Adv. Appl. Prob. **50,** 944–982 (2018)
*doi:*10.1017/apr.2018.42
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LARGE DEVIATIONS FOR RANDOMLY CONNECTED NEURAL NETWORKS: I. SPATIALLY EXTENDED SYSTEMS

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Abstract

In a series of two papers, we investigate the large deviations and asymptotic behavior of stochastic models of brain neural networks with random interaction coefficients. In this first paper, we take into account the spatial structure of the brain and consider first the presence of interaction delays that depend on the distance between cells and then the Gaussian random interaction amplitude with a mean and variance that depend on the position of the neurons and scale as the inverse of the network size. We show that the empirical measure satisfies a large deviations principle with a good rate function reaching its minimum at a unique spatially extended probability measure. This result implies an averaged convergence of the empirical measure and a propagation of chaos. The limit is characterized through a complex non-Markovian implicit equation in which the network interaction term is replaced by a nonlocal Gaussian process with a mean and covariance that depend on the statistics of the solution over the whole neural field.

Keywords: Large deviation; convergence of measure; neural network; disordered system

2010 Mathematics Subject Classification: Primary 60F10
Secondary 60B10; 92B20; 82C44

1. General introduction

In this two-part paper, we study the asymptotic behavior of neuronal networks with heterogeneous interconnections inspired by neurobiology. The model we consider describes the stochastic dynamics of N interconnected neurons in a random environment. Each neuron is described by a real state variable (e.g. its voltage)—the results generalize to multi-dimensional variables with obvious modifications—and the network satisfies the following stochastic differential equation (SDE):

$$\mathrm{d}X_t^{i,N} = \left(f(r_i, t, X_t^{i,N}) + \sum_{j=1}^{N} J_{ij} b(X_t^{i,N}, X_{t-\tau(r_i,r_j)}^{j,N}) \right) \mathrm{d}t + \lambda(r_i)\, \mathrm{d}W_t^i, \tag{1.1}$$

where $X_t^{i,N}$ is the state of neuron $i \in \{1, \dots, N\}$ at time t.

- The map f denotes each neuron's intrinsic dynamics, which may depend on time (e.g. to model nonconstant stimuli) and on the neuron's identity through a heterogeneity parameter $r_i \in D$ with D a compact subset of $\mathbb{R}^d$, in this paper modeling the spatial location of the neuron, but possibly representing other heterogeneities (e.g. intrinsic oscillatory frequency), an interpretation favored in the companion paper [13].

Received 18 January 2017; revision received 1 June 2018.
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https://doi.org/10.1017/apr.2018.42 Published online by Cambridge University Press

- The terms $(J_{ij}b(x^i, x^j))$ represent the impact of the neuron j with state x^j onto the neuron i with state x^i. The amplitude of these interaction coefficients $(J_{ij})_{(i,j)\in\{1,\dots,N\}^2}$, called *synaptic efficiencies*, are modeled as independent real Gaussian variables with mean and variance scaling as the inverse of the network size. Moreover, the time necessary to propagate and transmit the information is taken into account in the delay map with τ depending on the cells' locations.

- The stochastic fluctuations are driven by independent standard real-valued Brownian motions at the level of each cell $(W_t^i)_{t\geq 0}$, with positive diffusion coefficient depending on the cell location.

In this framework, the scaling of the variance of the coefficients J_{ij} of order $1/N$ decays much slower than in the more classical mean-field scaling $\mathcal{O}(1/N^2)$; see [22], [32], and [46]. The scaling chosen was shown to preserve a nontrivial contribution of the fluctuation of the synaptic weights in the thermodynamic limit, leading to rich behaviors depending on disorder levels. In particular, in the physics literature it has been shown that, under this scaling assumption, the disorder level governs the glassy transition in spin glass systems [41], a chaotic transition in randomly connected neural networks [42], and a transition in the Kuramoto model whose nature was debated in [18], [19], [43], and [44]. Ben Arous and Guionnet [2], [3], [28] developed a general mathematical methodology relying on large deviation techniques to characterize the thermodynamic limit of Langevin spin glass systems with bounded spins, linear interactions $b(x, y) = y$, and random connectivity coefficients with variance $1/N$. These methods were soon adapted to biological neural networks with sigmoidal interactions $b(x, y) = S(y)$, either in discrete time [15], [20], [25], [35], or in continuous time with multiple populations and delays [12]. All these cases did not take into account firstly the spatial structure of the network along with biologically relevant space-dependent delay, nor secondly the interactions depending on the state of both neurons (i.e. b having a nontrivial dependence in both its variables). Addressing these two issues is precisely the topic of this two-part paper.

In this first paper we address the specific difficulties associated with the spatial extension of the network by keeping $b(x, y) = S(y)$, while the companion paper [13] deals with the case of a general interaction term, in which the spatial dependence is ignored. Our proof follows the general methodology developed in [2]; the two papers are organized in a similar fashion and, to avoid repetition, the proofs that do not present additional difficulties in the second part will appear only in the first. Furthermore, treating these two independent difficulties separately notably allows us to obtain results valid throughout the present paper, while specific difficulties of the companion paper [13] will lead us to restrict our analysis to a finite-time interval, but in which we will prove stronger estimates.

The biological motivation for this first paper is to strengthen the link between the microscopic dynamics of (1.1) with the spatiotemporal patterns of activity recorded at the surface of the cortex. These patterns are biologically relevant as they correlate with cognitive processes such as memory [27], visual illusions [30], or the propagation of a localized stimulus [36]. Historically, an efficient—though heuristic—mathematical model was famously provided by Wilson and Cowan [49], [50] in order to describe the macroscopic dynamics of neural fields and to account for these spatial phenomena. The authors proposed an integro-differential equation involving the level of activity $u(r, t)$ of cells at location r, i.e.

$$\frac{\partial u}{\partial t} = -u(r,t) + \int_D J(r,r')S(u(r',t))\,\mathrm{d}r' + I(r,t), \tag{1.2}$$

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where $I(r,t)$ represents the input at location r, $J(r,r')$ is the averaged interconnection weight from neurons at location r' onto neurons at location r, and the nondecreasing map S associate the resulting firing rate to a level of activity u. While this equation has been very successful in describing a number of biological phenomena—in particular, working memory [31] and visual hallucination patterns [7], [24]—it lacked a rigorous derivation. Justifying the relationship between the macroscopic dynamics of neural fields and the properties of finite-sized networks is a prominent issue in computational neuroscience [6]. It has been the topic of numerous investigations using partial differential equation formalisms and kinetic equations with applications to the visual system [14], [38], [39], moment reductions and master equations [5], [33], but also the development of specific Markov chain models describing the dynamics of Wilson–Cowan systems in the thermodynamics limit [5], [8]–[11]. Spatially extended networks with connectivity coefficients having a variance scaling as $1/N^2$ were investigated in [32], [47], and [48] using coupling or compactness methods. We refer the reader to the general work on large deviations for interacting heterogeneous diffusions with nonrandom interconnections [17], a landmark work with numerous applications. Unfortunately, none of these techniques can handle cases where the synaptic couplings have a variance decaying in $1/N$, as we address here. This scaling has the advantage of enriching the original Wilson–Cowan equation by making the effect of heterogeneous connections explicit.

The organization of this paper is as follows. In Section 2 we provide the general notation and assumptions used here and in the companion paper [13]. In Section 3 we summarize the main results proved in this first paper. Section 4 is dedicated to the proof of a weak large deviations principle for the averaged system. In Section 5 we develop an alternative and more compact methodology compared to [2] and [28] for characterizing the limit under our regularity assumptions. Moreover, we combine all our results to conclude with the convergence of the empirical measure and the propagation of the chaos property.

2. Mathematical setting

In order to take into account the random spatial organization of the network, we consider that the neurons have independent locations $r_i \in D$ drawn according to a probability measure $\pi \in \mathcal{M}_1^+(D)$ representing the density of neurons over the cortex, and assumed to be absolutely continuous with respect to Lebesgue's measure. Moreover, the connectivity coefficients J_{ij} are Gaussian variables, independent conditionally on the location of the neurons, and with laws depending on the location of both neurons, i.e.

$$J_{ij} \sim \mathcal{N}\left(\frac{J(r_i,r_j)}{N}, \frac{\sigma^2(r_i,r_j)}{N}\right).$$

(To fix ideas, we will consider $J_{ij} = J(r_i,r_j)/N + (\sigma(r_i,r_j)/\sqrt{N})\xi_{ij}$ with ξ_{ij} independent standard Gaussian random variables independent of the variables $(r_i)_{i\in\{1,\dots,N\}}$.) There are thus two sources of randomness.

- *Random environment:* the locations of neurons and connectivity coefficients are random variables in a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}})$.
- *Stochastic dynamics:* the neuron states are driven by a collection of independent $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$-Brownian motions $(W_t^i)_{i\in\mathbb{N}}$ independent of the environment.

Hence, the dynamics of the $X^{i,N}$ depends both on the random environment (i.e. the realization of locations $\boldsymbol{r}$ and weights J) and noise (the realization of the Brownian motions). We denote

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by $\mathcal{E}$ the expectation over the environment (i.e. with respect to the probability distribution $\mathcal{P}$) and introduce the shorthand notation $\mathcal{P}_J$ and $\mathcal{E}_J$ for the probability and expectation over the synaptic weights matrix J only (i.e. $\mathcal{P}$ and $\mathcal{E}$ conditioned over the positions $\boldsymbol{r}$).

Throughout the paper, we make the following regularity assumptions.

- The function $f\colon D \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is K_f-Lipschitz continuous in its three variables.
- The map $b\colon \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is bounded ($\|b\|_\infty := \sup_{x,y\in\mathbb{R}\times\mathbb{R}} |b(x, y)| < \infty$) and K_b-Lipschitz continuous in all variables.
- The mean and variance maps of the weights J and σ are bounded and, respectively, K_J and K_σ-Lipschitz continuous with respect to both variables. We denote

$$\|J\|_\infty = \sup_{(r,r')\in D^2} |J(r, r')|, \qquad \|\sigma\|_\infty = \sup_{(r,r')\in D^2} \sigma(r, r').$$

- We have $\tau\colon D^2 \to \mathbb{R}^+$ is Lipschitz continuous, with constant K_τ. It is, in particular, bounded by the compactness of D. We denote by $\bar{\tau}$ its supremum.
- The diffusion coefficient $\lambda\colon D \to \mathbb{R}_+^*$ is a K_λ-Lipschitz continuous and uniformly lower bounded for all $r \in D$, $\lambda(r) \geq \lambda_* > 0$.

Throughout the paper, $\mathcal{M}_1^+(\Sigma)$ will denote the set of Borel probability measures on the Polish space Σ. We denote $\mathcal{C}_\tau := \mathcal{C}([-\bar{\tau}, 0], \mathbb{R})$. Moreover, we define the initial condition at each location through the map $\mu_0\colon D \to \mathcal{M}_1^+(\mathcal{C}_\tau)$ taking value in the space of probability distributions with bounded second moment, and assume that this map is continuous in the sense that there exists a random mapping $\bar{x}^0\colon D \to \mathcal{C}_\tau$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and $C_0 > 0$ such that, for all $r, r' \in D$,

$$\mathcal{L}(\bar{x}^0(r)) = \mu_0(r), \quad \mathbb{E}\Big[\sup_{-\bar{\tau}\leq s\leq 0} |\bar{x}_s^0(r) - \bar{x}_s^0(r')|^2\Big] \leq C_0\|r - r'\|_{\mathbb{R}^d}^2, \tag{2.1}$$

where '$\|\cdot\|_{\mathbb{R}^d}$' denotes the Euclidean norm on $\mathbb{R}^d$. We denote by $\|\cdot\|_{\tau,\infty}$ the supremum norm on $\mathcal{C}_\tau$. We assume that the network's initial conditions are independent realizations of μ_0, i.e.

$$\text{law of } (x_t)_{t\in[-\bar{\tau},0]} = \bigotimes_{i=1}^N \mu_0(r_i). \tag{2.2}$$

Since (1.1) is a standard delayed SDE with regular parameters, is classically well posed; see [16] and [34].

Proposition 2.1. *For each $\boldsymbol{r} \in D^N$, and $J \in \mathbb{R}^{N\times N}$ and $T > 0$, there exists a unique weak solution to system (1.1) defined on $[-\bar{\tau}, T]$ with initial condition (2.2). Moreover, this solution is square integrable.*

Remark 2.1. Note that if the initial condition is $(X_t^{i,N})_{t\in[-\bar{\tau},0]} = \zeta^i$ with $\zeta^i \overset{\mathcal{L}}{=} \mu_0(r_i)$, where '$\overset{\mathcal{L}}{=}$' denotes equality in law for random variables, we can, of course, prove strong existence and the uniqueness of solutions.

We now work with an arbitrary fixed time $T > 0$ and denote by $Q_{\boldsymbol{r}}^N(J)$ the unique in law solution of the network equations restricted to the σ-algebra $\sigma(X_s^{i,N},\ 1 \leq i \leq N,\ -\bar{\tau} \leq s \leq T)$. The solution $Q_{\boldsymbol{r}}^N(J)$ is a probability measure on $\mathcal{C}^N$, where $\mathcal{C} := \mathcal{C}([-\bar{\tau}, T], \mathbb{R})$

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is the space of real-valued continuous functions from $[-\bar{\tau}, T]$ to $\mathbb{R}$. This measure depends on the realizations of both the connectivity matrix J and the locations of neurons $\boldsymbol{r}$. For any $t \in [0, T]$, we denote by $\|\cdot\|_{\infty,t}$ the supremum norm on $\mathcal{C}([-\bar{\tau}, t], \mathbb{R})$. In order to characterize the behavior of the system as the network size grows, we will show a large deviations principle (LDP) for the empirical measure $\hat{\mu}_N \in \mathcal{M}_1^+(\mathcal{C} \times D)$ defined as

$$\hat{\mu}_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{(X^{i,N}, r_i)}, \tag{2.3}$$

where $\delta_{(x,r)} \in \mathcal{M}_1^+(\mathcal{C} \times D)$ denotes the Dirac measure at $(x, r) \in \mathcal{C} \times D$.

We will rely on Sanov's theorem which is central in large deviations techniques. This result states that the empirical measure of independent copies of the same law μ on a Polish space Σ satisfies a full LDP with a good rate function corresponding to the relative entropy $I(. \mid \mu)$ defined, for $\nu \in \mathcal{M}_1^+(\Sigma)$, by

$$I(\nu \mid \mu) := \begin{cases} \displaystyle\int \log \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \, \mathrm{d}\nu & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

Because of the interaction term and the lack of symmetry due to the particular realization of the coupling coefficients, it is clear that the variables $(X^{i,N}, r_i)$ satisfying (1.1) are neither independent nor identically distributed, and one cannot use naively Sanov's theorem to show an LDP for the empirical measure (2.3). Sanov's theorem will nevertheless apply in an averaged sense for a network with no interaction, and we will build upon this result and on a particular generalization of Varadhan's lemma (see [21] and [23]) to derive a weak LDP for the empirical measure of the original network (1.1) averaged on the environment variables.

The law of a neuron at location $r \in D$ in the absence of interaction, denoted $P_r \in \mathcal{M}_1^+(\mathcal{C})$, is the unique weak solution of the one-dimensional standard SDE, i.e.

$$\mathrm{d}X_t = f(r, t, X_t)\,\mathrm{d}t + \lambda(r)\,\mathrm{d}W_t, \qquad (X_t)_{t\in[-\bar{\tau},0]} \overset{\mathcal{L}}{=} \mu_0(r). \tag{2.4}$$

We remark that, by a direct application of Girsanov's theorem, $Q_{\boldsymbol{r}}^N(J)$ is absolutely continuous with respect to $P_{\boldsymbol{r}} := \bigotimes_{i=1}^N P_{r_i}$, with density

$$\begin{aligned} \frac{\mathrm{d}Q_{\boldsymbol{r}}^N(J)}{\mathrm{d}P_{\boldsymbol{r}}}(\boldsymbol{x}) = \exp\Bigg\{ & \sum_{i=1}^{N} \int_0^T \left(\frac{1}{\lambda(r_i)} \sum_{j=1}^{N} J_{ij} b(x_t^i, x_{t-\tau(r_i,r_j)}^j) \right) \mathrm{d}W_t(x^i, r_i) \\ & - \frac{1}{2} \int_0^T \left(\frac{1}{\lambda(r_i)} \sum_{j=1}^{N} J_{ij} b(x_t^i, x_{t-\tau(r_i,r_j)}^j) \right)^2 \mathrm{d}t \Bigg\}, \end{aligned} \tag{2.5}$$

where $\boldsymbol{x} = (x^1, \ldots, x^N) \in \mathcal{C}^N$ and, for all $(x, r) \in \mathcal{C} \times D$, $t \in [0, T]$,

$$W_t(x, r) := \frac{x_t - x_0}{\lambda(r)} - \int_0^t \frac{f(r, s, x_s)}{\lambda(r)} \, \mathrm{d}s. \tag{2.6}$$

We remark that, by (2.4), $(W_t(\cdot, r))_t$ is a P_r-Brownian motion. Moreover, under $P_{\boldsymbol{r}}$, the Brownian motions $(W_t(x^i, r_i),\ 0 \le t \le T)_{i\in\{1,\ldots,N\}}$ are independent.

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Under $P_{\boldsymbol{r}}$, neurons are independent but not identically distributed, while the pairs $(X^{i,N}, r_i)$ are, if we take into account the random character of the locations. We thus introduce the probability measure $P \in \mathcal{M}_1^+(\mathcal{C} \times D)$ defined by $\mathrm{d}P(x, r) := \mathrm{d}P_r(x)\,\mathrm{d}\pi(r)$; see Appendix A for details. We remark that Sanov's theorem does apply to the empirical measure (2.3) considered under $P^{\otimes N}$.

In order to obtain an analogous result for the full network, we construct its associated symmetric law.

Lemma 2.1. *Let $Q_{\boldsymbol{r}}^N := \mathcal{E}_J(Q_{\boldsymbol{r}}^N(J))$. Then the map*

$$\mathcal{Q}: \begin{cases} D^N \to \mathcal{M}_1^+(\mathcal{C}^N), \\ \boldsymbol{r} \to Q_{\boldsymbol{r}}^N \end{cases}$$

is continuous with respect to the weak topology. Moreover,

$$\mathrm{d}Q^N(\boldsymbol{x}, \boldsymbol{r}) := \mathrm{d}Q_{\boldsymbol{r}}^N(\boldsymbol{x})\,\mathrm{d}\pi^{\otimes N}(\boldsymbol{r})$$

defines a probability measure on $\mathcal{M}_1^+((\mathcal{C} \times D)^N)$.

Proof. See Appendix A. □

As mentioned in the introduction, we adopt the classical firing-rate formalism of [1], [49], and [50] in this first paper by keeping $b(x, y) = S(y)$. Here, S is a sigmoidal transformation assumed to be a smooth (at least continuously differentiable) increasing map tending to 0 at $-\infty$ and to 1 at ∞.

Equation (1.1) thus becomes

$$\mathrm{d}X_t^{i,N} = \left(f(r_i, t, X_t^{i,N}) + \sum_{j=1}^{N} J_{ij} S(X_{t-\tau(r_i, r_j)}^{j,N}) \right) \mathrm{d}t + \lambda(r)\,\mathrm{d}W_t^i. \tag{2.7}$$

We now state the main results for this dynamics.

3. Statement of the main results

Theorem 3.1. *It holds that $(Q^N(\hat{\mu}_N \in \cdot))_{N \in \mathbb{N}^*}$ satisfies a weak LDP of speed N and converges, as $N \to \infty$, towards a measure $\delta_Q \in \mathcal{M}_1^+(\mathcal{M}_1^+(\mathcal{C} \times D))$ for Q characterized in Theorem 3.4.*

Remark 3.1. Note that, for $T < \lambda_*^2/2\|\sigma\|_\infty^2\|b\|_\infty^2$, a full LDP can be proved, as we can readily prove exponential tightness of the averaged empirical measure; see, e.g. [2] and the companion paper [13]. As we are mainly interested in proving convergence of the empirical measure, a weak LDP will be sufficient in this first paper, allowing us to show the convergence results without the limitation of time.

Theorem 3.1 relies on two key quantitative estimates on the sequence of empirical measures: LDP and tightness, formally expressed as follows.

Theorem 3.2. *There exists a good rate function H on $\mathcal{M}_1^+(\mathcal{C} \times D)$ such that for any compact subset K of $\mathcal{M}_1^+(\mathcal{C} \times D)$,*

$$\limsup_{N \to \infty} \frac{1}{N} \log Q^N(\hat{\mu}_N \in K) \leq -\inf_K H.$$

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Theorem 3.3. *For any real number $\varepsilon > 0$, there exists a compact subset K_ε such that for any integer N,*

$$Q^N(\hat{\mu}_N \notin K_\varepsilon) \leq \varepsilon.$$

These two results imply convergence of the empirical measure towards the set of minima of the rate function H. The uniqueness and characterization of the limit is the subject of the next theorem, proved in Section 5.

Theorem 3.4. *The good rate function H achieves its minimal value at the unique probability measure $Q \in \mathcal{M}_1^+(\mathcal{C} \times D)$ satisfying*

$$Q \sim P, \qquad \frac{\mathrm{d}Q}{\mathrm{d}P}(x,r) = \mathcal{E}\left[\exp\left\{\frac{1}{\lambda(r)}\int_0^T G_t^Q(r)\,\mathrm{d}W_t(x,r) - \frac{1}{2\lambda(r)^2}\int_0^T G_t^Q(r)^2\,\mathrm{d}t\right\}\right],$$

where $(W_t)_{t\in[0,T]}$ is a P-Brownian motion, and $G^Q(r)$ is, under $\mathcal{P}$, a Gaussian process with mean and covariance

$$\mathcal{E}[G_t^Q(r)] = \int_{\mathcal{C}\times D} J(r,r')S(x_{t-\tau(r,r')})\,\mathrm{d}Q(x,r'), \tag{3.1}$$

$$\mathcal{E}[G_t^Q(r)G_s^Q(r)] = \int_{\mathcal{C}\times D} \sigma(r,r')^2 S(x_{t-\tau(r,r')})S(x_{s-\tau(r,r')})\,\mathrm{d}Q(x,r). \tag{3.2}$$

The convergence result of Theorem 3.1 also implies the propagation of chaos, thanks to a result due to Sznitman [45, Lemma 3.1].

Theorem 3.5. *The system enjoys the propagation of the chaos property. In other terms, Q^N is Q-chaotic, i.e. for any bounded continuous functions $\phi_1, \ldots, \phi_m \in \mathcal{C}_b(\mathcal{C} \times D)$ and any neuron indexes $(k_1, \ldots, k_m)$, we have*

$$\lim_{N\to\infty} \int \prod_{j=1}^m \phi_j(x^{k_j}, r_{k_j})\,\mathrm{d}Q^N(x,r) = \prod_{j=1}^m \int \phi_j(x,r)\,\mathrm{d}Q(x,r).$$

We now proceed to the proof of our results.

4. The LDP

In this section we establish the weak LDP result. We proceed by characterizing the good rate function, in the flavor of Varadhan's lemma, before demonstrating an upper bound result valid on compact sets, and the tightness for the collection of empirical measures. While the approach is classical, specific difficulties arise: Varadhan's lemma does not apply directly, as the map we derive is neither upper bounded nor continuous; moreover, the spatial framework introduces new difficulties, requiring the introduction of an appropriate distance on $\mathcal{C} \times D$.

4.1. Construction of the good rate function

Heuristically, the interaction term of (2.7),

$$G_t^{i,N}(\boldsymbol{x},\boldsymbol{r}) := \frac{1}{\lambda(r_i)}\sum_{j=1}^N J_{ij}S(x^j_{t-\tau(r_i,r_j)}),$$

under Boltzmann's molecular chaos approach, behaves as a Gaussian process in the large N limit (by virtue of a central limit theorem). With this heuristic in mind, we introduce,

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for $\mu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, the plausible mean and covariance functions of its limit, respectively defined on $[0, T]^2 \times D$ and $[0, T] \times D$, i.e.

$$K_\mu(s,t,r) := \frac{1}{\lambda(r)^2} \int_{\mathcal{C}\times D} \sigma(r,r')^2 S(x_{t-\tau(r,r')}) S(x_{s-\tau(r,r')}) \, \mathrm{d}\mu(x,r'),$$
$$m_\mu(t,r) := \frac{1}{\lambda(r)} \int_{\mathcal{C}\times D} J(r,r') S(x_{t-\tau(r,r')}) \, \mathrm{d}\mu(x,r').$$

Here, μ can be understood as the putative limit law of the couple (x^j, r_j) if it exists. The covariance and mean functions K_μ and m_μ are well defined, since for every fixed $r \in D$, the two maps

$$A_r : (x,r') \to \frac{1}{\lambda(r)} J(r,r') S(x_{\cdot-\tau(r,r')}),$$
$$\tilde{A}_r : (x,r') \to \frac{1}{\lambda(r)^2} \sigma(r,r')^2 S(x_{\cdot-\tau(r,r')}) S(x_{\cdot-\tau(r,r')}),$$

with $x_{\cdot-\tau(r,r')}$, the time-shifted map, i.e. $x_{\cdot-\tau(r,r')}(t) = x_{t-\tau(r,r')}$ are continuous for the classical product norm $\|(x,r')\|_{\mathcal{C}\times D} := \sup_{t\in[-\bar{\tau},T]} |x(t)| + \|r\|_{\mathbb{R}^d}$. Hence, they are Borel measurable and integrable with respect to every element of $\mathcal{M}_1^+(\mathcal{C} \times D)$. We remark that, since S takes value in $[0, 1]$, both functions are bounded, i.e.

$$|K_\mu(s,t,r)| \le \frac{\|\sigma\|_\infty^2}{\lambda_*^2} \quad \text{and} \quad |m_\mu(t,r)| \le \frac{\|J\|_\infty}{\lambda_*}.$$

Moreover, as μ is a measure with support contained in the space of continuous functions, K_μ and m_μ are continuous maps by the dominated convergence theorem.

By construction, K_μ has a covariance structure. As a consequence, we can define a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \gamma)$ and a family of independent stochastic processes $(G^\mu(r))_{\mu\in\mathcal{M}_1^+(\mathcal{C}\times D), r\in D}$ for any measure $\mu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, such that $G^\mu(r)$ is a centered Gaussian process with covariance $K_\mu(\cdot,\cdot,r)$ under γ. This ensures the continuity of the map $r \to \mathcal{L}(G^\mu(r))$. We denote by $\mathcal{E}_\gamma$ the expectation under this measure.

Remark 4.1. (i) As in [2] and [12], we could alternatively have defined a family of probability measure $(\gamma_\mu)_{\mu\in\mathcal{M}_1^+(\mathcal{C}\times D)}$, and a family of Gaussian processes $(G(r))_{r\in D}$ with covariance K_μ under γ_μ. Our approach is equivalent with the advantage of being adapted to Fubini's theorem.

(ii) The family of processes $(G_t^\mu(r))_{\mu,r}$ is intended to encompass possible candidates for the effective asymptotic interactions $\lim_N (G^{i,N}(\boldsymbol{x},\boldsymbol{r}))_{i\in\mathbb{N}^*}$. In these interactions, the Gaussian weights are independent for different particles, so that it seems natural to assume the independence of $(G_t^\mu(r))$ for different locations. Notably in our proof, we can swap from a continuous version of $G_t^\mu(r)$ to an independent one very easily, as they are never taken jointly under γ. Thus, we can literally choose their covariance structure. For the sake of measurability under any Borel measure of $\mathcal{M}_1^+(\mathcal{C} \times D)$, we will mainly work with the continuous version of $G^\mu(r)$, and explicitly introduce independent versions when independence is needed.

We recall a few general properties on the relative entropy that are used repeatedly throughout the paper. For p and q two probability measures on a Polish space E (see, e.g. [23, Lemma 3.2.13]), we have

$$I(q \mid p) = \sup\left\{ \int_E \Phi \, \mathrm{d}q - \log \int_E \exp\{\Phi\} \, \mathrm{d}p;\ \Phi \in \mathcal{C}_b(E) \right\},$$

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which implies, in particular, that for any bounded measurable function Φ on E,

$$\int_E \Phi \, \mathrm{d}q \leq I(q \mid p) + \log \int_E \exp\{\Phi\} \, \mathrm{d}p. \tag{4.1}$$

If Φ is a lower bounded (or upper bounded) measurable function, this inequality holds by monotone convergence.

We introduce some useful objects for the general scope of our proof. For any Gaussian process $(G_t)_{t\in[0,T]}$ of $(\hat{\Omega}, \hat{\mathcal{F}}, \gamma)$, and any $t \in [0, T]$,

$$\Lambda_t(G) := \frac{\exp\{-\int_0^t G_s^2 \, \mathrm{d}s/2\}}{\mathcal{E}_\gamma[\exp\{-\int_0^t G_s^2 \, \mathrm{d}u/2\}]}.$$

For any $t \in [0, T]$, $r \in D$, and $\nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, the following defines a probability measure on $(\hat{\Omega}, \hat{\mathcal{F}})$ (see [37]):

$$\mathrm{d}\gamma_{\widetilde{K}^t_{\nu,r}}(\omega) := \Lambda_t(G^\nu(\omega, r)) \, \mathrm{d}\gamma(\omega) \quad \text{for all } \omega \in \hat{\Omega}.$$

As proven in [37], $G^\nu(r)$ is still a centered Gaussian process under $\gamma_{\widetilde{K}^t_{\nu,r}}$, with covariance

$$\widetilde{K}^t_{\nu,r}(s, u) := \mathcal{E}_\gamma[G^\nu_u(r) G^\nu_s(r) \Lambda_t(G^\nu(r))].$$

We also define for any $\nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, $(x, r) \in \mathcal{C} \times D$, and $t \in [0, T]$, the processes

$$\begin{aligned} L^\nu_t(x, r) &:= \int_0^t G^\nu_s(r)(\mathrm{d}W_s(x, r) - m_\nu(s, r) \, \mathrm{d}s), \\ V^\nu_t(x, r) &:= W_t(x, r) - \int_0^t m_\nu(s, r) \, \mathrm{d}s. \end{aligned} \tag{4.2}$$

We now characterize a few properties of these objects.

Proposition 4.1. *There exists a constant $C_T > 0$ depending on T such that for any $\nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, $r \in D$, and $t \in [0, T]$,*

$$\sup_{0\leq s,u\leq t} \widetilde{K}^t_{\nu,r}(s, u) \leq C_T, \qquad \Lambda_t(G^\nu(r)) \leq C_T, \tag{4.3}$$

$$\mathcal{E}_\gamma\left[\exp\left\{-\frac{1}{2}\int_0^T G^\nu_t(r)^2 \, \mathrm{d}t\right\}\right] = \exp\left\{-\frac{1}{2}\int_0^T \widetilde{K}^t_{\nu,r}(t, t) \, \mathrm{d}t\right\}. \tag{4.4}$$

Moreover, if $(G_t)_{0\leq t\leq T}$ and $(G'_t)_{0\leq t\leq T}$ are two centered Gaussian processes of $(\hat{\Omega}, \hat{\mathcal{F}}, \gamma)$ with uniformly bounded covariance, then there exists $\tilde{C}_T > 0$ such that for all $t \in [0, T]$,

$$|\Lambda_t(G) - \Lambda_t(G')| \leq \tilde{C}_T \left\{\int_0^t \mathcal{E}_\gamma[(G_s - G'_s)^2]^{1/2} \, \mathrm{d}s + \int_0^t |G_s^2 - {G'_s}^2| \, \mathrm{d}s\right\}. \tag{4.5}$$

Proof. Observe that

$$\begin{aligned} \Lambda_t(G^\nu(r)) &\leq \mathcal{E}_\gamma\left[\exp\left\{-\frac{1}{2}\int_0^t G^\nu_s(r)^2 \, \mathrm{d}u\right\}\right]^{-1} \\ &\leq \exp\left\{\frac{1}{2}\int_0^t \mathcal{E}_\gamma[G^\nu_s(r)^2] \, \mathrm{d}u\right\} \\ &\leq \exp\left\{\frac{\|\sigma\|^2_\infty t}{2\lambda^2_*}\right\}, \end{aligned}$$

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where the second equality follows from Jensen's inequality. Then

$$\begin{aligned}\widetilde{K}^t_{\nu,r}(s,u) &= \mathcal{E}_\gamma[G^\nu_u(r)G^\nu_s(r)\Lambda_t(G^\nu(r))] \\ &\le \sqrt{K_\nu(s,s,r)K_\nu(t,t,r)}\exp\left\{\frac{\|\sigma\|^2_\infty t}{2\lambda^2_*}\right\} \\ &\le \frac{\|\sigma\|^2_\infty}{\lambda^2_*}\exp\left\{\frac{\|\sigma\|^2_\infty t}{2\lambda^2_*}\right\},\end{aligned}$$

where the first inequality follows from the Cauchy–Schwarz inequality. For (4.4), let $f(t) := \mathcal{E}_\gamma[\exp\{-\frac{1}{2}\int_0^t G^\nu_s(r)^2\,\mathrm{d}s\}]$. As $(t,\omega) \to G^\nu_t(\omega,r)^2\exp\{-\frac{1}{2}\int_0^t G^\nu_s(\omega,r)^2\,\mathrm{d}s\}$ is well defined, γ almost surely continuous, and integrable under γ, we have

$$f'(t) = -\frac{1}{2}\mathcal{E}_\gamma\left[G^\nu_t(r)^2\exp\left\{-\frac{1}{2}\int_0^t G^\nu_s(r)^2\,\mathrm{d}s\right\}\right] = -\frac{1}{2}\widetilde{K}^t_{\nu,r}(t,t)f(t),$$

so that integrating f'/f yields the result. Furthermore, letting $(G_t)_{0\le t\le T}$ and $(G'_t)_{0\le t\le T}$ be two centered γ-Gaussian processes with variance bounded by a common constant C_T, we have

$$|\Lambda_t(G) - \Lambda_t(G')| \le \frac{1}{2}\exp\left\{\frac{\|\sigma\|^2_\infty t}{\lambda^2_*}\right\}\left\{\int_0^t \mathcal{E}_\gamma[|G_s^2 - {G'_s}^2|]\,\mathrm{d}s + \int_0^t |G_s^2 - {G'_s}^2|\,\mathrm{d}s\right\},$$

where we have used the Lipschitz continuity of the exponential on $\mathbb{R}_-$. Consequently, relying on the Cauchy–Schwarz inequality, we obtain

$$|\Lambda_t(G) - \Lambda_t(G')| \le \tilde{C}_T\left\{\int_0^t \mathcal{E}_\gamma[(G_s - G'_s)^2]^{1/2}\,\mathrm{d}s + \int_0^t |G_s^2 - {G'_s}^2|\,\mathrm{d}s\right\}. \qquad \square$$

We now state a key result to our analysis.

Lemma 4.1. *It holds that*

$$\frac{\mathrm{d}Q^N}{\mathrm{d}P^{\otimes N}}(\boldsymbol{x},\boldsymbol{r}) = \exp\{N\bar{\Gamma}(\hat{\mu}_N)\},$$

where

$$\bar{\Gamma}(\hat{\mu}_N) := \frac{1}{N}\sum_{i=1}^N \log\mathcal{E}_\gamma\left[\exp\left\{\int_0^T (G_t^{\hat{\mu}_N}(r_i) + m_{\hat{\mu}_N}(t,r_i))\,\mathrm{d}W_t(x^i,r_i) - \frac{1}{2}\int_0^T (G_t^{\hat{\mu}_N}(r_i) + m_{\hat{\mu}_N}(t,r_i))^2\,\mathrm{d}t\right\}\right],$$

with W defined as in (2.6).

Proof. Let, for all $(\boldsymbol{x},\boldsymbol{r}) \in (\mathcal{C}\times D)^N$,

$$X^N_i(\boldsymbol{x},\boldsymbol{r}) := \int_0^T G^{i,N}_t(\boldsymbol{x},\boldsymbol{r})\,\mathrm{d}W_t(x^i,r_i) - \frac{1}{2}\int_0^T G^{i,N}_t(\boldsymbol{x},\boldsymbol{r})^2\,\mathrm{d}t.$$

From (2.5),

$$\frac{\mathrm{d}Q^N_{\boldsymbol{r}}(J)}{\mathrm{d}P_{\boldsymbol{r}}}(\boldsymbol{x}) = \exp\left\{\sum_{i=1}^N X^N_i(\boldsymbol{x},\boldsymbol{r})\right\}.$$

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Averaging on J and applying Fubini's theorem, we find that $Q_{\boldsymbol{r}}^N \ll P_{\boldsymbol{r}}$ with density

$$\frac{\mathrm{d}Q_{\boldsymbol{r}}^N}{\mathrm{d}P_{\boldsymbol{r}}}(\boldsymbol{x}) = \mathcal{E}_J\left[\exp\left\{\sum_{i=1}^N X_i^N(\boldsymbol{x},\boldsymbol{r})\right\}\right].$$

Moreover, $\mathrm{d}Q^N(\boldsymbol{x},\boldsymbol{r}) = \mathrm{d}Q_{\boldsymbol{r}}^N(\boldsymbol{x})\,\mathrm{d}\pi^{\otimes N}(\boldsymbol{r})$ and $\mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r}) = \mathrm{d}P_{\boldsymbol{r}}(\boldsymbol{x})\,\mathrm{d}\pi^{\otimes N}(\boldsymbol{r})$ yield

$$\frac{\mathrm{d}Q^N}{\mathrm{d}P^{\otimes N}}(\boldsymbol{x},\boldsymbol{r}) = \prod_{i=1}^N \mathcal{E}_J[\exp\{X_i^N(\boldsymbol{x},\boldsymbol{r})\}] = \exp\left\{\sum_{i=1}^N \log \mathcal{E}_J[\exp\{X_i^N(\boldsymbol{x},\boldsymbol{r})\}]\right\},$$

where we have used the independence of the synaptic weights J_{ij}. Note that here $\boldsymbol{x}$ are coordinates, thus independent of the J_{ij}, and the fact that $\{G_t^{i,N}(\boldsymbol{x},\boldsymbol{r}),\ 0 \le t \le T\}$ is, under $\mathcal{P}_J$, a Gaussian process with covariance $K_{\hat{\mu}_N}(t,s,r_i)$ and mean $m_{\hat{\mu}_N}(t,r_i)$. □

Varadhan's lemma motivates us to consider the map

$$\int_{\mathcal{C}\times D} \log \mathcal{E}_\gamma[\exp\{X^\mu(x,r)\}]\,\mathrm{d}\mu(x,r),$$

with for all $(x,r) \in \mathcal{C}\times D$ and $\mu \in \mathcal{M}_1^+(\mathcal{C}\times D)$,

$$X^\mu(x,r) := \int_0^T (G_t^\mu(r) + m_\mu(t,r))\,\mathrm{d}W_t(x,r) - \frac{1}{2}\int_0^T (G_t^\mu(r) + m_\mu(t,r))^2\,\mathrm{d}t. \tag{4.6}$$

The rigorous definition of this functional and a few useful properties are the subject of the next proposition.

Proposition 4.2. *The map*

$$\Gamma := \mu \in \mathcal{M}_1^+(\mathcal{C}\times D) \to \begin{cases} \displaystyle\int_{\mathcal{C}\times D} \log \mathcal{E}_\gamma[\exp\{X^\mu(x,r)\}]\,\mathrm{d}\mu(x,r) & \text{if } I(\mu \mid P) < \infty, \\ +\infty & \text{otherwise,} \end{cases}$$

satisfies

(i) $\Gamma \le I(\cdot \mid P)$;

(ii) *for all* $\iota \in (0,1)$, $e \ge 0$, $|\Gamma(\mu)| \le \iota I(\mu \mid P) + e$.

Proof. (i) Let $F_\mu := \log \mathcal{E}_\gamma[\exp\{X^\mu(x,r)\}]$. This is measurable as a continuous function of the maps $(x,r) \to (K_\mu(t,s,r),\ 0 \le t,\ s \le T)$, $(m_\mu(t,r),\ 0 \le t \le T)$, and $(W_t(x,r),\ 0 \le t \le T)$ which are continuous. Nevertheless, due to the contribution of the mean term m_μ (see the expression of Γ_2 in Proposition 4.3), it is not bounded from below, as was the case in [2]. We prove that it is still μ-integrable. In fact, for any $M > 0$,

$$\begin{aligned} -F_\mu^-(x,r) &\le F_\mu^+(x,r) - F_\mu^-(x,r) = F_\mu(x,r) \\ &\le \log(\mathcal{E}_\gamma[\exp\{X^\mu(x,r)\}] \vee M^{-1}) =: F_{\mu,M}(x,r), \end{aligned}$$

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where F_μ^+ and F_μ^- respectively denote the positive and negative parts of F_μ. As F_μ^- and $F_{\mu,M}$ are measurable and bounded from below, (4.1) applies. Let $\alpha \geq 1$. On the one hand,

$$\begin{aligned}\alpha &\int_{\mathcal{C}\times D} F_{\mu,M}(x,r)\,\mathrm{d}\mu(x,r)\\ &\leq I(\mu \mid P) + \log\left\{\int_{\mathcal{C}\times D} \exp\{\alpha F_{\mu,M}(x,r)\}\,\mathrm{d}P(x,r)\right\}\\ &\leq I(\mu \mid P) + \log\left\{M^{-\alpha} + \mathcal{E}_\gamma\left[\int_D \int_{\mathcal{C}} \exp\{\alpha X^\mu(x,r)\}\,\mathrm{d}P_r(x)\,\mathrm{d}\pi(r)\right]\right\},\end{aligned} \tag{4.7}$$

where the inequality follows from Jensen's inequality and Fubini's theorem, with the right-hand side of the two latter inequalities being possibly infinite. On the other hand,

$$\begin{aligned}\alpha &\int_{\mathcal{C}\times D} F_\mu^-(x,r)\,\mathrm{d}\mu(x,r)\\ &= \alpha \int_{\mathcal{C}\times D} (-\log \mathcal{E}_\gamma[\exp\{X^\mu(x,r)\}])^+\,\mathrm{d}\mu(x,r)\\ &\leq I(\mu \mid P) + \alpha C_T + \log\left\{\mathcal{E}_\gamma\left[\int_D \int_{\mathcal{C}} \exp\{-\alpha X^\mu(x,r)\}\,\mathrm{d}P_r(x)\,\mathrm{d}\pi(r)\right]\right\},\end{aligned} \tag{4.8}$$

where the second inequality follows from Jensen's inequality, Fubini's theorem, and (4.1). Moreover, $W(\cdot, r)$ being a P_r-Brownian motion, the martingale property yields

$$\begin{aligned}&\mathcal{E}_\gamma\left[\int_D \int_{\mathcal{C}} \exp\{\alpha X^\mu(x,r)\}\,\mathrm{d}P_r(x)\,\mathrm{d}\pi(r)\right]\\ &\quad\leq \exp\{\alpha C_T\} \int_D \mathcal{E}_\gamma\left[\exp\left\{\frac{\alpha^2-\alpha}{2}\int_0^T (G_t^\mu(r) + m_\mu(t,r))^2\,\mathrm{d}t\right\}\right]\mathrm{d}\pi(r).\end{aligned} \tag{4.9}$$

Letting $\alpha = 1$, we can see that F_μ is μ-integrable with

$$\begin{aligned}\int_{\mathcal{C}\times D} |F_\mu(x,r)|\,\mathrm{d}\mu(x,r) &= \int_{\mathcal{C}\times D} F_\mu^-(x,r) + F_{\mu,1}(x,r)\,\mathrm{d}\mu(x,r)\\ &\leq 2I(\mu \mid P) + C_T + \log(2).\end{aligned}$$

Moreover,

$$\Gamma(\mu) := \int_{\mathcal{C}\times D} F_\mu(x,r)\,\mathrm{d}\mu(x,r) \leq I(\mu \mid P) + \log\{M^{-1}+1\},$$

where the inequality follows from (4.7), so that letting $M \to +\infty$ yields the result.

(ii) For $\alpha \geq 1$, (4.7)–(4.9) ensure that

$$\begin{aligned}\alpha|\Gamma(\mu)| &\leq I(\mu \mid P) + \alpha C_T\\ &\quad + \left|\log\left\{\int_D \mathcal{E}_\gamma\left[\exp\left\{\frac{\alpha^2-\alpha}{2}\int_0^T (G_t^\mu(r) + m_\mu(t,r))^2\,\mathrm{d}t\right\}\right]\mathrm{d}\pi(r)\right\}\right|.\end{aligned}$$

We recall that basic Gaussian calculus yields

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\mathcal{N}(m,v)^2\right\}\right] = \frac{1}{\sqrt{1-v}}\exp\left\{\frac{m^2}{2(1-v)}\right\} = \exp\left\{\frac{1}{2}\left(\frac{m^2}{1-v} - \log(1-v)\right)\right\} \tag{4.10}$$

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as soon as $v < 1$. Jensen's inequality and Fubini's theorem yield

$$\mathcal{E}_\gamma\left[\exp\left\{\frac{(\alpha^2-\alpha)T}{2}\int_0^T (G_t^\mu(r)+m_\mu(t,r))^2\frac{\mathrm{d}t}{T}\right\}\right] \le \int_0^T \mathcal{E}_\gamma\left[\exp\left\{\frac{(\alpha^2-\alpha)T}{2}(G_t^\mu(r)+m_\mu(t,r))^2\right\}\right]\frac{\mathrm{d}t}{T}.$$

As $\sqrt{(\alpha^2-\alpha)T}(G_t^\mu(r)+m_\mu(t,r)) \sim \mathcal{N}(\sqrt{(\alpha^2-\alpha)T}m_\mu(t,r), (\alpha^2-\alpha)TK_\mu(t,t,r))$ under γ, then for small enough $(\alpha-1)$, there exists a constant C_T uniform in space such that

$$\mathcal{E}_\gamma\left[\exp\left\{\frac{(\alpha^2-\alpha)}{2}\int_0^T (G_t^\mu(r)+m_\mu(t,r))^2\,\mathrm{d}t\right\}\right] \le \exp\{(\alpha-1)C_T + \underbrace{o(\alpha-1)}_{\text{uniform in } r}\} \le \exp\{(\alpha-1)C_T\}.$$

This eventually yields

$$|\Gamma(\mu)| \le \iota I(\mu \mid P) + e,$$

with $\iota := 1/\alpha$ and $e := (2\alpha-1)C_T$. □

Remark 4.2. Note that for $\hat{\mu}_N \not\ll P$, $\Gamma(\hat{\mu}_N) = +\infty$ so that it is not equal to $\bar{\Gamma}(\hat{\mu}_N)$. These objects are different in nature, as the latter is random and must be considered under a proper probability measure on $(\mathcal{C}\times D)^N$, making sense of the stochastic integrals over the $(W_t(x^i, r_i),\ 0 \le t \le T)_{i\in\{1,\dots,N\}}$ (which are well defined under $P_{\boldsymbol{r}}$).

As $\mathcal{C}\times D$ and $\mathcal{M}_1^+(\mathcal{C}\times D)$ are Polish spaces, and as the $(X^{i,N}, r_i)$ are independent and identically distributed under $P^{\otimes N}$, Sanov's theorem ensures that $\hat{\mu}_N$ satisfies the LDP with a good rate function $I(\cdot \mid P)$. Furthermore, if Γ was bounded and continuous, Varadhan's lemma would, as a consequence of Lemma 4.1, entail a full LDP under Q^N, with a good rate function

$$H(\mu) := \begin{cases} I(\mu \mid P) - \Gamma(\mu) & \text{if } I(\mu \mid P) < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

At this point it would be easy to conclude, provided that Γ presented a few regularity properties. Unfortunately, Varadhan's lemma assumptions fail here, as Γ is neither continuous nor bounded from above. Obtaining a weak LDP as well as the convergence of the empirical measure requires us to return to the basics of large deviations theory.

Observe that Γ is a nonlinear function of μ, involving, in particular, an exponential term depending on the mean and covariance structure of the Gaussian process. In order to handle terms of this type, a key technique proposed by Ben Arous and Guionnet is to linearize this map by considering the terms in the exponential as depending on an additional variable $\nu \in \mathcal{M}_1^+(\mathcal{C}\times D)$; see [2] and [28]. In our case, this family of linearizations are the maps

$$\Gamma_\nu := \mu \in \mathcal{M}_1^+(\mathcal{C}\times D) \to \begin{cases} \displaystyle\int_{\mathcal{C}\times D} \log \mathcal{E}_\gamma[\exp\{X^\nu(x,r)\}]\,\mathrm{d}\mu(x,r) & \text{if } I(\mu \mid P) < \infty, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mu, \nu \in \mathcal{M}_1^+(\mathcal{C}\times D)$.

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Remark 4.3. Observe that Proposition 4.2 also applies to Γ_ν for every $\nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, and that $\Gamma(\mu) = \Gamma_\mu(\mu)$ for any $\mu \in \mathcal{M}_1^+(\mathcal{C} \times D)$.

Moreover, defining

$$\bar{\Gamma}_\nu(\delta_{(x,r)}) := \log \mathcal{E}_\gamma[\exp\{X^\nu(x,r)\}], \qquad \bar{\Gamma}_\nu(\hat{\mu}_N) := \frac{1}{N}\sum_{i=1}^N \log \mathcal{E}_\gamma[\exp\{X^\nu(x^i,r_i)\}],$$

we note that $\bar{\Gamma}_\nu(\hat{\mu}_N) = (1/N)\sum_{i=1}^N \bar{\Gamma}_\nu(\delta_{(x^i,r_i)})$. Introducing $Q_\nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$ by

$$\mathrm{d}Q_\nu(x,r) := \{\bar{\Gamma}_\nu(\delta_{(x,r)})\}\,\mathrm{d}P(x,r) = \mathcal{E}_\gamma[\exp\{X^\nu(x,r)\}]\,\mathrm{d}P(x,r), \tag{4.11}$$

we thus have

$$\mathrm{d}Q_\nu^{\otimes N}(\boldsymbol{x},\boldsymbol{r}) = \exp\{N\bar{\Gamma}_\nu(\hat{\mu}_N)\}\,\mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r}).$$

This equality highlights the fact that, applying again Sanov's theorem, the empirical measure satisfies a full LDP under $(Q_\nu)^{\otimes N}$ with a good rate function $I(\cdot \mid Q_\nu)$. On the other hand, Vardhan's lemma suggests that $\hat{\mu}_N$ satisfies, under the same measure, an LDP with rate function

$$H_\nu\colon \mu \to \begin{cases} I(\mu \mid P) - \Gamma_\nu(\mu) & \text{if } I(\mu \mid P) < +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

This is, for now, only a supposition, since just like its original counterpart, the map Γ_ν is neither bounded from above nor continuous, and $\Gamma_\nu(\hat{\mu}_N)$ and $\bar{\Gamma}_\nu(\hat{\mu}_N)$ are not equal. However, if the result holds then uniqueness of the good rate function would imply that H_ν is equal to $I(\cdot \mid Q_\nu)$.

We now rigorously demonstrate that this result is correct, justify the definition of Q_ν, and, in Theorem 4.1, prove that the latter equality holds.

To this end, it is useful to introduce the following decomposition of Γ_ν based on Gaussian calculus; see [2], [12], and [37].

Proposition 4.3. *For every $\nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, Γ_ν admits the decomposition*

$$\Gamma_\nu = \Gamma_{1,\nu} + \Gamma_{2,\nu}, \tag{4.12}$$

where, for all $\mu \in \mathcal{M}_1^+(\mathcal{C} \times D)$,

$$\Gamma_{1,\nu}(\mu) := -\frac{1}{2}\int_{\mathcal{C}\times D}\int_0^T \{\widetilde{K}^t_{\nu,r}(t,t) + m_\nu(t,r)^2\}\,\mathrm{d}t\,\mathrm{d}\mu(x,r),$$

and

$$\Gamma_{2,\nu}(\mu) := \begin{cases} \dfrac{1}{2}\displaystyle\int_{\mathcal{C}\times D}\int_{\hat{\Omega}} L_T^\nu(x,r)^2\,\mathrm{d}\gamma_{\widetilde{K}^T_{\nu,r}}\,\mathrm{d}\mu(x,r) \\ \quad + \displaystyle\int_{\mathcal{C}\times D}\int_0^T m_\nu(t,r)\,\mathrm{d}W_t(x,r)\,\mathrm{d}\mu(x,r) & \text{if } I(\mu \mid P) < \infty, \\ +\infty & \text{otherwise.} \end{cases}$$

In particular, we have $\Gamma = \Gamma_1 + \Gamma_2$, where $\Gamma_i(\mu) := \Gamma_{i,\mu}(\mu)$ for $i \in \{1,2\}$ and any $\mu \in \mathcal{M}_1^+(\mathcal{C} \times D)$.

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This decomposition has the effect of splitting the difficulties: while the first term will be relatively easy to handle (see Proposition 4.4), the local martingale term will require finer estimates based on Gaussian calculus and a number of tools from stochastic calculus theory. It is also useful to state the next lemma, central for our analysis; see [2, Lemma 5.15].

Lemma 4.2. *For any $r \in D$, the following equality holds P_r-almost surely:*

$$\mathcal{E}_\gamma[\exp\{X^\nu(x,r)\}] = \exp\left\{\int_0^T O_\nu(t,x,r)\,\mathrm{d}W_t(x,r) - \frac{1}{2}\int_0^T O_\nu^2(t,x,r)\,\mathrm{d}t\right\}, \tag{4.13}$$

where

$$O_\nu(t,x,r) := \mathcal{E}_\gamma[\Lambda_t(G^\nu(r))G_t^\nu(r)L_t^\nu(x,r)] + m_\nu(t,r).$$

We are now ready to state one of the main results of this section, in which we prove the intuitive equality between the two rate functions $I(\cdot \mid Q_\nu)$ and H_ν.

Theorem 4.1. *We have $H_\nu(\mu) = I(\mu \mid Q_\nu)$. In particular, H_ν is a good rate function.*

Proof. Recall (4.11) and further define for any $r \in D$,

$$\mathrm{d}Q_{\nu,r}(x) = \mathcal{E}_\gamma[\exp\{X^\nu(x,r)\}]\,\mathrm{d}P_r(x).$$

It is clear that Q_ν and $Q_{\nu,r}$ are positive measure of $\mathcal{C}\times D$ and $\mathcal{C}$, respectively. Moreover, Novikov's criterion ensures that

$$\left(\exp\left\{\int_0^t (G_s^\nu(r) + m_\nu(s,t))\,\mathrm{d}W_s(x,r) - \frac{1}{2}\int_0^t (G_s^\nu(r)+m_\nu(s,t))^2\,\mathrm{d}s\right\}\right)_{0\le t\le T}$$

is a uniformly integrable P_r-martingale, γ-almost surely so that Jensen's inequality ensures that $Q_\nu(\mathcal{C}\times D) = Q_{\nu,r}(\mathcal{C}) = 1$. Hence, Q_ν and $Q_{\nu,r}$ are probability measures satisfying $\mathrm{d}Q_\nu(x,r) = \mathrm{d}Q_{\nu,r}(x)\,\mathrm{d}\pi(r)$.

Fix $r \in D$ and define the probability measure $\bar{Q}_{\nu,r} \in \mathcal{M}_1^+(\mathcal{C})$ by

$$\frac{\mathrm{d}\bar{Q}_{\nu,r}}{\mathrm{d}P_r}(x) := \exp\left\{\int_0^T m_\nu(t,r)\,\mathrm{d}W_t(x,r) - \frac{1}{2}\int_0^T m_\nu(t,r)^2\,\mathrm{d}t\right\}$$

for which Novikov's criterion holds from the boundedness of m_ν. By Girsanov's theorem, $V^\nu(\cdot,r)$ (defined in (4.2)) is a $\bar{Q}_{\nu,r}$-Brownian motion, and we can use Novikov's criterion again to check that

$$\frac{\mathrm{d}P_r}{\mathrm{d}\bar{Q}_{\nu,r}}(x) := \exp\left\{-\int_0^T m_\nu(t,r)\,\mathrm{d}V_t^\nu(x,r) - \frac{1}{2}\int_0^T m_\nu(t,r)^2\,\mathrm{d}t\right\},$$

implying $\bar{Q}_{\nu,r} \sim P_r$. Moreover, let $\bar{Q}_\nu \in \mathcal{M}_1^+(\mathcal{C}\times D)$ be such that

$$\mathrm{d}\bar{Q}_\nu(x,r) = \mathrm{d}\bar{Q}_{\nu,r}(x)\,\mathrm{d}\pi(r).$$

Then $\bar{Q}_\nu \sim P$, and from the previous lemma $Q_\nu \ll \bar{Q}_\nu$ with density

$$\begin{aligned}
\frac{\mathrm{d}Q_\nu}{\mathrm{d}\bar{Q}_\nu}(x,r) &= \mathcal{E}_\gamma\left[\exp\left\{\int_0^T G_t^\nu(r)\,\mathrm{d}V_t^\nu(x,r) - \frac{1}{2}\int_0^T G_t^\nu(r)^2\,\mathrm{d}t\right\}\right] \\
&= \exp\left\{\frac{1}{2}\int_{\hat{\Omega}} L_T^\nu(x,r)^2\,\mathrm{d}\gamma_{\widetilde{K}_{\nu,r}^T} - \frac{1}{2}\int_0^T \widetilde{K}_{\nu,r}^t(t,t)\,\mathrm{d}t\right\} \\
&\ge \exp\{-C_T\} \\
&> 0,
\end{aligned}$$

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where the second equality follows from (4.12) and (4.13), and the inequality from (4.3). With these estimates at hand, we complete the proof in two steps.

Step 1. We first prove that $I(Q_{\nu,r} \mid \bar{Q}_{\nu,r})$ is finite. This will yield, by applying the exact same reasoning as in [2, Appendix B], the equality

$$\bar{H}_\nu(\mu) = I(\mu \mid Q_\nu) \quad \text{for all } \mu \in \mathcal{M}_1^+(\mathcal{C} \times D),$$

where

$$\bar{H}_\nu : \mu \to \begin{cases} I(\mu \mid \bar{Q}_\nu) - \displaystyle\int_{\mathcal{C}\times D} \log\left(\frac{\mathrm{d}Q_\nu}{\mathrm{d}\bar{Q}_\nu}(x,r)\right) \mathrm{d}\mu(x,r) & \text{if } I(\mu \mid \bar{Q}_\nu) < +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Step 2. We will then prove that for every $\mu \in \mathcal{M}_1^+(\mathcal{C} \times D)$,

$$\bar{H}_\nu(\mu) = H_\nu(\mu), \tag{4.14}$$

which will conclude the proof.

Step 1. For the first point, observe that Girsanov's theorem ensures that the process

$$\left(B_t^\nu(\cdot, r) := V_t^\nu(\cdot, r) - \int_0^t \mathcal{E}_\gamma[\Lambda_s(G^\nu(r))G_s^\nu(r)L_s^\nu(\cdot, r)]\,\mathrm{d}s\right)_{0\le t\le T}$$

is a $Q_{\nu,r}$-Brownian motion, so that

$$I(Q_{\nu,r} \mid \bar{Q}_{\nu,r}) = \frac{1}{2}\int_0^T \underbrace{\int_{\mathcal{C}} \mathcal{E}_\gamma\left[\Lambda_t(G^\nu(r))G_t^\nu(r)\left(\int_0^t G_s^\nu(r)\,\mathrm{d}V_s^\nu(x,r)\right)\right]^2 \mathrm{d}Q_{\nu,r}(x)}_{\varphi_\nu(t,r)}\,\mathrm{d}t.$$

We now find a uniform bound $\varphi_\nu(t, r)$ in order to obtain the result. Thus,

$$\begin{aligned}\varphi_\nu(t,r) &\le C_T\left\{\mathcal{E}_\gamma\left[\int_{\mathcal{C}}\int_0^t G_s^\nu(r)^2\,\mathrm{d}s\,\mathrm{d}Q_{\nu,r}(x)\right] + \int_0^t \varphi_\nu(s,r)\,\mathrm{d}s\right\} \\ &\le C_T\left\{1 + \int_0^t \varphi_\nu(s,r)\,\mathrm{d}s\right\},\end{aligned}$$

where the first inequality follows from the Cauchy–Schwarz inequality, Fubini's theorem, and (4.3), and the second from Fubini and (4.3). We have also used Itô isometry, and the fact that C_T is uniform in space. Gronwall's lemma ensures the uniform boundedness of $\sup_{0\le t\le T}\varphi_\nu(t,\cdot)$, implying the boundedness of $r \to I(Q_{\nu,r} \mid \bar{Q}_{\nu,r})$. Being positive and π-measurable, we can integrate this map on D to obtain

$$I(Q_\nu \mid \bar{Q}_\nu) = \int_D I(Q_{\nu,r} \mid \bar{Q}_{\nu,r})\,\mathrm{d}\pi(r) \le \tilde{C}_T < \infty.$$

We remark that the proof of Proposition 4.2 readily applies to show that there exists $0 < \iota < 1$ and $e > 0$ such that

$$\int_{\mathcal{C}\times D} \log\left(\frac{\mathrm{d}Q_\nu}{\mathrm{d}\bar{Q}_\nu}(x,r)\right)\mathrm{d}\mu(x,r) \le \iota I(\mu \mid \bar{Q}_\nu) + e.$$

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In particular, $\bar{H}_\nu$ is finite whenever $I(\cdot \mid \bar{Q}_\nu)$ is finite. Moreover, we can directly apply [2, Appendix B] to obtain

$$\bar{H}_\nu(\mu) = I(\mu \mid Q_\nu) \quad \text{for all } \mu \in \mathcal{M}_1^+(\mathcal{C} \times D),$$

Step 2. We now show that (4.14) holds. If both $I(\mu \mid P)$ and $I(\mu \mid \bar{Q}_\nu)$ are infinite, the result is trivial. We thus focus on the cases where at least one term is finite. We note that $\{I(\cdot \mid P) < +\infty\} = \{I(\cdot \mid \bar{Q}_\nu) < +\infty\}$. In fact, let $\mu \ll P \sim \bar{Q}_\mu$, suppose that $I(\mu \mid \bar{Q}_\nu)$ is finite, and observe that

$$I(\mu \mid P) = \int_{\mathcal{C}\times D} \left\{ \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\bar{Q}_\nu}(x,r) \right) + \log\left(\frac{\mathrm{d}\bar{Q}_\nu}{\mathrm{d}P}(x,r) \right) \right\} \mathrm{d}\mu(x,r).$$

On the one hand, $\log(\mathrm{d}\mu/\mathrm{d}\bar{Q}_\nu)$ is μ-integrable as

$$\int_{\mathcal{C}\times D} \left| \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\bar{Q}_\nu}(x,r) \right) \right| \mathrm{d}\mu(x,r) = \int_{\mathcal{C}\times D} \left| \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\bar{Q}_\nu}(x,r) \right) \frac{\mathrm{d}\mu}{\mathrm{d}\bar{Q}_\nu}(x,r) \right| \mathrm{d}\bar{Q}_\nu(x,r),$$

as for all $x \in \mathbb{R}_+$, $x \log(x) \geq -1/\exp(1)$, and as $I(\mu \mid \bar{Q}_\nu) < +\infty$. We now show that $\log(\mathrm{d}\bar{Q}_\nu/\mathrm{d}P)$ is also μ-integrable. In fact,

$$\begin{aligned}
\int_{\mathcal{C}\times D} \log\left(\frac{\mathrm{d}\bar{Q}_\nu}{\mathrm{d}P}(x,r) \right)^- \mathrm{d}\mu(x,r) &= \int_{\mathcal{C}\times D} \log\left(1 \vee \frac{\mathrm{d}P}{\mathrm{d}\bar{Q}_\nu}(x,r) \right) \mathrm{d}\mu(x,r) \\
&\leq I(\mu \mid \bar{Q}_\nu) + \log\left(1 + \int_{\mathcal{C}\times D} \frac{\mathrm{d}P}{\mathrm{d}\bar{Q}_\nu}(x,r)\, \mathrm{d}\bar{Q}_\nu(x,r) \right) \\
&\leq I(\mu \mid \bar{Q}_\nu) + \log(2) \\
&< +\infty,
\end{aligned}$$

where the first inequality follows from (4.1). However,

$$\begin{aligned}
&\int_{\mathcal{C}\times D} \log\left(\frac{\mathrm{d}\bar{Q}_\nu}{\mathrm{d}P}(x,r) \right)^+ \mathrm{d}\mu(x,r) \\
&\quad\leq I(\mu \mid \bar{Q}_\nu) + \log\left(1 + \int_{\mathcal{C}\times D} \frac{\mathrm{d}\bar{Q}_\nu}{\mathrm{d}P}(x,r)\, \mathrm{d}\bar{Q}_\nu \right) \\
&\quad\leq I(\mu \mid \bar{Q}_\nu) + \log\left(1 + \int_D \int_{\mathcal{C}} \exp\left\{ \int_0^T m_\nu(t,r)\, \mathrm{d}V_t^\nu(x,r) \right. \right. \\
&\qquad\qquad \left. \left. - \frac{1}{2} \int_0^T m_\nu(t,r)^2\, \mathrm{d}t \right\} \mathrm{d}\bar{Q}_{\nu,r}(x) \right. \\
&\qquad\qquad \left. \times \exp\left\{ \int_0^T m_\nu(t,r)^2\, \mathrm{d}t \right\} \mathrm{d}\pi(r) \right) \\
&\quad\leq I(\mu \mid \bar{Q}_\nu) + \log\left(1 + \exp\left\{ \frac{T \|\bar{J}\|_\infty^2}{2\lambda_*^2} \right\} \right) \\
&\quad< +\infty,
\end{aligned}$$

where the first inequality follows from (4.1). Hence, $I(\mu \mid \bar{Q}_\nu) < +\infty$ implies that

$$I(\mu \mid P) = I(\mu \mid \bar{Q}_\nu) + \int_{\mathcal{C}\times D} \log\left(\frac{\mathrm{d}\bar{Q}_\nu}{\mathrm{d}P}(x,r) \right) \mathrm{d}\mu(x,r) < \infty,$$

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and by symmetry, $I(\mu \mid P) < \infty$ implies the finiteness of $I(\mu \mid \bar{Q}_\nu)$ with the same equality. Moreover, we can apply a similar reasoning as in the proof of Proposition 4.2 to show that there exist constants $0 < \iota < 1$ and $e > 0$ such that

$$\left| \int_{\mathcal{C}\times D} \log\left(\frac{\mathrm{d}\bar{Q}_\nu}{\mathrm{d}P}(x,r)\right) \mathrm{d}\mu(x,r) \right| \leq \iota I(\mu \mid \bar{Q}_\nu) + e,$$

and

$$|\Gamma_\nu(\mu)| \leq \iota I(\mu \mid P) + e.$$

Hence, for $\mu \in \{I(\cdot \mid P) < +\infty\} = \{I(\cdot \mid \bar{Q}_\nu) < +\infty\}$ these quantities are finite. Moreover,

$$\Gamma_\nu(\mu) = \int_{\mathcal{C}\times D} \left\{ \log\left(\frac{\mathrm{d}Q_\nu}{\mathrm{d}\bar{Q}_\nu}(x,r)\right) + \log\left(\frac{\mathrm{d}\bar{Q}_\nu}{\mathrm{d}P}(x,r)\right) \right\} \mathrm{d}\mu(x,r)$$

and we can split this integral since $\mathrm{d}Q_\nu/\mathrm{d}\bar{Q}_\nu$ is bounded away from 0 and $\log(\mathrm{d}\bar{Q}_\nu/P)$ is μ-integrable to obtain

$$\Gamma_\nu(\mu) = \int_{\mathcal{C}\times D} \log\left(\frac{\mathrm{d}Q_\nu}{\mathrm{d}\bar{Q}_\nu}(x,r)\right) \mathrm{d}\mu(x,r) + \int_{\mathcal{C}\times D} \log\left(\frac{\mathrm{d}\bar{Q}_\nu}{\mathrm{d}P}(x,r)\right) \mathrm{d}\mu(x,r).$$

As a consequence, for any $\mu \in \{I(\cdot \mid P) < +\infty\}$, we have

$$I(\mu \mid P) - \Gamma_\nu(\mu) = I(\mu \mid \bar{Q}_\nu) - \int_{\mathcal{C}\times D} \log\left(\frac{\mathrm{d}Q_\nu}{\mathrm{d}\bar{Q}_\nu}(x,r)\right) \mathrm{d}\mu(x,r),$$

which concludes the proof. □

We have thus proved that H_ν is a good rate function, and we would like to extend this property to H: H_ν is seen in our proof as an intermediate tool, equal to H when $I(\mu \mid P) = \infty$, but differs from H by an amount $\Gamma - \Gamma_\nu$ otherwise. We control this difference below in Lemma 4.3. To this end, we introduce two preliminary objects that will appear in the obtained upper bound.

First, due to spatial extension, it is useful to introduce a proper distance on $\mathcal{C} \times D$.

Definition 4.1. Let

$$d_T : (\mathcal{C} \times D)^2 \to \mathbb{R}_+,$$
$$d_T : ((x,r),(y,r')) \to \left\{ \|r - r'\|_{\mathbb{R}^d}^2 + \sup_{a,b\in[-\bar{\tau},0],\, t\in[0,T],\, |b-a|\leq K_T\|r-r'\|_{\mathbb{R}^d}} |x_{t+a} - y_{t+b}|^2 \right\}^{1/2}.$$

Then we have the following:

(i) d_T is a distance on $\mathcal{C} \times D$;

(ii) $d_T((x^n, r_n), (x, r)) \to 0$ if and only if $\|x - x^n\|_{\infty,T} + \|r - r_n\|_{\mathbb{R}^d} \to 0$;

(iii) $(\mathcal{C} \times D, d_T)$ is complete.

Remark 4.4. In particular, d_T generates the natural Borel σ-field of $\mathcal{C} \times D$, and $(\mathcal{C} \times D, d_T)$ is a Polish space.

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Proof of Definition 1. (i) Symmetry and separation are straightforward to obtain. A triangular inequality is obtained by remarking that for any (x, r), (y, r'), and $(z, \tilde{r})$,

$$\sup_{a,b\in[-\bar{\tau},0],\, t\in[0,T],\, |b-a|\le K_\tau\|r-r'\|_{\mathbb{R}^d}} |x_{t+a} - y_{t+b}|$$
$$\le \sup_{\substack{a,c\in[-\bar{\tau},0],\, t\in[0,T]\\ |c-a|\le K_\tau\|r-\tilde{r}\|_{\mathbb{R}^d}}} |x_{t+a} - z_{t+c}| + \sup_{\substack{c,b\in[-\bar{\tau},0],\, t\in[0,T]\\ |b-c|\le K_\tau\|\tilde{r}-r'\|_{\mathbb{R}^d}}} |z_{t+c} - y_{t+b}|,$$

and the triangular inequality for the Euclidean norm on $\mathbb{R}^2$ yields

$$\{(a_1+b_1)^2 + (a_2+b_2)^2\}^{1/2} \le \{a_1^2 + a_2^2\}^{1/2} + \{b_1^2 + b_2^2\}^{1/2} \quad \text{for all } a_1, b_1, a_2, b_2 \in \mathbb{R}.$$

(ii) The first implication is easy to see since

$$\|r - r'\|_{\mathbb{R}^d} + \|x - y\|_{\infty,T} \le d_T((x, r), (y, r')). \tag{4.15}$$

Conversely, let $(x^n, r_n)_{n\in\mathbb{N}} \in (\mathcal{C} \times D)^{\mathbb{N}}$ and $(x, r) \in \mathcal{C} \times D$, suppose that $r_n \to r$ and $\|x^n - x\|_{\infty,T} \to 0$ and let $\eta > 0$ such that

$$|s - t| \le \eta \quad \Longrightarrow \quad |x_t - x_s| \le \varepsilon \quad \text{for all } s, t \in [-\bar{\tau}, T],$$

We can then find a n_0 such that, for all $n \ge n_0$,

$$\begin{aligned} d_T((x^n, r_n), (x, r))^2 &\le \|r - r_n\|^2_{\mathbb{R}^d} + 2\|x - x^n\|^2_{\infty,T} \\ &\quad + 2 \sup_{a,b\in[-\bar{\tau},0],\, t\in[0,T],\, |b-a|\le\eta} |x_{t+a} - x_{t+b}|^2 \\ &\le 5\varepsilon^2. \end{aligned}$$

(iii) The completeness of $(\mathcal{C}, \|\cdot\|_{\infty,T})$ and $(D, |\cdot|)$, along with (4.15), yields the result. □

We also define the 2-Vaserstein distance on $\mathcal{M}_1^+(\mathcal{C} \times D)$ associated with d_T as

$$d_T^V(\mu, \nu) := \inf_{\xi} \left\{ \int_{(\mathcal{C}\times D)^2} d_T((x, r), (y, r'))^2 \, \mathrm{d}\xi((x, r), (y, r')) \right\}^{1/2},$$

the infimum being taken on the laws $\xi \in \mathcal{M}_1^+((\mathcal{C} \times D)^2)$ with marginals μ and ν. In the following, we will, for any $t \in [0, T]$, denote by d_t and d_t^V the respective restrictions of d_T and d_T^V on $(\mathcal{C}([-\bar{\tau}, t], \mathbb{R}) \times D)^2$.

We now present a few regularity properties of the covariance and mean of our Gaussian interactions.

Proposition 4.4. *There exists $C_T > 0$ such that for any $\mu, \nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, $r \in D$, $t \in [0, T]$, and $u, s \in [0, t]$,*

$$|m_\mu(t, r) - m_\nu(t, r)| + |K_\mu(t, s, r) - K_\nu(t, s, r)| + |\widetilde{K}^t_{\mu,r}(s, u) - \widetilde{K}^t_{\nu,r}(s, u)| \le C_T d_T^V(\mu, \nu). \tag{4.16}$$

Proof. Let $\xi \in \mathcal{M}_1^+((\mathcal{C} \times D)^2)$ with marginals μ and ν, and let (G, G') be, under γ, a family of independent bi-dimensional centered Gaussian processes with covariance $K_\xi(s, t, r)$, given by

$$\frac{1}{\lambda(r)^2} \int_{(\mathcal{C}\times D)^2} \begin{pmatrix} \sigma^2_{rr'} S(x_{s-\tau_{rr'}}) S(x_{t-\tau_{rr'}}) & \sigma_{rr'}\sigma_{r\tilde{r}'} S(x_{s-\tau_{rr'}}) S(y_{t-\tau_{r\tilde{r}'}}) \\ \sigma_{rr'}\sigma_{r\tilde{r}'} S(y_{s-\tau_{r\tilde{r}'}}) S(x_{t-\tau_{rr'}}) & \sigma^2_{r\tilde{r}'} S(y_{s-\tau_{r\tilde{r}'}}) S(y_{t-\tau_{r\tilde{r}'}}) \end{pmatrix} \times \mathrm{d}\xi((x, r'), (y, \tilde{r}')) \tag{4.17}$$

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with the shorthand notation $\sigma_{rr'} := \sigma(r, r')$, $\tau_{rr'} = \tau(r, r')$. First,

$$|m_\mu(t,r) - m_\nu(t,r)| = \left| \frac{1}{\lambda(r)} \int_{\mathcal{C}\times D} J(r,r') S(x_{t-\tau(r,r')}) \, \mathrm{d}(\mu-\nu)(x,r') \right| \\ \leq C_T \left\{ \int_{(\mathcal{C}\times D)^2} d_T((x,r'),(y,\tilde{r}'))^2 \, \mathrm{d}\xi((x,r'),(y,\tilde{r}')) \right\}^{1/2},$$

where the inequality follows from the Cauchy–Schwarz inequality. Moreover,

$$|K_\mu(t,s,r) - K_\nu(t,s,r)| = |\mathcal{E}_\gamma[G_s G_t - G'_s G'_t]| \\ \leq C_T g \{ \mathcal{E}_\gamma[(G_t - G'_t)^2]^{1/2} + \mathcal{E}_\gamma[(G_s - G'_s)^2]^{1/2} g \},$$

and

$$|\widetilde{K}^t_{\mu,r}(s,u) - \widetilde{K}^t_{\nu,r}(s,u)| \\ \leq C_T \{ \mathcal{E}_\gamma[(\Lambda_t(G) - \Lambda_t(G'))^2]^{1/2} + \mathcal{E}_\gamma[(G_s - G'_s)^2]^{1/2} + \mathcal{E}_\gamma[(G_u - G'_u)^2]^{1/2} \} \\ \leq C_T \left\{ \left(\int_0^t \mathcal{E}_\gamma[(G_v - G'_v)^2] \, \mathrm{d}v \right)^{1/2} + \mathcal{E}_\gamma[(G_s - G'_s)^2]^{1/2} + \mathcal{E}_\gamma[(G_u - G'_u)^2]^{1/2} \right\},$$

where the inequalities follow from (4.3) and (4.5), respectively. We have

$$\mathcal{E}_\gamma[(G_t - G'_t)^2] \\ = \frac{1}{\lambda(r)^2} \int_{(\mathcal{C}\times D)^2} (\sigma(r,r') S(x_{t-\tau(r,r')}) - \sigma(r,\tilde{r}') S(y_{t-\tau(r,\tilde{r}')}))^2 \, \mathrm{d}\xi((x,r'),(y,\tilde{r}')) \\ \leq C_T \int_{(\mathcal{C}\times D)^2} d_t((x,r'),(y,\tilde{r}'))^2 \, \mathrm{d}\xi((x,r'),(y,\tilde{r}')). \tag{4.18}$$

Taking the infimum on ξ yields (4.16). □

We now have all the elements needed to state the following technical lemma concluding on the fact that H is a good rate function.

Lemma 4.3. *Let $\mu, \nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$.*

(i) *There exists a positive constant C_T such that:*

(a) $|\Gamma_{1,\nu}(\mu) - \Gamma_1(\mu)| \leq C_T d_T^V(\mu, \nu)$;

(b) $|\Gamma_{2,\nu}(\mu) - \Gamma_2(\mu)| \leq C_T(1 + I(\mu \mid P)) d_T^V(\mu, \nu)$.

(ii) *H is a good rate function.*

Proof. The main techniques were introduced in [2, Lemmas 3.3 and 3.4] and used in a neuroscience setting in [12, Lemma 5], but spatiality induces new issues essentially impacting the proof of point (i)(b). For the sake of completeness, we will here reproduce these techniques, and address the specific spatial difficulties.

(i)(a) We define

$$\Gamma_1(\mu, r) := -\frac{1}{2} \int_0^T (\widetilde{K}^t_{\mu,r}(t,t) + m_\mu(t,r)^2) \, \mathrm{d}t,$$

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so that

$$\begin{aligned}|\Gamma_{1,\nu}(\mu)-\Gamma_1(\mu)| &= \left|\int_{\mathcal{C}\times D}(\Gamma_1(\nu,r)-\Gamma_1(\mu,r))\,\mathrm{d}\mu(x,r)\right| \\ &\le \frac{1}{2}\int_0^T |(m_\mu(t,r)-m_\nu(t,r))(m_\mu(t,r)+m_\nu(t,r))| \\ &\qquad + |\widetilde{K}^t_{\nu,r}(t,t)-\widetilde{K}^t_{\mu,r}(t,t)|\,\mathrm{d}t \\ &\le C_T d_T^V(\mu,\nu), \end{aligned} \tag{4.19}$$

where the last line follows from (4.16).

(i)(b) Note that if $I(\mu \mid P)=\infty$, the inequality is obvious. Then let $\mu \in \mathcal{M}_1^+(\mathcal{C}\times D)$ with $I(\mu \mid P)<\infty$ implying $\mu \ll P$ and the finiteness of $\Gamma_\nu(\mu)$ and $\Gamma(\mu)$. This also implies that μ has a measurable density ρ_μ with respect to $\mathcal{B}(\mathcal{C}\times D)$, i.e.

$$\mathrm{d}\mu(x,r)=\rho_\mu(x,r)\,\mathrm{d}P(x,r)=\rho_\mu(x,r)\,\mathrm{d}P_r(x)\,\mathrm{d}\pi(r).$$

Hence, for $r\in D$ such that $c_\mu(r):=\int_{\mathcal{C}}\rho_\mu(x,r)\,\mathrm{d}P_r(x)\notin\{0,+\infty\}$, we can properly define $\mu_r\in\mathcal{M}_1^+(\mathcal{C})$ by $\mathrm{d}\mu_r(x):=(\rho_\mu(x,r)/c_\mu(r))\,\mathrm{d}P_r(x)$. Of course $\mu_r\ll P_r$, and

$$\mathrm{d}\mu(x,r)=\mathrm{d}\mu_r(x)c_\mu(r)\,\mathrm{d}\pi(r).$$

We remark that c_μ is a measurable function of space such that $\int_D c_\mu(r)\,\mathrm{d}\pi(r)=1$, and that the set $\{r\in D, c_\mu(r)\in\{0,+\infty\}\}$ does not impact the value of integrals over μ.

In order to obtain the proper inequality, we can split our interest into different terms, i.e.

$$\begin{aligned}|\Gamma_{2,\nu}(\mu)-\Gamma_2(\mu)| &\le \frac{1}{2}\left|\int_{\mathcal{C}\times D}\left\{\int_{\hat\Omega}L_T^\nu(x,r)^2\,\mathrm{d}\gamma_{\widetilde{K}^T_{\nu,r}}-\int_{\hat\Omega}L_T^\mu(x,r)^2\,\mathrm{d}\gamma_{\widetilde{K}^T_{\mu,r}}\right\}\mathrm{d}\mu(x,r)\right| \\ &\quad+\left|\int_{\mathcal{C}\times D}\int_0^T(m_\nu-m_\mu)(t,r)\,\mathrm{d}W_t(x,r)\,\mathrm{d}\mu(x,r)\right| \\ &\le \frac{1}{2}\left|\int_{\mathcal{C}\times D}\int_{\hat\Omega}L_T^\nu(x,r)^2\,\mathrm{d}(\gamma_{\widetilde{K}^T_{\nu,r}}-\gamma_{\widetilde{K}^T_{\mu,r}})\,\mathrm{d}\mu(x,r)\right| \\ &\quad+\frac{1}{2}\left|\int_{\mathcal{C}\times D}\int_{\hat\Omega}\{L_T^\mu(x,r)^2-L_T^\nu(x,r)^2\}\,\mathrm{d}\gamma_{\widetilde{K}^T_{\mu,r}}\,\mathrm{d}\mu(x,r)\right| \\ &\quad+\left|\int_{\mathcal{C}\times D}\int_0^T(m_\nu-m_\mu)(t,r)\,\mathrm{d}W_t(x,r)\,\mathrm{d}\mu(x,r)\right|. \end{aligned}$$

Let $\xi\in\mathcal{M}_1^+((\mathcal{C}\times D)^2)$ with the marginals μ and ν be such that

$$\int_{(\mathcal{C}\times D)^2}d_T((x,r'),(y,\tilde r'))^2\,\mathrm{d}\xi((x,r'),(y,\tilde r'))\le(d_T^V(\mu,\nu)+\varepsilon)^2.$$

Moreover, let $(G(r),G'(r))_{r\in D}$ be, under γ, a family of independent bi-dimensional centered Gaussian processes with covariance $K_\xi(s,t,r)$ as defined in (4.17). Note that for any map f,

$$\mathcal{E}_\gamma[f(G^\mu(r))]-\mathcal{E}_\gamma[f(G^\nu(r))]=\mathcal{E}_\gamma[f(G(r))-f(G'(r))]$$

and as proved in Proposition 4.4,

$$\mathcal{E}_\gamma[(G_t(r)-G'_t(r))^2]\le C_T(d_T^V(\mu,\nu)+\varepsilon)^2.$$

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In addition, let

$$L_t(x,r) := \int_0^t G_s(r)\,\mathrm{d}V_s^{\mu}(x,r), \qquad L'_t(x,r) := \int_0^t G'_s(r)\,\mathrm{d}V_s^{\nu}(x,r).$$

Using (4.3), we then obtain

$$\begin{aligned}
&|\Gamma_{2,\nu}(\mu) - \Gamma_2(\mu)| \\
&\quad \le C_T \Bigg\{ \underbrace{\int_{\mathcal{C}\times D} \mathcal{E}_\gamma[|\Lambda_T(G(r)) - \Lambda_T(G'(r))| L'_T(x,r)^2]\,\mathrm{d}\mu(x,r)}_{B_1} \\
&\qquad + \underbrace{\prod_{\varepsilon=\pm 1} \left(\int_{\mathcal{C}\times D} \mathcal{E}_\gamma \left[\left(\int_0^T (G_t(r) + \varepsilon G'_t)\,\mathrm{d}V_t^{\mu}(x,r) \right)^2 \right] \mathrm{d}\mu(x,r) \right)^{1/2}}_{B_2} \\
&\qquad + \underbrace{\left| \int_{\mathcal{C}\times D} \mathcal{E}_\gamma \left[\left(\int_0^T G_t(r)\,\mathrm{d}V_t^{\mu}(x,r) \right)^2 - \left(\int_0^T G_t(r)\,\mathrm{d}V_t^{\nu}(x,r) \right)^2 \right] \mathrm{d}\mu(x,r) \right|}_{B_3} \\
&\qquad + \underbrace{\left(\int_{\mathcal{C}\times D} \left| \int_0^T (m_\nu - m_\mu)(t,r)\,\mathrm{d}W_t(x,r) \right|^2 \mathrm{d}\mu(x,r) \right)^{1/2}}_{B_4} \Bigg\}
\end{aligned}$$

using the Cauchy–Schwarz inequality. Before bounding these four terms, we prove a useful inequality. For any $h, m \in L^2([0;\,T], \mathrm{d}t)$ with m bounded, and any $r \in D$ with $c_\mu(r) \notin \{0, +\infty\}$,

$$\begin{aligned}
&\int_{\mathcal{C}} \left(\int_0^T h_t(\mathrm{d}W_t(x,r) - m(t)\,\mathrm{d}t) \right)^2 \mathrm{d}\mu_r(x) \\
&\qquad \le 2 \left\{ \int_{\mathcal{C}} \left(\int_0^T h_t\,\mathrm{d}W_t(x,r) \right)^2 + \left(\int_0^T h_t m_t\,\mathrm{d}t \right)^2 \mathrm{d}\mu_r(x) \right\}. \qquad (4.20)
\end{aligned}$$

Moreover, supposing that $h \neq 0_{L^2([0;\,T],\mathrm{d}t)}$ then $\Phi_h(x) = (\int_0^T h_t\,\mathrm{d}W_t(x,r))^2/4(\int_0^T h_t^2\,\mathrm{d}t)$ is a well-defined, positive, and measurable function of the σ-algebra $\mathcal{B}(\mathcal{C})$, so that resorting to (4.1), we obtain

$$\int_{\mathcal{C}} \Phi_h(x)\,\mathrm{d}\mu_r(x) \le I(\mu_r \mid P_r) + \log \int_{\mathcal{C}} \exp\{\Phi_h(x)\}\,\mathrm{d}P_r(x).$$

As $W(\cdot, r)$ is a Brownian motion under P_r, $\Phi_h \sim \mathcal{N}(0,\,1/4)^2$, so Gaussian calculus yields for any $C > 2$,

$$\int_{\mathcal{C}} \left(\int_0^T h_t\,\mathrm{d}W_t(x,r) \right)^2 \mathrm{d}\mu_r(x) \le C(I(\mu_r \mid P_r) + 1) \left(\int_0^T h_t^2\,\mathrm{d}t \right).$$

Note that this inequality obviously holds when $h = 0_{L^2([0;\,T],\mathrm{d}t)}$. Applying this result to

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(4.20), we eventually arrive at

$$\int_{\mathcal{C}}\left(\int_0^T h_t(\mathrm{d}W_t(x,r)-m(t)\,\mathrm{d}t)\right)^2\mathrm{d}\mu_r(x)\le 2(C(1+I(\mu_r\mid P_r))+m_\infty^2T)\left(\int_0^T h_t^2\,\mathrm{d}t\right)$$
$$\le C_T(1+I(\mu_r\mid P_r))\left(\int_0^T h_t^2\,\mathrm{d}t\right), \tag{4.21}$$

where the first inequality follows from the Cauchy–Schwarz inequality.

With this result in mind, we now control the first term. Recall that, by (4.5),

$$|\Lambda_T(G(r))-\Lambda_T(G'(r))|$$
$$\le C_T\left(\int_0^T|G_t(r)^2-{G'}_t(r)^2|\,\mathrm{d}t+\int_0^T\mathcal{E}_\gamma[(G_t(r)-{G'}_t(r))^2]^{1/2}\,\mathrm{d}t\right).$$

Now, relying on the decomposition of μ, we find that

$$B_1=\int_D\mathcal{E}_\gamma\left[|\Lambda_T(G(r))-\Lambda_T(G'(r))|\left\{\int_{\mathcal{C}}L'_T(x,r)^2\,\mathrm{d}\mu_r(x)\right\}\right]c_\mu(r)\,\mathrm{d}\pi(r)$$
$$\le C_T\left(\int_D I(\mu_r|P_r)c_\mu(r)\,\mathrm{d}\pi(r)+1\right)(d_T^V(\mu,\nu)+\varepsilon),$$

where the equality follows from Fubini's theorem, and the inequality comes from the Cauchy–Schwarz inequality, Isserlis' theorem, and (4.21). Observe that

$$\int_D I(\mu_r\mid P_r)c_\mu(r)\mathrm{d}\pi(r)=\int_D\int_{\mathcal{C}}\log\left(\frac{\rho_\mu(x,r)}{c_\mu(r)}\right)\mathrm{d}\mu_r(x)c_\mu(r)\,\mathrm{d}\pi(r)$$
$$=\int_{\mathcal{C}\times D}\log(\rho_\mu(x,r))\,\mathrm{d}\mu(x,r)-\int_D\log(c_\mu(r))c_\mu(r)\,\mathrm{d}\pi(r)$$
$$\le I(\mu\mid P),$$

where last line follows from Jensen's inequality. As a consequence,

$$B_1\le C_T(1+I(\mu\mid P))(d_T^V(\mu,\nu)+\varepsilon).$$

Similarly, there exists a constant c_T such that

$$B_2\le C_T(1+I(\mu\mid P))(d_T^V(\mu,\nu)+\varepsilon).$$

To bound B_3, we first use the Cauchy–Schwarz inequality to obtain

$$B_3\le\prod_{\varepsilon=\pm1}\left\{\int_{\mathcal{C}\times D}\mathcal{E}_\gamma\left[\left|\int_0^T G_t(r)((1+\varepsilon)\,\mathrm{d}W_t(x,r)-(m_\nu(t,r)+\varepsilon m_\mu(t,r))\,\mathrm{d}t)\right|^2\right]\right.$$
$$\left.\times\,\mathrm{d}\mu(x,r)\right\}^{1/2}. \tag{4.22}$$

Then, again by Cauchy–Schwarz and (4.16), we observe that

$$\mathcal{E}_\gamma\left[\left|\int_0^T G_t(r)(m_\mu(t,r)-m_\nu(t,r))\,\mathrm{d}t\right|^2\right]\le C_Td_T^V(\mu,\nu)^2.$$

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Moreover, (4.21) yields

$$\int_{\mathcal{C}} \left\{ \int_0^T 2G_t(r) \left(\mathrm{d}W_t(x,r) - \frac{m_\mu(t,r) + m_\nu(t,r)}{2} \, \mathrm{d}t \right) \right\}^2 \mathrm{d}\mu_r(x) \\ \leq c_T (1 + I(\mu_r \mid P_r)) \int_0^T G_t^2(r) \, \mathrm{d}t.$$

Using Jensen's inequality and inserting the last two inequalities into (4.22) yields

$$B_3 \leq C_T (1 + I(\mu \mid P))^{1/2} d_T^V(\mu, \nu) \leq C_T (1 + I(\mu \mid P)) d_T^V(\mu, \nu) \quad \text{as } I(\cdot \mid P) \geq 0.$$

For the last term, we have

$$\begin{aligned} B_4 &\leq \left(\int_D c_T (1 + I(\mu_r \mid P_r)) \left\{ \int_0^T (m_\mu(t,r) - m_\nu(t,r))^2 \, \mathrm{d}t \right\} c_\mu(r) \, \mathrm{d}\pi(r) \right)^{1/2} \\ &\leq C_T (1 + I(\mu \mid P))^{1/2} d_T^V(\mu, \nu) \\ &\leq C_T (1 + I(\mu \mid P)) d_T^V(\mu, \nu), \end{aligned}$$

where the first two inequalities follow from (4.21) and (4.16), respectively. Hence, we conclude that there exists a constant C_T satisfying

$$|\Gamma_{2,\nu}(\mu) - \Gamma_2(\mu)| \leq C_T (1 + I(\mu \mid P))(d_T^V(\mu, \nu) + \varepsilon).$$

The result follows when $\varepsilon \to 0$.

(ii) We proceed exactly as in [12, Lemma 5(vi)], remarking that $(\mathcal{C} \times D, d_T)$ is a Polish space. □

4.2. Upper bound and tightness

We have proved that $H = I(\cdot \mid P) - \Gamma$ is a good rate function, and we now want to show that it is indeed associated with an LDP. We prove here a weak LDP relying on an upper bound inequality for compact subsets, and tightness of the family $(Q^N(\hat{\mu}_N \in \cdot))_N$. To prove the first point, we take advantage of the full LDP followed by $\hat{\mu}_N$ under $(Q_\nu)^{\otimes N}$, with an error term to control. The second point will rely on the exponential tightness of $P^{\otimes N}$. These proofs are inspired from those of Guionnet in a nonspatial spin glass model; see [28].

Theorem 4.2. *For any compact subset K of $\mathcal{M}_1^+(\mathcal{C} \times D)$,*

$$\limsup_{N \to \infty} \frac{1}{N} \log Q^N(\hat{\mu}_N \in K) \leq - \inf_K H.$$

Proof. Let $\delta > 0$. As $(\mathcal{M}_1^+(\mathcal{C} \times D), d_T^V)$ is a Polish space, we can find an integer M and a family $(\nu_i)_{1 \leq i \leq M}$ of $\mathcal{M}_1^+(\mathcal{C} \times D)$ such that

$$K \subset \bigcup_{i=1}^M B(\nu_i, \delta),$$

where $B(\nu_i, \delta) := \{\mu \mid d_T^V(\mu, \nu_i) < \delta\}$. A classical result (see, e.g. [21, Lemma 1.2.15]), ensures that

$$\limsup_{N \to \infty} \frac{1}{N} \log Q^N(\hat{\mu}_N \in K) \leq \max_{1 \leq i \leq M} \limsup_{N \to \infty} \frac{1}{N} \log Q^N(\hat{\mu}_N \in K \cap B(\nu_i, \delta)).$$

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An application of Lemma 4.1 yields

$$\begin{aligned}
&Q^N(\hat{\mu}_N \in K \cap B(\nu,\delta)) \\
&\quad = \int_{\hat{\mu}_N \in K\cap B(\nu,\delta)} \exp\{N\bar{\Gamma}(\hat{\mu}_N)\}\,\mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r}) \\
&\quad = \int_{\hat{\mu}_N \in K\cap B(\nu,\delta)} \exp\{N(\bar{\Gamma}(\hat{\mu}_N) - \bar{\Gamma}_\nu(\hat{\mu}_N))\}\exp\{N\bar{\Gamma}_\nu(\hat{\mu}_N)\}\,\mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r}).
\end{aligned}$$

Recall definition (4.6) and let $(\tilde{X}_i^\mu)_{1\le i\le N}$ be a family of independent variables of $(\hat{\Omega},\hat{\mathcal{F}},\gamma)$ with the same law as $(X_i^\mu)_{1\le i\le N}$. We denote by $(\tilde{G}^\mu(r_i))_{1\le i\le N}$ the associated independent Gaussian processes. Then, for any conjugate exponents (p,q),

$$\begin{aligned}
&Q^N(\hat{\mu}_N \in K \cap B(\nu,\delta)) \\
&\quad = \int_{\hat{\mu}_N \in K\cap B(\nu,\delta)} \exp\{N(\bar{\Gamma}(\hat{\mu}_N) - \bar{\Gamma}_\nu(\hat{\mu}_N))\}\,\mathrm{d}Q_\nu^{\otimes N}(\boldsymbol{x},\boldsymbol{r}) \\
&\quad \le Q_\nu^{\otimes N}(\hat{\mu}_N \in K \cap B(\nu,\delta))^{1/p} \\
&\qquad \times \left(\int_{\hat{\mu}_N \in K\cap B(\nu,\delta)} \exp\{qN(\bar{\Gamma}(\hat{\mu}_N) - \bar{\Gamma}_\nu(\hat{\mu}_N))y\}\,\mathrm{d}Q_\nu^{\otimes N}(\boldsymbol{x},\boldsymbol{r})\right)^{1/q} \\
&\quad \le Q_\nu^{\otimes N}(\hat{\mu}_N \in K \cap B(\nu,\delta))^{1/p} \\
&\qquad \times \left(\int_{\hat{\mu}_N \in K\cap B(\nu,\delta)} \prod_{i=1}^N \mathcal{E}_\gamma\left(\frac{\exp\{\tilde{X}_i^{\hat{\mu}_N}\}}{\mathcal{E}_\gamma(\exp\{\tilde{X}_i^\nu\})}\right)^q \mathrm{d}Q_\nu^{\otimes N}(\boldsymbol{x},\boldsymbol{r})\right)^{1/q} \\
&\quad \le Q_\nu^{\otimes N}(\hat{\mu}_N \in K \cap B(\nu,\delta))^{1/p} \\
&\qquad \times \left(\int_{\hat{\mu}_N \in K\cap B(\nu,\delta)} \mathcal{E}_\gamma\left[\left(\prod_{i=1}^N \exp\{q(\tilde{X}_i^{\hat{\mu}_N} - \tilde{X}_i^\nu)\}\right)\frac{\prod_{i=1}^N \exp\{\tilde{X}_i^\nu\}}{\mathcal{E}_\gamma[\prod_{i=1}^N \exp\{\tilde{X}_i^\nu\}]}\right]\mathrm{d}Q_\nu^{\otimes N}\right)^{1/q} \\
&\quad \le Q_\nu^{\otimes N}(\hat{\mu}_N \in K \cap B(\nu,\delta))^{1/p} \\
&\qquad \times \underbrace{\left(\int_{\hat{\mu}_N \in K\cap B(\nu,\delta)} \mathcal{E}_\gamma\left[\prod_{i=1}^N \exp\{q(\tilde{X}_i^{\hat{\mu}_N} - \tilde{X}_i^\nu) + \tilde{X}_i^\nu\}\right]\mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r})\right)^{1/q}}_{B_N},
\end{aligned} \tag{4.23}$$

where we use Jensen's inequality in the penultimate inequality.

The first term oo the right-hand side of (4.23) can be controlled by large deviation estimates. Controlling the second term is the object of the following lemma. With these two results, we can now conclude as in [2, Lemma 4.7]. □

Lemma 4.4. *For any real number $q > 1$, there exists a strictly positive real number $\delta_q > 0$ and a function $C_q\colon (0,\delta_q) \to \mathbb{R}$ such that $\lim_{\delta\to 0} C_q(\delta) = 0$ and*

$$B_N \le \exp\{C_q(\delta)N\}.$$

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Proof. Using Hölder inequality with conjugate exponents (σ, η), we find that

$$B^N \le \underbrace{\mathcal{E}_\gamma\left[\int_{(\mathcal{C}\times D)^N} \exp\left\{\sum_{i=1}^N \sigma \tilde{X}_i^\nu\right\} \mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r})\right]^{1/\sigma}}_{(B_1^N)^{1/\sigma}}$$
$$\times \underbrace{\mathcal{E}_\gamma\left[\int_{\hat{\mu}_N \in B(\nu,\delta)} \prod_{i=1}^N \exp\{q\eta(\tilde{X}_i^{\hat{\mu}_N} - \tilde{X}_i^\nu)\} \mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r})\right]^{1/\eta}}_{(B_2^N)^{1/\eta}}.$$

The first term is controlled by the martingale property, i.e.

$$\begin{aligned}
B_1^N &= \mathcal{E}_\gamma\Bigg[\int_{D^N} \exp\Bigg\{\sum_{i=1}^N \frac{\sigma^2-\sigma}{2}\int_0^T (\tilde{G}_t^\nu(r_i) + m_\nu(t,r_i))^2\,\mathrm{d}t\Bigg\} \\
&\quad \times \int_{\mathcal{C}^N} \exp\Bigg\{\sum_{i=1}^N \sigma\int_0^T (\tilde{G}_t^\nu(r_i) + m_\nu(t,r_i))\,\mathrm{d}W_t(x^i,r_i) \\
&\qquad - \frac{\sigma^2}{2}\int_0^T (\tilde{G}_t^\nu(r_i) + m_\nu(t,r_i))^2\,\mathrm{d}t\Bigg\} \mathrm{d}P_{\boldsymbol{r}}(\boldsymbol{x})\,\mathrm{d}\pi^{\otimes N}(\boldsymbol{r})\Bigg] \\
&\le \int_{D^N}\prod_{i=1}^N\Bigg\{\int_0^T \mathcal{E}_\gamma\Bigg[\exp\Bigg\{\frac{\sigma(\sigma-1)T}{2}(\tilde{G}_t^\nu(r_i) + m_\nu(t,r_i))^2\Bigg\}\Bigg]\frac{\mathrm{d}t}{T}\Bigg\}\mathrm{d}\pi^{\otimes N}(\boldsymbol{r}) \\
&\le \exp\{c_T(\sigma-1)N\}
\end{aligned}$$

with c_T uniform in space, where we use Jensen's inequality and Fubini's theorem in the penultimate inequality, and (4.10) in the final inequality.

We now control the second term, denoting $\kappa = q\eta$ and supposing that δ is small enough. Using the Cauchy–Schwarz inequality and Fubini's theorem, we have

$$\begin{aligned}
B_2^N &\le \mathcal{E}_\gamma\Bigg[\int_{(\mathcal{C}\times D)^N}\prod_{i=1}^N \exp\Bigg\{2\kappa\int_0^T (\tilde{G}_t^{\hat{\mu}_N}(r_i) - \tilde{G}_t^\nu(r_i) \\
&\qquad + (m_{\hat{\mu}_N}(t,r_i) - m_\nu(t,r_i)))\,\mathrm{d}W_t(x^i,r_i) \\
&\qquad - 2\kappa^2\int_0^T (\tilde{G}_t^{\hat{\mu}_N}(r_i) - \tilde{G}_t^\nu(r_i) \\
&\qquad + (m_{\hat{\mu}_N}(t,r_i) - m_\nu(t,r_i)))^2\,\mathrm{d}t\Bigg\}\mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r})\Bigg]^{1/2} \\
&\times \Bigg\{\int_{\hat{\mu}_N\in B(\nu,\delta)} \mathcal{E}_\gamma\Bigg[\prod_{i=1}^N \exp\Bigg\{2\kappa^2\int_0^T (\tilde{G}_t^{\hat{\mu}_N}(r_i) - \tilde{G}_t^\nu(r_i) \\
&\qquad + g(m_{\hat{\mu}_N}(t,r_i) - m_\nu(t,r_i)))^2\,\mathrm{d}t \\
&\qquad - \kappa\int_0^T (\tilde{G}_t^{\hat{\mu}_N}(r_i) + m_{\hat{\mu}_N}(t,r_i))^2 \\
&\qquad - (\tilde{G}_t^\nu(r_i) + m_\nu(t,r_i))^2\,\mathrm{d}t\Bigg\}\Bigg]\mathrm{d}P^{\otimes N}(\boldsymbol{x},\boldsymbol{r})\Bigg\}^{1/2}.
\end{aligned}$$

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The first term is equal to 1 by the martingale property. For the second term, we remark that

$$
\begin{aligned}
&-\int_0^T (\tilde{G}_t^{\hat{\mu}_N}(r_i) + m_{\hat{\mu}_N}(t, r_i))^2 - (\tilde{G}_t^{\nu}(r_i) + m_{\nu}(t, r_i))^2 \,\mathrm{d}t \\
&\quad \le \frac{\delta^{1/2}}{2}\Bigg(\frac{1}{\delta}\int_0^T (\tilde{G}_t^{\hat{\mu}_N}(r_i) - \tilde{G}_t^{\nu}(r_i) + (m_{\hat{\mu}_N}(t, r_i) - m_{\nu}(t, r_i)))^2 \,\mathrm{d}t \\
&\qquad\quad + \int_0^T (\tilde{G}_t^{\hat{\mu}_N}(r_i) + \tilde{G}_t^{\nu}(r_i) + (m_{\hat{\mu}_N}(t, r_i) + m_{\nu}(t, r_i)))^2 \,\mathrm{d}t\Bigg)
\end{aligned}
$$

so that, using the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
B_2^N \le &\left\{\int_{\hat{\mu}_N \in B(\nu,\delta)} \mathcal{E}_\gamma\left[\prod_{i=1}^N \exp\left\{(4\kappa^2 + \kappa\delta^{1/2})\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \times \int_0^T (\tilde{G}_t^{\hat{\mu}_N}(r_i) - \tilde{G}_t^{\nu}(r_i) + (m_{\hat{\mu}_N} - m_{\nu})(t, r_i))^2 \,\mathrm{d}t\right\}\right] \right. \\
&\qquad \left. \times \,\mathrm{d}P^{\otimes N}(\boldsymbol{x}, \boldsymbol{r})\right\}^{1/4} \\
&\times \left\{\int_{(\mathcal{C}\times D)^N} \prod_{i=1}^N \underbrace{\mathcal{E}_\gamma\left[\exp\left\{\kappa\delta^{1/2}\int_0^T (\tilde{G}_t^{\hat{\mu}_N}(r_i) + \tilde{G}_t^{\nu}(r_i) + (m_{\hat{\mu}_N} + m_{\nu})(t, r_i))^2 \,\mathrm{d}t\right\}\right]}_{\le \exp\{c_T\kappa\delta^{1/2}\} \text{ (via (4.10))}} \right. \\
&\qquad \left. \times \,\mathrm{d}P^{\otimes N}(\boldsymbol{x}, \boldsymbol{r})\right\}^{1/4}.
\end{aligned}
$$

We now control the first term of the product by taking advantage of the fact that $\hat{\mu}_N \in B(\nu, \delta)$. We have, for any $\xi \in \mathcal{M}_1^+((\mathcal{C} \times D)^2)$ with marginals $\hat{\mu}_N$ and ν,

$$
|m_{\hat{\mu}_N} - m_{\nu}|(t, r_i) \le C_T d_T^V(\hat{\mu}_N, \nu) \le C_T\delta
$$

using (4.16) and, similarly,

$$
\mathcal{E}_\gamma[(\tilde{G}_t^{\hat{\mu}_N}(r_i) - \tilde{G}_t^{\nu}(r_i))^2] \le C\delta^2.
$$

Moreover, an application of Jensen's inequality yields

$$
(\tilde{G}_t^{\hat{\mu}_N}(r_i) - \tilde{G}_t^{\nu}(r_i) + (m_{\hat{\mu}_N} - m_{\nu})(t, r_i))^2 \le C\delta^2 + 2(\tilde{G}_t^{\hat{\mu}_N}(r_i) - \tilde{G}_t^{\nu}(r_i))^2,
$$

so that from the independence of the $\tilde{G}$ for different locations and (4.10),

$$
\begin{aligned}
&\mathcal{E}_\gamma\left[\prod_{i=1}^N \exp\left\{(4\kappa^2 + \kappa\delta^{-1/2})\int_0^T (\tilde{G}_t^{\hat{\mu}_N}(r_i) - \tilde{G}_t^{\nu}(r_i) + (m_{\hat{\mu}_N} - m_{\nu})(t, r_i))^2 \,\mathrm{d}t\right\}\right] \\
&\quad \le \exp\{C_T(4\kappa^2 + \kappa\delta^{-1/2})\delta^2 N\}.
\end{aligned}
$$

Hence,

$$
B_2^N \le \exp\{C_\kappa(\delta)N\}
$$

with $C_\kappa(\delta) \to 0$ as $\delta \to 0$. □

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Theorem 4.3. (Tightness.) *For any real number $\varepsilon > 0$, there exists a compact set K_ε of $\mathcal{M}_1^+(\mathcal{C} \times D)$ such that for any integer N,*

$$Q^N(\hat{\mu}_N \notin K_\varepsilon) \leq \varepsilon.$$

Proof. The proof consists in using the relative entropy inequality (4.1) and the exponential tightness of the sequence $(P^{\otimes N})_N$. We refer the reader to [12, Theorem 2], and remark that, using (4.1) and then (4.2),

$$\begin{aligned}\frac{\mathrm{d}Q^N}{\mathrm{d}P^{\otimes N}}(\boldsymbol{x}, \boldsymbol{r}) &= \prod_{i=1}^N \mathcal{E}_\gamma[\exp\{X^{\hat{\mu}_N}(x^i, r_i)\}] \\ &= \prod_{i=1}^N \exp\left\{\int_0^T O_{\hat{\mu}_N}(t, x^i, r_i)\,\mathrm{d}W_t(x^i, r_i) - \frac{1}{2}\int_0^T O_{\hat{\mu}_N}(t, x^i, r_i)^2\,\mathrm{d}t\right\},\end{aligned}$$

where

$$O_{\hat{\mu}_N}(t, x, r) := \mathcal{E}_\gamma[\Lambda_t(G^{\hat{\mu}_N}(r))G_t^{\hat{\mu}_N}(r)L_t^{\hat{\mu}_N}(x, r)] + m_{\hat{\mu}_N}(t, r).$$

Thus,

$$\begin{aligned}&I(Q^N \mid P^{\otimes N}) \\ &\quad= N\int_{(\mathcal{C}\times D)^N}\left\{\int_0^T O_{\hat{\mu}_N}(t, x^1, r_1)\,\mathrm{d}W_t(x^1, r_1) - \frac{1}{2}\int_0^T O_{\hat{\mu}_N}(t, x^1, r_1)^2\,\mathrm{d}t\right\}\mathrm{d}Q^N(\boldsymbol{x}, \boldsymbol{r}) \\ &\quad\leq N\left\{\int_{D^N}\int_0^T \underbrace{\int_{\mathcal{C}^N} \mathcal{E}_\gamma[\Lambda_t(G^{\hat{\mu}_N}(r_1))G_t^{\hat{\mu}_N}(r_1)L_t^{\hat{\mu}_N}(x^1, r_1)]^2\,\mathrm{d}Q_{\boldsymbol{r}}^N(\boldsymbol{x})}_{\varphi(t,\boldsymbol{r})}\,\mathrm{d}t\,\mathrm{d}\pi^{\otimes N}(\boldsymbol{r})\right. \\ &\qquad\left. + \frac{\|\bar{J}\|_\infty^2 T}{\lambda_*^2}\right\},\end{aligned}$$

where the inequality follows by use of Fubini's theorem.

We then bound $\varphi(t, \boldsymbol{r})$ uniformly in space to conclude with

$$\sup_{t\leq T}\varphi(t, \boldsymbol{r}) \leq 2\frac{\|\sigma\|_\infty^4 T}{\lambda_*^4}\exp\left\{2\frac{\|\sigma\|_\infty^4 T}{\lambda_*^4}\right\}. \qquad \square$$

5. Identification of the mean-field equations

We have seen that the series of empirical measures $(\hat{\mu}_N)_N$ satisfies an LDP of speed N, and with a good rate function H. In order to identify the limit of the system, we now study the minima of H. In the spin glass model investigated in [2], existence and uniqueness of solutions were made difficult by the fact that the drift was not considered to be Lipschitz continuous. Moreover, the characterization of the possible minima of the good rate function H was achieved through an intricate variational study. Here, we propose another approach that substantially simplifies this characterization. Moreover, due to the regularity of our dynamics, we propose an original contraction argument to show that the good rate function H admits a unique minimum, the proof of which has yet to be developed in the context of the neuronal equations. (For instance, in the nonspatialized case treated in [12], the authors used a strong assumption of linearity of the intrinsic dynamics (our function f) which implied that the solutions were Gaussian, a special case for which moment methods were used; see [26].)

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Lemma 5.1. *Let μ be a probability measure on $\mathcal{C} \times D$ which minimizes H. Then*

$$\mu \sim P, \qquad \mu = Q_\mu, \tag{5.1}$$

where Q_μ is defined by (4.11).

Proof. Let $\mu \in \mathcal{M}_1^+(\mathcal{C} \times D)$ minimize H. The probability measure Q_μ is equivalent to P by Theorem 4.1. Since H is a good rate function, we have $H(\mu) = I(\mu \mid P) - \Gamma(\mu) = 0$, so that $I(\mu \mid P) = \Gamma(\mu) < +\infty$ by Proposition 4.2, yielding $\mu \ll P$. Theorem 4.1 combined with the classical properties of the relative entropy (see [23]) ensure that, for all $\nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, $H_\mu(\nu) = I(\nu \mid Q_\mu) = I(\nu \mid P) - \Gamma_\mu(\nu)$. In particular,

$$I(\mu \mid Q_\mu) = H_\mu(\mu) = I(\mu \mid P) - \Gamma_\mu(\mu) = I(\mu \mid P) - \Gamma(\mu) = H(\mu) = 0,$$

so that $\mu = Q_\mu$. □

We now prove that there exists a unique probability measure satisfying (5.1).

Theorem 5.1. *It holds that $\mu \to Q_\mu$ admits a unique fixed point on $\mathcal{M}_1^+(\mathcal{C} \times D)$.*

Proof. After showing a stochastic process $(x_t^\mu(r))_{t\geq 0}$ (see Lemma 5.2) with law $Q_{\mu,r}$ (defined above through the identity $\mathrm{d}Q_\mu(x,r) := \mathrm{d}Q_{\mu,r}(x)\,\mathrm{d}\pi(r)$), we break the proof in two steps. In Step 1 we control the distance between $(x_t^\mu(r))_{t\geq 0}$ and $(x_t^\nu(r))_{t\geq 0}$ for $\mu, \nu \in \mathcal{M}_1^+(\mathcal{C} \times D)$ and for fixed r, and in Step 2 we control the regularity with respect to r of $(x_t^\mu(r))_{t\geq 0}$.

This proof relies on the definition of a particular stochastic process with law $Q_{\mu,r}$. For any $\mu \in \mathcal{M}_1^+(\mathcal{C} \times D)$ and $r \in D$, Lemma 4.2 yields

$$\frac{\mathrm{d}Q_\mu}{\mathrm{d}P}(x,r) = \exp\left\{ \int_0^T O_\mu(t,x,r)\,\mathrm{d}W_t(x,r) - \frac{1}{2}\int_0^T O_\mu(t,x,r)^2\,\mathrm{d}t \right\},$$

where

$$O_\mu(t,x,r) := \mathcal{E}_\gamma[\Lambda_t(G^\mu(r))G_t^\mu(r)L_t^\mu(x,r)] + m_\mu(t,r).$$

By a direct application of Girsanov's theorem, for any $\mu \in \mathcal{M}_1^+(\mathcal{C} \times D)$ and $r \in D$, $Q_{\mu,r}(x)$ is the law of $(x_t^\mu(r))_{t\in[0,T]}$, the unique strong solution (see Lemma 5.2) of

$$\mathrm{d}x_t^\mu(r) = f(r,t,x_t^\mu(r))\,\mathrm{d}t + \lambda(r)O_\mu(t,x^\mu(r),r)\,\mathrm{d}t + \lambda(r)\,\mathrm{d}W_t,$$
$$(x_s^\mu(r))_{-\bar\tau\leq s\leq 0} = (\bar x_s^0(r))_{-\bar\tau\leq s\leq 0},$$

where W is a $\mathbb{P}$-Brownian motion and $\bar x_0(r) \in \mathbb{R}$ is the realization of the continuous version for the family of initial laws $(\mu_0(r))_{r\in D}$ evaluated at r; see (2.1).

Note that

$$W_t(x^\mu(r),r) = \int_0^t O_\mu(s,x^\mu(r),r)\,\mathrm{d}s + W_t,$$

and

$$L_t^\mu(x^\mu(r),r) = \int_0^t G_s^\mu(r)\underbrace{(O_\mu(s,x^\mu(r),r) - m_\mu(s,r))}_{\hat O_\mu(s,x^\mu(r),r)}\,\mathrm{d}s + \int_0^t G_s^\mu(r)\,\mathrm{d}W_s. \tag{5.2}$$

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Step 1. Control of the distance between $x_t^\mu(r)$ and $x_t^\nu(r)$. We have

$$(x_t^\mu(r) - x_t^\nu(r)) = \int_0^t (f(r,s,x_s^\mu(r)) - f(r,s,x_s^\nu(r)) + \lambda(r)(m_\mu(s,r) - m_\nu(s,r)))\,\mathrm{d}s + \lambda(r)\int_0^t (\hat{O}_\mu(s,x^\mu(r),r) - \hat{O}_\nu(s,x^\nu(r),r))\,\mathrm{d}s. \tag{5.3}$$

As in Proposition 4.4, we can obtain

$$\lambda(r)(m_\mu(s,r) - m_\nu(s,r)) \le C_T d_s^V(\mu,\nu).$$

Let $\xi \in \mathcal{M}_1^+((\mathcal{C}\times D)^2)$ with marginals μ and ν, and let (G, G') be a bi-dimensional centered Gaussian process on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \gamma)$ with covariance $K_\xi(\cdot,\cdot,r)$ given in (4.17). By the Cauchy–Schwarz inequality, we see that

$$\begin{aligned}
&(\hat{O}_\mu(s,x^\mu(r),r) - \hat{O}_\nu(s,x^\nu(r),r))\\
&\quad = \mathcal{E}_\gamma[\Lambda_t(G)G_tL_t - \Lambda_t(G')G_t'L_t']\\
&\quad \le \mathcal{E}_\gamma[L_t^2]^{1/2}(\mathcal{E}_\gamma[(\Lambda_t(G)-\Lambda_t(G'))^2G_t^2]^{1/2} + \mathcal{E}_\gamma[\Lambda_t(G')^2(G_t-G_t')^2]^{1/2})\\
&\qquad + \mathcal{E}_\gamma[\Lambda_t(G')^2G_t'^2]^{1/2}\mathcal{E}_\gamma[(L_t-L_t')^2]^{1/2},
\end{aligned} \tag{5.4}$$

where $L_t := \int_0^t G_s(\mathrm{d}W_s(x^\mu,r) - m_\mu(s,r)\,\mathrm{d}s)$ and $L_t' := \int_0^t G_s'(\mathrm{d}W_s(x^\nu,r) - m_\nu(s,r)\,\mathrm{d}s)$.

In the remainder of the proof, we use the shorthand notation $\hat{O}(t)$ for $\hat{O}_\mu(t,x^\mu(r),r)$ and $\hat{O}'(t)$ for $\hat{O}_\nu(t,x^\nu(r),r)$.

On the one hand, relying on (4.3), (4.5), and Isserlis' theorem, we can show, as in Proposition 4.4, that there exists $C_T > 0$ such that

$$\begin{aligned}
&\mathcal{E}_\gamma[(\Lambda_t(G)-\Lambda_t(G'))^2G_t^2] + \mathcal{E}_\gamma[\Lambda_t(G')^2(G_t-G_t')^2]\\
&\qquad \le C_T\left(\int_{(\mathcal{C}\times D)^2} d_t((y,r'),(z,\tilde{r}))^2\,\mathrm{d}\xi((y,r'),(z,\tilde{r}))\right),
\end{aligned} \tag{5.5}$$

and

$$\lambda(r)\mathcal{E}_\gamma[\Lambda_t(G')^2G_t'^2]^{1/2} \le C_T.$$

On the other hand, we remark that

$$\begin{aligned}
\int_0^t \hat{O}^2(s)\,\mathrm{d}s &\le 2\int_0^t \mathcal{E}_\gamma[G_s^2\Lambda_s(G)^2]\mathcal{E}_\gamma[L_s^2]\,\mathrm{d}s\\
&\le C_T\left\{\int_0^t \mathcal{E}_\gamma\left[\left(\int_0^s G_u\,\mathrm{d}W_u\right)^2\right]\mathrm{d}s + \int_0^t\int_0^s \hat{O}^2(u)\,\mathrm{d}u\,\mathrm{d}s\right\}\\
&\le C_T\int_0^t \mathcal{E}_\gamma\left[\left(\int_0^s G_u\,\mathrm{d}W_u\right)^2\right]\mathrm{d}s,
\end{aligned}$$

where the inequalities follow from Cauchy–Schwarz, (5.2) and Cauchy–Schwarz, and Gronwall's lemma, respectively. Consequently, Cauchy–Schwarz and (5.2) yield

$$\mathcal{E}_\gamma[L_t^2] \le C_T\int_0^t \mathcal{E}_\gamma\left[\left(\int_0^s G_u\,\mathrm{d}W_u\right)^2\right]\mathrm{d}s,$$

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so that, using Jensen and Fubini for the first inequality,

$$\begin{aligned}\mathcal{E}_\gamma[(L_t - L'_t)^2] &\le 2\mathcal{E}_\gamma\left[\left(\int_0^t G_s - G'_s \,\mathrm{d}W_s\right)^2\right] \\ &\quad + 4t\int_0^t \{\mathcal{E}_\gamma[(G_s - G'_s)^2]\hat{O}(s)^2 + \mathcal{E}_\gamma[G'^2_s](\hat{O}(s) - \hat{O}'(s))^2\}\,\mathrm{d}s \\ &\le C_T\left\{\mathcal{E}_\gamma\left[\left(\int_0^t G_s - G'_s \,\mathrm{d}W_s\right)^2\right]\right. \\ &\quad + \left(1 + \int_0^t \mathcal{E}_\gamma\left[\left(\int_0^s G_u \,\mathrm{d}W_u\right)^2\right]\mathrm{d}s\right) \\ &\quad \left.\times\left(\int_{(\mathcal{C}\times D)^2} d_t((y,r'),(z,\tilde{r}))^2\,\mathrm{d}\xi((y,r'),(z,\tilde{r}))\right)\right\},\end{aligned}$$

where the last inequality relies on Gronwall's lemma, as well as (4.16), (5.4), and (5.5). Substituting these results into (5.3), we obtain

$$\begin{aligned}\|x^\mu(r) - x^\nu(r)\|^2_{\infty,t} &\le C_T\int_0^t \left\{\|x^\mu(r) - x^\nu(r)\|^2_{\infty,s} + \mathcal{E}_\gamma\left[\sup_{v\le s}\left(\int_0^v G_u - G'_u \,\mathrm{d}W_u\right)^2\right]\right. \\ &\quad + \left(1 + \mathcal{E}_\gamma\left[\sup_{v\le s}\left(\int_0^v G_u \,\mathrm{d}W_u\right)^2\right]\right) \\ &\quad \left.\times\left(\int_{(\mathcal{C}\times D)^2} d_s((y,r'),(z,\tilde{r}))^2\,\mathrm{d}\xi((y,r'),(z,\tilde{r}))\right)\right\}\mathrm{d}s,\end{aligned}$$

so that using Gronwall's lemma again yields

$$\begin{aligned}&d_t((x^\mu(r),r),(x^\nu(r),r))^2 \\ &\quad\le C_T\int_0^t \left\{\mathcal{E}_\gamma\left[\sup_{v\le s}\left(\int_0^v G_u - G'_u \,\mathrm{d}W_u\right)^2\right]\right. \\ &\qquad + \left(1 + \mathcal{E}_\gamma\left[\sup_{v\le s}\left(\int_0^v G_u \,\mathrm{d}W_u\right)^2\right]\right) \\ &\qquad \left.\times\left(\int_{(\mathcal{C}\times D)^2} d_s((y,r'),(z,\tilde{r}))^2\,\mathrm{d}\xi((y,r'),(z,\tilde{r}))\right)\right\}\mathrm{d}s.\end{aligned}$$

Taking the expectation over the Brownian path and initial condition, and using Fubini's theorem, (4.18), and the Burkholder–Davis–Gundy inequality, we obtain

$$\begin{aligned}&\mathbb{E}[d_t((x^\mu(r),r),(x^\nu(r),r))^2] \\ &\quad\le C_T\int_0^t\left(\int_{(\mathcal{C}\times D)^2} d_s((y,r'),(z,\tilde{r}))^2\,\mathrm{d}\xi((y,r'),(z,\tilde{r}))\right)\mathrm{d}s. \qquad (5.6)\end{aligned}$$

Step 2. Regularity with respect to the spatial variable r. We now show that we can integrate the term on the left-hand side with respect to the measure π. To this end, we show that the

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process is continuous with respect to r. Let $r', r \in D$ be two distinct locations. We have

$$\begin{aligned}|x_t^\mu(r) - x_t^\mu(r')| &\le |\bar{x}_0^0(r) - \bar{x}_0^0(r')| + (K_f + K_\lambda |W_t|)\|r - r'\|_{\mathbb{R}^d} \\ &\quad + \int_0^t (K_f \|x^\mu(r) - x^\mu(r')\|_{\infty,s} + C_T |\lambda(r) m_\mu(s,r) - \lambda(r') m_\mu(s,r')|)\,\mathrm{d}s \\ &\quad + \int_0^t |\mathcal{E}_\gamma[\lambda(r)\Lambda_t(G^\mu(r)) G_t^\mu(r) L_t^\mu(x^\mu(r), r) \\ &\qquad - \lambda(r')\Lambda_t(G^\mu(r')) G_t^\mu(r') L_t^\mu(x^\mu(r'), r')]|\,\mathrm{d}s,\end{aligned}$$

where the Gaussian process $(G^\mu(r), G^\mu(r'))$ has the covariance structure

$$\int_{\mathcal{C}\times D} \begin{pmatrix} \dfrac{\sigma(r,\tilde{r})^2}{\lambda(r)^2} S(y_{s-\tau(r,\tilde{r})}) S(y_{t-\tau(r,\tilde{r})}) & \dfrac{\sigma(r,\tilde{r})\sigma(r',\tilde{r})}{\lambda(r)\lambda(r')} S(y_{s-\tau(r,\tilde{r})}) S(y_{t-\tau(r',\tilde{r})}) \\ \dfrac{\sigma(r,\tilde{r})\sigma(r',\tilde{r})}{\lambda(r)\lambda(r')} S(y_{s-\tau(r',\tilde{r})}) S(y_{t-\tau(r,\tilde{r})}) & \dfrac{\sigma(r',\tilde{r})^2}{\lambda(r')^2} S(y_{s-\tau(r',\tilde{r})}) S(y_{t-\tau(r',\tilde{r})}) \end{pmatrix} \mathrm{d}\mu(y,\tilde{r}).$$

First, observe that

$$\begin{aligned}&|\lambda(r) m_\mu(s,r) - \lambda(r') m_\mu(s,r')| \\ &\quad = \left| \int_{\mathcal{C}\times D} (J(r,\tilde{r}) S(y_{s-\tau(r,\tilde{r})}) - J(r',\tilde{r}) S(y_{s-\tau(r',\tilde{r})}))\,\mathrm{d}\mu(y,\tilde{r}) \right| \\ &\quad \le K_J \|r - r'\|_{\mathbb{R}^d} + \|\bar{J}\|_\infty K_S \int_{\mathcal{C}\times D} |y_{s-\tau(r,\tilde{r})} - y_{s-\tau(r',\tilde{r})}|\,\mathrm{d}\mu(y,\tilde{r}) \\ &\quad \le C \int_{\mathcal{C}\times D} d_s((y,r),(y,r'))\,\mathrm{d}\mu(y,\tilde{r}).\end{aligned}$$

Moreover, as in Proposition 4.4, we can show that

$$\begin{aligned}&\mathcal{E}_\gamma[(\lambda(r)\Lambda_t(G^\mu(r)) - \lambda(r')\Lambda_t(G^\mu(r')))^2 G_t^\mu(r)^2] \\ &\quad + \mathcal{E}_\gamma[\lambda(r')^2 \Lambda_t(G^\mu(r'))^2 (G_t^\mu(r) - G_t^\mu(r'))^2] \\ &\quad \le C_T \int_{\mathcal{C}\times D} d_t((y,r),(y,r'))^2\,\mathrm{d}\mu(y,\tilde{r}).\end{aligned}$$

As a consequence, using the same decomposition as in (5.4), we obtain

$$\begin{aligned}&\|x^\mu(r) - x^\mu(r')\|_{\infty,t}^2 \\ &\quad \le C_T \Bigg\{ \|\bar{x}^0(r) - \bar{x}^0(r')\|_{\tau,\infty}^2 + (1 + (W_t^*)^2)\|r - r'\|_{\mathbb{R}^d}^2 + \int_0^t \|x^\mu(r) - x^\mu(r')\|_{\infty,s}^2\,\mathrm{d}s \\ &\qquad + \int_0^t \left(\int_{\mathcal{C}\times D} d_s((y,r),(y,r'))^2\,\mathrm{d}\mu(y,\tilde{r}) \right) \\ &\qquad \times \left(1 + \mathcal{E}_\gamma\left[\sup_{v\le s} \left(\int_0^v G_u^\mu(r)\,\mathrm{d}W_u \right)^2 \right] \right) \mathrm{d}s \\ &\qquad + \int_0^t \mathcal{E}_\gamma\left[\sup_{v\le s} \left(\int_0^v G_u^\mu(r) - G_u^\mu(r')\,\mathrm{d}W_u \right)^2 \right] \mathrm{d}s \Bigg\},\end{aligned}$$

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where $W_t^* = \sup_{0\le s\le t} |W_s|$. Using Gronwall's lemma, taking the expectation, and relying again on Fubini's theorem and the Burkholder–Davis–Gundy inequality, we obtain

$$\mathbb{E}[\|x^\mu(r) - x^\mu(r')\|_{\infty,t}^2] \le C_T \Bigg\{ \mathbb{E}[\|\bar{x}^0(r) - \bar{x}^0(r')\|_{\tau,\infty}^2] + (1 + \mathbb{E}[(W_t^*)^2]) \|r - r'\|_{\mathbb{R}^d}^2 + \int_0^t \int_{\mathcal{C}\times D} d_s((y,r),(y,r'))^2 \, \mathrm{d}\mu(y,\tilde{r}) \, \mathrm{d}s \Bigg\}.$$

Hence, $\mathbb{E}[\|x^\mu(r) - x^\mu(r')\|_{\infty,t}^2] \to 0$ as $\|r' - r\|_{\mathbb{R}^d} \searrow 0$, using (2.1) and the monotone convergence theorem. Now, observe that

$$\phi_t^{\mu,\nu}(r) := \mathbb{E}[d_t((x^\mu(r), r), (x^\nu(r), r))^2] = \mathbb{E}[\|x^\mu(r) - x^\nu(r)\|_{\infty,t}^2].$$

As a consequence, by the Cauchy–Schwarz inequality and (5.6),

$$\begin{aligned} |\phi_t^{\mu,\nu}(r) - \phi_t^{\mu,\nu}(r')| &= |\mathbb{E}[\|x^\mu(r) - x^\nu(r)\|_{\infty,t}^2 - \|x^\mu(r') - x^\nu(r')\|_{\infty,t}^2]| \\ &\le C_T d_T^V(\mu,\nu) \sqrt{\mathbb{E}[\|x^\mu(r) - x^\mu(r')\|_{\infty,t}^2] + \mathbb{E}[\|x^\nu(r) - x^\nu(r')\|_{\infty,t}^2]}. \end{aligned}$$

In particular, $r \to \phi_t^{\mu,\nu}(r)$ is continuous as soon as $d_T^V(\mu,\nu) < +\infty$ (which is, for instance, the case as soon as the two measures have a bounded second moment). We can thus integrate (5.6) over space, and obtain, for any ξ with marginals μ and ν,

$$\mathbb{E}\Bigg[\int_D d_t((x^\mu(r), r), (x^\nu(r), r))^2 \, \mathrm{d}\pi(r)\Bigg] \le C_T \int_0^t \Bigg(\int_{(\mathcal{C}\times D)^2} d_s((y,r'),(z,\tilde{r}))^2 \, \mathrm{d}\xi((y,r'),(z,\tilde{r})) \Bigg) \mathrm{d}s. \tag{5.7}$$

Conclusion of the proof. We now have all elements to conclude the proof using Picard's iterations. Define by $Q_{\mu,\nu}$ the law of $((x^\mu(r), r), (x^\nu(r), r))$, where $\mathcal{L}(r) = \pi$. It has marginal Q_μ and Q_ν so that for any $\xi \in \mathcal{M}_1^+((\mathcal{C}\times D)^2)$ with marginals μ and ν, using (5.7),

$$\begin{aligned} &\mathbb{E}\Bigg[\int_D d_t((x^{Q_\mu}(r), r), (x^{Q_\nu}(r), r))^2 \, \mathrm{d}\pi(r)\Bigg] \\ &\quad\le C_T \int_0^t \Bigg(\int_{(\mathcal{C}\times D)^2} d_s((y,r'),(z,\tilde{r}))^2 \, \mathrm{d}Q_{\mu,\nu}((y,r'),(z,\tilde{r})) \Bigg) \mathrm{d}s \\ &\quad\le C_T^2 \int_0^t \int_0^s \Bigg(\int_{(\mathcal{C}\times D)^2} d_u((y,r'),(z,\tilde{r}))^2 \, \mathrm{d}\xi((y,r'),(z,\tilde{r})) \Bigg) \mathrm{d}u \, \mathrm{d}s. \end{aligned}$$

Letting, for any $\mu \in \mathcal{M}_1^+(\mathcal{C}\times D)$, $Q_\mu^0 := \mu$, and by recurrence $Q_\mu^n := Q_{Q_\mu^{n-1}}$ for any $n \ge 1$, we can iterate this inequality to obtain for any $\xi \in \mathcal{M}_1^+((\mathcal{C}\times D)^2)$ with marginals μ and ν, using (5.7) in the second inequality,

$$\begin{aligned} d_t^V(Q_\mu^n, Q_\nu^n)^2 &\le \mathbb{E}\Bigg[\int_D d_t((x^{Q_\mu^n}(r), r), (x^{Q_\nu^n}(r), r))^2 \, \mathrm{d}\pi(r)\Bigg] \\ &\le C_T^{n+1} \int_0^t \int_0^{t_1} \cdots \int_0^{t_n} \Bigg(\int_{(\mathcal{C}\times D)^2} d_{t_{n+1}}((y,r'),(z,\tilde{r}))^2 \, \mathrm{d}\xi((y,r'),(z,\tilde{r})) \Bigg) \\ &\qquad\qquad \times \mathrm{d}t_{n+1} \cdots \mathrm{d}t_1 \, \mathrm{d}s \\ &\le \frac{C_T^{n+1} t^{n+1}}{(n+1)!} \Bigg(\int_{(\mathcal{C}\times D)^2} d_t((y,r'),(z,\tilde{r}))^2 \, \mathrm{d}\xi((y,r'),(z,\tilde{r})) \Bigg). \end{aligned}$$

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Taking the infimum on ξ now yields

$$d_t^V(Q_\mu^n, Q_\nu^n)^2 \le \frac{C_T^{n+1} t^{n+1}}{(n+1)!} d_t^V(\mu, \nu)^2.$$

Hence, for large enough n, $\mu \to Q_\mu^n$ is a contraction so that we conclude the proof using classical arguments. □

Lemma 5.2. *For any $r \in D$ and $\mu \in \mathcal{M}_1^+(\mathcal{C} \times D)$, there exists a unique strong solution to the SDE*

$$\mathrm{d}x_t^\mu(r) = f(r, t, x_t^\mu(r))\,\mathrm{d}t + \lambda(r) O_\mu(t, x^\mu, r)\,\mathrm{d}t + \lambda(r)\,\mathrm{d}W_t, \qquad (x_s^\mu(r))_{-\bar\tau \le s \le 0} = \bar x^0(r),$$

where W is a $\mathbb{P}$-Brownian motion and $\bar x^0(r) \in \mathcal{C}_\tau$ is the continuous realization of the initial law $\mu_0(r)$ of (2.1).

Proof. The proof relies on Picard's iterations. Let $x^0 \in \mathcal{C}_\tau$ with $x^0 = \bar x^0(r)$, and define recursively the sequence $(x_t^n,\ 0 \le t \le T)_{n \in \mathbb{N}^*}$ by $(x_s^n)_{-\bar\tau \le s \le 0} = \bar x^0(r)$ and

$$x_t^{n+1} = \bar x_0^0(r) + \int_0^t f(r, s, x_s^n)\,\mathrm{d}s + \int_0^t \lambda(r) O_\mu(s, x^n, r)\,\mathrm{d}s + \lambda(r) W_t \quad \text{for all } t \in [0, T].$$

We have, for any $n \ge 1$, $t \in [0, T]$,

$$x_t^{n+1} - x_t^n = \int_0^t (f(r, s, x_s^n) - f(r, s, x_s^{n-1}))\,\mathrm{d}s + \lambda(r) \int_0^t O_\mu(s, x^n, r) - O_\mu(s, x^{n-1}, r)\,\mathrm{d}s.$$

We then proceed as in the proof of Theorem 5.1 to show that

$$\mathbb{E}\Big[\sup_{s \le t} |x_s^{n+1} - x_s^n|^2\Big] \le C_T \int_0^t \mathbb{E}\Big[\sup_{u \le s} |x_u^n - x_u^{n-1}|^2\Big]\,\mathrm{d}s.$$

The conclusion relies on classical arguments. □

5.1. Convergence of the process

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. For δ a strictly positive real number and $B(Q, \delta)$ the open ball of radius δ centered in Q for the Wasserstein distance, we prove that $Q^N(\hat\mu_N \notin B(Q, \delta))$ tends to 0 as $N \to \infty$. Indeed, for K_ε a compact set defined in Theorem 4.3, we have for any $\varepsilon > 0$,

$$Q^N(\hat\mu_N \notin B(Q, \delta)) \le \varepsilon + Q^N(\hat\mu_N \in B(Q, \delta)^c \cap K_\varepsilon).$$

The set $B(Q, \delta)^c \cap K_\varepsilon$ is a compact set, and Theorem 4.2 now ensures that

$$\limsup_{N \to \infty} \frac{1}{N} \log Q^N(\hat\mu_N \in B(Q, \delta)^c \cap K_\varepsilon) \le - \inf_{B(Q,\delta)^c \cap K_\varepsilon} H$$

and, eventually, Theorem 3.4 ensures that the right-hand side of the inequality is strictly negative, which implies that

$$\lim_{N \to \infty} Q^N(\hat\mu_N \notin B(Q, \delta)) \le \varepsilon,$$

i.e.

$$\lim_{N \to \infty} Q^N(\hat\mu_N \notin B(Q, \delta)) = 0.$$ □

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Appendix A. Proof of Lemma 2.1: regularity of the solutions for the limit equation

In this appendix we prove the regularity in space of the solutions to the result of Lemma 2.1. We start by showing a technical lemma on the uncoupled system before proceeding to the proof of that result.

Lemma A.1. (i) *The map*

$$\mathcal{P}: \begin{cases} D \to \mathcal{M}_1^+(\mathcal{C}), \\ r \to P_r \end{cases} \tag{A.1}$$

is continuous with respect to the Borel topology on D and the weak topology on $\mathcal{M}_1^+(\mathcal{C})$, e.g. $r_n \to r$ implies that $P_{r_n} \xrightarrow{\mathcal{L}} P_r$.

(ii) *It holds that P is a well-defined probability measure on $\mathcal{C} \times D$.*

Proof. The first point is the consequence of a coupling argument. Let W be a P-Brownian motion, $\bar{x}^0 \colon D \to \mathcal{C}_\tau$ be as in (2.1), and $(r_n)_{n\in\mathbb{N}} \in D^{\mathbb{N}}$ a sequence of positions that converges toward $r \in D$. We consider X^n and X, the respective strong solutions of the SDEs

$$\begin{aligned} \mathrm{d}X_t^n &= f(r_n, t, X_t^n)\,\mathrm{d}t + \lambda(r_n)\,\mathrm{d}W_t, \qquad (X_t^n)_{t\in[-\bar{\tau},0]} = \bar{x}^0(r_n), \\ \mathrm{d}X_t &= f(r, t, X_t)\,\mathrm{d}t + \lambda(r)\,\mathrm{d}W_t, \qquad (X_t)_{t\in[-\bar{\tau},0]} = \bar{x}^0(r) \end{aligned}$$

driven by the same Brownian motion W.

Then, by Gronwall's lemma, letting $W_T^* = \sup_{t\in[-\bar{\tau},T]} |W_t|$,

$$\|X^n - X\|_{\infty,T} \le (\|\bar{x}^0(r_n) - \bar{x}^0(r)\|_{\tau,\infty} + \|r - r_n\|_{\mathbb{R}^d} K_f T + K_\lambda \|r - r_n\|_{\mathbb{R}^d} W_T^*) e^{\{K_f T\}}.$$

Hence, by (2.1),

$$\mathbb{E}[\|X^n - X\|_{\infty,T}^2] \to 0 \quad \text{as } r_n \to r,$$

so that $P_{r_n} = \mathcal{L}(X_n)$ implies that $\mathcal{L}(X) = P_r$ as $r_n \to r$.

In order to prove the second point, let W_r be the unique strong solution of

$$\mathrm{d}X_t = \lambda(r)\,\mathrm{d}W_t, \qquad (X_t)_{t\in[-\bar{\tau},0]} = \bar{x}^0(r).$$

Following [40, Exercise (2.10)], we remark that, by the Lipschitz continuity of f, explosion of P_r almost surely never occurs in finite time, so that Girsanov's theorem applies, i.e.

$$P_r \ll W_r, \qquad \frac{\mathrm{d}P_r}{\mathrm{d}W_r} = \exp\left\{ \int_0^T \frac{f(r,t,X_t)}{\lambda(r)}\,\mathrm{d}X_t - \frac{1}{2}\int_0^T \left(\frac{f(r,t,X_t)}{\lambda(r)}\right)^2 \mathrm{d}t \right\}.$$

Consequently, for all $\mathcal{A} \in \mathcal{B}(\mathcal{C})$,

$$P_r(A) = \mathbb{E}_{W_r}\left(\frac{\mathrm{d}P_r}{\mathrm{d}W_r}\mathbf{1}_A\right),$$

so that $P_r(A) = 0$ as soon as $W_r(A) = 0$. As $\lambda(r) \ge \lambda_*$, $W_r(\partial\mathcal{B}(y,\varepsilon)) = 0$ for any $y \in \mathcal{C}$ and $\varepsilon > 0$, so that $r \to P_r(\mathcal{B}(y,\varepsilon))$ is a continuous map by the Portmanteau theorem and the first point. We can thus unequivocally define $P(\mathcal{B}(y,\varepsilon) \times B) := \int_B P_r(\mathcal{B}(y,\varepsilon))\,\mathrm{d}\pi(r)$ for any $y \in \mathcal{C}$, $\varepsilon > 0$, and $B \in \mathcal{B}(D)$. As $\{\mathcal{B}(y,\varepsilon) \times B,\ y \in \mathcal{C},\ \varepsilon > 0,\ B \in \mathcal{B}(D)\}$ forms a Π-system that generates $\mathcal{B}(\mathcal{C} \times D)$, P is a well-defined probability measure on $\mathcal{C} \times D$. □

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We now proceed to prove Lemma 2.1 which we restate.

Lemma A.2. *(Lemma 2.1 restated.) The map*

$$\mathcal{Q}: \begin{cases} D^N \to \mathcal{M}_1^+(\mathcal{C}^N), \\ \boldsymbol{r} \to Q_{\boldsymbol{r}}^N, \end{cases}$$

where $Q_{\boldsymbol{r}}^N := \mathcal{E}_J(Q_{\boldsymbol{r}}^N(J))$*, is continuous with respect to the weak topology. Moreover,*

$$\mathrm{d}Q^N(\boldsymbol{x}, \boldsymbol{r}) := \mathrm{d}Q_{\boldsymbol{r}}^N(\boldsymbol{x})\,\mathrm{d}\pi^{\otimes N}(\boldsymbol{r})$$

defines a probability measure on $\mathcal{M}_1^+((\mathcal{C} \times D)^N)$.

Remark A.1. Note that $\mathcal{Q}$ maps the positions $\boldsymbol{r}$ to the Gaussian averaged of the solutions $Q_{\boldsymbol{r}}^N(J)$, so that its continuity seems to be a consequence of the Cauchy–Lipschitz theorem with parameter $\boldsymbol{r}$. Yet, the equation depends on $\boldsymbol{r}$ through the Gaussian synaptic weights J_{ij} which satisfy only a continuity in law. Meanwhile, although the proof is not difficult, it must rely on another argument. The one developed here is a coupling method.

Proof of Lemma 2.1. Before proceeding to the coupling method, we remark that the Gaussian synaptic connections display some regularity in space. In fact, although the variables J_{ij} and $J_{i'j'}$ are independent for $i \neq i'$ or $j \neq j'$, their probability distribution continuously depends on the spatial location of the cells, in the sense that one can find a version $\tilde{J}_{i'j'}$ of $J_{i'j'}$ such that

$$\mathcal{E}_J(|\tilde{J}_{i'j'} - J_{ij}|) \leq \frac{C}{\sqrt{N}}(\|r_i - r_{i'}\|_{\mathbb{R}^d} + \|r_j - r_{j'}\|_{\mathbb{R}^d}) \tag{A.2}$$

for some $C > 0$ independent of the location of the neurons. This is a consequence of the Lipschitz continuity and boundedness of the mean and variance maps J and σ.

We now proceed to the proof, and note the fact that N remains constant. Fix a deterministic sequence $(\boldsymbol{r}^n = (r_i^n)_{1\leq i\leq N})_{n\in\mathbb{N}^*} \to_n \boldsymbol{r} = (r_i)_{1\leq i\leq N} \in D^N$, let $(W_t^i,\ 0 \leq t \leq T)_{i\in[[1,N]]}$ be a family of independent $\mathbb{P}$-Brownian motions, and $\bar{x}^{0,i}: D \to \mathcal{C}_\tau$, $1 \leq i \leq N$, be N independent initial conditions as in (2.1). Let $X_{\boldsymbol{r}^n}^N = (X_{\boldsymbol{r}^n}^{i,N})_{i\in[[1,N]]}$ and $X_{\boldsymbol{r}}^N = (X_{\boldsymbol{r}}^{i,N})_{i\in[[1,N]]}$ be the respective strong solutions of the following two SDEs:

$$\mathrm{d}X_{\boldsymbol{r}^n}^{i,N}(t) = \left(f(r_i^n, t, X_{\boldsymbol{r}^n}^{i,N}(t)) + \sum_{j=1}^N \tilde{J}_{ij}^{\boldsymbol{r}^n} S(X_{\boldsymbol{r}^n}^{j,N}(t - \tau_{r_i^n r_j^n}))\right)\mathrm{d}t + \lambda(r_i^n)\,\mathrm{d}W_t^i,$$
$$(X_{\boldsymbol{r}^n}^N(t))_{t\in[-\bar{\tau},0]} = (\bar{x}^{0,i}(r_i^n))_{1\leq i\leq N},$$

and

$$\mathrm{d}X_{\boldsymbol{r}}^{i,N}(t) = \left(f(r_i, t, X_{\boldsymbol{r}}^{i,N}(t)) + \sum_{j=1}^N J_{ij}^{\boldsymbol{r}} S(X_{\boldsymbol{r}}^{j,N}(t - \tau_{r_i r_j}))\right)\mathrm{d}t + \lambda(r_i)\,\mathrm{d}W_t^i,$$
$$(X_{\boldsymbol{r}}^N(t))_{t\in[-\bar{\tau},0]} = (\bar{x}^{0,i}(r_i))_{1\leq i\leq N},$$

where $J_{ij}^{\boldsymbol{r}} \sim \mathcal{N}(J(r_i, r_j)/N, \sigma(r_i, r_j)^2/N)$ and $\tilde{J}_{ij}^{\boldsymbol{r}^n} \sim J_{ij}^{\boldsymbol{r}^n}$ satisfy (A.2), and where we use the shorthand notation $\tau_{rr'} := \tau(r, r')$. In particular, $X_{\boldsymbol{r}^n}^{i,N}$ has law $Q_{\boldsymbol{r}^n}^N(J^{\boldsymbol{r}^n})$, and $X_{\boldsymbol{r}}^{i,N}$ has law $Q_{\boldsymbol{r}}^N(J^{\boldsymbol{r}})$.

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Let $W_T^{*,i} = \sup_{t\in[0,T]} |W_t^i|$. Then using the Lipschitz continuity of f, λ, S, the fact that $|S| \le 1$, and taking the supremum in time, we obtain

$$\begin{aligned}
&\|X^{i,N}_{\boldsymbol{r}^n} - X^{i,N}_{\boldsymbol{r}}\|_{\infty,t} \\
&\quad \le \|\bar{x}^{0,i}(r_i^n) - \bar{x}^{0,i}(r_i)\|_{\tau,\infty} + (K_f t + K_\lambda W_T^{*,i})\|r_i^n - r_i\|_{\mathbb{R}^d} \\
&\qquad + \int_0^t \left\{ K_f \|X^{i,N}_{\boldsymbol{r}^n} - X^{i,N}_{\boldsymbol{r}}\|_{\infty,s} + K_S \sum_{j=1}^N |J^{\boldsymbol{r}}_{ij}| \|X^{j,N}_{\boldsymbol{r}^n} - X^{j,N}_{\boldsymbol{r}}\|_{\infty,s} \right\} \mathrm{d}s \\
&\qquad + \sum_{j=1}^N \Bigg\{ t|\tilde{J}^{\boldsymbol{r}^n}_{ij} - J^{\boldsymbol{r}}_{ij}| \\
&\qquad\qquad + K_S \int_0^t |J^{\boldsymbol{r}}_{ij}| \sup_{a,b\in[-\bar{\tau},0],\, |a-b|\le 2K_\tau \|\boldsymbol{r}^n - \boldsymbol{r}\|_\infty} |X^{j,N}_{\boldsymbol{r}}(s+a) - X^{j,N}_{\boldsymbol{r}}(s+b)| \,\mathrm{d}s \Bigg\},
\end{aligned}$$

where $\|\boldsymbol{r}\|_\infty = \sup_{1\le i\le N} \|r_i\|_{\mathbb{R}^d}$. Denote, for any $X = (X^i)_{1\le i\le N} \in \mathcal{C}^N$, $t \in [-\bar{\tau}, T]$, $\|X\|^1_{\infty,t} = \sum_{1\le i\le N} \|X^i\|_{\infty,t}$, and for any $\boldsymbol{r} \in D^N$, $\|\boldsymbol{r}\|_1 = \sum_{1\le i\le N} \|r_i\|_{\mathbb{R}^d}$. Summing over $i \in [[1, N]]$ and using Gronwall's inequality now yields

$$\begin{aligned}
&\|X^N_{\boldsymbol{r}^n} - X^N_{\boldsymbol{r}}\|^1_{\infty,t} \\
&\quad \le C_T \exp\left\{ C_T \sum_{i,j=1}^N |J^{\boldsymbol{r}}_{ij}| \right\} \\
&\qquad \times \Bigg(\|\bar{\boldsymbol{x}}^0(\boldsymbol{r}^n) - \bar{\boldsymbol{x}}^0(\boldsymbol{r})\|^1_{\tau,\infty} + \Big(\sup_{1\le i\le N} W_T^{*,i} \Big) \|\boldsymbol{r}^n - \boldsymbol{r}\|_1 + \sum_{i,j=1}^N t|\tilde{J}^{\boldsymbol{r}^n}_{ij} - J^{\boldsymbol{r}}_{ij}| \\
&\qquad\quad + \Big(\sum_{i,j=1}^N |J^{\boldsymbol{r}}_{ij}| \Big) \int_0^t \sup_{1\le j\le N} \left\{ \sup_{\substack{a,b\in[-\bar{\tau},0] \\ |a-b|\le 2K_\tau \|\boldsymbol{r}^n - \boldsymbol{r}\|_\infty}} |X^{j,N}_{\boldsymbol{r}}(s+a) - X^{j,N}_{\boldsymbol{r}}(s+b)| \right\} \mathrm{d}s \Bigg).
\end{aligned} \tag{A.3}$$

The synaptic connections being Gaussian, observe that there exists a map $\chi : D^N \times D^N \to \mathbb{R}$ such that $\chi(\boldsymbol{r}^n, \boldsymbol{r}) \to 0$ when $n \to \infty$, and a constant $C_{T,N}$ such that

$$\mathcal{E}_J\left[\exp\left\{ 2C_T \sum_{i,j=1}^N |J^{\boldsymbol{r}}_{ij}| \right\} \right]^{1/2} + \mathbb{E}\left[\Big(\sup_{1\le i\le N} W_T^{*,i} \Big)^2 \right] + \mathcal{E}_J\left[\Big(\sum_{i,j=1}^N |J^{\boldsymbol{r}}_{ij}| \Big)^2 \right] \le C_{T,N},$$

$$\mathbb{E}[(\|\bar{\boldsymbol{x}}^0(\boldsymbol{r}^n) - \bar{\boldsymbol{x}}^0(\boldsymbol{r})\|^1_{\tau,\infty})^2] + (\|\boldsymbol{r}^n - \boldsymbol{r}\|_1)^2 + \sum_{i,j=1}^N t^2 \mathcal{E}_J[(\tilde{J}^{\boldsymbol{r}^n}_{ij} - J^{\boldsymbol{r}}_{ij})^2] \le C_{T,N} \chi(\boldsymbol{r}^n, \boldsymbol{r})$$

using (2.1) and (A.2). Denoting $\mathbb{E}_J[\cdot] := \mathbb{E}[\mathcal{E}_J[\cdot]]$, we find that, taking the expectation in (A.3) and relying on the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}_J[[\|X^N_{\boldsymbol{r}^n} - X^N_{\boldsymbol{r}}\|^1_{\infty,t}] \\
&\quad \le \tilde{C}_{T,N} \left(\chi(\boldsymbol{r}^n, \boldsymbol{r}) + \mathbb{E}_J\left[\sup_{1\le j\le N} \sup_{\substack{a,b\in[-\bar{\tau},0],\, s\in[0,t] \\ |a-b|\le 2K_\tau \|\boldsymbol{r}^n - \boldsymbol{r}\|_\infty}} |X^{j,N}_{\boldsymbol{r}}(s+a) - X^{j,N}_{\boldsymbol{r}}(s+b)|^2 \right] \right)^{1/2}.
\end{aligned}$$

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As the solutions are $\mathbb{P}$-almost surely continuous, and N remains (here) finite, the monotone convergence theorem ensures that the right-hand side tends toward 0 when $n \to \infty$. It implies, in particular, that $Q^N_{\boldsymbol{r}^n}$ converges in law toward $Q^N_{\boldsymbol{r}}$ when $n \to \infty$, so that the map $\boldsymbol{r} \to \mathcal{E}_J[\int_{\mathcal{C}^N} \phi(\boldsymbol{x})\, \mathrm{d}Q^N_{\boldsymbol{r}}(\boldsymbol{x})]$ is continuous and integrable with respect to $\pi^{\otimes N}$. In particular, $\mathrm{d}Q^N(\boldsymbol{x}, \boldsymbol{r}) := \mathrm{d}Q^N_{\boldsymbol{r}}(\boldsymbol{x})\, \mathrm{d}\pi^{\otimes N}(\boldsymbol{r})$ defines a probability measure on $(\mathcal{C} \times D)^N$. □

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