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Control Approach to an Ill-Posed Variational Inequality

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Abstract. We are concerned with the proof of a generalized solution to an ill-posed variational inequality. This is determined as a solution to an appropriate minimization problem involving a nonconvex functional, treated by an optimal control technique.

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Mathematics Subject Classification: 35K55, 47J06, 49J20, 76SXX1

1. Introduction

In this paper we aim to provide a generalized solution to an ill-posed variational inequality by using an optimal control technique. More exactly, we deal with the problem

$$y_t - \Delta y \in \partial I_K(y) + f, \text{ in } Q = (0,T) \times \Omega, \tag{1.1}$$

$$-\frac{\partial y}{\partial \nu} = g, \qquad \text{on } \Sigma = (0,T) \times \Gamma, \tag{1.2}$$

$$y(0) = y_0, \text{ in } \Omega, \tag{1.3}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, with a C^1 boundary Γ, T is finite, $\frac{\partial}{\partial \nu}$ is the normal derivative to Γ, f, g and y_0 are given in Q, Σ, and Ω, respectively. All over in the paper, ϕ_t denotes the time derivative of a generic function ϕ. Here, I_K is the indicator function of a closed convex subset K of $\mathbb{R}$,

$$I_K(r) = 0 \text{ if } r \in K, \ I_K(r) = +\infty \text{ if } r \notin K,$$

and $\partial I_K(r)$ is its subdifferential, that is

$$\partial I_K(r) = \{\eta \in \mathbb{R};\ \eta(r - \overline{r}) \geq 0,\ \forall \overline{r} \in K\}.$$

In the following we assume that $K = [b,c]$, $b, c \in \mathbb{R}$, $b \geq 0$, $c > 0$, and so

$$\partial I_K(r) = \begin{cases} (-\infty, 0], & \text{if } r = b, \\ 0, & \text{if } r \in (b,c), \\ [0, +\infty), & \text{if } r = c. \end{cases}$$

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For a later use, we also introduce the maximal monotone graph

$$H_{b,c}(r) = \begin{cases} b, & \text{if } r < 0, \\ [b,c], & \text{if } r = 0, \\ c, & \text{if } r > 0, \end{cases}$$

which clearly is the inverse of ∂I_K.

The model (1.1)-(1.3) may describe an absorption-desorption process, as for example oxygen absorbed by biological tissues (see e.g., [4], [7], [2], p. 140). Also, we can refer to a fluid absorbed into the pores of a porous material (e.g., water in the soil), or radiation or heat retained or ceded by materials or tissues. In this model, y is the absorbed or desorbed fluid concentration at the rate f, whenever it is in the interval (b,c).

Equations (1.1)-(1.3) describe in fact the evolution of a physical system in time dependent evolving domains, which can occur as subsets of Ω, namely

$$\Omega_{b,c}(t) = \{x \in \Omega;\ b < y(t,x) < c\},$$
$$\Omega_b(t) = \{x \in \Omega;\ y(t,x) = b\},\ \Omega_c(t) = \{x \in \Omega;\ y(t,x) = c\}$$

with the boundaries $\Gamma_b(t) = \partial\Omega_b(t)$, $\Gamma_c(t) = \partial\Omega_c(t)$.

Roughly speaking (1.1)-(1.3) reduces to

$$y_t - \Delta y = f \text{ in } Q_{b,c}(t) = (0,T) \times \Omega_{b,c}(t), \tag{1.4}$$

$$y_t - \Delta y \le f, \text{ in } Q_b(t) = (0,T) \times \Omega_b(t), \tag{1.5}$$

$$y_t - \Delta y \ge f, \text{ in } Q_c(t) = (0,T) \times \Omega_c(t), \tag{1.6}$$

$$-\frac{\partial y}{\partial \nu} = 0, \text{ on } \Sigma_{b,c}(t) = (0,T) \times \Gamma_{b,c}(t), \tag{1.7}$$

$$-\frac{\partial y}{\partial \nu} = g, \text{ on } \Sigma = (0,T) \times \Gamma, \tag{1.8}$$

$$y(0) = y_0, \text{ in } \Omega. \tag{1.9}$$

If y were a smooth solution, then (1.4)-(1.9) would follow in the sense of distributions by a direct calculation. But, problem (1.1)-(1.3) with the maximal monotone graph $\partial I_K(y)$ on the right-hand side of equation (1.1) is ill-posed, so it might have only a local in time solution. That is why, as a heuristically observation we can say that in the free boundary problem described by (1.4)-(1.9), the time evolving subset $(0,T) \times \Omega_c(t)$ represents the domain where the concentration remains at the high constant level c, even if it should still increase due to a positive jump of the supply rate $u + f > f$, the subset $(0,T) \times \Omega_b(t)$ represents the inferior limit of the desorption domain, where the concentration remains at the low level b, even if it should still decrease due to a supply rate $u + f < f$, and the process evolves such that the fluid concentration remains between the levels b and c, at the rate f. Here we denoted $u(t,x) \in \partial I_K(y(t,x))$ a.e. $(t,x) \in Q$.

Since (1.1)-(1.3) is ill-posed we aim to find, by an optimal control technique, a function y which can be viewed as an approximating solution to (1.1)-(1.3). For solving (1.1)-(1.3) in terms of a control problem one should search for a solution which minimizes the distance between y and the graph $H_{b,c}(u)$, roughly speaking to study the minimization problem

$$\text{Min } \left\{ \frac{\lambda_1}{2} \int_Q (y^u(t,x) - H_{b,c}(u(t,x)))^2 dxdt + \frac{\lambda_2}{2} \int_Q u^2(t,x) dxdt \right\},$$

with the controller u. However, in this problem we do not have a sufficient regularity for u to prove existence, so that we have to change the control into another one v, and deduce u from an additional equation which guarantees the desired regularity. To this end, we define

$$J(v,w) = \frac{\lambda_1}{2} \int_Q (y^{v,w}(t,x) - w(t,x))^2 dxdt + \frac{\lambda_2}{2} \int_Q v^2(t,x) dxdt, \tag{1.10}$$

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and introduce the following minimization problem for the nonconvex functional J,

$$\text{Min } \{J(v,w);\ v \in L^2(Q),\ w \in L^\infty(Q),\ w(t,x) \in H_{b,c}(u^{v,w}(t,x)) \text{ a.e. } (t,x) \in Q\}, \tag{1.11}$$

subject to the system

$$\begin{aligned} y_t^{v,w} - \Delta y^{v,w} &= u^{v,w} + f, \text{ in } Q, \\ -\frac{\partial y^{v,w}}{\partial \nu} &= g, \quad \text{on } \Sigma, \\ y^{v,w}(0) &= y_0, \quad \text{in } \Omega \end{aligned} \tag{1.12}$$

$$\begin{aligned} u_t^{v,w} - \Delta u^{v,w} &= v, \quad \text{in } Q, \\ u^{v,w} &= 0, \quad \text{on } \Sigma, \\ u^{v,w}(0) &= u_0, \text{ in } \Omega. \end{aligned} \tag{1.13}$$

We specify that u_0 is arbitrary, but fixed. In particular it can be taken 0.

Problem (1.11) will be treated as a control problem with two controllers, v and w. Actually, it can be viewed as a control problem for a singular system (see e.g., appropriate examples in [6]). The functions $u^{v,w}$ and $y^{v,w}$ are the states (indicated by the superscripts ^{v,w}) controlled by v and w. For simplicity, later we shall drop these superscripts. The function u is determined by (1.13) in order to get the regularity necessary to prove the existence for (1.11). The positive constants λ_1 and λ_2 can be chosen such that to induce more importance to one term or another in (1.10).

A solution to (1.11) realizes the minimum in J, and as a matter of fact if J computed at this solution were zero, then the corresponding state y would satisfy (1.1). Since the existence of a null minimizer cannot be proved, it follows that the state y corresponding to a solution to (1.11) only approximates a section of $H_{b,c}(u)$. Therefore, we call the state y provided by a solution to (1.11) *a generalized solution* to (1.1)-(1.3). The existence of such a generalized solution, in other words the existence of a minimum of J, is proved in Section 2, Theorem 2.1. Since this solution cannot be directly characterized, due to the presence of the graph $H_{b,c}$, we shall introduce in Section 3 an approximating minimization problem (indexed upon ε) which converges in some sense to (1.11), as $\varepsilon \to 0$. The existence of a minimizer $\{v_\varepsilon^*, w_\varepsilon^*\}$ and the convergence result will be given in Section 3, Proposition 3.1 and Theorem 3.2. The calculation of the necessary conditions that should be satisfied by $\{v_\varepsilon^*, w_\varepsilon^*\}$ as a minimizer for the approximating problem will be provided in Section 4, by using a further approximation (indexed on σ) and an adapted penalization method (see [2]).

2. Existence of the control

In this section we prove the existence of a solution to the minimization problem (1.11).

We denote by $H^1(\Omega)$ and $H^2(\Omega)$ the standard Sobolev spaces and by $H_0^1(\Omega)$ the space of the functions belonging to $H^1(\Omega)$ with vanishing traces on the boundary.

We also note that if $v \in L^2(Q)$, $u_0 \in L^2(\Omega)$ then (1.13) has a unique solution

$$u \in W^{1,2}([0,T]; H^{-1}(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega)),$$

and if $f \in L^2(Q)$ and $g \in L^2(\Sigma)$, problem (1.12) has a unique solution

$$y \in W^{1,2}([0,T]; (H^1(\Omega))') \cap L^2(0,T; H^1(\Omega)) \cap C([0,T]; L^2(\Omega)),$$

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where $H^{-1}(\Omega)$ and $(H^1(\Omega))'$ are the dual spaces of $H_0^1(\Omega)$ and $H^1(\Omega)$, respectively.

Theorem 2.1. *Let $y_0 \in L^2(\Omega)$, $u_0 \in L^2(\Omega)$, $f \in L^2(Q)$, $g \in L^2(\Sigma)$. Then, problem (1.11) has at least one solution.*

Proof. Let us denote $d = \inf_{v,w} J(v,w)$. Consider a minimizing sequence $\{v_n, w_n\}$ such that

$$d \le \frac{\lambda_1}{2}\int_Q (y_n - w_n)^2 dxdt + \frac{\lambda_2}{2}\int_Q v_n^2 dxdt \le d + \frac{1}{n},\ n \ge 1, \tag{2.1}$$

where y_n and u_n are the unique solutions to (1.12)-(1.13) corresponding to $v = v_n$. We can select subsequences (denoted still by $_n$) such that, as $n \to \infty$, we have

$$v_n \to v \text{ weakly in } L^2(Q),$$

$$w_n \to w \text{ weak* in } L^\infty(Q).$$

By (1.13) and (1.12) we deduce that

$$u_n \to u \text{ weakly in } W^{1,2}([0,T]; H^{-1}(\Omega)) \cap L^2(0,T; H_0^1(\Omega)),$$

$$y_n \to y \text{ weakly in } W^{1,2}([0,T]; (H^1(\Omega))') \cap L^2(0,T; H^1(\Omega)),$$

and

$$u_n \to u,\ y_n \to y \text{ strongly in } L^2(Q),$$

by the Lions theorem (see [5], p. 58). Then, it is obvious that y and u are the solutions to (1.12)-(1.13) corresponding to v. We recall that $w_n(t,x) \in H_{b,c}(u_n(t,x))$ a.e. $(t,x) \in Q$, and since the realization of $H_{b,c}$ on $L^2(Q)$ is maximal monotone, hence strongly-weakly closed, we get that $w(t,x) \in H_{b,c}(u(t,x))$ a.e. $(t,x) \in Q$. Finally, we pass to the limit in (2.1) and on the basis of the weak lower semicontinuity of the convex integrands we get that $d = J(v,w)$.

□

3. The approximating problems

Since J is not differentiable we cannot determine the optimality conditions directly for (1.11) and we have to introduce an approximating minimization problem in which the graph $H_{b,c}$ is approximated. Next we show its convergence to problem (1.11).

First, we define $j : \mathbb{R} \to \mathbb{R}$,

$$j(r) = \begin{cases} br, & r < 0, \\ cr, & r \ge 0 \end{cases}$$

and introduce its conjugate $j^* : \mathbb{R} \to \mathbb{R}$, by $j^*(\omega) = \sup_{r\in\mathbb{R}}(\omega r - j(r))$, getting that

$$j^*(\omega) = I_K(\omega),\ \text{for any } \omega \in \mathbb{R}.$$

We note that $\partial j(r) = H_{b,c}(r)$, for all $r \in \mathbb{R}$. We recall that two conjugate functions j and j^* satisfy the Legendre-Fenchel relations (see e.g., [3], p. 6)

$$\begin{aligned} &j(r) + j^*(\omega) \ge r\omega,\ \text{for all } r,\ \omega \in R, \\ &j(r) + j^*(\omega) = r\omega,\ \text{iff } \omega \in \partial j(r), \end{aligned} \tag{3.1}$$

where ∂j is the subdifferential of j.

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Let $\varepsilon > 0$. We define

$$J_\varepsilon(v,w) = \frac{\lambda_1}{2}\int_Q (y(t,x) - w(t,x))^2 dxdt + \frac{\lambda_2}{2}\int_Q v^2(t,x)dxdt \tag{3.2}$$
$$+\frac{1}{\varepsilon}\int_Q (j(u(t,x) + j^*(w(t,x)) - u(t,x)w(t,x))dxdt,$$

and introduce the approximating minimization problem

$$\text{Min } \{J_\varepsilon(v,w);\ v \in L^2(Q),\ w \in L^\infty(Q),\ b \le w(t,x) \le c \text{ a.e. } (t,x) \in Q\} \tag{3.3}$$

subject to (1.12)-(1.13).

Roughly speaking, if the minimum in (3.3) were zero, then the last integral in (3.2) would be zero and so, by (3.1) one would obtain that $w(t,x) \in \partial j(u(t,x)) = H_{b,c}(u(t,x))$ a.e. $(t,x) \in Q$. The fact that (3.3) is an appropriate approximate for (1.11) is rigorously proved in Theorem 3.2.

It is obvious that, by considering (3.3) we can write $j^*(w(t,x)) = 0$ in $J_\varepsilon(v,w)$.

Proposition 3.1. *Let $y_0 \in L^2(\Omega)$, $u_0 \in L^2(\Omega)$, $f \in L^2(Q)$, $g \in L^2(\Sigma)$. Then, problem* (3.3) *has at least one solution.*

Proof. Let us denote $d_\varepsilon = \inf_{v,w} J_\varepsilon(v,w)$ and consider a minimizing sequence $\{v_\varepsilon^n, w_\varepsilon^n\}$ such that

$$d \le \frac{\lambda_1}{2}\int_Q (y_\varepsilon^n - w_\varepsilon^n)^2 dxdt + \frac{\lambda_2}{2}\int_Q (v_\varepsilon^n)^2 dxdt \tag{3.4}$$
$$+\frac{1}{\varepsilon}\int_Q (j(u_\varepsilon^n) - u_\varepsilon^n w_\varepsilon^n)dxdt \le d + \frac{1}{n},$$

where y_ε^n and u_ε^n are the unique solutions to (1.12)-(1.13) corresponding to $v = v_\varepsilon^n$. As in Theorem 2.1, Section 2, it follows on subsequences ($n \to \infty$) that

$$v_\varepsilon^n \to v_\varepsilon \text{ weakly in } L^2(Q),$$
$$w_\varepsilon^n \to w_\varepsilon \text{ weak* in } L^\infty(Q),\ w_\varepsilon(t,x) \in [b,c] \text{ a.e. } (t,x) \in Q.$$

By (1.13) and (1.12) we deduce

$$u_\varepsilon^n \to u_\varepsilon \text{ weakly in } W^{1,2}([0,T]; H^{-1}(\Omega)) \cap L^2(0,T; H_0^1(\Omega)),$$
$$y_\varepsilon^n \to y_\varepsilon \text{ weakly in } W^{1,2}([0,T]; (H^1(\Omega))') \cap L^2(0,T; H^1(\Omega)),$$
$$u_\varepsilon^n \to u_\varepsilon,\ y_\varepsilon^n \to y_\varepsilon \text{ strongly in } L^2(Q),$$

and y_ε and u_ε are the solutions to (1.12)-(1.13) corresponding to v_ε. Then,

$$\lim_{n\to\infty}\int_Q y_\varepsilon^n w_\varepsilon^n dxdt = \int_Q y_\varepsilon w_\varepsilon dxdt$$

and since j is continuous and $u_\varepsilon^n \to u_\varepsilon$ a.e. in Q, we have $j(u_\varepsilon^n(t,x)) \to j(u_\varepsilon(t,x))$, a.e., as $n \to \infty$. Now, we pass to the limit in (3.4). In order to be allowed to apply the Fatou lemma we recall first that a proper, convex l.s.c. function is bounded below by an affine function (see [3], p. 5), hence

$$j(t,x,r) \ge k_1(t,x)r + k_2(t,x),\ k_1, k_2 \in L^\infty(Q).$$

We replace j in (3.4) by $\widetilde{j}(r) = j(r) - k_1 r - k_2$, which is nonnegative, and have

$$d + \frac{1}{\varepsilon}\int_Q u_\varepsilon^n w_\varepsilon^n dxdt - \int_Q (k_1 u_\varepsilon^n + k_2)dxdt$$
$$\le \frac{\lambda_1}{2}\int_Q (y_\varepsilon^n - w_\varepsilon^n)^2 dxdt + \frac{\lambda_2}{2}\int_Q (v_\varepsilon^n)^2 dxdt$$
$$+\frac{1}{\varepsilon}\int_Q \widetilde{j}(u_\varepsilon^n)dxdt \le d + \frac{1}{n} + \frac{1}{\varepsilon}\int_Q u_\varepsilon^n w_\varepsilon^n dxdt - \int_Q (k_1 u_\varepsilon^n + k_2)dxdt.$$

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We pass to the limit as $n \to \infty$, applying the Fatou lemma and finally getting that $d_\varepsilon = J_\varepsilon(v_\varepsilon, w_\varepsilon)$. This proves that $\{v_\varepsilon, w_\varepsilon\}$ is a minimizer for J_ε. □

Next we should deal with the optimality conditions for problem (3.3), which are the necessary conditions that the optimal controller $\{v_\varepsilon^*, w_\varepsilon^*\}$ must satisfy in order to be a solution to (3.3). Because J_ε is not differentiable due to j, we shall introduce a second approximating problem in order to regularize j. This is based on the Moreau regularization defined as

$$j_\sigma(r) = \inf_{s\in\mathbb{R}} \left\{ \frac{|r-s|^2}{2\sigma} + j(s) \right\}, \text{ for any } r \in \mathbb{R},\ \sigma > 0, \tag{3.5}$$

which can be still written

$$j_\sigma(r) = \frac{1}{2\sigma} \left|(I + \sigma\partial j)^{-1} r - r\right|^2 + j((I + \sigma\partial j)^{-1} r), \tag{3.6}$$

where I is the identity on $\mathbb{R}$. The function j_σ is convex, Lipschitz continuous and has the properties (see [3], p. 48)

$$\begin{aligned} j_\sigma(r) &\le j(r), \text{ for any } r \in \mathbb{R}, \\ \lim_{\sigma\to 0} j_\sigma(r) &= j(r), \text{ for any } r \in \mathbb{R}. \end{aligned} \tag{3.7}$$

Let $\{v_\varepsilon^*, w_\varepsilon^*\}$ be optimal in (3.3).

We introduce the approximating adapted penalized problem (see [2])

$$\begin{aligned} \text{Min} \Big\{ J_{\varepsilon,\sigma}(u,v,\eta) &= \frac{\lambda_1}{2} \int_Q (y(t,x) - w(t,x))^2 dxdt + \frac{\lambda_2}{2} \int_Q v^2(t,x) dxdt \\ &+ \frac{1}{\varepsilon} \int_Q (j_\sigma(u(t,x)) - u(t,x)w(t,x)) dxdt \\ &+ \frac{1}{2} \int_Q (v(t,x) - v_\varepsilon^*(t,x))^2 dxdt + \frac{1}{2} \int_Q (w(t,x) - w_\varepsilon^*(t,x))^2 dxdt \Big\} \end{aligned} \tag{3.8}$$

subject to (1.12)-(1.13), for all $v \in L^2(Q)$, $w \in L^\infty(Q)$, $w(t,x) \in [b,c]$ a.e. $(t,x) \in Q$.

It is immediate (by a similar proof as in Proposition 3.1) that problem (3.8) has at least one solution. Next, we prove that this problem indexed upon σ converges to the approximating problem indexed upon ε, as $\sigma \to 0$.

Theorem 3.2. *Let $(v_\varepsilon^*, w_\varepsilon^*)$ be optimal in (3.3) and $(v_{\varepsilon,\sigma}^*, w_{\varepsilon,\sigma}^*)$ be optimal in (3.8). Then,*

$$\begin{aligned} v_{\varepsilon,\sigma}^* &\to v_\varepsilon^* \text{ strongly in } L^2(Q), \text{ as } \sigma \to 0, \\ w_{\varepsilon,\sigma}^* &\to w_\varepsilon^* \text{ strongly in } L^2(Q), \text{ as } \sigma \to 0, \end{aligned} \tag{3.9}$$

and the corresponding states $\{u_{\varepsilon,\sigma}^*, y_{\varepsilon,\sigma}^*\}_\sigma$ converge to $\{u_\varepsilon^*, y_\varepsilon^*\}$, the corresponding states in (3.3),

$$\begin{aligned} u_{\varepsilon,\sigma}^* \to u_\varepsilon^* &\text{ weakly in } W^{1,2}([0,T]; H^{-1}(\Omega)) \cap L^2(0,T; H_0^1(\Omega)), \\ &\text{strongly in } L^2(0,T; L^2(\Omega)), \end{aligned} \tag{3.10}$$

$$\begin{aligned} y_{\varepsilon,\sigma}^* \to y_\varepsilon^* &\text{ weakly in } W^{1,2}([0,T]; (H^1(\Omega))') \cap L^2(0,T; H^1(\Omega)), \\ &\text{strongly in } L^2(0,T; L^2(\Omega)). \end{aligned} \tag{3.11}$$

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Proof. We write that $(v^*_{\varepsilon,\sigma}, w^*_{\varepsilon,\sigma})$ is optimal in (3.8), that is

$$\begin{aligned}
&\frac{\lambda_1}{2}\int_Q (y^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma})^2 dxdt + \frac{\lambda_2}{2}\int_Q (v^*_{\varepsilon,\sigma})^2 dxdt\\
&+\frac{1}{\varepsilon}\int_Q (j_\sigma(u^*_{\varepsilon,\sigma}) - w^*_{\varepsilon,\sigma}u^*_{\varepsilon,\sigma})dxdt\\
&+\frac{1}{2}\int_Q (v^*_{\varepsilon,\sigma} - v^*_\varepsilon)^2 dxdt + \frac{1}{2}\int_Q (w^*_{\varepsilon,\sigma} - w^*_\varepsilon)^2 dxdt\\
&\leq \frac{\lambda_1}{2}\int_Q (y-w)^2 dxdt + \frac{\lambda_2}{2}\int_Q v^2 dxdt + \frac{1}{\varepsilon}\int_Q (j_\sigma(u) - wu)dxdt\\
&+\frac{1}{2}\int_Q (v - v^*_\varepsilon)^2 dxdt + \frac{1}{2}\int_Q (w - w^*_\varepsilon)^2 dxdt,
\end{aligned}$$

for all $v \in L^2(Q)$, $w \in L^\infty(Q)$, $w(t,x) \in [b,c]$ a.e., with y, u the solutions to (1.12)-(1.13) corresponding to v.

In particular, we set $v = v^*_\varepsilon$, $w = w^*_\varepsilon$ which provide by (1.12)-(1.13) the optimal states y^*_ε and u^*_ε. We have

$$\begin{aligned}
&\frac{\lambda_1}{2}\int_Q (y^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma})^2 dxdt + \frac{\lambda_2}{2}\int_Q (v^*_{\varepsilon,\sigma})^2 dxdt\\
&+\frac{1}{\varepsilon}\int_Q (j_\sigma(u^*_{\varepsilon,\sigma}) - w^*_{\varepsilon,\sigma}u^*_{\varepsilon,\sigma})dxdt\\
&+\frac{1}{2}\int_Q (v^*_{\varepsilon,\sigma} - v^*_\varepsilon)^2 dxdt + \frac{1}{2}\int_Q (w^*_{\varepsilon,\sigma} - w^*_\varepsilon)^2 dxdt \qquad (3.12)\\
&\leq \frac{\lambda_1}{2}\int_Q (y^*_\varepsilon - w^*_\varepsilon)^2 dxdt + \frac{\lambda_2}{2}\int_Q (v^*_\varepsilon)^2 dxdt + \frac{1}{\varepsilon}\int_Q (j_\sigma(u^*_\varepsilon) - w^*_\varepsilon u^*_\varepsilon)dxdt\\
&\leq \frac{\lambda_1}{2}\int_Q (y^*_\varepsilon - w^*_\varepsilon)^2 dxdt + \frac{\lambda_2}{2}\int_Q (v^*_\varepsilon)^2 dxdt + \frac{1}{\varepsilon}\int_Q (j(u^*_\varepsilon) - w^*_\varepsilon u^*_\varepsilon)dxdt
\end{aligned}$$

by (3.7). The left-hand side is bounded independently of σ, because u^*_ε is bounded in $L^2(Q)$, by (1.13). Consequently, by selecting subsequences (denoted still by the subscript $_\sigma$) we get

$$\begin{gathered}
v^*_{\varepsilon,\sigma} \to v_\varepsilon \text{ weakly in } L^2(Q),\\
w^*_{\varepsilon,\sigma} \to w_\varepsilon \text{ weak* in } L^\infty(Q),
\end{gathered}$$

$$u^*_{\varepsilon,\sigma} \to u_\varepsilon \text{ weakly in } W^{1,2}([0,T]; H^{-1}(\Omega)) \cap L^2(0,T; H^1_0(\Omega)),$$

$$y^*_{\varepsilon,\sigma} \to y_\varepsilon \text{ weakly in } W^{1,2}([0,T]; (H^1(\Omega))') \cap L^2(0,T; H^1(\Omega)),$$

$$u^*_{\varepsilon,\sigma} \to u_\varepsilon,\ y^*_{\varepsilon,\sigma} \to y_\varepsilon \text{ strongly in } L^2(Q),$$

where $\{y_\varepsilon, u_\varepsilon\}$ is the solution to (1.12)-(1.13) corresponding to v_ε.

Now, we prove that

$$\liminf_{\sigma\to 0}\int_Q j_\sigma(u^*_{\varepsilon,\sigma})dxdt \geq \int_Q j(u_\varepsilon)dxdt, \qquad (3.13)$$

where $u_\varepsilon = \lim_{\sigma\to 0} u^*_{\varepsilon,\sigma}$. By (3.6) we have

$$\frac{1}{2\sigma}\int_Q \left|(I+\sigma\partial j)^{-1}u^*_{\varepsilon,\sigma} - u^*_{\varepsilon,\sigma}\right|^2 dxdt \leq \int_Q j_\sigma(u^*_{\varepsilon,\sigma})dxdt \leq \text{ constant},$$

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by (3.12), which implies that

$$\lim_{\sigma\to 0}\left\|(I+\sigma\partial j)^{-1}u^*_{\varepsilon,\sigma}-u^*_{\varepsilon,\sigma}\right\|_{L^2(Q)}=0.$$

We deduce that

$$\lim_{\sigma\to 0}(I+\sigma\partial j)^{-1}u^*_{\varepsilon,\sigma}=u_\varepsilon \text{ strongly in } L^2(Q). \tag{3.14}$$

Again by (3.6) we can write

$$\liminf_{\sigma\to 0}\int_Q j_\sigma(u^*_{\varepsilon,\sigma})dxdt\geq\liminf_{\sigma\to 0}\int_Q j((I+\sigma\partial j)^{-1}u^*_{\varepsilon,\sigma})dxdt=\int_Q j(u_\varepsilon)dxdt,$$

by Fatou's lemma (following the argument given in Proposition 3.1), the continuity of j and (3.14). Here I is the identity on $L^2(Q)$.

Then, passing to the limit (in the first and last terms) in (3.12), as $\sigma\to 0$ we get

$$\begin{aligned}&\frac{\lambda_1}{2}\int_Q(y_\varepsilon-w_\varepsilon)^2dxdt+\frac{\lambda_2}{2}\int_Q^2 v_\varepsilon^2dxdt+\frac{1}{\varepsilon}\int_Q(j(u_\varepsilon)-w_\varepsilon u_\varepsilon)dxdt\\&+\frac{1}{2}\int_Q(v_\varepsilon-v^*_\varepsilon)^2dxdt+\frac{1}{2}\int_Q(w_\varepsilon-w^*_\varepsilon)^2dxdt\\&\leq\frac{\lambda_1}{2}\int_Q(y^*_\varepsilon-w^*_\varepsilon)^2dxdt+\frac{\lambda_2}{2}\int_Q(v^*_\varepsilon)^2dxdt+\frac{1}{\varepsilon}\int_Q(j(u^*_\varepsilon)-w^*_\varepsilon u^*_\varepsilon)dxdt\\&\leq\frac{\lambda_1}{2}\int_Q(y_\varepsilon-w_\varepsilon)^2dxdt+\lambda_2\int_Q v_\varepsilon^2dxdt+\frac{1}{\varepsilon}\int_Q(j(u_\varepsilon)-w_\varepsilon u_\varepsilon)dxdt.\end{aligned}$$

The last inequality can be written because $(v^*_\varepsilon,w^*_\varepsilon)$ is optimal in (3.3). From here we get that

$$v_\varepsilon=v^*_\varepsilon,\ w_\varepsilon=w^*_\varepsilon \text{ a.e. on } Q,$$

and so $u_\varepsilon=u^*_\varepsilon$ and $y_\varepsilon=y^*_\varepsilon$ a.e. on Q. Actually, by (3.12) we obtain (3.9) and the strong convergences in (3.10)-(3.11). This ends the proof. □

4. Optimality conditions

Let $\{v^*_{\varepsilon,\sigma},w^*_{\varepsilon,\sigma}\}$ (with the corresponding $\{u^*_{\varepsilon,\sigma},y^*_{\varepsilon,\sigma}\}$) be a solution to (3.8) and $\{v^*_\varepsilon,w^*_\varepsilon\}$ (with the corresponding $\{u^*_\varepsilon,y^*_\varepsilon\}$) be a solution to (3.3).

Proposition 4.1. *Let $y_0\in L^2(\Omega)$, $u_0\in H^1_0(\Omega)$, $f\in L^2(Q)$, $g\in L^2(\Sigma)$ and let $\{v^*_{\varepsilon,\sigma},w^*_{\varepsilon,\sigma}\}$ be an optimal controller in* (3.8). *Then, the optimality conditions for* (3.8) *read as*

$$w^*_{\varepsilon,\sigma}(t,x)=\begin{cases}b, & \text{in } \{(t,x);a^*_{\varepsilon,\sigma}(t,x)\leq b\},\\ a^*_{\varepsilon,\sigma}(t,x), & \text{in } \{(t,x);b<a^*_{\varepsilon,\sigma}(t,x)<c\},\\ c, & \text{in } \{(t,x);a^*_{\varepsilon,\sigma}(t,x)\geq c\},\end{cases} \tag{4.1}$$

$$v^*_{\varepsilon,\sigma}(t,x)=\frac{1}{1+\lambda_2}(v^*_\varepsilon+z^*_{\varepsilon,\sigma})(t,x), \tag{4.2}$$

where

$$a^*_{\varepsilon,\sigma}(t,x)=\frac{1}{1+\lambda_1}(\lambda_1y^*_{\varepsilon,\sigma}+\frac{1}{\varepsilon}u^*_{\varepsilon,\sigma}+w^*_\varepsilon)(t,x), \tag{4.3}$$

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$z^*_{\varepsilon,\sigma}$ is the solution to

$$\begin{aligned} (z^*_{\varepsilon,\sigma})_t + \Delta z^*_{\varepsilon,\sigma} &= p_{\varepsilon,\sigma} + \frac{1}{\varepsilon}(\xi^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma}), \ \textit{in } Q, \\ z^*_{\varepsilon,\sigma} &= 0, \qquad \textit{on } \Sigma, \\ (z^*_{\varepsilon,\sigma})(T) &= 0, \qquad \textit{in } \Omega, \end{aligned} \tag{4.4}$$

$$\xi^*_{\varepsilon,\sigma}(t,x) = j'_\sigma(u^*_{\varepsilon,\sigma}(t,x)), \ \textit{a.e. } (t,x) \in Q, \tag{4.5}$$

and $p^*_{\varepsilon,\sigma}$ is the solution to

$$\begin{aligned} (p_{\varepsilon,\sigma})_t + \Delta p_{\varepsilon,\sigma} &= -\lambda_1(y^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma}), \ \textit{in } Q, \\ \frac{\partial p_{\varepsilon,\sigma}}{\partial \nu} &= 0, \qquad \textit{on } \Sigma, \\ p_{\varepsilon,\sigma}(T) &= 0, \qquad \textit{in } \Omega. \end{aligned} \tag{4.6}$$

Proof. Let $\lambda \in [0,1]$ and write $w^\lambda_{\varepsilon,\sigma} = w^*_{\varepsilon,\sigma} + \lambda W$, $W = w - w^*_{\varepsilon,\sigma}$, for any $w \in L^\infty(Q)$, $w(t,x) \in K$ a.e. $(t,x) \in Q$. We write that $v^*_{\varepsilon,\sigma}, w^*_{\varepsilon,\sigma}$ is optimal in (3.8), that is

$$J_{\varepsilon,\sigma}(v^*_{\varepsilon,\sigma}, w^*_{\varepsilon,\sigma}) \le J_{\varepsilon,\sigma}(\widehat{v}, \widehat{w}), \ \text{for all } \widehat{v} \in L^2(Q), \ \widehat{w}(t,x) \in [b,c], \ \text{a.e. } (t,x) \in Q,$$

and set $\widehat{v} = v^*_{\varepsilon,\sigma}$ and $\widehat{w} = w^\lambda_{\varepsilon,\sigma}$. We obtain after a few calculations

$$\int_Q \left(-\lambda_1(y^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma}) - \frac{1}{\varepsilon}u^*_{\varepsilon,\sigma} + w^*_{\varepsilon,\sigma} - w^*_\varepsilon \right) W dxdt \ge 0,$$

whence we get

$$\int_Q (w^*_{\varepsilon,\sigma} - w) \left(\lambda_1(y^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma}) + \frac{1}{\varepsilon}u^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma} + w^*_\varepsilon \right) dxdt \ge 0, \tag{4.7}$$

for all $w \in L^\infty(Q)$, $w(t,x) \in [b,c]$ a.e. $(t,x) \in Q$. This yields

$$w^*_{\varepsilon,\sigma} + \frac{1}{1+\lambda_1}\partial I_K(w^*_{\varepsilon,\sigma}) \ni \frac{1}{1+\lambda_1}\left(\lambda_1 y^*_{\varepsilon,\sigma} + \frac{1}{\varepsilon}u^*_{\varepsilon,\sigma} + w^*_\varepsilon \right), \tag{4.8}$$

which still reads

$$w^*_{\varepsilon,\sigma} = P_K\left(\frac{1}{1+\lambda_1}\left(\lambda_1 y^*_{\varepsilon,\sigma} + \frac{1}{\varepsilon}u^*_{\varepsilon,\sigma} + w^*_\varepsilon \right) \right),$$

where $P_K(z)$ is the projection of the point z on the set K. From here we get (4.1).

Now, let $\lambda > 0$ and consider the variation $v^\lambda_{\varepsilon,\sigma} = v^*_{\varepsilon,\sigma} + \lambda V$, for all $V \in L^2(Q)$. If we denote by $y^\lambda_{\varepsilon,\sigma}$ and $u^\lambda_{\varepsilon,\sigma}$ the solutions to (1.12)-(1.13) corresponding to $v = v^\lambda_{\varepsilon,\sigma}$, by U the solution to (1.13) corresponding to $v = V$ and by Y the weak limit in $L^2(Q)$ defined as

$$Y(t,x) = \lim_{\lambda \to 0} \frac{y^{v^\lambda_{\varepsilon,\sigma}}(t,x) - y^{v^*_{\varepsilon,\sigma}}(t,x)}{\lambda},$$

we get by a direct calculation that Y and U satisfy the systems in variations

$$\begin{aligned} Y_t - \Delta Y &= U, \ \text{in } Q, \\ -\frac{\partial Y}{\partial \nu} &= 0, \ \ \text{on } \Sigma, \\ Y(0) &= 0, \ \ \text{in } \Omega, \end{aligned} \tag{4.9}$$

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$$\begin{aligned} U_t - \Delta U &= V, \text{ in } Q, \\ U &= 0, \quad \text{on } \Sigma, \\ U(0) &= 0, \quad \text{in } \Omega. \end{aligned} \tag{4.10}$$

Obviously, each of these systems has a unique solution

$$U \in C([0,T]; H_0^1(\Omega)] \cap W^{1,2}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)),$$

$$Y \in C([0,T]; H^1(\Omega)) \cap W^{1,2}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)).$$

We introduce the dual systems (4.6),

$$\begin{aligned} (p_{\varepsilon,\sigma})_t + \Delta p_{\varepsilon,\sigma} &= -\lambda_1(y^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma}), \text{ in } Q, \\ \frac{\partial p_{\varepsilon,\sigma}}{\partial \nu} &= 0, \qquad \text{on } \Sigma, \\ p_{\varepsilon,\sigma}(T) &= 0, \qquad \text{in } \Omega, \end{aligned}$$

and

$$\begin{aligned} (q_{\varepsilon,\sigma})_t + \Delta q_{\varepsilon,\sigma} &= -\frac{1}{\varepsilon}\left(\xi^*_{\varepsilon,\sigma} - w^*_{\varepsilon}\right), \text{ in } Q, \\ q_{\varepsilon,\sigma} &= 0, \qquad \text{on } \Sigma, \\ q_{\varepsilon,\sigma}(T) &= 0, \qquad \text{in } \Omega, \end{aligned} \tag{4.11}$$

where

$$\xi^*_{\varepsilon,\sigma} = j'_\sigma(u^*_{\varepsilon,\sigma}).$$

Systems (4.6) and (4.11) have a unique solution

$$p_{\varepsilon,\sigma} \in C([0,T]; H^1(\Omega)] \cap W^{1,2}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)),$$

$$q_{\varepsilon,\sigma} \in C([0,T]; H_0^1(\Omega)] \cap W^{1,2}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)).$$

We write again that $w^*_{\varepsilon,\sigma}$, $v^*_{\varepsilon,\sigma}$ is optimal in (3.8),

$$J_{\varepsilon,\sigma}(v^*_{\varepsilon,\sigma}, w^*_{\varepsilon,\sigma}) \le J_{\varepsilon,\sigma}(\widehat{v}, \widehat{w}), \text{ for all } \widehat{v} \in L^2(Q), \ \widehat{w}(t,x) \in [b,c] \text{ a.e. } (t,x) \in Q.$$

In particular, we set $\widehat{v} = v^\lambda_{\varepsilon,\sigma}$, $\widehat{w} = w^*_{\varepsilon,\sigma}$ and after dividing by λ and letting it to go to 0, we get

$$\int_Q \lambda_1(y^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma})Y dxdt + \int_Q ((\lambda_2+1)v^*_{\varepsilon,\sigma} - v^*_\varepsilon)V dxdt + \frac{1}{\varepsilon}\int_Q \left(\xi^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma}\right) U dxdt \ge 0,$$

for all $V \in L^2(Q)$. Changing λ into $(-\lambda)$ we finally obtain

$$\int_Q \lambda_1(y^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma})Y dxdt + \int_Q ((\lambda_2+1)v^*_{\varepsilon,\sigma} - v^*_\varepsilon)V dxdt + \frac{1}{\varepsilon}\int_Q \left(\xi^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma}\right) U dxdt = 0, \tag{4.12}$$

for all $V \in L^2(Q)$.

Next, we multiply (4.9) by $p_{\varepsilon,\sigma}$, add with (4.10) multiplied by $q_{\varepsilon,\sigma}$, and integrate over Q. We obtain

$$-\int_Q ((p_{\varepsilon,\sigma})_t + \Delta p_{\varepsilon,\sigma})Y dxdt - \int_Q ((q_{\varepsilon,\sigma})_t + \Delta q_{\varepsilon,\sigma})U dxdt = \int_Q (Up_{\varepsilon,\sigma} + Vq_{\varepsilon,\sigma})dxdt,$$

and further

$$\int_Q \lambda_1(y^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma})Y dxdt + \frac{1}{\varepsilon}\int_Q \left(\xi^*_{\varepsilon,\sigma} - w^*_{\varepsilon,\sigma}\right) U dxdt = \int_Q (Up_{\varepsilon,\sigma} + Vq_{\varepsilon,\sigma})dxdt. \tag{4.13}$$

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Plugging this relation in (4.12) we get

$$\int_Q Up_{\varepsilon,\sigma}dxdt + \int_Q (q_{\varepsilon,\sigma} + \lambda_2 v^*_{\varepsilon,\sigma} + v^*_{\varepsilon,\sigma} - v^*_\varepsilon)Vdxdt = 0, \text{ for all } V \in L^2(Q). \tag{4.14}$$

By (4.10) we write the solution representation

$$U(t) = \int_0^t e^{\Delta(t-s)}V(s)ds, \ t \geq 0,$$

where $e^{\Delta t}$ is the semigroup generated by the operator Laplace with Dirichlet boundary conditions, and replace it in (4.14), obtaining

$$\int_Q (q_{\varepsilon,\sigma} + (\lambda_2+1)v^*_{\varepsilon,\sigma} - v^*_\varepsilon)Vdxdt + \int_0^T \int_\Omega p_{\varepsilon,\sigma}(t)e^{\Delta t}\int_0^t e^{-\Delta s}V(s)dsdxdt = 0.$$

Changing the order of integrals in the second term we finally deduce

$$\int_0^T \int_\Omega V(t)\left(\int_t^T e^{-\Delta(t-s)}p_{\varepsilon,\sigma}(s)ds + q_{\varepsilon,\sigma} + (\lambda_2+1)v^*_{\varepsilon,\sigma} - v^*_\varepsilon\right)dxdt = 0, \tag{4.15}$$

for all $V \in L^2(Q)$, which implies

$$(1+\lambda_2)v^*_{\varepsilon,\sigma} - v^*_\varepsilon + q_{\varepsilon,\sigma} = -\int_t^T e^{-\Delta(t-s)}p_{\varepsilon,\sigma}(s)ds. \tag{4.16}$$

Actually, denoting $z^*_{\varepsilon,\sigma} = (1+\lambda_2)v^*_{\varepsilon,\sigma} - v^*_\varepsilon$ we have that $z^*_{\varepsilon,\sigma}$ is provided by the equation

$$\begin{aligned} (z^*_{\varepsilon,\sigma})_t + (q_{\varepsilon,\sigma})_t + \Delta z^*_{\varepsilon,\sigma} + \Delta q_{\varepsilon,\sigma} &= p_{\varepsilon,\sigma}, \text{ in } Q, \\ z^*_{\varepsilon,\sigma} + q_{\varepsilon,\sigma} &= 0, \quad \text{on } \Sigma, \\ (z^*_{\varepsilon,\sigma} + q_{\varepsilon,\sigma})(T) &= 0, \quad \text{in } \Omega, \end{aligned} \tag{4.17}$$

whence taking into account (4.11) we get (4.4).

Problem (4.4) has a unique solution

$$z^*_{\varepsilon,\sigma} = (1+\lambda_2)v^*_{\varepsilon,\sigma} - v^*_\varepsilon \in C([0,T];H^1_0(\Omega)] \cap W^{1,2}([0,T];L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)). \tag{4.18}$$

In conclusion, recalling that $\{v^*_{\varepsilon,\sigma}, w^*_{\varepsilon,\sigma}\}$ denotes an optimal controller in (3.8), with $\{u^*_{\varepsilon,\sigma}, y^*_{\varepsilon,\sigma}\}$ the corresponding states, we get (4.1)-(4.6) as claimed. □

Finally, as proved in Theorem 3.2, each solution v^*_ε, w^*_ε to (3.3) is the strong limit of the sequences $\{v^*_{\varepsilon,\sigma}\}_{\varepsilon>0}$ and $\{w^*_{\varepsilon,\sigma}\}_{\varepsilon>0}$ having the expressions given by (4.1) and (4.2), so that we can conclude with the next Proposition.

Proposition 4.2. *Let $y_0 \in L^2(\Omega)$, $u_0 \in L^2(\Omega)$, $f \in L^2(Q)$, $g \in L^2(\Sigma)$ and let $\{v^*_\varepsilon, w^*_\varepsilon\}$ be an optimal controller in* (3.3). *Then, the optimality conditions for* (3.3) *read as*

$$w^*_\varepsilon(t,x) = \begin{cases} b, & \text{in } \{(t,x); a^*_\varepsilon(t,x) \leq b\}, \\ a^*_\varepsilon(t,x), & \text{in } \{(t,x); b < a^*_\varepsilon(t,x) < c\}, \\ c, & \text{in } \{(t,x); a^*_\varepsilon(t,x) \geq c\}, \end{cases} \tag{4.19}$$

where

$$a^*_\varepsilon(t,x) = y^*_\varepsilon(t,x) + \frac{1}{\varepsilon\lambda_1}u^*_\varepsilon(t,x), \tag{4.20}$$

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and

$$v_\varepsilon^*(t,x) = \frac{z_\varepsilon^*(t,x)}{\lambda_2}, \tag{4.21}$$

where z_ε^* *is a solution to the problem*

$$\begin{aligned} (z_\varepsilon^*)_t + \Delta z_\varepsilon^* &\in p_\varepsilon + \frac{1}{\varepsilon}(H_{b,c}(u_\varepsilon^*) - w_\varepsilon^*), \ \text{in } Q, \\ z_\varepsilon^* &= 0, \qquad \text{on } \Sigma, \\ (z_\varepsilon^*)(T) &= 0, \qquad \text{in } \Omega, \end{aligned} \tag{4.22}$$

and p_ε *is the unique solution to the problem*

$$\begin{aligned} (p_\varepsilon)_t + \Delta p_\varepsilon &= -\lambda_1(y_\varepsilon^* - w_\varepsilon^*), \ \text{in } Q, \\ \frac{\partial p_\varepsilon}{\partial \nu} &= 0, \qquad \text{on } \Sigma, \\ p_\varepsilon(T) &= 0, \qquad \text{in } \Omega. \end{aligned} \tag{4.23}$$

Proof. By Theorem 3.2, we have (3.9) and that

$$u_{\varepsilon,\sigma}^* \to u_\varepsilon^*, \quad y_{\varepsilon,\sigma}^* \to y_\varepsilon^* \text{ strongly in } L^2(Q), \text{ as } \sigma \to 0.$$

Then, we pass to the limit in (4.1) and deduce

$$w_\varepsilon^*(t,x) = \begin{cases} b, & \text{on } \{(t,x); a_\varepsilon^*(t,x) = b\}, \\ a_\varepsilon^*(t,x), & \text{on } \{(t,x); b < a_\varepsilon^*(t,x) < c\}, \\ c, & \text{on } \{(t,x); a_\varepsilon^*(t,x) = c\}, \end{cases}$$

where $a_\varepsilon^*(t,x) = \frac{1}{1+\lambda_1}(\lambda_1 y_\varepsilon^* + \frac{1}{\varepsilon}u_\varepsilon^* + w_\varepsilon^*)(t,x)$. After a few algebra this implies (4.19). We note that

$$\xi_{\varepsilon,\sigma}^* \to \xi_\varepsilon^* \text{ weak}^* \text{ in } L^\infty(Q), \ \xi_\varepsilon^*(t,x) \in \partial j(u_\varepsilon^*(t,x)) \text{ a.e. } (t,x) \in Q.$$

Indeed,

$$\int_Q (j_\sigma(u_{\varepsilon,\sigma}^*) - j_\sigma(z))dxdt \le \int_Q j_\sigma'(u_{\varepsilon,\sigma}^*)(u_{\varepsilon,\sigma}^* - z)dxdt,$$

for any $z \in L^2(Q)$. Passing to the limit as $\sigma \to 0$ and recalling (3.13) and (3.7) we get

$$\int_Q (j(u_\varepsilon^*) - j(z))dxdt \le \int_Q \xi_\varepsilon^*(u_\varepsilon^* - z)dxdt,$$

whence $\xi_\varepsilon^*(t,x) \in \partial j(u_\varepsilon^*(t,x))$ a.e. $(t,x) \in Q$.

By (4.6) we get that

$$\begin{aligned} p_{\varepsilon,\sigma} \to p_\varepsilon \ &\text{weakly in } W^{1,2}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)), \\ &\text{weak}^* \text{ in } L^\infty(0,T; H^1(\Omega)), \\ &\text{strongly in } L^2(0,T; H^1(\Omega)), \text{ as } \sigma \to 0, \end{aligned}$$

and so, p_ε satisfies (4.23).

By (4.4) we have,

$$\begin{aligned} z_{\varepsilon,\sigma}^* \to z_\varepsilon^* \ &\text{weakly in } W^{1,2}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)), \\ &\text{weak}^* \text{ in } L^\infty(0,T; H_0^1(\Omega)), \\ &\text{strongly in } L^2(0,T; H_0^1(\Omega)), \text{ as } \sigma \to 0, \end{aligned}$$

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and

$$z^*_{\varepsilon,\sigma}(t) \to z^*_\varepsilon(t) \text{ strongly in } L^2(\Omega), \text{ uniformly in } t \text{ on compacts},$$

by the Arzelà-Ascoli theorem. Moreover, z^*_ε satisfies the equation

$$\begin{aligned} (z^*_\varepsilon)_t + \Delta z^*_\varepsilon &= p_\varepsilon + \frac{1}{\varepsilon}(\xi^*_\varepsilon - w^*_\varepsilon), \text{ in } Q, \\ z^*_\varepsilon &= 0, \qquad \text{on } \Sigma, \\ (z^*_\varepsilon)(T) &= 0, \qquad \text{in } \Omega, \end{aligned} \tag{4.24}$$

where $\xi^*_\varepsilon(t,x) \in \partial j(u^*_\varepsilon(t,x))$ a.e. $(t,x) \in Q$. Then, by (4.2) we get (4.21). □

5. Numerical results

In the last section we present some numerical simulations based on an algorithm developed for computing the approximating solution to (3.3). For simplicity we shall not write the subscript ε for $v^*_\varepsilon, w^*_\varepsilon$ and the other functions.

Fix ε_{stop} and indicate the problem data.

Give $y_0(x)$, $u_0(x)$.

Step 0. Choose the "0"-order values for the controls w^*_0 and v^*_0. Set $k := 0$.

Step 0$_1$. Determine u^*_k by (1.13) with $v = v^*_k$ (here v^*_0),

$$\begin{aligned} (u^*_k)_t - \Delta u^*_k &= v^*_k, \text{ in } Q, \\ u^*_k &= 0, \quad \text{on } \Sigma, \\ u^*_k(0) &= u_0, \text{ in } \Omega. \end{aligned}$$

Determine y^*_k by (1.12), with $u = u^*_k$,

$$\begin{aligned} (y^*_k)_t - \Delta y^*_k &= u^*_k + f, \text{ in } Q, \\ -\frac{\partial y^*_k}{\partial \nu} &= g, \quad \text{on } \Sigma, \\ y^*_k(0) &= y_0, \text{ in } \Omega, \end{aligned}$$

and $J_k(v^*_k, w^*_k)$ by (3.2),

$$J_k(v^*_k, w^*_k) = \frac{\lambda_1}{2}\int_Q (y^*_k - w^*_k)^2 dxdt + \frac{\lambda_2}{2}\int_Q (v^*_k)^2 dxdt + \frac{1}{\varepsilon}\int_Q (j(u^*_k) - u^*_k w^*_k) dxdt.$$

Step 0$_2$. Compute (for using in the next step) the following: a^*_k by (4.20),

$$a^*_k = y^*_k + \frac{1}{\varepsilon\lambda_1} u^*_k,$$

p_k by (4.23),

$$\begin{aligned} (p_k)_t + \Delta p_k &= -\lambda_1(y^*_k - w^*_k), \text{ in } Q, \\ \frac{\partial p_k}{\partial \nu} &= 0, \qquad \text{on } \Sigma, \\ p_k(T) &= 0, \qquad \text{in } \Omega, \end{aligned}$$

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z_k^* by (4.22),

$$\begin{aligned}(z_k^*)_t + \Delta z_k^* &\in p_k + \frac{1}{\varepsilon}(\xi_\varepsilon^* - w_k^*), \text{ in } Q,\\ z_k^* &= 0, \qquad \text{on } \Sigma,\\ (z_k^*)(T) &= 0, \qquad \text{in } \Omega,\\ \xi_\varepsilon^*(t,x) &\in H_{b,c}(u_k^*(t,x)) \text{ a.e. } (t,x) \in Q.\end{aligned}$$

Finally, compute the function given by (4.19), which here is denoted

$$W_k(t,x) = \begin{cases} b, & \text{in } \{(t,x); a_k^*(t,x) \le b\},\\ a_k^*(t,x), & \text{in } \{(t,x); b < a_k^*(t,x) < c\},\\ c, & \text{in } \{(t,x); a_k^*(t,x) \ge c\},\end{cases}$$

and the function given by (4.21), denoted

$$V_k(t,x) = \frac{z_k^*(t,x)}{\lambda_2}.$$

This ends *Step 0*.
Step $k+1$. Make $k := k+1$.
Step k_1. Following Rosen algorithm (see [1], p. 44) compute λ which realizes

$$\min\{J_{k+1}(v_{k+1}^\lambda, w_{k+1}^\lambda);\ \lambda \in (0,1]\}, \tag{5.1}$$

where the variations $w_{k+1}^\lambda, v_{k+1}^\lambda$ are determined using the functions W_k and V_k, i.e.,

$$w_{k+1}^\lambda(t,x) = w_k^*(t,x) + \lambda W_k(t,x),\ \lambda \in (0,1], \tag{5.2}$$

$$v_{k+1}^\lambda(t,x) = v_k^*(t,x) + \lambda V_k(t,x). \tag{5.3}$$

In the expression of J_{k+1}, u_{k+1}^λ is given by (1.13) with $v = v_{k+1}^\lambda$ and y_{k+1}^λ is determined by (1.12) with $u = u_{k+1}^\lambda$.

In order to achieve (5.1) we make the following remark. Step k_1 is run for a sequence of p values $\lambda_p = p{*}\text{step}_\lambda$, $\text{step}_\lambda \in (0,1)$, $p = 1, ..., 1/\text{step}_\lambda$.
Step k_2. Pick up that λ_p which realizes the minimum in (5.1), denote it $\lambda_{\min}$, set

$$J_{k+1}(v_{k+1}^{\lambda_{\min}}, w_{k+1}^{\lambda_{\min}}) = \min\{J_{k+1}(v_{k+1}^\lambda, w_{k+1}^\lambda);\ \lambda \in (0,1]\}, \tag{5.4}$$

and then set

$$\begin{aligned}v_{k+1}^*(t,x) = v_{k+1}^{\lambda_{\min}}(t,x),\ w_{k+1}^*(t,x) = w_{k+1}^{\lambda_{\min}}(t,x),\\ u_{k+1}^*(t,x) = u_{k+1}^{\lambda_{\min}}(t,x),\ y_{k+1}^*(t,x) = y_{k+1}^{\lambda_{\min}}(t,x),\\ J_{k+1}(v_{k+1}^*, w_{k+1}^*) = J_{k+1}(v_{k+1}^{\lambda_{\min}}, y_{k+1}^{\lambda_{\min}}).\end{aligned}$$

Step k_3. Denote

$$err_{k+1} = J_{k+1}(v_{k+1}^*, w_{k+1}^*) - J_k(v_k^*, w_k^*).$$

If $|err_{k+1}| \le \varepsilon_{stop}$, then the algorithm stops and set $\{v_{k+1}^*, w_{k+1}^*\}$ and $\{u_{k+1}^*, y_{k+1}^*\}$ as optimal. If not, go to the next step.
Step k_4. Compute (for using in a possible next step) a_{k+1}^* by (4.20), W_{k+1} by (4.19), p_{k+1} by (4.23), z_{k+1}^* by (4.22), V_{k+1} by (4.21) and continue from *Step* k_1.

The 1D numerical simulations have been obtained with the software Mathematica (license L4698-0010), for the following data:

$$\begin{aligned}\Omega = [0, L],\ L = 2,\ T = 1,\ \lambda_1 = 1,\ \lambda_2 = 0.1,\\ b = 0.1,\ c = 1,\ g = 0,\ \varepsilon = 0.1,\\ step_\lambda = 0.001,\ p = 10,\ \varepsilon_{stop} = 0.1.\end{aligned}$$

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The computation of ξ_ε^* was done by approximating it by $\xi_{\varepsilon,\sigma}^*$ (see (4.5)) with $\sigma = \varepsilon_H = 0.00001$.

Case 1. We considered:

$$y_0(x) = 0.9x^2(L-x)^2,\ u_0(x) = 0,\ w_0^*(x) = 0.5,\ v_0^*(x) = 1,\ f = 0.$$

The surfaces $z = y_k^*(t,x)$ and $z = w_k^*(t,x)$ obtained after $k = 9$ steps are represented in Figure 1.1 and Figure 1.2.

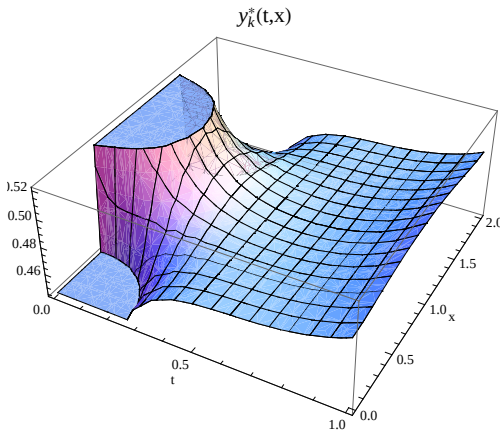


FIGURE 1.1: Case 1. $y_k^*(t,x)$, $k = 9$

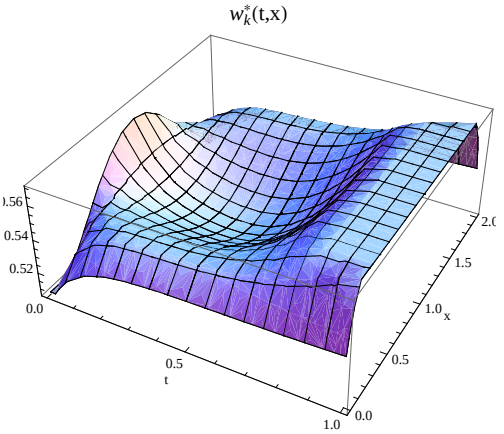


FIGURE 1.2: Case 1. $w_k^*(t,x)$, $k = 9$

The contour plots of $y_k^*(t,x)$ and $w_k^*(t,x)$ (after 9 steps) are plotted in Figure 1.3 and Figure 1.4, for $t \in [0,T]$. One can observe there the evolution of the level curves from $t = 0$ to $t = T$.

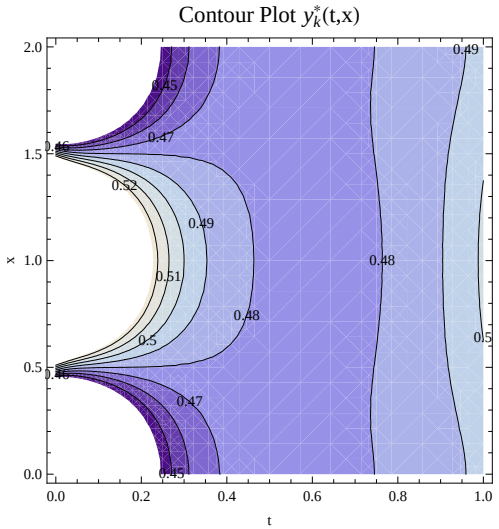


FIGURE 1.3: Case 1. $y_k^*(t,x)$, $t \in [0,T]$

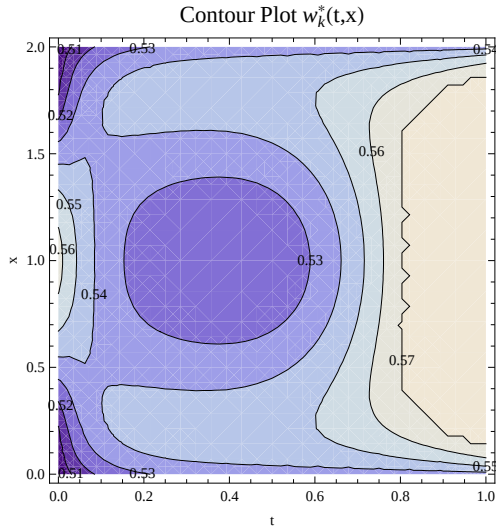


FIGURE 1.4: Case 1. $w_k^*(t,x)$, $t \in [0,T]$

The contour plots of $y_k^*(t,x)$ and $w_k^*(t,x)$ for $t \in [0,T/10]$ are given in Figure 1.5 and Figure 1.6. These figures make a zoom in the time interval $[0,0.1]$ and put into evidence the regions Q_b in the left corners of Figures 1.5 and 1.6. Figures 1.3 - 1.6 show the evolution of the regions $Q_{b,c}(t)$, $Q_b(t)$ ($Q_c(t)$ being not formed).

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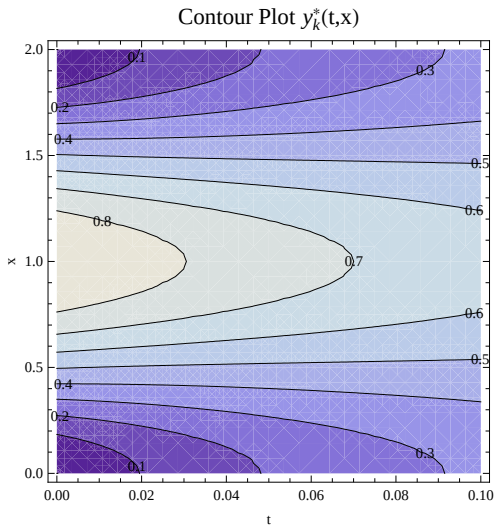


FIGURE 1.5: Case 1. $y_k^*(t,x)$, $t \in [0, T/10]$

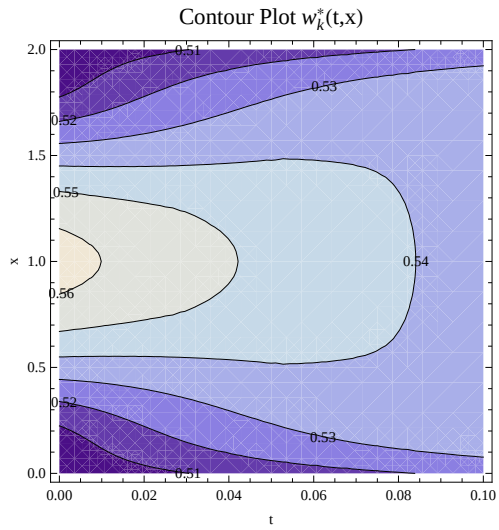


FIGURE 1.6: Case 1. $w_k^*(t,x)$, $t \in [0, T/10]$

The values of $J_k(v_k^*, w_k^*)$ for $k = 0, ..., 9$ are : $15.5428, 14.9529, 14.3786, 13.8208$, 13.2773, 12.7492, 12.2395, 11.9325, 11.8396, 11.8308.

Case 2 presents in Figures 2.1 - 2.6 similar graphics obtained after $k = 8$ steps for

$$y_0(x) = 0.5,\ u_0(x) = 0,\ w_0^*(x) = 0.5,\ v_0^*(x) = 1,\ f = 0.$$

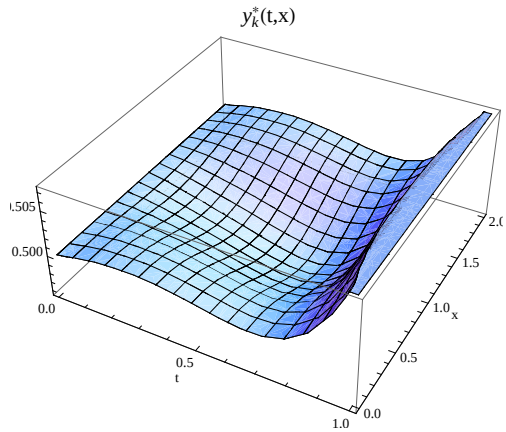


FIGURE 2.1: Case 2. $y_k^*(t,x)$, $k = 8$

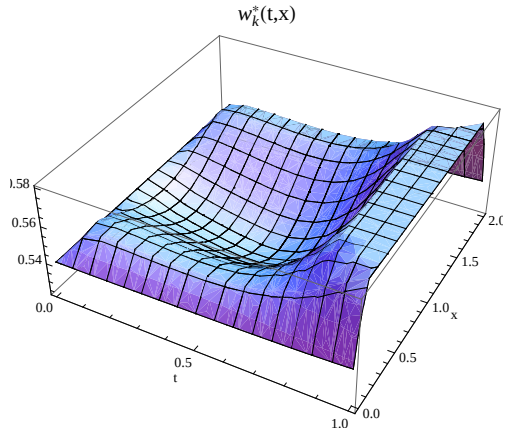


FIGURE 2.2: Case 2. $w_k^*(t,x)$, $k = 8$

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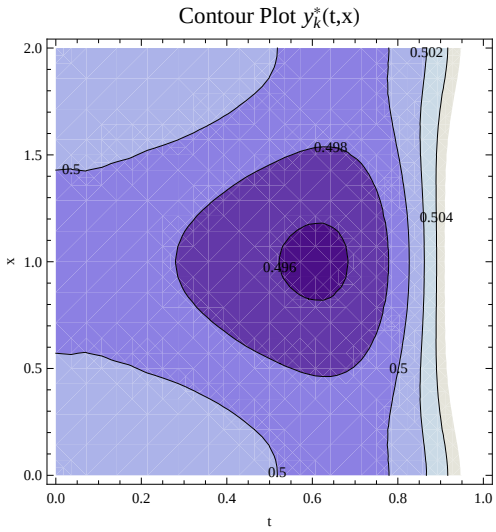


FIGURE 2.3: Case 2. $y_k^*(t,x)$, $t \in [0,T]$

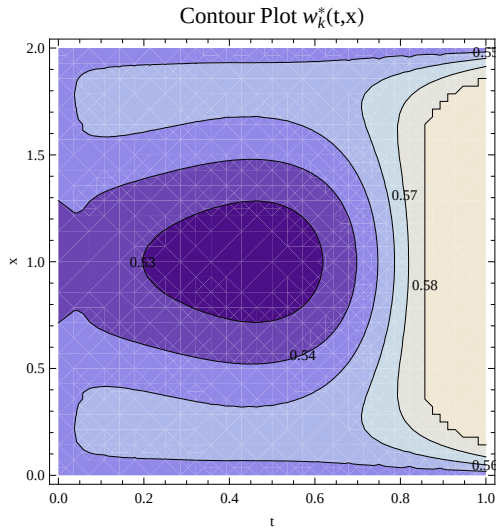


FIGURE 2.4: Case 2. $w_k^*(t,x)$, $t \in [0,T]$

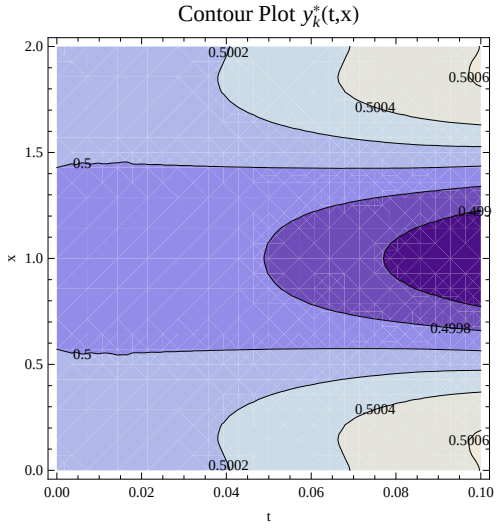


FIGURE 2.5: Case 2. $y_k^*(t,x)$, $t \in [0,T/10]$

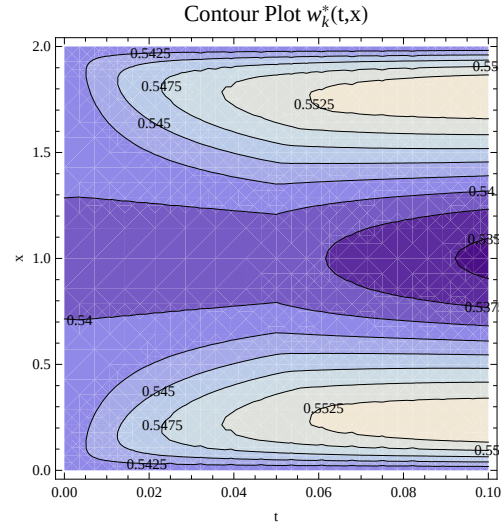


FIGURE 2.6: Case 2. $w_k^*(t,x)$, $t \in [0,T/10]$

The values of $J_k(v_k^*, w_k^*)$ for $k = 0, ..., 8$ are : 14.8746, 14.2868, 13.7165, 13.1631, 12.6256, 12.107, 11.5939, 11.2742, 11.1873.

We notice in Figures 2.3 - 2.4 and Figures 2.5 - 2.6 a better matching between the contour plots of $y_k^*(t,x)$ and $w_k^*(t,x)$, both for smaller time and for larger time. This might be explained by the initial choice $w_0^* = y_0^*$.

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References

[1] V. Arnăutu, P. Neittaanmäki. Optimal control from theory to computer programs. Kluwer Academic Publishers. Dordrecht, 2003.
[2] V. Barbu. Optimal control of variational inequalities. Pitman. Massachusets, 1984.
[3] V. Barbu. Nonlinear differential equations of monotone types in Banach spaces. Springer. New York, 2010.

https://doi.org/10.1051/mmnp/20149410 Published online by Cambridge University Press

[4] C.M. Elliott, J.R. Ockendon. Weak and variational methods for moving boundary problems. Research Notes in Mathematics 59. Pitman, 1982.
[5] J.L. Lions. Quelques méthodes de resolution des problèmes aux limites non linéaires. Dunod. Paris, 1969.
[6] J.L. Lions. Contrôle des systèmes distribués singuliers. Bordas. Paris, 1983.
[7] E. Magenes. Topics in parabolic equations: *Some typical free boundary problems*. In: Boundary value Problems for Linear Evolution Partial Differential Equations (ed. H.G. Garnir). D. Reidel (1977), 239-312.

https://doi.org/10.1051/mmnp/20149410 Published online by Cambridge University Press