$$

Combining (1), (2) and (3) we have

$$g(m,j) > h(m,j).$$

As n^* was the least value of n for which this inequality held, we have obtained a contradiction.

https://doi.org/10.1017/S0305004100038573 Published online by Cambridge University Press

THEOREM 2. *For* $N > 2, f(N) \leqslant 6\log_2(3N-1)-10 < 6\log_2 N$.

Proof. This follows from Lemma 1 and Theorem 1. For

$$N > 2, \quad f(N) \leqslant 2g(\tfrac{1}{2}\{N-i\}, i)+2 \leqslant 2h(\tfrac{1}{2}\{N-i\}, i)+2$$

$$= \begin{cases} 6\log_2[\frac{1}{2}(3N-1)]-4 & (i=1) \\ 6\log_2[\frac{1}{2}(3N-4)]-4 & (i=2) \\ 6\log_2[\frac{1}{2}(3N-2)]-6 & (i=4). \end{cases}$$

Thus $f(N) \leqslant 6\log_2(3N-1)-10 < 6\log_2 N$.

I should like to thank the referee for suggesting a neat approach used in the presentation of this paper.

REFERENCES

(1) BROOKS, R. L., SMITH, C. A. B., STONE, A. H. and TUTTE, W. T. *Duke Math. J.* **7** (1940), 312–340.
(2) CONWAY, J. H. *Proc. Cambridge Philos. Soc.* **60** (1964), 363–368.
(3) DUDENEY, H. E. *Amusements in mathematics* (Dover; New York, 1959).
(4) MORÓN, Z. *Przeglad Mat.-Fiz.* **3** (1925), 152–153.
(5) MOSER, L. In *Proceedings of the* 1963 *number theory conference* (Boulder, Colorado, 1963).
(6) SPRAGUE, R. *Math. Z.* **45** (1939), 607–608.
(7) WILLCOCKS, T. H. *Canad. J. Math.* **3** (1951), 304–308.

https://doi.org/10.1017/S0305004100038573 Published online by Cambridge University Press