Combinatorics, Probability and Computing (2001) 10, 187–201. Printed in the United Kingdom
© 2001 Cambridge University Press

Descent Functions and Random Young Tableaux

RON M. ADIN and YUVAL ROICHMAN†

Department of Mathematics and Computer Science,
Bar-Ilan University, Ramat-Gan 52900, Israel
(e-mail: radin@math.biu.ac.il, yuvalr@math.biu.ac.il)

Received 7 October 1999; revised 13 October 2000

The expectation of the descent number of a random Young tableau of a fixed shape is given, and concentration around the mean is shown. This result is generalized to the major index and to other descent functions. The proof combines probabilistic arguments together with combinatorial character theory. Connections with Hecke algebras are mentioned.

1. Introduction

1.1. Background

In the late sixties Erdős and Turán published a classical series of papers on random permutations. Since then there has been a resurgence of interest in probabilistic aspects of combinatorial parameters of permutations and related objects.

This paper deals principally with two classical combinatorial parameters: descent number and major index. These parameters were originally studied in the context of permutations. The study of the descent number of a permutation started with Euler; the major index was introduced by MacMahon [14]. Foata and Schützenberger [6, 7], Garsia and Gessel [9], and others, carried out an extensive study of these parameters. The definitions of descent number and major index for permutations led to definitions of the same concepts for Young tableaux. These parameters on permutations – as well as on tableaux – play significant roles in algebraic combinatorics: the Solomon descent algebra [15, Chapter 9], Schur functions [19, Chapter 7], and combinatorial character formulas [17]. These concepts have also been applied to sorting [13, Section 5.1] and card shuffling [4].

In this paper we study the distribution of these parameters for random Young tableaux

† Both authors supported in part by the Israel Science Foundation and by internal research grants from Bar-Ilan University.

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

of a given shape. Proofs of the main results are obtained by a combination of probabilistic arguments and combinatorial character theory.

1.2. Main results

Let λ be a partition of n (for definitions of basic concepts see Section 2). We shall be concerned with *random* (standard Young) tableaux, assumed to be chosen uniformly with prescribed shape λ. A *descent* in a standard Young tableau T is an entry i such that $i+1$ is strictly south (and weakly west) of i. Denote the set of all descents in T by $D(T)$.

For any function $f : \boldsymbol{N} \to \boldsymbol{R}$ we introduce the corresponding *descent function* d_f on standard Young tableaux:

$$d_f(T) := \sum_{i \in D(T)} f(i).$$

This concept generalizes two classical combinatorial parameters, the *descent number* and the *major index* (for tableaux):

$$\mathrm{des}(T) := \sum_{i \in D(T)} 1, \qquad \mathrm{maj}(T) := \sum_{i \in D(T)} i.$$

In this paper we prove the following result.

Theorem 1.1. *Let λ be a fixed partition of n, and let $E_\lambda[d_f]$ be the expected value of a descent function d_f on random standard Young tableaux of shape λ. Then*

$$E_\lambda[d_f] = c(\lambda) \cdot \sum_{i=1}^{n-1} f(i),$$

where $c(\lambda) := [\binom{n}{2} - \sum_i \binom{\lambda_i}{2} + \sum_j \binom{\lambda'_j}{2}]/n(n-1)$. (Here λ_i is the length of the ith row in the Young diagram of shape λ, and λ'_i is the length of the ith column.)

See Theorem 4.1 below.

Under mild conditions the descent function is concentrated around its mean. A function $f : \boldsymbol{N} \to \boldsymbol{R}$ has *strictly polynomial growth* if there exist constants $0 < c_1 < c_2$ and $\alpha > 0$, such that $c_1 \leqslant \frac{f(n)}{n^\alpha} \leqslant c_2$ for sufficiently large n.

Theorem 1.2. *Let $0 < \delta < 1$ and $\varepsilon > 0$ be fixed constants, and let λ be a partition of n with $\lambda_1 \leqslant \delta n$. Then, for any function f with strictly polynomial growth, and for a random standard Young tableau T of shape λ,*

$$d_f(T) = (1 + O(n^{-\frac{1}{2}+\varepsilon}))E_\lambda[d_f]$$

almost surely (i.e., with probability tending to 1 as n tends to infinity), uniformly on f and λ as above.

See Theorem 5.1 below. For other work, following the current paper, see [11].

The rest of the paper is organized as follows. Definitions, notation and necessary preliminaries are given in Section 2. In Section 3 the expectation and variance of the

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

major index are evaluated. Here Stanley's hook formula plays a crucial role. Results obtained in Section 3 are extended to general descent functions in Sections 4 and 5 by combining probabilistic arguments with combinatorial character formulas. We end the paper with remarks on connections of the statistics of descent functions with the spectra of so-called 'good' elements in Hecke algebras.

2. Preliminaries

2.1. Young tableaux

Let n be a positive integer. A *partition* of n is a vector of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k$ and $\lambda_1 + \cdots + \lambda_k = n$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ define the *conjugate partition* $\lambda' = (\lambda'_1, \ldots, \lambda'_t)$ by letting λ'_i be the number of parts of λ that are $\geqslant i$.

The *dominance* (*partial*) *order* on partitions is defined as follows. For any two partitions of n, μ and λ, μ *dominates* λ if and only if, for any i, $\sum_{j=0}^{i} \lambda_j \leqslant \sum_{j=0}^{i} \mu_j$. For example, $\lambda = (4, 4, 2, 1)$ is a partition of 11. Then $\lambda' = (4, 3, 2, 2)$, and λ dominates λ'.

The set $\{(i, j) \mid i, j \in \boldsymbol{Z},\ 0 < i \leqslant k,\ 0 < j \leqslant \lambda_i\}$ is called the *Young diagram* of *shape* λ, (i, j) being the *cell* in row i and column j. The diagram of the conjugate shape λ' may be obtained from the diagram of shape λ by interchanging rows and columns.

A *Young tableau* of shape λ is obtained by inserting the integers $1, 2, \ldots, n$ as *entries* in the cells of the Young diagram of shape λ, allowing no repetitions. A *standard Young tableau* of shape λ is a Young tableau whose entries increase along rows and columns.

We shall draw Young tableaux as in the following example.

Example 1.

$$\begin{array}{ccccc} 1 & 3 & 4 & 6 & 9 \\ 2 & 7 & 8 & 12 & \\ 5 & 11 & & & \\ 10 & & & & \end{array}$$

The *hook length* of a cell (i, j) in the diagram of shape λ is defined by

$$h_{i,j} := \lambda_i + \lambda'_j - i - j + 1.$$

Let f^λ denote the number of standard Young tableaux of shape λ. A famous combinatorial formula describes this number in term of hook lengths.

The Frame–Robinson–Thrall hook formula ([18], Theorem 3.1.2).

$$f^\lambda = \frac{n!}{\prod\limits_{(i,j)\in\lambda} h_{ij}}.$$

2.2. Descents

A *descent* in a standard Young tableau T is an entry i such that $i + 1$ is strictly south (and weakly west) of i. Denote the set of all descents in T by $D(T)$. The *descent number* and the *major index* (for tableaux) are defined as follows:

$$\operatorname{des}(T) := \sum_{i\in D(T)} 1, \qquad \operatorname{maj}(T) := \sum_{i\in D(T)} i.$$

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

Example 2. Let T be the standard Young tableau drawn in Example 1. Then $D(T) = \{1, 4, 6, 9\}$, $\mathrm{des}(T) = 4$, and $\mathrm{maj}(T) = 1 + 4 + 6 + 9 = 20$.

The following theorem describes the generating function for the major index of standard Young tableaux.

The Stanley hook formula ([19], Corollary 21.5).

$$\sum_{\mathrm{shape}(T)=\lambda} q^{\mathrm{maj}(T)} = q^{\sum_i (i-1)\lambda_i} \cdot \frac{\prod_{k=1}^{n} [k]_q}{\prod_{(i,j)\in\lambda} [h_{ij}]_q},$$

where the sum is taken over all standard Young tableaux of shape λ, h_{ij} are the hook lengths in the diagram of λ, and, for any positive integer m,

$$[m]_q := 1 + q + q^2 + \cdots + q^{m-1}.$$

For $q = 1$ this formula reduces to the Frame–Robinson–Thrall hook formula for the number of standard Young tableaux of a given shape. No such formula is known for the descent number of tableaux.

2.3. Characters

A (complex) representation of a group G is a homomorphism $\rho : G \to GL_n(\boldsymbol{C})$. The character $\chi^\rho : G \to \boldsymbol{C}$ is the trace of $\rho(g)$, $g \in G$. By definition, the character is a class function on the group (*i.e.*, invariant under conjugation). An *irreducible* representation is one that has no nontrivial subspace invariant under all $\rho(g)$, $g \in G$.

The conjugacy classes of the symmetric group S_n are described by their cycle type; thus, by the partitions of n. The irreducible representations of S_n are also indexed by these partitions: see, for instance, [18].

Let λ and μ be partitions of n. Let χ^λ_μ denote the value at a conjugacy class of cycle type μ of the character of the irreducible representation indexed by λ.

The following combinatorial formula represents the irreducible characters of S_n in terms of descents of standard Young tableaux. This formula is a special case of [16, Theorem 4].

Theorem 2.1.

$$\chi^\lambda_\mu = \sum_{\mathrm{shape}(T)=\lambda} \mathrm{weight}_\mu(T),$$

where the sum is taken over all standard tableaux of shape λ, and the weight $\mathrm{weight}_\mu(T) \in \{\pm 1, 0\}$ *is defined as follows:*

$$\mathrm{weight}_\mu(T) := \prod_{\substack{1\le i\le k \\ i\notin B(\mu)}} f_\mu(i, T),$$

where $B(\mu) = \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \cdots + \mu_t\}$, *and*

$$f_\mu(i, T) := \begin{cases} -1, & i \in D(T), \\ 0, & i \notin D(T), i+1 \in D(T) \text{ and } i+1 \notin B(\mu), \\ 1, & \text{otherwise.} \end{cases}$$

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

3. Major index

In this section we apply Stanley's hook formula (see Section 1.2) to evaluate the expectation and variance of the major index of random Young tableaux of a given shape.

Proposition 3.1. *Let λ be a fixed partition of n, and let $E_\lambda[\mathrm{maj}]$ be the expected value of the major index on random standard Young tableaux of shape λ. Then*

$$E_\lambda[\mathrm{maj}] = \frac{1}{2}\left[\binom{n}{2} - \sum_i \binom{\lambda_i}{2} + \sum_j \binom{\lambda'_j}{2}\right].$$

Proof. Let

$$f(q) := \sum_{\mathrm{shape}(T)=\lambda} q^{\mathrm{maj}(T)}.$$

Then

$$q \cdot f'(q) = \sum_{\mathrm{shape}(T)=\lambda} \mathrm{maj}(T) \cdot q^{\mathrm{maj}(T)}.$$

Hence

$$(\log f(q))'|_{q=1} = \left.\frac{f'(q)}{f(q)}\right|_{q=1} = \frac{\sum_{\mathrm{shape}(T)=\lambda} \mathrm{maj}(T)}{\sum_{\mathrm{shape}(T)=\lambda} 1} = E_\lambda[\mathrm{maj}]. \tag{3.1}$$

In order to evaluate the expected value of the major index we need the following elementary limit:

$$\lim_{q\to 1} \frac{[m]'_q}{[m]_q} = \frac{\binom{m}{2}}{m} = \frac{m-1}{2}. \tag{3.2}$$

Substituting Stanley's hook formula in (3.1), and using (3.2), we obtain

$$E_\lambda[\mathrm{maj}] = (\log f(q))'|_{q=1} = \sum_i (i-1)\lambda_i + \sum_{k=1}^{n} \frac{k-1}{2} - \sum_{(i,j)\in\lambda} \frac{h_{ij}-1}{2}. \tag{3.3}$$

Note that

$$\sum_i (i-1)\lambda_i = \sum_j \binom{\lambda'_j}{2} \tag{3.4}$$

and

$$\sum_{(i,j)\in\lambda} \frac{h_{ij}-1}{2} = \frac{1}{2}\left[\sum_i \binom{\lambda_i}{2} + \sum_j \binom{\lambda'_j}{2}\right]. \tag{3.5}$$

Substituting (3.4) and (3.5) in (3.3) completes the proof. □

Proposition 3.2. *Let λ be a fixed partition of n, and let $\mathrm{Var}_\lambda[\mathrm{maj}]$ be the variance of the major index on random standard Young tableaux of shape λ. Then*

$$\mathrm{Var}_\lambda[\mathrm{maj}] = \frac{1}{12}\left[\sum_{k=1}^{n} k^2 - \sum_{(i,j)\in\lambda} h_{ij}^2\right].$$

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

Proof. Let f denote, as in the previous proof, the generating function of the major index of Young tableaux of a given shape. Then

$$\sum_{\mathrm{shape}(T)=\lambda} (\mathrm{maj}(T))^2 \cdot q^{\mathrm{maj}(T)} = q^2 \cdot f''(q) + q \cdot f'(q).$$

Hence

$$\begin{aligned} \mathrm{Var}_\lambda[\mathrm{maj}] &= E_\lambda[\mathrm{maj}^2] - (E_\lambda[\mathrm{maj}])^2 \\ &= \left[\frac{q^2 f'' + qf'}{f} - \left(q\frac{f'}{f}\right)^2\right]\Bigg|_{q=1} \\ &= \left[q^2\left(\frac{f'}{f}\right)' + q\frac{f'}{f}\right]\Bigg|_{q=1}. \end{aligned} \tag{3.6}$$

Now

$$\frac{f'}{f} = (\log f)' = \sum_i (i-1)\lambda_i \cdot q^{-1} + \sum_{k=1}^{n} \frac{[k]_q'}{[k]_q} - \sum_{(i,j)\in\lambda} \frac{[h_{ij}]_q'}{[h_{ij}]_q}$$

and

$$\begin{aligned} \lim_{q\to 1}\left(\frac{[m]_q'}{[m]_q}\right)' &= \lim_{q\to 1}\left[\frac{[m]_q''}{[m]_q} - \left(\frac{[m]_q'}{[m]_q}\right)^2\right] \\ &= \lim_{q\to 1}\left[\frac{\sum_{k=2}^m (k-1)(k-2)q^{k-3}}{\sum_{k=1}^m q^{k-1}} - \left(\frac{\sum_{k=1}^m (k-1)q^{k-2}}{\sum_{k=1}^m q^{k-1}}\right)^2\right] \\ &= \frac{\sum_{k=2}^m (k-1)(k-2)}{m} - \left(\frac{\sum_{k=1}^m (k-1)}{m}\right)^2 \\ &= \frac{2\binom{m}{3}}{m} - \left(\frac{m-1}{2}\right)^2 = \frac{(m-1)(m-5)}{12}. \end{aligned}$$

Hence

$$\lim_{q\to 1}\left(\frac{f'}{f}\right)' = -\sum_i (i-1)\lambda_i + \sum_{k=1}^{n} \frac{(k-1)(k-5)}{12} - \sum_{(i,j)\in\lambda} \frac{(h_{ij}-1)(h_{ij}-5)}{12}. \tag{3.7}$$

Substituting (3.7) and (3.3) into the right-hand side of (3.6), we obtain the desired result. □

Corollary 3.3. *Let $0 < \delta < 1$ and $\varepsilon > 0$ be fixed constants, and let λ be a partition of n with $\lambda_1 \leqslant \delta n$. Then*

$$\mathrm{maj}(T) = (1 + O(n^{-\frac{1}{2}+\varepsilon}))E_\lambda[\mathrm{maj}]$$

almost surely (i.e., with probability tending to 1 as n tends to infinity) uniformly on λ as above.

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

Proof. By Chebyshev's inequality [5, (6.2)],

$$\Pr\left(|\mathrm{maj}(T) - E_\lambda[\mathrm{maj}]| \geqslant tE_\lambda[\mathrm{maj}]\right) \leqslant \frac{1}{t^2} \cdot \frac{\mathrm{Var}_\lambda[\mathrm{maj}]}{E_\lambda[\mathrm{maj}]^2}.$$

Now, by Proposition 3.2,

$$0 \leqslant \mathrm{Var}_\lambda[\mathrm{maj}] \leqslant \frac{1}{12}\sum_{k=1}^{n} k^2 = O(n^3).$$

In order to bound $E_\lambda[\mathrm{maj}]$ from below, note that

$$\sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda'_j}{2} = \sum_{(i,j)\in\lambda} (j-i). \tag{3.8}$$

Thus this expression is a monotone increasing function of λ with respect to the dominance order of partitions (see also [21]). So, under the restriction $\lambda_1 \leqslant \delta n$ (we may assume that $\delta > \frac{1}{2}$), this expression is maximized when $\lambda_1 = \delta n, \lambda_2 = (1-\delta)n$. Hence, by Proposition 3.1,

$$\begin{aligned} E_\lambda[\mathrm{maj}] &= \frac{1}{2}\left[\binom{n}{2} - \sum_i \binom{\lambda_i}{2} + \sum_j \binom{\lambda'_j}{2}\right] = \frac{1}{2}\left[\binom{n}{2} - \sum_{(i,j)\in\lambda}(j-i)\right] \\ &\geqslant \frac{1}{2}\left[\binom{n}{2} - \binom{\delta n}{2} - \binom{(1-\delta)n}{2}\right] = \Omega(n^2). \end{aligned}$$

Therefore,

$$\frac{1}{t^2} \cdot \frac{\mathrm{Var}_\lambda[\mathrm{maj}]}{E_\lambda[\mathrm{maj}]^2} \leqslant O\left(\frac{1}{t^2 n}\right) = O(n^{-2\varepsilon}),$$

provided that $\frac{1}{t} = O(n^{1/2-\varepsilon})$. We conclude that, for such λ and t,

$$(1-t)E_\lambda[\mathrm{maj}] < \mathrm{maj}(T) < (1+t)E_\lambda[\mathrm{maj}]$$

with probability tending to 1 as n tends to infinity. □

4. Expectation of descent functions

In this section we generalize Proposition 3.1 to an arbitrary descent function. This is done by combining probabilistic arguments together with combinatorial character formulas.

Recall the definition of descent functions from Section 1.

Theorem 4.1. *Let λ be a fixed partition of n, let $f : \boldsymbol{N} \to \boldsymbol{R}$ be an arbitrary function, and let $E_\lambda[d_f]$ be the expected value of d_f on random standard Young tableaux of shape λ. Then*

$$E_\lambda[d_f] = c(\lambda) \cdot \sum_{i=1}^{n-1} f(i),$$

where $c(\lambda) := [\binom{n}{2} - \sum_i \binom{\lambda_i}{2} + \sum_j \binom{\lambda'_j}{2}]/n(n-1)$.

Substituting $f(i) = 1$ and $f(i) = i$ in Theorem 4.1 gives the expectations for the

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

descent number and for the major index of a random tableau, respectively. In particular, Proposition 3.1 is a special case of Theorem 4.1.

To prove Theorems 4.1 and 5.1 we shall use a variant of the character formula given in Theorem 2.1. For a fixed $1 \leqslant i < n$, define the i-weight

$$\mathrm{weight}^{i}_{(21\ldots1)}(T) := \begin{cases} -1, & \text{if } i \in D(T), \\ 1, & \text{if } i \notin D(T). \end{cases}$$

For a fixed $1 \leqslant i < n-1$, define another i-weight

$$\mathrm{weight}^{i}_{(31\ldots1)}(T) := \begin{cases} -1, & \text{if } i \in D(T) \text{ and } i+1 \notin D(T), \\ 1, & \text{if } i, i+1 \in D(T) \text{ or } i, i+1 \notin D(T), \\ 0, & \text{if } i \notin D(T) \text{ and } i+1 \in D(T). \end{cases}$$

For a fixed pair $1 \leqslant i < j < n$ with $j - i > 1$, define the ij-weight

$$\mathrm{weight}^{ij}_{(221\ldots1)}(T) := \begin{cases} 1, & \text{if } i, j \in D(T) \text{ or } i, j \notin D(T), \\ -1, & \text{otherwise.} \end{cases}$$

The i-weight (or ij-weight) of a standard tableau T depends on i (or i, j). However, the sum of i-weights (or ij-weights) over all standard tableaux of shape λ is independent of i and j and gives the corresponding character.

Lemma 4.2. *For any partition λ of n,*

(i) $$\chi^{\lambda}_{(21\ldots1)} = \sum_{\mathrm{shape}(T)=\lambda} \mathrm{weight}^{i}_{(21\ldots1)}(T) \qquad (1 \leqslant i < n),$$

(ii) $$\chi^{\lambda}_{(31\ldots1)} = \sum_{\mathrm{shape}(T)=\lambda} \mathrm{weight}^{i}_{(31\ldots1)}(T) \qquad (1 \leqslant i < n-1),$$

(iii) $$\chi^{\lambda}_{(221\ldots1)} = \sum_{\mathrm{shape}(T)=\lambda} \mathrm{weight}^{ij}_{(221\ldots1)}(T) \qquad (1 \leqslant i < j-1 < n-1).$$

For proofs and more details see [16, Section 7].

Recall also that

$$\chi^{\lambda}_{(1\ldots1)} = \sum_{\mathrm{shape}(T)=\lambda} 1 = f^{\lambda}.$$

Proof of Theorem 4.1. Let T be a random standard Young tableau of shape λ. For $1 \leqslant i < n$, let X_i be the random variable defined by

$$X_i := \begin{cases} 1, & \text{if } i \in D(T), \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$E_{\lambda}[d_f] = E_{\lambda}\left[\sum_{i=1}^{n-1} f(i) X_i\right] = \sum_{i=1}^{n-1} f(i) E_{\lambda}[X_i]. \tag{4.1}$$

Now, by definition,

$$\mathrm{weight}^{i}_{(21\ldots1)} = 1 - 2X_i$$

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

and therefore, by Lemma 4.2(i),

$$1 - 2E_\lambda[X_i] = E_\lambda[\text{weight}^i_{(21\ldots1)}] = \chi^\lambda_{(21\ldots1)}/\chi^\lambda_{(1\ldots1)}. \tag{4.2}$$

In particular, note that $E_\lambda[X_i]$ is independent of i (see [19, Proposition 7.19.9]).

A classical formula of Frobenius shows that

$$\chi^\lambda_{(21\ldots1)}/\chi^\lambda_{(1\ldots1)} = \frac{1}{\binom{n}{2}}\left[\sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda'_j}{2}\right] \tag{4.3}$$

(see [12]).

Combining (4.1), (4.2) and (4.3) completes the proof. □

5. Concentration

In this section Corollary 3.3 is generalized to descent functions satisfying certain mild conditions.

Recall that a function $f : \boldsymbol{N} \to \boldsymbol{R}$ has *strictly polynomial growth* if there exist constants $0 < c_1 < c_2$ and $\alpha > 0$, such that $c_1 \leqslant \frac{f(n)}{n^\alpha} \leqslant c_2$ for sufficiently large n.

Theorem 5.1. *Let $0 < \delta < 1$ and $\varepsilon > 0$ be fixed constants, and let λ be a partition of n with $\lambda_1 \leqslant \delta n$. Then, for any function f with strictly polynomial growth, and for a random standard Young tableau T of shape λ,*

$$d_f(T) = (1 + O(n^{-\frac{1}{2}+\varepsilon}))E_\lambda[d_f]$$

almost surely (i.e., with probability tending to 1 as n tends to infinity), uniformly on f and λ as above.

This holds, in particular, if d_f is either a descent number or a major index.

Theorem 5.1 is proved by estimating the asymptotic behaviour of the variance. This is done by expressing the variance in terms of S_n-characters evaluated at 'small' conjugacy classes, and showing that $\text{Var}_\lambda[d_f]/E_\lambda[d_f]^2$ is independent of the function f, up to a multiplicative constant.

Let

$$\begin{aligned} r_2^\lambda &:= \chi^\lambda_{21\ldots1}/\chi^\lambda_{1\ldots1}, \\ r_3^\lambda &:= \chi^\lambda_{31\ldots1}/\chi^\lambda_{1\ldots1}, \\ r_{22}^\lambda &:= \chi^\lambda_{221\ldots1}/\chi^\lambda_{1\ldots1}. \end{aligned}$$

These are the values of the normalized irreducible character corresponding to λ at conjugacy classes of types $(21\ldots1), (31\ldots1)$ and $(221\ldots1)$ respectively.

The following lemma plays a crucial role in the proof of Theorem 5.1.

Lemma 5.2.

(i) $$\Pr[X_i = 1] = \tfrac{1}{2}(1 - r_2^\lambda) \qquad (1 \leqslant i < n),$$

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

(ii) $\Pr[X_i = 1 \text{ and } X_{i+1} = 1] = \frac{1}{2}(1 - r_2^\lambda) - \frac{1}{3}(1 - r_3^\lambda) \qquad (1 \leqslant i < n-1),$

(iii) $\Pr[X_i = 1 \text{ and } X_j = 1] = \frac{1}{2}(1 - r_2^\lambda) - \frac{1}{4}(1 - r_{22}^\lambda) \qquad (1 \leqslant i < j-1 < n-1),$

where the probability $\Pr[\cdot]$ *is taken in the probability space of all standard Young tableaux of a given shape, defined in Section 1.*

Proof. **(i)** Since X_i is a 0–1 variable,

$$E_\lambda[X_i] = \Pr[X_i = 1].$$

Using equation (4.2) and the definition of r_2^λ, the desired result follows.

(ii) By Lemma 4.2(ii), for $1 \leqslant i < n-1$,

$$r_3^\lambda = \Pr[X_i = 1 \wedge X_{i+1} = 1] + \Pr[X_i = 0 \wedge X_{i+1} = 0] - \Pr[X_i = 1 \wedge X_{i+1} = 0].$$

Therefore,

$$r_3^\lambda = (1 - \Pr[X_i = 1 \wedge X_{i+1} = 0] - \Pr[X_i = 0 \wedge X_{i+1} = 1]) - \Pr[X_i = 1 \wedge X_{i+1} = 0]. \quad (5.1)$$

Now

$$\begin{aligned} \Pr[X_i = 1 \wedge X_{i+1} = 0] &= \Pr[X_i = 1] - \Pr[X_i = 1 \wedge X_{i+1} = 1], \\ \Pr[X_i = 0 \wedge X_{i+1} = 1] &= \Pr[X_{i+1} = 1] - \Pr[X_i = 1 \wedge X_{i+1} = 1], \end{aligned}$$

and therefore, using (i) above,

$$\begin{aligned} \Pr[X_i = 1 \wedge X_{i+1} = 0] &= \Pr[X_i = 0 \wedge X_{i+1} = 1] \\ &= \tfrac{1}{2}(1 - r_2^\lambda) - \Pr[X_i = 1 \wedge X_{i+1} = 1]. \end{aligned}$$

Thus, from (5.1),

$$\begin{aligned} 1 - r_3^\lambda &= 2\Pr[X_i = 1 \wedge X_{i+1} = 0] + \Pr[X_i = 0 \wedge X_{i+1} = 1] \\ &= \tfrac{3}{2}(1 - r_2^\lambda) - 3\Pr[X_i = 1 \wedge X_{i+1} = 1], \end{aligned}$$

and (ii) follows.

(iii) By Lemma 4.2(iii), for any $1 \leqslant i < j-1 < n-1$,

$$\begin{aligned} r_{22}^\lambda &= \Pr[X_i = 1 \wedge X_j = 1] + \Pr[X_i = 0 \wedge X_j = 0] \\ &\quad - \Pr[X_i = 0 \wedge X_j = 1] - \Pr[X_i = 1 \wedge X_j = 0]. \end{aligned}$$

We now continue as in the proof of (ii) above. □

Let

$$\begin{aligned} P_{2,\lambda} &:= \Pr[X_i = 1], \\ P_{3,\lambda} &:= \Pr[X_i = 1 \text{ and } X_{i+1} = 1], \\ P_{22,\lambda} &:= \Pr[X_i = 1 \text{ and } X_j = 1] \qquad (j - i > 1). \end{aligned}$$

By Lemma 5.2, these probabilities are well defined (independent of i and j).

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

Proof of Theorem 5.1. By Chebyshev's inequality,

$$\Pr\left(|d_f - E_\lambda[d_f]| \geqslant tE_\lambda[d_f]\right) \leqslant \frac{1}{t^2} \cdot \frac{\mathrm{Var}_\lambda[d_f]}{E_\lambda[d_f]^2}.$$

In order to prove Theorem 5.1, it is sufficient to give an effective upper bound on $\mathrm{Var}_\lambda[d_f]/E_\lambda[d_f]^2$.

For a random tableau T of shape λ,

$$d_f(T) = \sum_{i=1}^{n-1} f(i)X_i.$$

Thus,

$$\begin{aligned}\mathrm{Var}_\lambda[d_f] &= E_\lambda[d_f^2] - E_\lambda[d_f]^2 = E_\lambda\left[\left(\sum_{i=1}^{n-1} X_i f(i)\right)^2\right] - \left(E_\lambda\left[\sum_{i=1}^{n-1} X_i f(i)\right]\right)^2 \qquad (5.2)\\ &= \sum_{i=1}^{n-1} E_\lambda[X_i^2]f(i)^2 + 2\sum_{i=1}^{n-2} E_\lambda[X_iX_{i+1}]f(i)f(i+1)\\ &\quad + 2\sum_{j-i>1} E_\lambda[X_iX_j]f(i)f(j) - \left(\sum_{i=1}^{n-1} E_\lambda[X_i]f(i)\right)^2.\end{aligned}$$

Since X_i is a 0–1 variable,

$$E_\lambda[X_i^2] = E_\lambda[X_i] = P_{2,\lambda} \qquad (1 \leqslant i < n), \qquad (5.3)$$

$$E_\lambda[X_iX_{i+1}] = P_{3,\lambda} \qquad (1 \leqslant i < n-1), \qquad (5.4)$$

and

$$E_\lambda[X_iX_j] = P_{22,\lambda} \qquad (j - i > 1). \qquad (5.5)$$

Substituting (5.3)–(5.5) into the right-hand side of (5.2), we obtain

$$\begin{aligned}\mathrm{Var}_\lambda[d_f] &= P_{2,\lambda}\sum_{i=1}^{n-1} f(i)^2 + 2P_{3,\lambda}\sum_{i=1}^{n-2} f(i)f(i+1)\\ &\quad + 2P_{22,\lambda}\sum_{j-i>1} f(i)f(j) - P_{2,\lambda}^2\left(\sum_{i=1}^{n-1} f(i)\right)^2.\end{aligned}$$

Let

$$\begin{aligned}\Sigma_1 &:= \sum_{i=1}^{n-1} f(i),\\ \Sigma_2 &:= \sum_{i=1}^{n-1} f(i)^2,\\ \Sigma_3 &:= 2\sum_{i=1}^{n-2} f(i)f(i+1).\end{aligned}$$

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

Obviously

$$2\sum_{j-i>1} f(i)f(j) = \Sigma_1^2 - \Sigma_2 - \Sigma_3,$$

and therefore

$$\begin{aligned}\mathrm{Var}_\lambda[d_f] &= P_{2,\lambda}\Sigma_2 + P_{3,\lambda}\Sigma_3 + P_{22,\lambda}(\Sigma_1^2 - \Sigma_2 - \Sigma_3) - P_{2,\lambda}^2\Sigma_1^2 \\ &= (P_{2,\lambda} - P_{22,\lambda})\Sigma_2 + (P_{3,\lambda} - P_{22,\lambda})\Sigma_3 + (P_{22,\lambda} - P_{2,\lambda}^2)\Sigma_1^2.\end{aligned}$$

Also

$$E_\lambda[d_f] = P_{2,\lambda}\Sigma_1,$$

and consequently, using Lemma 5.2(i)–(iii),

$$\frac{\mathrm{Var}_\lambda[d_f]}{E_\lambda[d_f]^2} = \frac{1-r_{22}^\lambda}{(1-r_2^\lambda)^2}\cdot\frac{\Sigma_2}{\Sigma_1^2} + \frac{4r_3^\lambda - 3r_{22}^\lambda - 1}{3(1-r_2^\lambda)^2}\cdot\frac{\Sigma_3}{\Sigma_1^2} + \frac{r_{22}^\lambda - (r_2^\lambda)^2}{(1-r_2^\lambda)^2}. \tag{5.6}$$

Now f has strictly polynomial growth. It follows that

$$\frac{\Sigma_2}{\Sigma_1^2} = O(n^{-1}) \qquad \text{and} \qquad \frac{\Sigma_3}{\Sigma_1^2} = O(n^{-1}), \tag{5.7}$$

where the constants in $O(\cdot)$ depend only on c_1, c_2 and α.

It follows from formula (4.3) and the proof of Corollary 3.3 that, for any fixed $\frac{1}{2} < \delta < 1$,

$$\begin{aligned}\max_{\lambda_1 \leqslant \delta n} r_2^\lambda &= \max_{\lambda_1 \leqslant \delta n} \frac{1}{\binom{n}{2}}\left[\sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda'_j}{2}\right] \\ &\leqslant \frac{1}{\binom{n}{2}}\cdot\left[\binom{\delta n}{2} + \binom{(1-\delta)n}{2}\right] \leqslant \delta^2 + (1-\delta)^2,\end{aligned}$$

where the maximum is taken over all partitions λ of n with $\lambda_1 \leqslant \delta n$.

Hence

$$\frac{1}{(1-r_2^\lambda)^2} \leqslant \frac{1}{2\delta(1-\delta)}. \tag{5.8}$$

The absolute value of a normalized character of a finite group is bounded above by 1. Combining this elementary fact with (5.8) implies that there exist constants $c_1(\delta), c_2(\delta)$ independent of n, so that

$$c_1(\delta) \leqslant \frac{1-r_{22}^\lambda}{(1-r_2^\lambda)^2} \leqslant c_2(\delta) \qquad \text{and} \qquad c_1(\delta) \leqslant \frac{4r_3^\lambda - 3r_{22}^\lambda - 1}{3(1-r_2^\lambda)^2} \leqslant c_2(\delta). \tag{5.9}$$

To complete the proof it still remains to estimate the asymptotics of $r_{22}^\lambda - (r_2^\lambda)^2$.

Using the classical Frobenius character formula, it may be shown that

$$r_{22}^\lambda = \frac{4}{(n-2)(n-3)} + \frac{1}{\binom{n}{2,2,n-4}}\cdot\left[\left(\sum_{(i,j)\in\lambda}(j-i)\right)^2 - 3\sum_{(i,j)\in\lambda}(j-i)^2\right]$$

(see [12] and [21]).

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

Using this formula, (4.3) and (3.8), we obtain

$$|r_{22}^{\lambda} - (r_2^{\lambda})^2| \leqslant \frac{4}{(n-2)(n-3)} + \left[\frac{\binom{n}{2}^2}{\binom{n}{2,2,n-4}} - 1\right](r_2^{\lambda})^2 \tag{5.10}$$
$$+ \frac{3}{\binom{n}{2,2,n-4}} \sum_{(i,j)\in\lambda} (j-i)^2 \leqslant O(n^{-2}) + O(n^{-1})(r_2^{\lambda})^2 + O(n^{-1}) = O(n^{-1}).$$

Substituting (5.7)–(5.10) into the right-hand side of (5.6) shows that

$$\frac{\mathrm{Var}_{\lambda}[d_f]}{E_{\lambda}[d_f]^2} = O(n^{-1}).$$

This completes the proof. □

6. Exponents in Hecke algebras

Surprisingly, the expectations that appear in Proposition 3.1 and Theorem 4.1 turn out to be the exponents of q in eigenvalues of irreducible Hecke algebra representations. In particular, $E_{\lambda}[\mathrm{des}]$ and $E_{\lambda}[\mathrm{maj}]$ are the exponents of q for the Hecke algebra elements corresponding to a Coxeter element and the longest element of S_n, respectively. This follows from a well-known result of Benson and Curtis [3].

The Hecke algebra $\mathscr{H}_n(q)$ of type A is the algebra over $\boldsymbol{F} := \boldsymbol{C}(q^{\frac{1}{2}})$ generated by $n-1$ generators $T_1, \ldots, T_{n-1}$, satisfying the Moore–Coxeter relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad (1 \leqslant i < n-1)$$

and

$$T_i T_j = T_j T_i, \qquad \text{if } |i-j| > 1,$$

as well as the following 'deformed involution' relation:

$$T_i^2 = (1-q)T_i + q \qquad (1 \leqslant i < n).$$

Note that the third relation is slightly nonstandard. This is done in order to have a more elegant formulation of Proposition 6.1.

Let $w_0 := (1,n)(2,n-1)\cdots$ be the longest permutation in the symmetric group S_n, and $c_n := (12\ldots n)$ be a full cycle (also known as a Coxeter element).

Proposition 6.1.

(i) *The eigenvalues of T_{w_0} in the irreducible representation indexed by λ are $\pm q^{E_{\lambda}[\mathrm{maj}]}$.*

(ii) *The eigenvalues of T_{c_n} in the irreducible representation indexed by λ are*

$$\omega^{\mathrm{maj}(T)} q^{E_{\lambda}[\mathrm{des}]},$$

where T ranges over all standard tableaux of shape λ, and $\omega = e^{\frac{2\pi i}{n}}$.

Proof. For self-containment of the paper we recall the proof of [3, corrections and additions]. See also [10, Lemma 4.1]. It is well known that $T_{w_0}^2$ lies in the centre of $\mathscr{H}_n(q)$.

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

Hence, letting ρ^λ denote the irreducible representation of $\mathscr{H}_n(q)$ indexed by λ, $\rho^\lambda(T_{w_0}^2)$ is a scalar operator.

On the other hand, for each generator $T_i, 1 \leqslant i < n$, $\rho^\lambda(T_i)$ has two eigenvalues, 1 and $-q$, with multiplicities $\frac{1}{2}(f^\lambda + \chi_2^\lambda)$ and $\frac{1}{2}(f^\lambda - \chi_2^\lambda)$ respectively. Here f^λ is the degree of ρ^λ and $\chi_2^\lambda = \mathrm{tr}\rho^\lambda(T_i)|_{q=1}$. Hence

$$\det\rho^\lambda(T_i) = (-q)^{\frac{1}{2}(f^\lambda - \chi_2^\lambda)} = \pm q^{\frac{1}{2}(f^\lambda - \chi_2^\lambda)}.$$

T_{w_0} is a product of $\binom{n}{2}$ generators T_i. Hence

$$\det\rho^\lambda(T_{w_0}^2) = \det\rho^\lambda(T_i)^{n(n-1)} = q^{\binom{n}{2}(f^\lambda - \chi_2^\lambda)}.$$

This shows that the eigenvalues of the scalar operator $\rho^\lambda(T_{w_0}^2)$ are all equal to

$$q^{\binom{n}{2}\frac{f^\lambda - \chi_2^\lambda}{f^\lambda}},$$

and those of T_{w_0} are

$$\pm q^{\binom{n}{2}\frac{f^\lambda - \chi_2^\lambda}{2f^\lambda}} = \pm q^{\binom{n}{2}\frac{1-r_2^\lambda}{2}} = \pm q^{E_\lambda[\mathrm{maj}]}.$$

The last equality follows from Proposition 3.1 and formula (4.3).

To prove the second part of Proposition 6.1 recall that

$$T_{c_n}^n = T_{w_0}^2.$$

So, the eigenvalues of T_{c_n} are the complex nth roots of the eigenvalues of $\rho^\lambda(T_{w_0}^2)$. Combining this fact with Theorem 4.1 gives the corresponding exponent of q. For a calculation of the exponents of ω see [20]. □

Proposition 6.1 may be useful in the study of probabilistic interpretations of the Hecke algebra, and of related asymmetric random walks.

References

[1] Adin, R. M. and Roichman, Y. (1999) On random Young tableaux (extended abstract). In *Paul Erdős and his Mathematics: Research Communications*, János Bolyai Mathematical Society, Budapest, pp. 4–6.

[2] Alon, N. and Spencer, J. H. (1992) *The Probabilistic Method* (with an appendix by Paul Erdős), Wiley, New York.

[3] Benson, C. T. and Curtis, C. W. (1972) On the degrees and rationality of certain characters of finite Chevalley groups. *Trans. Amer. Math. Soc.* **165** 251–273. Corrections and additions, *ibid.* **202** (1975) 405–406.

[4] Diaconis, P., McGreath, M. and Pitman, J. (1995) Riffle shuffles, cycles, and descents. *Combinatorica* **15** 11–29.

[5] Feller, W. (1970) *An Introduction to Probability Theory and its Applications, Vol. I*, Wiley, New York.

[6] Foata, D. (1968) On the Netto inversion number of a sequence. *Proc. Amer. Math. Soc.* **19** 236–240.

[7] Foata, D. and Schützenberger, M. P. (1978) Major index and inversion number of permutations. *Math. Nachr.* **83** 143–159.

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press

[8] Fulman, J. (1998) The distribution of descents in fixed conjugacy classes of the symmetric groups. *J. Combin. Theory Ser. A* **84** 171–180.
[9] Garsia, A. and Gessel, I. (1979) Permutation statistics and partitions. *Adv. Math.* **31** 288–305.
[10] Geck, M. and Michel, J. (1997) 'Good' elements of finite Coxeter groups and representations of Iwahori–Hecke algebras. *Proc. London Math. Soc.* **74** 275–305.
[11] Hästö, P. A. (2000) On descents in standard Young tableaux. *Electron. J. Combin.* **7** #59.
[12] Ingram, R. E. (1950) Some characters of the symmetric group. *Proc. Amer. Math. Soc.* **1** 358–369.
[13] Knuth, D. (1973) *The Art of Computer Programming, Vol. 3: Sorting and Searching*, Addison-Wesley, Reading, MA.
[14] MacMahon, P. A. (1916) *Combinatory Analysis, Vols I–II*, Cambridge University Press, London/New York. (Reprinted by Chelsea, New York (1960).)
[15] Reutenauer, C. (1993) *Free Lie Algebras*, Vol. 7 of *London Math. Soc. Monographs, New Series*, Oxford University Press.
[16] Roichman, Y. (1997) A recursive rule for Kazhdan–Lusztig characters. *Adv. Math.* **129** 25–45.
[17] Roichman, Y. (2000) On permutation statistics and Hecke algebra characters. In *Combinatorial Methods in Representation Theory* (K. Koike *et al.*, eds), Vol. 28 of *Adv. Stud. Pure Math.*, Math. Soc. Japan, pp. 287–304.
[18] Sagan, B. E. (1991) *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Wadsworth and Brooks/Cole, CA.
[19] Stanley, R. P. (1999) *Enumerative Combinatorics, Vol. II*, Cambridge University Press.
[20] Stembridge, J. (1989) On the eigenvalues of representations of reflection groups and wreath products. *Pacific J. Math.* **140** 359–396.
[21] Suzuki, M. (1987) The values of irreducible characters of the symmetric group. In *The Arcata Conference on Representations of Finite Groups*, Vol. 47 of *Proceedings of Symposia in Pure Mathematics*, Part 2, Amer. Math. Soc., pp. 317–319.

https://doi.org/10.1017/S0963548301004679 Published online by Cambridge University Press