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INFERENCE IN INSTRUMENTAL VARIABLE MODELS WITH HETEROSKEDASTICITY AND MANY INSTRUMENTS*

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This paper proposes novel inference procedures for instrumental variable models in the presence of many, potentially weak instruments that are robust to the presence of heteroskedasticity. First, we provide an Anderson–Rubin-type test for the entire parameter vector that is valid under assumptions weaker than previously proposed Anderson–Rubin-type tests. Second, we consider the case of testing a subset of parameters under the assumption that a consistent estimator for the parameters not under test exists. We show that under the null, the proposed statistics have Gaussian limiting distributions and derive alternative chi-square approximations. An extensive simulation study shows the competitive finite sample properties in terms of size and power of our procedures. Finally, we provide an empirical application using college proximity instruments to estimate the returns to education.

1. INTRODUCTION

The performance of test statistics based on instrumental variable (IV) models crucially depends on the quality and quantity of said IVs. In the presence of weak instruments, standard test statistics tend to deliver unreliable results. It is also well known that the number of instruments used in the construction of such tests plays a key role (see, e.g., Kleibergen, 2002, and references therein).

The Anderson–Rubin test (Anderson and Rubin, 1949, henceforth AR) is one of the most widely used statistics in the context of IV. Notoriously, this approach

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has the advantage of being robust to the presence of weak instruments. However, when the number of instruments grows larger than the number of parameters, the performance of the AR test starts deteriorating (e.g., Anatolyev and Gospodinov, 2011). The presence of heteroskedasticity may exacerbate the problem.

Over the years, a number of improvements on the basic formulation of the AR test have been introduced (see, e.g., Staiger and Stock, 1997; Wang and Zivot, 1998; Zivot, Startz, and Nelson, 1998; Kleibergen, 2002; Stock, Wright, and Yogo, 2002; Andrews, Moreira, and Stock, 2006, Moreira, 2009; Andrews, Marmer, and Yu, 2019). However, those tests do not consider the framework when the number of instruments grows with the sample size.

Anatolyev and Gospodinov (2011) study the limiting behavior of the AR and Sargan statistic under Bekker's many instruments framework (Bekker, 1994). Under conditional homoskedasticity, they find that their test statistics are asymptotically normal and that the resulting limiting distributions depend on $\lambda = \lim_{n\to\infty} k/n$, $0 < \lambda < 1$, where k is the number of instruments and n is the sample size. Since the tests may display some size distortion when λ is close to either zero or one, the authors propose a suitable chi-square approximation. Donald, Imbens, and Newey (2003) and Andrews and Stock (2007) obtain similar results where the instruments are allowed to grow at slower rates. Phillips and Gao (2017) also consider the many instruments case under conditional homoskedasticity and derive asymptotic and finite sample results for a partially restricted version of the AR test. One of the advantages of this approach is that their test is informative for local alternatives that converge to the null at a rate faster than those for which the unrestricted version is informative.

Probably, the paper closest to ours is that of Chao et al. (2014), where the authors propose an overidentification test for many (weak) instruments and heteroskedasticity that exploits the properties of the jackknife IV estimator (see Hausman et al., 2012). The framework in Chao et al. (2014) is sufficiently general to include the Bekker's many instruments case, the many weak instruments case of Chao and Swanson (2005) and instruments that are either weak or strong. Furthermore, the ratio k/n is bounded and the number of instruments cannot grow faster than the square of the concentration parameter.

Newey and Windmeijer (2009) study generalized empirical likelihood and generalized method of moments methods in a model with moment restrictions and show that the tests of Guggenberger and Smith (2005) and Kleibergen (2005) have canonical chi-square limits even when the number of instruments goes to infinity. However, the rate of growth of the instruments is slower than that in Chao et al. (2014). In a recent paper, Bun, Farbmacher, and Poldermans (2019) proposed a general version of the AR test based on an Edgeworth expansion argument both for k fixed and, in the homoskedastic linear model case, for $k \to \infty$. We are not aware of any other studies that generalize the AR test to the case of many IVs and heteroskedasticity.

The objective of this paper is to construct test statistics for the parameter vector of a linear IV model in the presence of many, potentially weak instruments and

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heteroskedasticity. The starting point of our work is the paper by Bekker and Crudu (2015). The analysis is closely related to the papers by Hausman et al. (2012) and Chao et al. (2014).

First of all, we show that the many-instrument results in Anatolyev and Gospodinov (2011) are no longer valid under heteroskedasticity. Then, we propose a test statistic to test null hypotheses on the full vector of parameters associated with both endogenous and exogenous variables and a test statistic to test null hypotheses on a subset of the parameters of the model (see, e.g., Guggenberger et al., 2012). In the latter case, we assume the existence of a plug-in estimator that is consistent under the null hypothesis. We also allow for heteroskedasticity of unknown form. In this sense, our test statistics may be seen as generalizations of the AR test. The first statistic we introduce refers to the whole parameter vector and, under the null, does not explicitly depend on the convergence properties of the concentration parameter. On the other hand, the second test statistic is built to test a subset of the parameter vector and relies on a consistent plug-in estimator. In this case, when the plug-in is an IV estimator, the concentration parameter plays a role in the limiting properties of the test. The assumptions on the concentration parameter match those in Bekker and Crudu (2015) and are rather similar to those in Hausman et al. (2012). To the best of our knowledge, there is no other test on only a subset of the parameter vector in IV models with many, potentially weak instruments that allow for the presence of heteroskedasticity (for the fixed instruments model, see, e.g., Guggenberger et al., 2012).

The plan of the paper is as follows. Section 2 introduces the model, Section 3 describes the test statistics, the main asymptotic results and the associated assumptions. Sections 4 and 5 contain the simulation results and an empirical application using the college proximity instruments of Card (1995), respectively. Section 6 concludes the paper. Proofs, auxiliary results, and figures are relegated to the Appendix and some additional material is available in the online Supplementary Material associated with this article.

2. THE IV MODEL

Let us consider the model

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \tag{1}$$

$$\boldsymbol{X} = \boldsymbol{Z}\boldsymbol{\Pi} + \boldsymbol{U}, \tag{2}$$

where $\boldsymbol{y}$ is a vector of dimension n and $\boldsymbol{X}$ is an $n \times g$ matrix. Throughout the paper, it is assumed that the $n \times k$ matrix of instruments $\boldsymbol{Z}$ is nonstochastic and $\mathrm{E}[\boldsymbol{X}] = \boldsymbol{Z}\boldsymbol{\Pi}$, where the components of $\boldsymbol{\Pi}$ are allowed to vary with the sample size n. Such assumptions are made for convenience and may be generalized.[1]

[1] We may, for example, consider $\boldsymbol{Z}$ to be stochastic and in this case $\mathrm{E}[\boldsymbol{X}]$ should be interpreted as a conditional expectation with respect to $\boldsymbol{Z}$. The linearity of $\mathrm{E}[\boldsymbol{X}]$ may also be relaxed as suggested in, for example, Bekker (1994) and Chao et al. (2014).

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The rows of the disturbance couple $(\boldsymbol{\varepsilon}, \boldsymbol{U})$, say $(\varepsilon_i, \boldsymbol{U}_i')\, i = 1, \ldots, n$, are independent with zero mean and covariance matrices

$$\boldsymbol{\Sigma}_i = \begin{pmatrix} \sigma_i^2 & \boldsymbol{\sigma}_{i12} \\ \boldsymbol{\sigma}_{i21} & \boldsymbol{\Sigma}_{i22} \end{pmatrix} \tag{3}$$

while the covariance matrix of the rows $(y_i, \boldsymbol{X}_i')$ are

$$\boldsymbol{\Omega}_i = \begin{pmatrix} 1 & \boldsymbol{\beta}' \\ \mathbf{0} & \boldsymbol{I}_g \end{pmatrix} \boldsymbol{\Sigma}_i \begin{pmatrix} 1 & \mathbf{0} \\ \boldsymbol{\beta} & \boldsymbol{I}_g \end{pmatrix}. \tag{4}$$

3. ASYMPTOTIC RESULTS

In this section, we introduce a set of assumptions that are used to prove our asymptotic results. Furthermore, we generalize a result due to Anatolyev and Gospodinov (2011) to the heteroskedastic case and we introduce our main results. In addition to that, we compare our assumptions with those introduced in other related papers and we comment on the behavior of the proposed tests when some critical assumptions are violated.

The assumptions we use are similar to those in Bekker and Crudu (2015). Additional assumptions are included to generalize some results due to Anatolyev and Gospodinov (2011). In what follows, it is understood that the generic positive constant c_u may be different in different situations.

Assumption 1. The generic diagonal element P_{ii} of the projection matrix $\boldsymbol{P} = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1}\boldsymbol{Z}'$ satisfies $\max_i P_{ii} \leq 1 - 1/c_u$, with $1 < c_u < \infty$. In addition, $k \to \infty$ as $n \to \infty$.

Assumption 2. The variances satisfy $\sigma_i^2 \geq \underline{\sigma}^2$ with $0 < \underline{\sigma}^2 < \infty$, for any i.

Assumption 3. $\mathrm{E}[\varepsilon_i^4] \leq c_u$ and $\mathrm{E}\left[\|\boldsymbol{U}_i\|^4\right] \leq c_u$ with $0 < c_u < \infty$, for any i.

Assumption 1 is a technical condition on the projection matrix $\boldsymbol{P}$. It requires the main diagonal elements of $\boldsymbol{P}$ to be bounded away from 1. This assumption is rather standard in the literature (e.g., Hausman et al., 2012; Bekker and Crudu, 2015) and is strictly weaker than the so-called asymptotic balanced design (see Anatolyev, 2018) imposed, for example, in Anatolyev and Gospodinov (2011) and Bun et al. (2019) according to which all the diagonal elements of the projection matrix converge to the same constant. The assumption $k \to \infty$ as $n \to \infty$ formalizes the many instruments idea in a way that is known as Bekker asymptotics. Assumptions 2 and 3 are standard regularity conditions; the former bounds variances of the disturbances away from zero, while the latter bounds the fourth moments of the errors.

3.1. The AR Test Under Heteroskedasticity

In this section, we study the limiting distribution of the AR test statistics in the presence of heteroskedasticity. In addition, our derivation implies that the test statistics we propose in Section 3.2 are also valid under homoskedasticity.

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The AR statistic is a popular choice to test a null hypothesis defined as $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$. The statistic is defined as

$$AR = (n-k)\,\frac{\boldsymbol{\varepsilon}_0' \boldsymbol{P} \boldsymbol{\varepsilon}_0}{\boldsymbol{\varepsilon}_0' (\boldsymbol{I}_n - \boldsymbol{P})\, \boldsymbol{\varepsilon}_0} \tag{5}$$

and, under certain assumptions, it is asymptotically chi-square distributed with k degrees of freedom. In the many instruments context and in the presence of homoskedasticity, the behavior of the AR test has been studied by Andrews and Stock (2007) and Anatolyev and Gospodinov (2011), among others. The following result generalizes the results in Lem. 1 of Anatolyev and Gospodinov (2011) to the heteroskedastic case. Let us define $\overline{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n \sigma_i^2$ and $W_n = \frac{2}{k}\sum_{i \neq j} P_{ij}^2 \sigma_i^2 \sigma_j^2$.

PROPOSITION 1. *Suppose that Assumption 3 is satisfied,* $\lambda = \lim_{n\to\infty} \frac{k}{n} < 1$ *exists and* $\frac{1}{k}\sum_{i=1}^n (P_{ii} - \frac{k}{n})^2 \to 0$, $\frac{1}{\sqrt{k}}\sum_{i=1}^n (P_{ii} - \frac{k}{n})\sigma_i{}^2 \to 0$ *hold.*[2] *In addition, assume that* $\lim_{n\to\infty} \overline{\sigma}_n^2 = \sigma_0^2$ *and* $\lim_{n\to\infty} W_n = W_0$ *exist. Then the statistic* $AR_{AG} = \sqrt{k}\left(\frac{AR}{k} - 1\right)$ *proposed by Anatolyev and Gospodinov (2011) has the limit*[3]

$$AR_{AG} \xrightarrow{d} \mathcal{N}\left(0, \frac{W_0}{\sigma_0^4 (1-\lambda)^2}\right).$$

Remark 1. The asymptotic distribution result in Proposition 1 has two important implications. First, the asymptotic size of this test is

$$\begin{aligned}\Pr\left(AR_{AG} > \Phi^{-1}(1-\alpha)\right) &= \Pr\left(\frac{\sigma_0^2 (1-\lambda)}{\sqrt{W_0}} AR_{AG} < \frac{\sigma_0^2 (1-\lambda)}{\sqrt{W_0}} \Phi^{-1}(\alpha)\right) \\ &\to \Phi\left(\frac{\sigma_0^2 (1-\lambda)}{\sqrt{W_0}} \Phi^{-1}(\alpha)\right).\end{aligned}$$

Second, the test statistic T_1 proposed in Section 3.2 has broader applicability than that proposed by Anatolyev and Gospodinov even under homoskedasticity. This is because its asymptotic distribution requires the assumption that the main diagonal elements P_{ii}, $i = 1, \ldots, n$, of the projection matrix $\boldsymbol{P}$ should be bounded away from 1. The test statistic proposed by Anatolyev and Gospodinov (2011) requires the stronger assumption that the main diagonal elements of $\boldsymbol{P}$ converge to λ. This difference in the assumptions comes from the fact that the former test statistic does

[2] The assumption $\frac{1}{\sqrt{k}}\sum_{i=1}^n (P_{ii} - \frac{k}{n})\sigma_i{}^2 \to 0$ is needed here in order for the expected value of the statistic to converge to 0 because this does not always hold. In Example B.1 in the Supplemental Appendix, we provide an instance when this property is violated in the context of indicator instruments (Bekker and Van der Ploeg, 2005).

[3] We note that under homoskedasticity

$$\frac{\overline{\sigma}_n^2}{\sqrt{W_n}} \to \frac{\sigma^2}{\sqrt{2(1-\lambda)}\sigma^2} = \frac{1}{\sqrt{2(1-\lambda)}}, \text{ so } \sqrt{k}\left(\frac{AR}{k} - 1\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{2}{1-\lambda}\right),$$

which is exactly as in Lem. 1 of Anatolyev and Gospodinov (2011).

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not involve the diagonal elements of $\boldsymbol{P}$ while the latter statistic does. The following example clarifies this concept.

Example 1. *Consider indicator instruments with unequal group sizes (Bekker and Van der Ploeg, 2005). Anatolyev and Yaskov (2017, Sect. 5.1) show that in this case, the main diagonal elements of* $\boldsymbol{P}$ *do not converge to* λ. *In the Supplemental Appendix, we show that under homoskedasticity, the convergence in distribution* $\sqrt{k}\left(\frac{AR}{k}-1\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{2}{1-\lambda}\right)$ *is violated.*

3.2. Inference with Heteroskedasticity and Many Instruments

In this section, we present the main results. First, we present our test on the entire parameter vector. Then, we consider the more challenging case where we test the null on a subset of the coefficients; in this case, we assume that a consistent plug-in estimator exists for the parameters not under test. Furthermore, we study our tests when the number of instruments is fixed. Finally, we briefly discuss the behavior of our "subset" test for some commonly encountered specific plug-in estimators and in some pathological situations.

The test statistics proposed in this paper are related to the symmetric jackknife instrumental variable estimator (SJIVE) proposed by Bekker and Crudu (2015). The SJIVE estimates consistently, in the many (weak) instruments sense, the parameter vector $\boldsymbol{\beta}$ and it is defined as

$$\widehat{\boldsymbol{\beta}}_{SJIVE} = \arg\min_{\boldsymbol{\beta}} Q_{SJIVE}(\boldsymbol{\beta}) = \arg\min_{\boldsymbol{\beta}} \frac{(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})'\boldsymbol{C}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})}{(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})'\boldsymbol{B}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})} \tag{6}$$

and, given the projection matrix $\boldsymbol{P}$ and the diagonal matrix $\boldsymbol{D}$ containing the diagonal elements of $\boldsymbol{P}$,

$$\boldsymbol{C}=\boldsymbol{A}-\boldsymbol{B}, \qquad \boldsymbol{A}=\boldsymbol{P}+\boldsymbol{\Delta}, \qquad \boldsymbol{B}=(\boldsymbol{I}_n-\boldsymbol{P})\boldsymbol{D}(\boldsymbol{I}_n-\boldsymbol{D})^{-1}(\boldsymbol{I}_n-\boldsymbol{P}),$$
$$\boldsymbol{\Delta}=\boldsymbol{P}\boldsymbol{D}(\boldsymbol{I}_n-\boldsymbol{D})^{-1}\boldsymbol{P}-\frac{1}{2}\boldsymbol{P}\boldsymbol{D}(\boldsymbol{I}_n-\boldsymbol{D})^{-1}-\frac{1}{2}\boldsymbol{D}(\boldsymbol{I}_n-\boldsymbol{D})^{-1}\boldsymbol{P}.$$

Consider now testing the null hypothesis $H_0 : \boldsymbol{\beta}=\boldsymbol{\beta}_0$, where $\boldsymbol{\beta}$ is the true parameter vector.

The test statistic we propose is based on the numerator of the objective function in equation (6), namely,

$$Q(\boldsymbol{\beta}) = (\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})'\boldsymbol{C}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}), \tag{7}$$

and it is defined as

$$T_1 = \frac{1}{\sqrt{k}}\frac{(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}_0)'\boldsymbol{C}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta}_0)}{\sqrt{\widehat{V}(\boldsymbol{\beta}_0)}}, \quad \widehat{V}(\boldsymbol{\beta}_0)=\frac{2}{k}\boldsymbol{\varepsilon}_0^{(2)'}\boldsymbol{C}^{(2)}\boldsymbol{\varepsilon}_0^{(2)}, \tag{8}$$

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where $\boldsymbol{\varepsilon}_0 = \boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}_0$ and the superscript "$^{(2)}$" indicates the element-wise product of two conformable matrices or vectors. The following theorem provides the asymptotic distribution of the T_1 test statistic.[4]

THEOREM 1. *If Assumptions 1, 2, 3 are satisfied, then under* $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$, *we have* $T_1 \xrightarrow{d} \mathcal{N}(0,1)$.

Let us now consider a nominal level α and let z_α be the αth quantile of the normal distribution. Then, the null hypothesis is rejected if $T_1 \geq z_{1-\alpha}$.

Sometimes one is interested only in performing inference on a subset of parameters. In particular, we would like to test the coefficients associated with the endogenous variables. Let us now define the parameter vector as $\boldsymbol{\beta} = \left(\boldsymbol{\beta}_1', \boldsymbol{\beta}_2'\right)'$ and suppose we want to test the following null hypothesis

$$H_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_{10} \tag{9}$$

in the model

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon},$$

where the dimensions of $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are $n \times g_1$ and $n \times g_2$, respectively, with $g = g_1 + g_2$. Let $\boldsymbol{y}_0 = \boldsymbol{y} - \boldsymbol{X}_1\boldsymbol{\beta}_1$ so that under the null hypothesis the model becomes

$$\boldsymbol{y}_0 = \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}.$$

Accordingly, the reduced form model corresponding to $\boldsymbol{X}_2$ is

$$\boldsymbol{X}_2 = \boldsymbol{Z}\boldsymbol{\Pi}_2 + \boldsymbol{U}_2,$$

where $\boldsymbol{\Pi}_2$ and $\boldsymbol{U}_2$ both have g_2 columns. Furthermore, let

$$\boldsymbol{H} = \boldsymbol{\Pi}'\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\Pi} = \begin{pmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} \\ \boldsymbol{H}_{12}' & \boldsymbol{H}_{22} \end{pmatrix},$$

denote the signal matrix and let $\boldsymbol{H}_{22} = \boldsymbol{\Pi}_2'\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\Pi}_2$, which has dimension $g_2 \times g_2$.

We assume that a consistent estimator for $\boldsymbol{\beta}_2$, say $\widetilde{\boldsymbol{\beta}}_2$, exists. If the variables associated with $\boldsymbol{\beta}_2$ are exogenous, the ordinary least squares (OLS) estimator is a valid candidate. However, if this is not the case, we need a suitable IV estimator. Under the null, a consistent estimator is, for example, the SJIVE. For the null hypothesis $H_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_{10}$, consider $\widetilde{\boldsymbol{\beta}} = \left(\boldsymbol{\beta}_{10}', \widetilde{\boldsymbol{\beta}}_2'\right)'$, $\widetilde{\boldsymbol{\varepsilon}} = \boldsymbol{y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}}$ and let the modified test statistic, denoted as T_2, be

$$T_2 = \frac{1}{\sqrt{k}} \frac{\widetilde{\boldsymbol{\varepsilon}}'\boldsymbol{C}\widetilde{\boldsymbol{\varepsilon}}}{\sqrt{\widehat{V}(\widetilde{\boldsymbol{\beta}})}}, \quad \text{where } \widehat{V}(\widetilde{\boldsymbol{\beta}}) = \frac{2}{k}\widetilde{\boldsymbol{\varepsilon}}^{(2)\prime}\boldsymbol{C}^{(2)}\widetilde{\boldsymbol{\varepsilon}}^{(2)}. \tag{10}$$

[4] We could apply the same type of analysis by replacing $\boldsymbol{C}$ with $\boldsymbol{P} - \boldsymbol{D}$ as in Chao et al. (2014). We do not pursue that avenue since, as suggested in Bekker and Crudu (2015), $\boldsymbol{C}$ allows us to retain the whole signal matrix.

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Let now $r_{\min} = \lambda_{\min}(\boldsymbol{H}_{22})$ and $r_{\max} = \lambda_{\max}(\boldsymbol{H}_{22})$ be the smallest eigenvalue and the largest eigenvalue of $\boldsymbol{H}_{22}$, respectively. Moreover, let us define a generic constant κ such that $0 \leq \kappa < \infty$.

Assumption 4. $k/r_{\min} \to \kappa$, $r_{\max}/k \to \kappa$ when $n \to \infty$.

Assumption 5. $r_{\max}/k \to \kappa$, $r_{\min}/k \to 0$, $\sqrt{k}/r_{\min} \to 0$ when $n \to \infty$.

We have two remarks on these assumptions. First, Assumptions 4 and 5 are used in conjunction with Assumption 1 (specifically, $k \to \infty$ as $n \to \infty$), and therefore, either of them implies that $r_{\min} \to \infty$ and $r_{\max} \to \infty$ as $n \to \infty$. Second, Assumptions 4 and 5 regulate the convergence of the plug-in IV estimator. When the growth rates of $r_{\min}$ and $r_{\max}$ are the same, we are either in the many instruments framework of Bekker (1994) or in the many weak instruments framework of Chao and Swanson (2005). As in Chao et al. (2014), the growth rates of $r_{\min}$ and $r_{\max}$ are allowed to vary.

The following theorem provides the asymptotic distribution of the T_2 test.

THEOREM 2. *If Assumptions 1, 2, 3 and either 4 (many strong instruments case), or 5 (many weak instruments case) are satisfied, then* $T_2 \xrightarrow{d} \mathcal{N}(0,1)$.

Analogously to the T_1 case, the null hypothesis is rejected if $T_2 \geq z_{1-\alpha}$.

It is important to derive the limiting distribution of our tests in case the number of instruments does not grow with the sample size. The following theorem provides the limiting distribution of T_1 and T_2 under the assumption that k is fixed and the error couple $(\boldsymbol{\varepsilon}, \boldsymbol{U})$ is homoskedastic.

THEOREM 3. *Let the disturbance couple* $(\boldsymbol{\varepsilon}, \boldsymbol{U})$ *be zero mean and homoskedastic and let Assumption 3 hold. Furthermore, assume (i) k fixed and* $n \to \infty$, *(ii) as* $n \to \infty$, $\frac{\boldsymbol{Z}'\boldsymbol{Z}}{n} \to \boldsymbol{\Sigma}_{ZZ}$ *a full rank nonstochastic matrix, (iii) as* $n \to \infty$, $\frac{\boldsymbol{X}'\boldsymbol{Z}}{n} \to_p \boldsymbol{\Sigma}_{XZ}$ *a nonstochastic matrix with* $\text{rank}(\boldsymbol{\Sigma}_{XZ}) = g$, *(iv) as* $n \to \infty$, $\frac{\boldsymbol{Z}'\boldsymbol{\varepsilon}}{\sqrt{n}} \to_d \mathcal{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{\Sigma}_{ZZ})$. *Then* $\sqrt{2k}T_1 + k \to_d \chi^2_k$. *If* $\widetilde{\boldsymbol{\beta}}_2$ *is the two-stage least-squares estimator, then* $\sqrt{2k}T_2 + k \to_d \chi^2_{k-g_2}$.

Although the Gaussian approximation may work well in finite samples, it does not allow us to control for the number of instruments. This, as stressed in Anatolyev and Gospodinov (2011), may be an important issue. The following corollary shows how to obtain a chi-square approximation for T_1 and T_2.

COROLLARY 1. *If the assumptions of Theorem 1 hold true, then (i)* $\sqrt{k}T_1 + k \to_d \chi^2_k$. *If the assumptions of Theorem 2 hold true, then (ii)* $\sqrt{k}T_2 + k \to_d \chi^2_k$, *(iii)* $\sqrt{k}T_2 + k \to_d \chi^2_{k-g_2}$, *or (iv)* $\sqrt{k-g_2}T_2 + k - g_2 \to_d \chi^2_{k-g_2}$.

Corollary 1 shows that there are different possible chi-square approximations for T_2. Although approximation (*iii*) seems to be a natural candidate, also because

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it matches the result in Theorem 3, it may not deliver the best results in finite samples. We expect, for example, that, when k is small, approximations (*ii*) and (*iv*) enjoy better finite sample properties.[5]

The convergence properties of T_1 are determined by the behavior of the diagonal elements of $\boldsymbol{P}$ and by the properties of the disturbances. The T_2 test also depends on the properties of the plug-in estimator of the parameters not under test. When the OLS estimator $\widetilde{\boldsymbol{\beta}}_2 = \left(\boldsymbol{X}_2'\boldsymbol{X}_2\right)^{-1}\boldsymbol{X}_2'\boldsymbol{y}_0$ is a consistent plug-in estimator, that is, $\boldsymbol{X}_2$ is exogenous, it is easy to show that T_2 converges to a standard normal basically under the same assumptions as those imposed in Theorem 1. No further assumptions on the strength of the instruments need to be imposed (see Theorem B.1 in the Supplemental Appendix for a formal treatment). In the Supplemental Appendix (see Theorem B.2), we derive the limiting distribution of T_2 for the case when $\boldsymbol{X}_2$ is endogenous and the JIV1 estimator is used as plug-in. The convenient expression of the JIV1 estimator allows us to explain why underrejection of the null hypothesis occurs in most cases (see Remark B.1 in the Supplemental Appendix) and to better link the weak instrument assumption $\sqrt{k}/r_{\min} \to 0$ to the asymptotic distribution of T_2 (see Remark B.2 in the Supplemental Appendix). Our derivation suggests that the assumption $\sqrt{k}/r_{\min} \to 0$ is likely to be necessary for the asymptotic standard normality of the statistic T_2 (see Remark B.2). In Section 4.2 below, we further discuss the behavior of T_2 in relation with the limiting behavior of $\sqrt{k}/r_{\min}$. Specifically, we illustrate that for relatively large $\sqrt{k}/r_{\min}$ the histogram of T_2 differs substantially from the standard normal density. Finally, in the case when the plug-in estimator converges slowly to the true value, we find that the density of T_2 is shifted to the right causing the test to overreject (see Example B.2 in the Supplemental Appendix).

3.3. Comparison with Other Tests

In this section, we compare our set of assumptions with those used in some closely related papers. Some papers provide a broad range of results and a certain degree of variation in the specification of the assumptions. Therefore, for ease of presentation, some assumptions considered here are stronger than necessary (e.g., Newey and Windmeijer, 2009). In Table 1, we report the different assumptions imposed on the rate of convergence of the number of instruments and the concentration parameter, and whether or not they are robust to heteroskedasticity. Moreover, we distinguish between test statistics that consider null hypotheses on the full set of parameters or on a subset. We also report whether they allow for instruments with unbalanced design. Finally, we only consider the case where the model contains one endogenous variable. Thus, $r_{\min} = r_{\max} = r$ and $r = \boldsymbol{\pi}'\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{\pi}$, where r is the scalar version of the signal matrix $\boldsymbol{H}$ and is proportional to the concentration parameter.

There is a certain degree of heterogeneity in the type of assumptions that we show in Table 1. For example, Anatolyev and Gospodinov (2011) and Bun et al.

[5] See Section C in the Supplemental Appendix for some Monte Carlo evidence.

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TABLE 1. Comparison of assumptions in the many instruments framework.

	Subset	k/n	r	Heteroskedasticity	Unbalanced instruments
Anatolyev and Gospodinov (2011)	No	$\frac{k}{n} \to \lambda$, $0 < \lambda < 1$	$\frac{r}{n} \to \kappa$, $\kappa \in (0, \infty)$	No	No
Andrews and Stock (2007)	No	$\frac{k^3}{n} \to 0$	$\frac{r}{k^\zeta} \to \kappa_\zeta$, $\kappa_\zeta \in [0, \infty)$ $\zeta \in (0, \infty)$	No	Yes
Bun et al. (2019)	No	$\frac{k}{n} \to \lambda$, $0 < \lambda < 1$	–	No	No
Newey and Windmeijer (2009)	No	$\frac{k^2}{n} \to 0$ or $\frac{k^3}{n} \to 0$	$\frac{n}{r} \to \kappa$ or $\frac{r}{n} \to 0$ $\frac{k}{r}$ bounded	Yes	Yes
T_1	No	$\frac{k}{n}$ bounded	–	Yes	Yes
T_2	Yes	$\frac{k}{n}$ bounded	$\frac{k}{r} \to \kappa$ or $\frac{r}{k} \to 0$, $\frac{\sqrt{k}}{r} \to 0$	Yes	Yes

Note: For simplicity, we refer to the single endogenous variable case, where $r = \boldsymbol{\pi}'\mathbf{Z}'\mathbf{Z}\boldsymbol{\pi} = r_{\min} = r_{\max}$ and restrict ourselves to tests that use $k \to \infty$. Bun et al. (2019) also propose tests for the fixed k case that are robust to heteroskedasticity. Andrews and Stock (2007) and Newey and Windmeijer (2009) impose a different set of assumptions depending on the problem considered and the ones reported here might be stronger than necessary in some cases.

(2019) use Bekker's framework. Our assumptions, on the other hand, are more in line with those in Chao et al. (2014), with the difference that in our case r is bounded by k, while in Chao et al. (2014) it is bounded by n.[6] The assumptions in Andrews and Stock (2007) and Newey and Windmeijer (2009) are to some extent similar to ours but generally their rates tend to be slower. We also notice that only Newey and Windmeijer (2009) consider AR-type tests that are robust to heteroskedasticity. Moreover, no test other than T_2 seems to explicitly consider the subset null hypothesis presented in equation (9).

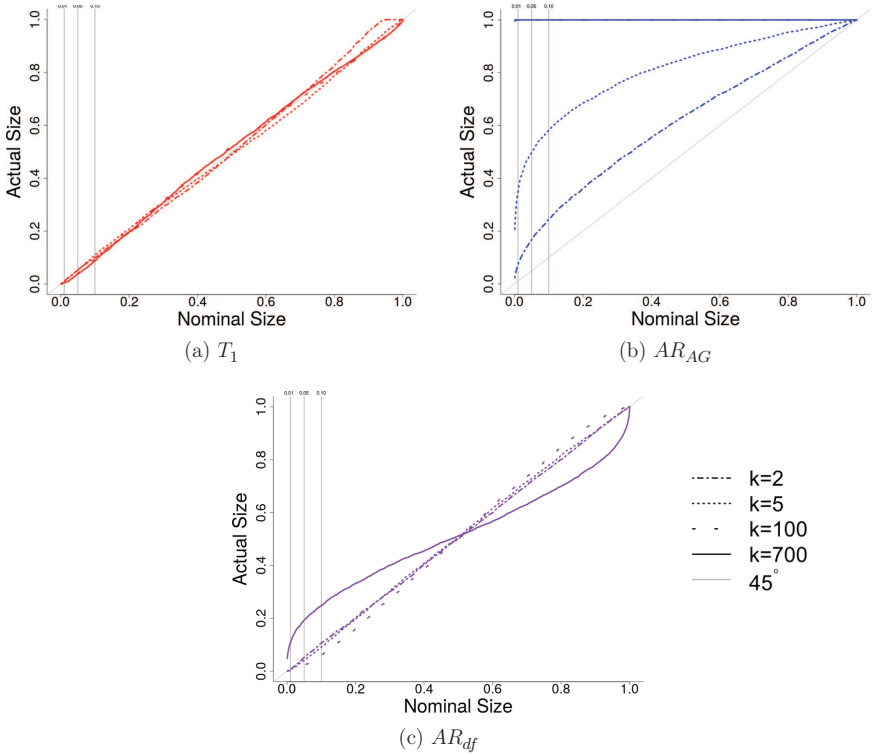
4. MONTE CARLO SIMULATIONS

In this section, we study the finite sample properties of the T_1 and T_2 tests in terms of size and power (see Figures 1 and 2 for the results on size and Figures 3–6 for the results on power).[7] Further Monte Carlo results may be found in the

[6] They assume either $\frac{n}{r} \to \kappa$ or $\frac{n}{r} \to 0$ and $\frac{\sqrt{k}}{r} \to 0$.

[7] The size properties of T_1 and T_2 are investigated by means of PP-plots as described in Davidson and MacKinnon (1998).

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(a) T_1 (b) AR_{AG} (c) AR_{df}

FIGURE 1. PP-plots with heteroskedasticity, $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$.

Supplemental Appendix. We make inference on the full parameter vector and on the sole parameter associated with the endogenous variable. The proposed tests are compared with the version of the *AR* test proposed by Anatolyev and Gospodinov (2011), denoted as AR_{AG}, and the *AR* test introduced in Bun et al. (2019) and defined as

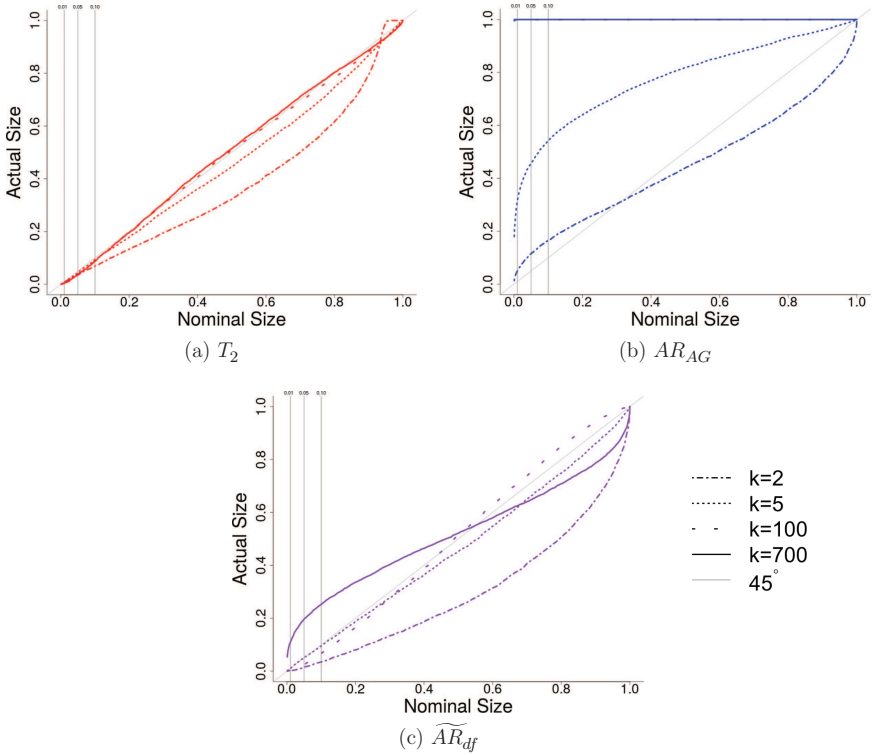
$$\widetilde{AR}_{df} = n\widehat{g}(\boldsymbol{\beta})'\widetilde{\boldsymbol{\Omega}}_{df}(\boldsymbol{\beta})^{-1}\widehat{g}(\boldsymbol{\beta}), \tag{11}$$

where $\widetilde{\boldsymbol{\Omega}}_{df}(\boldsymbol{\beta}) = \frac{n}{n-k}\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\beta})$, $\widetilde{\boldsymbol{\Omega}}(\boldsymbol{\beta}) = \widehat{\boldsymbol{\Omega}}(\boldsymbol{\beta}) - \widehat{g}(\boldsymbol{\beta})\widehat{g}(\boldsymbol{\beta})'$ and $\widehat{\boldsymbol{\Omega}}(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{g}(\boldsymbol{\beta})\boldsymbol{g}(\boldsymbol{\beta})'$. The moment condition model is defined as $\boldsymbol{g}_i(\boldsymbol{\beta}) = \boldsymbol{Z}_i(y_i - \boldsymbol{X}_i'\boldsymbol{\beta})$ and $\widehat{g}(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{g}_i(\boldsymbol{\beta})$. In the case of T_1, T_2, and AR_{AG}, we use the corresponding chi-square asymptotic distribution.[8]

This comparison is interesting for a number of reasons. First, we get a clearer idea on how much we gain by using our tests in a heteroskedastic context. Second, we get some important insights on how the considered test statistics work in the

[8]Due to the results in Figure C.2 in the Supplementary Appendix, for T_2, we use approximation (*ii*) in Corollary 1.

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(a) T_2 (b) AR_{AG}

(c) $\widetilde{AR}_{df}$

FIGURE 2. PP-plots with heteroskedasticity, $H_0 : \beta_1 = \beta_{10}$.

extreme cases where $\frac{k}{n} \approx 0$ and $\frac{k}{n} \approx 1$. A priori, we may expect the AR_{AG} to work well under homoskedasticity and for moderately large values of $\frac{k}{n}$, while it is probable that $\widetilde{AR}_{df}$ performs well also in the heteroskedastic case.

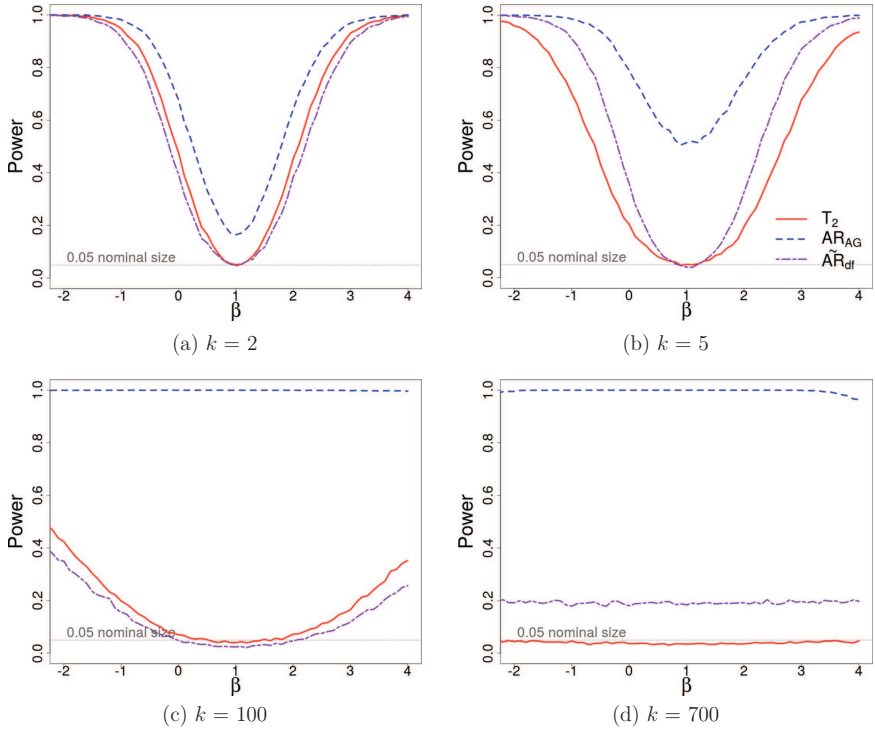
Furthermore, we introduce a two parameter model with two endogenous regressors; this model is used to study the role played by the boundary condition $\sqrt{k}/r_{\min} \to 0$ and by the plug-in estimator in determining the behavior of T_2 (see Figures 7 and 8 in Appendix B).

4.1. Data Generating Processes

Let us consider the Monte Carlo setup of Hausman et al. (2012). One of the features of this experiment is that the sum of the diagonal elements of $\boldsymbol{P}$ does not converge to $\lambda = \lim \frac{k}{n}$, as shown in Anatolyev and Yaskov (2017). The DGP is given by

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{\iota}\gamma + \boldsymbol{x}\beta + \boldsymbol{\varepsilon} \\ \boldsymbol{x} &= \boldsymbol{z}\pi + \boldsymbol{v}, \end{aligned} \tag{12}$$

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(a) $k = 2$ (b) $k = 5$

(c) $k = 100$ (d) $k = 700$

FIGURE 3. Power curves with heteroskedasticity and $\pi = 0.1$, $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$.

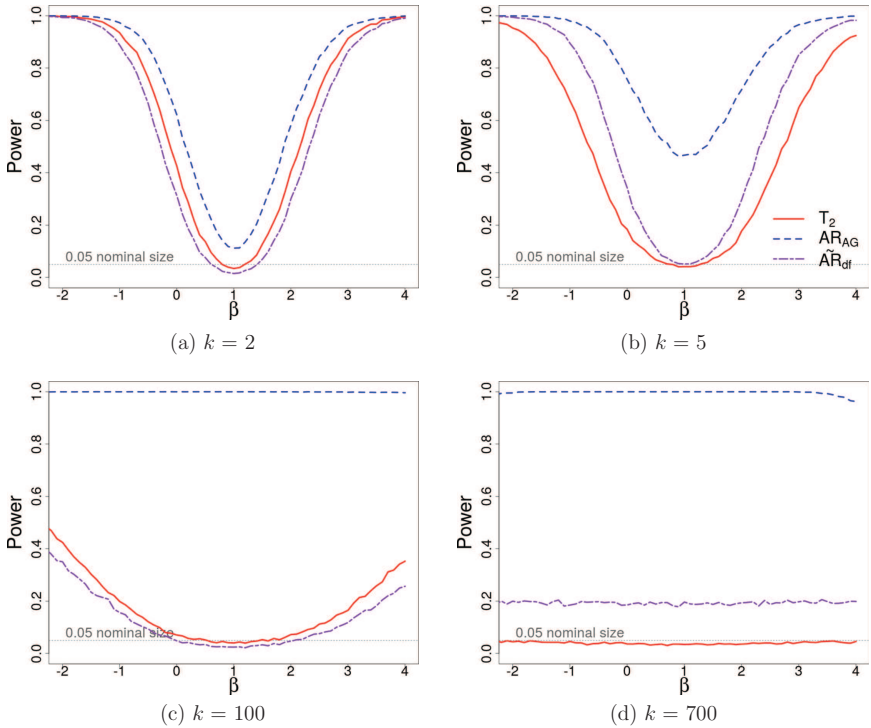
where $\gamma = \beta = 1$, while $\pi = 0.1$ in the analysis of size and $\pi \in \{0.1, 1\}$ in the analysis of power. The sample size is $n = 800$, $\boldsymbol{z} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_n)$ and independently $\boldsymbol{v} \sim \mathcal{N}\left(\boldsymbol{0}, 0.1^2 \times \boldsymbol{I}_n\right)$. The disturbances vector $\boldsymbol{\varepsilon}$ is generated as

$$\boldsymbol{\varepsilon} = \rho \boldsymbol{v} + \sqrt{\frac{1-\rho^2}{\phi^2 + \psi^4}}(\phi \boldsymbol{w}_1 + \psi \boldsymbol{w}_2), \tag{13}$$

where $\rho = 0.3$, $\psi = 0.86$ and conditional on $\boldsymbol{z}$, independent of $\boldsymbol{v}$, $\boldsymbol{w}_1 \sim \mathcal{N}\left(\boldsymbol{0}, \text{Diag}(\boldsymbol{z})^2\right)$, where $\text{Diag}(\boldsymbol{z})$ is a diagonal matrix where the diagonal elements are the elements of $\boldsymbol{z}$ and $\boldsymbol{w}_2 \sim \mathcal{N}\left(\boldsymbol{0}, \psi^2 \boldsymbol{I}_n\right)$. Notice that, $\phi = 1.38072$ implies heteroskedasticity, while $\phi = 0$ corresponds to the homoskedastic case. The instrument matrix $\boldsymbol{Z}$ is given by matrices with rows $(1, z_i, z_i^2, z_i^3, z_i^4)$ and $(1, z_i, z_i^2, z_i^3, z_i^4, z_i b_{1i}, \ldots, z_i b_{\ell i})$, $\ell = 95,695$, where, independent of other random variables, the elements $b_{1i}, \ldots, b_{\ell i}$ are i.i.d. Bernoulli distributed with $p = 1/2$.[9] We consider also two rather extreme situations: $k = 2$ and $k = 700$. We replicate

[9]The same set of instruments is used throughout the various repetitions.

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(a) $k = 2$ (b) $k = 5$

(c) $k = 100$ (d) $k = 700$

FIGURE 4. Power curves with heteroskedasticity and $\pi = 0.1$, $H_0 : \beta_1 = \beta_{10}$.

our experiments 5,000 times. When using the T_1 test and the T_2 test, we consider $H_0 : (\gamma, \beta)' = (1, 1)'$ and $H_0 : \beta = 1$ respectively.[10]

The following DGP is used to explore the properties of the T_2 test when the boundary condition $\sqrt{k}/r_{\min} \to 0$ is violated and when the plug-in estimator is inconsistent. Let us consider the following model

$$\boldsymbol{y} = \boldsymbol{x}\beta + \boldsymbol{w}\gamma + \boldsymbol{\varepsilon} \tag{14}$$
$$\boldsymbol{x} = \boldsymbol{Z}\boldsymbol{\pi}_x + \boldsymbol{u}_x, \quad \boldsymbol{w} = \boldsymbol{Z}\boldsymbol{\pi}_w + \boldsymbol{u}_w.$$

Let us now suppose we want to test the null $H_0 : \beta = \beta_0$, define $\boldsymbol{\eta}_i = (\varepsilon_i, u_{xi}, u_{wi})'$ and assume that

$$\boldsymbol{\eta}_i \sim \mathcal{N}\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1 & \rho & \xi\\ \rho & 1 & 0\\ \xi & 0 & 1\end{pmatrix}\right), \quad \boldsymbol{Z}_i \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_k), i = 1, \ldots, n. \tag{15}$$

[10] We computed results also for $\ell = 5, 15, 35, 55, 75,$ and we noticed that the p-value curves would converge from the p-value curve associated with $k = 5$ to the p-value curve with $k = 100$. This result replicates in all cases, including the power curves.

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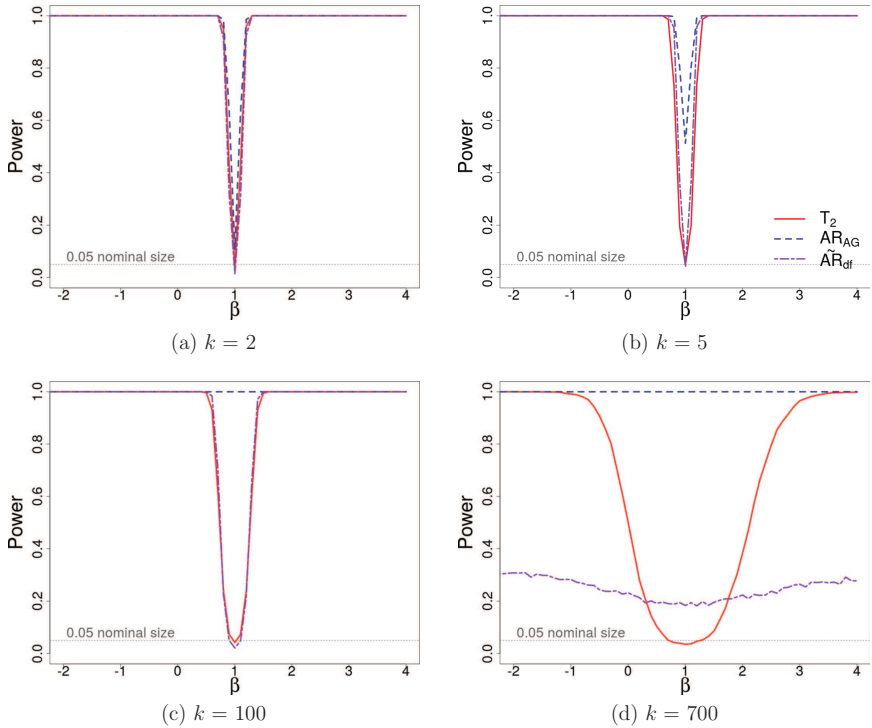


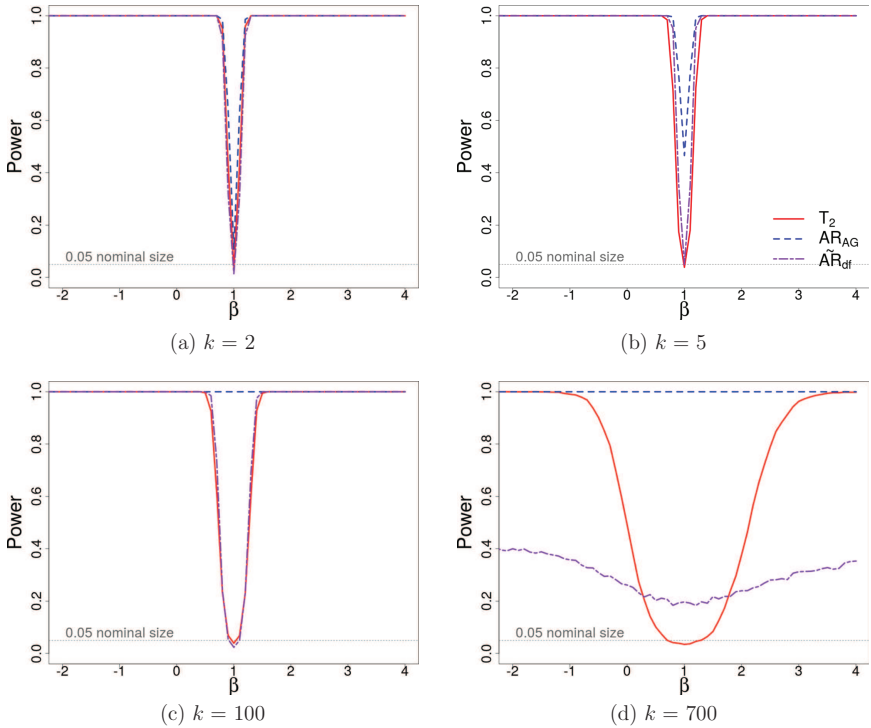
FIGURE 5. Power curves with heteroskedasticity and $\pi = 1$, $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$.

We consider two cases.

1. In the first case, we consider $n = 400$. In addition to that, we set $\rho = 0.2$, $\xi = 0.3$, $\boldsymbol{\pi}_x = \boldsymbol{\pi}_w = \sqrt{\frac{R^2}{k(1-R^2)}}\boldsymbol{\iota}_k$, $k = n/2$ and R^2 is chosen in such a way that $\frac{\sqrt{k}}{n\boldsymbol{\pi}_w'\boldsymbol{\pi}_w} = 0.1$ and $\frac{\sqrt{k}}{n\boldsymbol{\pi}_w'\boldsymbol{\pi}_w} = 1$.[11] Finally, as a plug-in estimator, we use the JIV1 estimator. The number of replications is 5,000.
2. The sample size is set to $n = 400$. Moreover, $\rho = 0.2$, $\xi \in \{0, 0.1, 0.2, 0.3\}$, $k \in \{2, 20, 200\}$, $\boldsymbol{\pi}_x = \boldsymbol{\pi}_w = \sqrt{\frac{R^2}{k(1-R^2)}}\boldsymbol{\iota}_k$ with $R^2 = 0.2$. Finally, as a plug-in estimator, we use the OLS estimator. The number of replications is 5,000.

[11] The condition $\frac{\sqrt{k}}{n\boldsymbol{\pi}_w'\boldsymbol{\pi}_w} = 1$ replicates the idea that the boundary condition $\sqrt{k}/r_{\min} \to 0$ is violated. We did run simulations also for $\frac{\sqrt{k}}{n\boldsymbol{\pi}_w'\boldsymbol{\pi}_w} = 10$ and $n = 100, 200$ finding similar results.

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(a) $k = 2$ (b) $k = 5$

(c) $k = 100$ (d) $k = 700$

FIGURE 6. Power curves with heteroskedasticity and $\pi = 1$, $H_0 : \beta_1 = \beta_{10}$.

4.2. Simulation Results

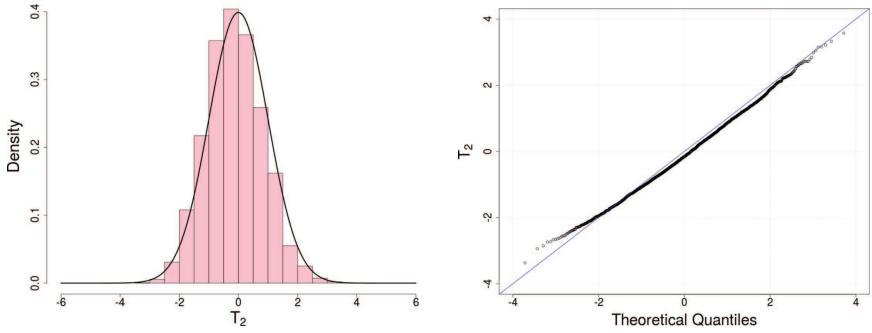
We first provide some interpretation of the simulations by separately analyzing the results on size and power. Then we discuss the behavior of T_2 when an inconsistent plug-in is used.

Size. Analyzing Figures 1 and 2, we observe that, in general, T_1 and T_2 work well in all the considered cases.[12] The $\widetilde{AR}_{df}$ test, on the other hand, works well for most of the cases but tends to overreject when k is large. Finally, as expected, the AR_{AG} test overrejects for any value of k and its performance deteriorates as k increases.

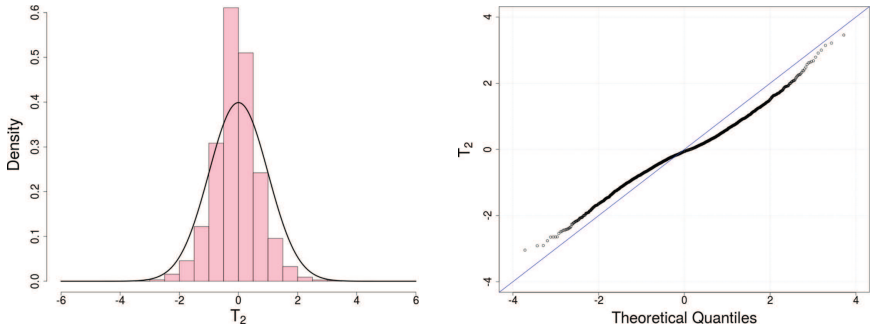
Power. The power properties of the various test statistics display some interesting patterns. When $k = 2, 5$, the T_1 and the T_2 tests along with the $\widetilde{AR}_{df}$ test are able to discriminate among alternatives (Figures 3–Figure 6(a) and (b)). To some extent, the same could be said about the case where $k = 100$ (Figures 3–Figure 6(c)). Finally, when $k = 700$, $\pi = 0.1$, the T_1 and the T_2 tests are unable

[12]It is worth noticing that, in general, for the hypothesis $H_0 : \beta_1 = \beta_{10,}$ all the tests tend to underreject for small values of k.

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(a) $n = 400$, $\frac{\sqrt{k}}{n\boldsymbol{\pi}_w'\boldsymbol{\pi}_w} = 0.1$



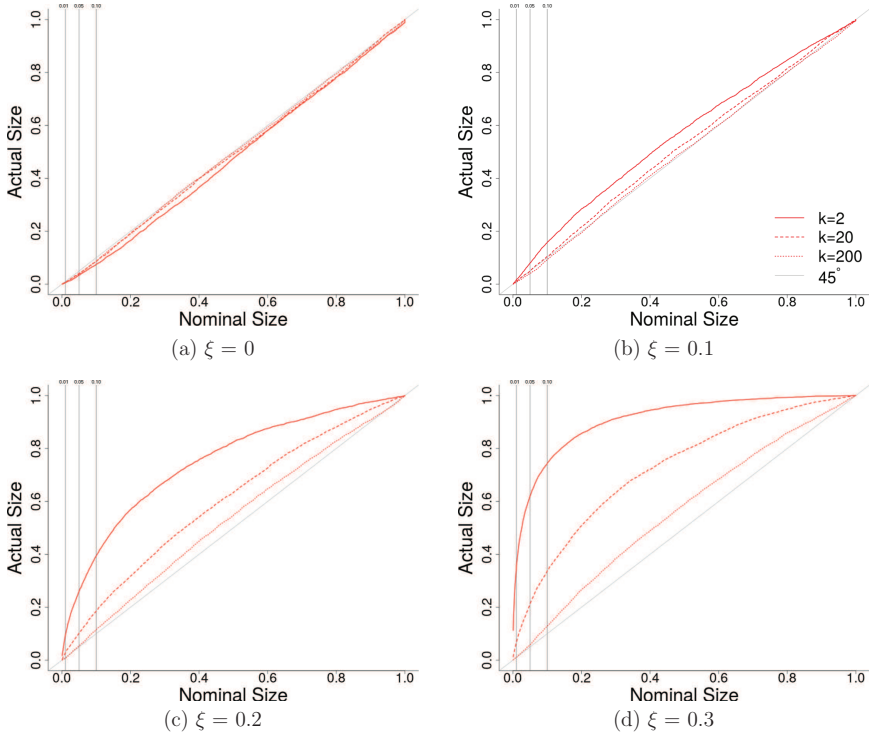
(b) $n = 400$, $\frac{\sqrt{k}}{n\boldsymbol{\pi}_w'\boldsymbol{\pi}_w} = 1$

FIGURE 7. Histograms and QQ-plots for T_2 and JIV1 plug-in. The black curve superimposed on the histograms is the standard normal distribution.

to discriminate among alternatives. More precisely, no test statistic among those considered seems to work well in this case. However, when $\pi = 1$, the T_1 and T_2 tests tend to outperform their competitors (Figures 3–Figure 6(d)). It is interesting to notice that the properties of T_1 and T_2 are affected by a trade-off between size and power with respect to k: as k grows the empirical size approaches the nominal size, but the power curves tend to get wider. This may be a problem when the instruments are weak as the tests may eventually have no power for k large. When the instruments are stronger, the effect of such a trade-off is less severe and our tests work well even in the extreme case with $k = 700$.

Over/underrejection. The comparison of the histograms and QQ–plots in Figure 7 displays how the T_2 test behaves when the boundary condition $\sqrt{k}/r_{\min} \to 0$ is violated. In particular, we notice that the (empirical) density tends to be more leptokurtic with respect to its asymptotic counterpart. This feature induces the test to underreject. On the other hand, the plots in Figure 8 show the behavior of the T_2 statistic when a slow plug-in estimator is used. We notice that the use of OLS instead of a more appropriate IV estimator causes T_2 to overreject. In particular,

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(a) $\xi = 0$ (b) $\xi = 0.1$

(c) $\xi = 0.2$ (d) $\xi = 0.3$

FIGURE 8. PP-plots for T_2 with a slow (inconsistent) plug-in.

T_2 overrejects more as ξ increases. Furthermore, we notice that the overrejection tendency is mitigated by the increased number of instruments.

Violation of the boundary condition. Figure 7 illustrates the behavior of the T_2 test with a small $\sqrt{k}/r_{\min}$ (Figure 7(a)) and a large $\sqrt{k}/r_{\min}$ (Figure 7(b)). In the latter case, the histogram of T_2 differs substantially from the one of a standard normal density, suggesting that $\sqrt{k}/r_{\min} \to 0$ is important for the asymptotic normality of our test.

5. EMPIRICAL APPLICATION

In this section, we apply our methods to the data from the National Longitudinal Survey of Young Men used by Card (1995) to estimate the returns to education. The data set includes 3,010 observations and 35 variables.[13]

We consider two different models to estimate the returns of education. Both models assume that the log of wages (*wage*) is a linear function of education

[13]The data are from the R package ivmodel of Jiang, Kang, and Small (2016).

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measured in years of schooling (*school*) and a set of exogenous variables $\boldsymbol{x}$, namely

$$\log(wage_i) = \beta school_i + \boldsymbol{x}_i'\boldsymbol{\gamma} + \varepsilon_i.$$

Similar to Kleibergen (2004), $\boldsymbol{x}$ includes a constant and binary variables for race, residence in a metropolitan area, and residence in the south of the United States as

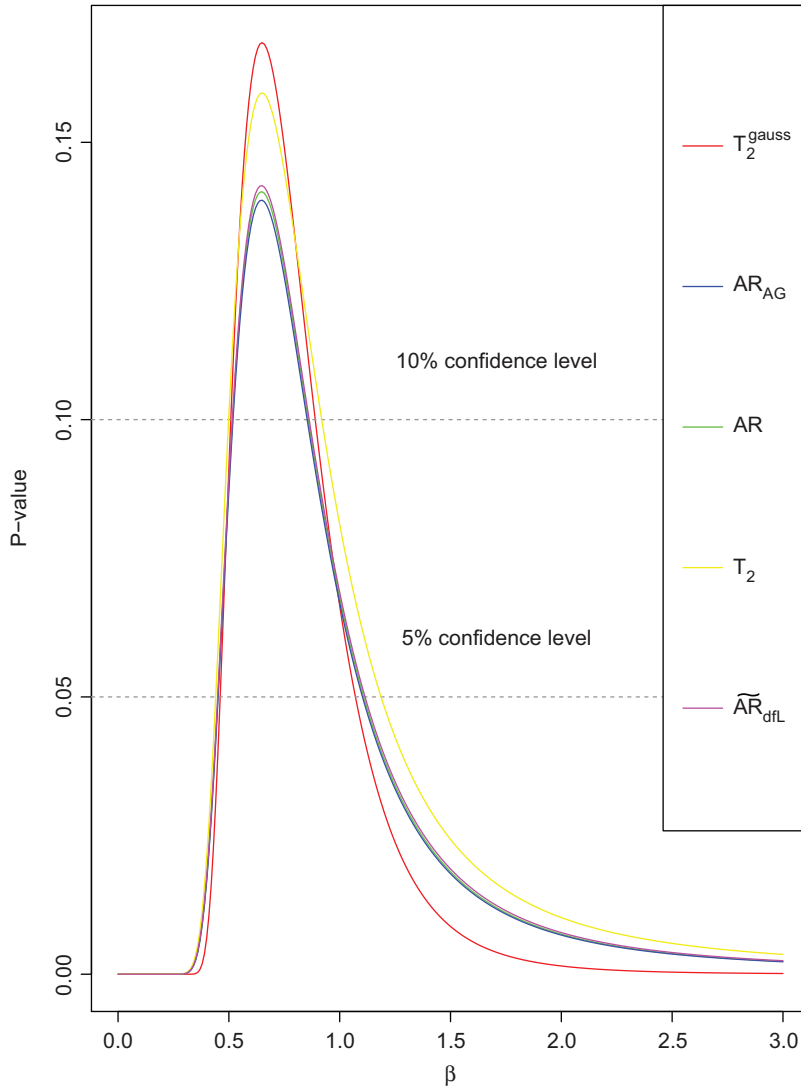


FIGURE 9. T_2, AR_{AG} and AR P-values for different values of β for the model with four instruments.

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well as IQ test score. As experience is measured simply as $age - school - 6$ in this data, we do not use it as a control variable in our models.[14] For the instruments, following once again Kleibergen (2004), in our first specifications, we use age and

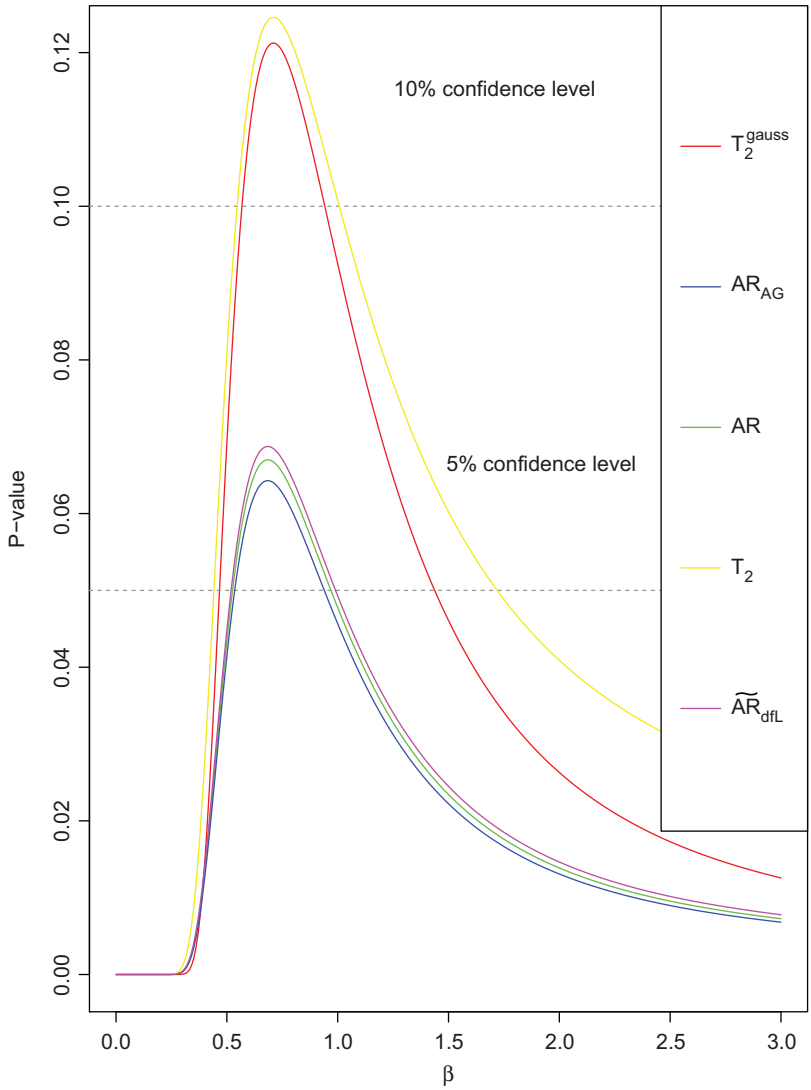


FIGURE 10. T_2, AR_{AG} and AR P-values for different values of β for the model with fourteen instruments.

[14] Another reason not to control for experience when estimating returns of education, at least in this data, is that experience is mechanically an outcome of education and it is therefore a bad control as discussed, for example, in Angrist and Pischke (2008).

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age square and two variables that indicate college proximity. The exogeneity of the college proximity instruments is somewhat questionable for several reasons. For example, areas with a high prevalence of people with high unobserved ability may be more likely to have a college nearby. Card (1995) argues that including other observable characteristics, as we do, should mitigate this issue. However, we cannot completely exclude the potential endogeneity of our instruments. In our second specification, we generate additional excluded instruments by interacting age, age squared, and the two college proximity variables with the geographical indicators and race. In the first specification, the instrument set includes four variables, while in the second it includes 14 variables.

It is very likely that the variance of the error depends on the exogenous variables, which motivates the use of our T_2 test for inference. For example, it appears very plausible that the conditional variance of the unobservables driving wages differs by college proximity, location as well as race. We run our T_2 statistic, using both the chi-square (T_2) and Gaussian (T_2^{gauss}) approximations, the AR_{AG} statistic of Anatolyev and Gospodinov (2011), the $\widetilde{AR}_{df}$ statistics of Bun et al. (2019), and the standard AR statistic to test 301 equidistant values in the interval $[0, 3]$ for the coefficient of education β. The results for the model with four instruments are reported in Figure 9. With only four excluded instruments, all the tests give very similar results, in particular, they are not able to reject values in the (approximate) interval $[.5, 1.5]$, at the 5% significance level, and in the (approximate) interval $[.5, .1]$, at the 10% significance level. The implied effects of education on wages are much higher than the one found in Card's study who, however, includes experience (which is arguably a "bad control") in his model. However, the large effects implied by our models are in line with the one found in Imbens and Rubin (1997).

The results with 14 instruments are reported in Figure 10. Probably due to the presence of heteroskedasticity, adding instruments deteriorates the performances of the AR, AR_{AG}, and $\widehat{AR}_{df}$ tests, which reject every single value of β at the 10% significance level. On the other hand, increasing the number of instruments does not seem to have a big impact on our T_2 and T_2^{gauss} tests. These results are in line with what we find in our simulation study where, with strong heteroskedasticity, the performances of the AR_{AG} deteriorates when we increase the number of instruments and the $\widehat{AR}_{df}$ tends to overreject with many instruments.

6. CONCLUSION

This paper introduces two test statistics for the parameters of a linear model in the presence of endogeneity, heteroskedasticity and many, potentially weak, instruments. The tests are easy to build as they are based on the numerator of the SJIVE estimator proposed by Bekker and Crudu (2015). We prove that, after appropriate rescaling, the limiting distribution of the test statistics are standard normal. Moreover, simulation evidence shows that, in finite samples, the proposed tests generally outperform their competitors, such as the AR tests proposed in Anatolyev and Gospodinov (2011) and in Bun et al. (2019).

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In our empirical application, the standard *AR* test and its modification by Anatolyev and Gospodinov (2011), probably due to the presence of heteroskedasticity, reject every single value chosen for the null when we increase the number of instruments from four to fourteen. On the other hand, our proposed statistic provides similar results independently of the number of instruments used.

The tests we propose can be applied broadly to any linear overidentified IV model and they are particularly appealing for the growing literature using genetic markers as instruments, see, for example, Von Hinke et al. (2016). In this literature, the number of instruments is potentially very large and the instruments are typically weak, a framework where our tests potentially outperform existing methods. Another potential field of application for our tests is the framework of Kang et al. (2016) and Windmeijer et al. (2017) where inference is carried out after a potentially large set of valid instruments is selected via LASSO.

SUPPLEMENTARY MATERIAL

To view supplementary material for this article, please visit http://dx.doi.org/10.1017/S026646662000016X

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APPENDIX

A: A PROOFS OF MAIN RESULTS

This section contains the proofs of the main theorems and some auxiliary results. In what follows it is understood that $\boldsymbol{O}$ is a conformable matrix of zeros and that the abbreviations law of large numbers (LLN), central limit theorem (CLT), and independent and identically distributed (IID) stand for law of large numbers, central limit theorem and independently and identically distributed respectively. In addition to that, $\sum_{i\neq j}$ is a double sum for $i,j = 1,\ldots,n$ that excludes the same index elements and $\sum_{i,j,k,\ell}$ replaces the quadruple sum $\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\ell=1}^{n}$. Triple sums are defined similarly.

Proof of Proposition 1. Under $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_{0,}$ we have

$$\sqrt{k}\left(\frac{AR}{k}-1\right)=\frac{\frac{1}{\sqrt{k}}\left(\frac{n-k}{k}\boldsymbol{\varepsilon}'\boldsymbol{P}\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}'\left(\boldsymbol{I}-\boldsymbol{P}\right)\boldsymbol{\varepsilon}\right)}{\frac{1}{k}\boldsymbol{\varepsilon}'\left(\boldsymbol{I}-\boldsymbol{P}\right)\boldsymbol{\varepsilon}}=\frac{n}{k}\frac{\frac{1}{\sqrt{k}}\left(\boldsymbol{\varepsilon}'\boldsymbol{P}\boldsymbol{\varepsilon}-\frac{k}{n}\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}\right)}{\frac{1}{k}\boldsymbol{\varepsilon}'\left(\boldsymbol{I}-\boldsymbol{P}\right)\boldsymbol{\varepsilon}}. \tag{A.1}$$

Note that

$$\frac{1}{\sqrt{k}}\left(\boldsymbol{\varepsilon}'\boldsymbol{P}\boldsymbol{\varepsilon}-\frac{k}{n}\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}\right)=\frac{1}{\sqrt{k}}\sum_{i\neq j}P_{ij}\varepsilon_i\varepsilon_j+\frac{1}{\sqrt{k}}\sum_{i=1}^{n}\left(P_{ii}-\frac{k}{n}\right)\varepsilon_i^2\equiv E_1+E_2. \tag{A.2}$$

We can apply the CLT from (Chao et al., 2012, Lem. A2) to the quadratic form

$$R=\sum_{i\neq j}P_{ij}\varepsilon_i\varepsilon_j$$

involved in E_1. We obtain that

$$\frac{R}{\sqrt{kW_n}}\xrightarrow{d}\mathcal{N}(0,1),$$

where

$$W_n=\frac{\mathrm{Var}[R]}{k}=\frac{2}{k}\sum_{i\neq j}P_{ij}^2\sigma_i^2\sigma_j^2$$

with the property that

$$\frac{1}{k}\mathrm{Var}[R]=\frac{2}{k}\sum_{i\neq j}P_{ij}^2\sigma_i^2\sigma_j^2\geq\frac{2\sigma^4}{k}\sum_{i\neq j}P_{ij}^2\geq\frac{2\sigma^4}{c_u},$$

(the latter inequality comes from (A.7)), which is bounded away from 0. Consequently, W_n is bounded between two positive numbers. We obtain that $E_1/\sqrt{W_n}\xrightarrow{d}\mathcal{N}(0,1)$. Regarding E_2, by the assumption $\frac{1}{\sqrt{k}}\sum_{i=1}^{n}(P_{ii}-\frac{k}{n})\sigma_i{}^2\to 0$, we have

$$\mathrm{E}[E_2]=\frac{1}{\sqrt{k}}\sum_{i=1}^{n}\left(P_{ii}-\frac{k}{n}\right)\sigma_i^2\to 0.$$

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Furthermore, by Assumption 3,

$$\mathrm{Var}[E_2] = \frac{1}{k}\sum_i \left(P_{ii} - \frac{k}{n}\right)^2 \mathrm{Var}\left[\varepsilon_i^2\right] \leq \frac{c_u}{k}\sum_{i=1}^n \left(P_{ii} - \frac{k}{n}\right)^2.$$

Using the assumption $\frac{1}{k}\sum_{i=1}^n \left(P_{ii} - \frac{k}{n}\right)^2 \to 0$, we obtain that $\mathrm{Var}[E_2] = o(1)$. Then by Chebyshev's inequality $E_2 = o_p(1)$. Therefore,

$$\frac{E_1 + E_2}{\sqrt{W_n}} \xrightarrow{d} \mathcal{N}(0,1). \tag{A.3}$$

Regarding the denominator involved in (A.1), we observe that

$$\frac{1}{k}\boldsymbol{\varepsilon}'(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{\varepsilon} = \frac{1}{k}\left(1-\frac{k}{n}\right)\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} - \frac{1}{k}\boldsymbol{\varepsilon}'\left(\boldsymbol{P}-\frac{k}{n}\boldsymbol{I}\right)\boldsymbol{\varepsilon}.$$

The second term is just the expression from (A.2) divided by $\sqrt{k}$, that is,

$$\frac{1}{k}\boldsymbol{\varepsilon}'(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{\varepsilon} = \frac{1}{k}\left(1-\frac{k}{n}\right)\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} - \frac{1}{\sqrt{k}}(E_1+E_2) = \frac{1}{k}\left(1-\frac{k}{n}\right)\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} + O_p\left(\frac{1}{\sqrt{k}}\right).$$

Using Assumption 3 and the LLN, using the notation

$$\overline{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n \sigma_i^2,$$

we have that

$$\frac{1}{n}\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon} - \overline{\sigma}_n^2 = O_p\left(\frac{1}{\sqrt{k}}\right). \tag{A.4}$$

Consequently,

$$\frac{1}{k}\boldsymbol{\varepsilon}'(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{\varepsilon} = \frac{n}{k}\left(1-\frac{k}{n}\right)\overline{\sigma}_n^2 + O_p\left(\frac{1}{\sqrt{k}}\right).$$

Now, from equation (A.1) and the fact that $\frac{n}{k}\left(1-\frac{k}{n}\right)\overline{\sigma}_n^2$ is bounded between two positive numbers, we have

$$\begin{aligned}\sqrt{k}\left(\frac{AR}{k}-1\right) &= \frac{n}{k}\frac{\frac{1}{\sqrt{k}}\left(\boldsymbol{\varepsilon}'\boldsymbol{P}\boldsymbol{\varepsilon} - \frac{k}{n}\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}\right)}{\frac{n}{k}\left(1-\frac{k}{n}\right)\overline{\sigma}_n^2} + \frac{n}{k}\frac{\frac{1}{\sqrt{k}}\left(\boldsymbol{\varepsilon}'\boldsymbol{P}\boldsymbol{\varepsilon} - \frac{k}{n}\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}\right)}{\frac{n}{k}\left(1-\frac{k}{n}\right)\overline{\sigma}_n^2}\left(\frac{\frac{n}{k}\left(1-\frac{k}{n}\right)\overline{\sigma}_n^2}{\frac{1}{k}\boldsymbol{\varepsilon}'(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{\varepsilon}} - 1\right)\\ &= \frac{E_1+E_2}{\left(1-\frac{k}{n}\right)\overline{\sigma}_n^2} + o_p(1).\end{aligned}$$

Therefore, collecting the above results, we obtain that

$$\left(1-\frac{k}{n}\right)\frac{\overline{\sigma}_n^2}{\sqrt{W_n}}\sqrt{k}\left(\frac{AR}{k}-1\right) = \frac{E_1+E_2}{\sqrt{W_n}} + o_p(1),$$

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which by (A.3) implies that

$$\left(1-\frac{k}{n}\right)\frac{\overline{\sigma}_n^2}{\sqrt{W_n}}\sqrt{k}\left(\frac{AR}{k}-1\right)\xrightarrow{d}\mathcal{N}(0,1). \tag{A.5}$$

Since we assume that $\lim_{n\to\infty}\overline{\sigma}_n^2=\sigma_0^2$ and $\lim_{n\to\infty}W_n=W_0$ exist, we obtain the result. □

In the proof of Theorem 1, we use the following CLT, which, as argued by Bekker and Crudu (2015, appendix A.4) can be proved in a way similar to Lem. A2 from Chao et al. (2012).

LEMMA A.1. *Consider the quadratic form* $Q=\sum_{i\neq j}C_{ij}\varepsilon_i\varepsilon_j$, *where* C_{ij} *is the* (i,j) *element of matrix* $\boldsymbol{C}$ *that is symmetric and has zero main diagonal elements. Suppose that there is a matrix* $\boldsymbol{P}$ *that is symmetric, idempotent,* $P_{ii}\leq c_u<1$, $|C_{ij}|\leq c_u|P_{ij}|$ *for any* $i\neq j$, *and rank*$(\boldsymbol{P})=k$, *where* $k\to\infty$ *as* $n\to\infty$, *and the following properties hold:* (*a*) $\mathrm{E}[\varepsilon_i]=0$ *and* $\varepsilon_1,\ldots,\varepsilon_n$ *are independent;* (*b*) $\mathrm{E}\left[\varepsilon_i^4\right]<\infty$; (*c*) $\frac{1}{k}\mathrm{Var}[Q]\geq c_u>0$. *Then,*

$$\frac{Q}{\sqrt{\mathrm{Var}[Q]}}\xrightarrow{d}\mathcal{N}(0,1).$$

LEMMA A.2. *Let* $\widehat{V}\left(\boldsymbol{\beta}_0\right)=\frac{2}{k}\boldsymbol{\varepsilon}_0^{(2)\prime}\boldsymbol{C}^{(2)}\boldsymbol{\varepsilon}_0^{(2)}$. *If Assumptions 1, 3 hold,* $\widehat{V}\left(\boldsymbol{\beta}_0\right)-V_n=O_p\left(\frac{1}{\sqrt{k}}\right)$; *consequently* $\widehat{V}\left(\boldsymbol{\beta}_0\right)-V_n\xrightarrow{p}0$, *where*

$$V_n=\frac{2}{k}\sum_{i=1}^{n}\sum_{j=1}^{n}C_{ij}^2\sigma_i^2\sigma_j^2.$$

(For a proof see the online Supplemental Appendix A.)

Proof of Theorem 1. Under the null hypothesis, we have

$$\mathrm{E}\left[\boldsymbol{\varepsilon}_0'\boldsymbol{C}\boldsymbol{\varepsilon}_0\right]=0,$$
$$\mathrm{Var}\left[\boldsymbol{\varepsilon}_0'\boldsymbol{C}\boldsymbol{\varepsilon}_0\right]=\mathrm{E}\left[\left(\boldsymbol{\varepsilon}_0'\boldsymbol{C}\boldsymbol{\varepsilon}_0\right)^2\right]=2\sum_{i=1}^{n}\sum_{j=1}^{n}C_{ij}^2\sigma_i^2\sigma_j^2\equiv kV_n.$$

We verify the conditions of the CLT stated in Lemma A.1 for $\boldsymbol{C}$ and $\boldsymbol{P}$ defined in Section 2. The properties of $\boldsymbol{C}$ and $\boldsymbol{P}$ hold by definition, Assumption 1 and the fact that $|C_{ij}|=\frac{|P_{ij}|}{2}\left(\frac{1}{1-P_{ii}}+\frac{1}{1-P_{jj}}\right)\leq c_u|P_{ij}|$ for any i,j, (see the proof of Lemma A.2 in the Supplemental Appendix).

Furthermore, (*a*) is clearly satisfied; (*b*) is satisfied due to Assumption 3. Regarding (*c*), note that by Assumption 2,

$$\frac{1}{k}\mathrm{Var}[Q]\equiv V_n=\frac{2}{k}\sum_{i=1}^{n}\sum_{j=1}^{n}C_{ij}^2\sigma_i^2\sigma_j^2\geq\frac{2\underline{\sigma}^4}{k}\sum_{i=1}^{n}\sum_{j=1}^{n}C_{ij}^2,$$

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where

$$\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}^2 = \sum_{i\neq j} \frac{P_{ij}^2}{4}\left(\frac{1}{1-P_{ii}}+\frac{1}{1-P_{jj}}\right)^2 \geq \sum_{i\neq j}\frac{P_{ij}^2}{4}(1+1)^2 = \sum_{i\neq j} P_{ij}^2$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} P_{ij}^2 - \sum_{i=1}^{n} P_{ii}^2 = tr(\boldsymbol{P}) - \sum_{i=1}^{n} P_{ii}^2 = k - \sum_{i=1}^{n} P_{ii}^2. \quad \textbf{(A.6)}$$

By Assumption 1,

$$\sum_{i=1}^{n} P_{ii}^2 \leq \max P_{ii} \sum_{i=1}^{n} P_{ii} \leq (1-1/c_u)\, tr(\boldsymbol{P}) = (1-1/c_u)\, k. \quad \textbf{(A.7)}$$

So

$$\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}^2 \geq k/c_u,$$

therefore,

$$\frac{1}{k}\mathrm{Var}[Q] \geq \frac{2\underline{\sigma}^4}{c_u},$$

which is bounded away from 0. In this case, we can apply the CLT in Lemma A.1 and complete the proof by using Lemma A.2. □

For the proof of Theorem 2, we need the following results (for proofs see Supplemental Appendix A).

LEMMA A.3. *Let* $\widehat{V}\left(\widetilde{\boldsymbol{\beta}}\right) = \frac{2}{k}\widetilde{\boldsymbol{\varepsilon}}^{(2)\prime}\boldsymbol{C}^{(2)}\widetilde{\boldsymbol{\varepsilon}}^{(2)}$. *If* $\widetilde{\boldsymbol{\beta}} \xrightarrow{p} \boldsymbol{\beta}$ *and Assumptions 1, 3 hold, then* $\widehat{V}\left(\widetilde{\boldsymbol{\beta}}\right) - V_n \xrightarrow{p} 0$.

LEMMA A.4. *Under Assumptions 1, 3*

1. $\mathrm{E}\left[\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{X}_2\right] = \boldsymbol{H}_{22}$, $\mathrm{Var}\left[\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{X}_2\right] \leq c_u\boldsymbol{H}_{22} + c_u k\boldsymbol{I}_{g_2} + c_u r_{\max}\boldsymbol{I}_{g_2}$,
2. $\mathrm{E}\left[\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}\right] = \boldsymbol{0}$, $\mathrm{Var}\left[\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}\right] \leq c_u\boldsymbol{H}_{22} + c_u k\boldsymbol{I}_{g_2}$.

Before proceeding to the proof of Theorem 2, we present some general facts that are used in the proofs of several results below. Consider a plug-in estimator $\widetilde{\boldsymbol{\beta}}_2$ of $\boldsymbol{\beta}_2$ and, as above, let $\widetilde{\boldsymbol{\beta}} = \left(\boldsymbol{\beta}_1', \widetilde{\boldsymbol{\beta}}_2'\right)'$. Notice that under $H_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_{10}$ it holds that

$$\boldsymbol{X}\left(\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \boldsymbol{X}_1\left(\boldsymbol{\beta}_{10} - \boldsymbol{\beta}_1\right) + \boldsymbol{X}_2\left(\widetilde{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2\right) = \boldsymbol{X}_2\left(\widetilde{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2\right),$$

so the residual vector can be written as

$$\widetilde{\boldsymbol{\varepsilon}} = \boldsymbol{y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}} = \boldsymbol{\varepsilon} - \boldsymbol{X}\left(\widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \boldsymbol{\varepsilon} - \boldsymbol{X}_2\left(\widetilde{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2\right).$$

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The statistic T_2 can be rewritten as

$$T_2 = \frac{1}{\sqrt{k}} \frac{\boldsymbol{\varepsilon}' \boldsymbol{C} \boldsymbol{\varepsilon}}{\sqrt{\widehat{V}(\boldsymbol{\beta})}} \left(\frac{\sqrt{\widehat{V}(\boldsymbol{\beta})}}{\sqrt{\widehat{V}(\widetilde{\boldsymbol{\beta}})}} - 1 \right) + \frac{1}{\sqrt{k}} \frac{\Delta}{\sqrt{\widehat{V}(\widetilde{\boldsymbol{\beta}})}} + \frac{1}{\sqrt{k}} \frac{\boldsymbol{\varepsilon}' \boldsymbol{C} \boldsymbol{\varepsilon}}{\sqrt{\widehat{V}(\beta)}} \tag{A.8}$$

$$\equiv B_1 + B_2 + B_3,$$

where

$$\Delta = (\widetilde{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2)' \boldsymbol{X}_2' \boldsymbol{C} \boldsymbol{X}_2 (\widetilde{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2) - 2 (\widetilde{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2)' \boldsymbol{X}_2' \boldsymbol{C} \boldsymbol{\varepsilon}. \tag{A.9}$$

The first term is equal to

$$B_1 = \frac{1}{\sqrt{k}} \frac{\boldsymbol{\varepsilon}' \boldsymbol{C} \boldsymbol{\varepsilon}}{\sqrt{\widehat{V}(\boldsymbol{\beta})}} \frac{\sqrt{\widehat{V}(\boldsymbol{\beta})} - \sqrt{\widehat{V}(\widetilde{\boldsymbol{\beta}})}}{\sqrt{\widehat{V}(\widetilde{\boldsymbol{\beta}})}},$$

where from Lemma A.3 and the consistency of the plug-in it follows that $\sqrt{\widehat{V}(\boldsymbol{\beta})} - \sqrt{\widehat{V}(\widetilde{\boldsymbol{\beta}})} = o_p(1)$, while since V_n is bounded away from 0 by Assumption 3, it follows that $1/\sqrt{\widehat{V}(\widetilde{\boldsymbol{\beta}})} = O_p(1)$. Theorem 1 implies that $\frac{1}{\sqrt{k}} \frac{\boldsymbol{\varepsilon}' \boldsymbol{C} \boldsymbol{\varepsilon}}{\sqrt{\widehat{V}(\boldsymbol{\beta})}} = O_p(1)$, so $B_1 = o_p(1)$. Regarding B_3, from Theorem 1, we have that $B_3 \to_d \mathcal{N}(0,1)$.

Consequently, if the plug-in estimator $\widetilde{\boldsymbol{\beta}}$ is consistent then under Assumptions 1, 3, we have $B_1 = o_p(1)$ and $B_3 \to_d \mathcal{N}(0,1)$. In order to derive the asymptotic distribution of T_2 we need to study the term B_2.

Proof of Theorem 2. Note that the first term from Δ in (A.9) is

$$(\widetilde{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2)' \boldsymbol{X}_2' \boldsymbol{C} \boldsymbol{X}_2 (\widetilde{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2) = (\widetilde{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2)' \boldsymbol{H}_{22}^{1/2} \boldsymbol{H}_{22}^{-1/2} \boldsymbol{X}_2' \boldsymbol{C} \boldsymbol{X}_2 \boldsymbol{H}_{22}^{-1/2} \boldsymbol{H}_{22}^{1/2} (\widetilde{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2). \tag{A.10}$$

First, we show that $\boldsymbol{H}_{22}^{-1/2} \boldsymbol{X}_2' \boldsymbol{C} \boldsymbol{X}_2 \boldsymbol{H}_{22}^{-1/2} \xrightarrow{p} \boldsymbol{I}_{g_2}$. Lemma A.4 implies that

$$\mathrm{E}\left[\boldsymbol{H}_{22}^{-1/2} \boldsymbol{X}_2' \boldsymbol{C} \boldsymbol{X}_2 \boldsymbol{H}_{22}^{-1/2} \right] = \boldsymbol{I}_{g_2} \tag{A.11}$$

and

$$\mathrm{Var}\left[\boldsymbol{H}_{22}^{-1/2} \boldsymbol{X}_2' \boldsymbol{C} \boldsymbol{X}_2 \boldsymbol{H}_{22}^{-1/2} \right] \leq \frac{1}{r_{\min}} \boldsymbol{H}_{22}^{-1/2} \left(c_u \boldsymbol{H}_{22} + c_u k \boldsymbol{I}_{g_2} + c_u r_{\max} \boldsymbol{I}_{g_2} \right) \boldsymbol{H}_{22}^{-1/2}$$

$$= \frac{1}{r_{\min}} \left(c_u + c_u \frac{k}{r_{\min}} + c_u \frac{r_{\max}}{r_{\min}} \right) \boldsymbol{I}_{g_2},$$

where the inequality holds due to $\boldsymbol{H}_{22}^{-1} \leq \frac{1}{r_{\min}} \boldsymbol{I}_{g_2}$. Therefore, Assumption 4 (many strong instruments case) implies that $\mathrm{Var}\left[\boldsymbol{H}_{22}^{-1/2} \boldsymbol{X}_2' \boldsymbol{C} \boldsymbol{X}_2 \boldsymbol{H}_{22}^{-1/2} \right] = O\left(\frac{1}{k} \right)$ while Assumption 5 (many weak instruments case) implies that $\mathrm{Var}\left[\boldsymbol{H}_{22}^{-1/2} \boldsymbol{X}_2' \boldsymbol{C} \boldsymbol{X}_2 \boldsymbol{H}_{22}^{-1/2} \right] = o(1)$. In either case, we obtain that $\mathrm{Var}\left[\boldsymbol{H}_{22}^{-1/2} \boldsymbol{X}_2' \boldsymbol{C} \boldsymbol{X}_2 \boldsymbol{H}_{22}^{-1/2} \right] \to 0$, which together with (A.11) implies that $\boldsymbol{H}_{22}^{-1/2} \boldsymbol{X}_2' \boldsymbol{C} \boldsymbol{X}_2 \boldsymbol{H}_{22}^{-1/2} \xrightarrow{p} \boldsymbol{I}_{g_2}$.

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Note that under Assumption 4 $\boldsymbol{H}_{22}^{1/2}\left(\widetilde{\boldsymbol{\beta}}_2-\boldsymbol{\beta}_2\right)=O_p(1)$ while under Assumption 5 $\frac{1}{\sqrt{k}}\boldsymbol{H}_{22}\left(\widetilde{\boldsymbol{\beta}}_2-\boldsymbol{\beta}_2\right)=O_p(1)$ (see Sect. 4 in Bekker and Crudu, 2015). Therefore, under either Assumption 4 or Assumption 5, from (A.10) we conclude that

$$\frac{1}{\sqrt{k}}\left(\widetilde{\boldsymbol{\beta}}_2-\boldsymbol{\beta}_2\right)'\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{X}_2\left(\widetilde{\boldsymbol{\beta}}_2-\boldsymbol{\beta}_2\right)=o_p(1). \tag{A.12}$$

The second term from Δ in (A.9) involves

$$\left(\widetilde{\boldsymbol{\beta}}_2-\boldsymbol{\beta}_2\right)'\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}=\left(\widetilde{\boldsymbol{\beta}}_2-\boldsymbol{\beta}_2\right)'\boldsymbol{H}_{22}\boldsymbol{H}_{22}^{-1}\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}.$$

Lemma A.4 implies that

$$\mathrm{E}\left[\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}\right]=\boldsymbol{0} \tag{A.13}$$

and

$$\mathrm{Var}\left[\boldsymbol{H}_{22}^{-1}\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}\right]\leq c_u\left(\frac{1}{r_{\min}}+\frac{k}{r_{\min}^2}\right)\boldsymbol{I}_{g_2}, \tag{A.14}$$

where the latter inequality is due to $\boldsymbol{H}_{22}^{-1}\leq\frac{1}{r_{\min}}\boldsymbol{I}_{g_2}$. We also obtain that

$$\mathrm{Var}\left[\frac{1}{\sqrt{k}}\boldsymbol{H}_{22}^{-1/2}\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}\right]\leq c_u\left(\frac{1}{k}+\frac{1}{r_{\min}}\right)\boldsymbol{I}_{g_2}. \tag{A.15}$$

Under Assumption 4 (many strong instruments case), we get $\mathrm{Var}\left[\frac{1}{\sqrt{k}}\boldsymbol{H}_{22}^{-1/2}\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}\right]=O\left(\frac{1}{k}\right)$, which together with (A.13) implies that $\frac{1}{\sqrt{k}}\boldsymbol{H}_{22}^{-1/2}\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}=o_p(1)$. Since $\boldsymbol{H}_{22}^{1/2}\left(\widetilde{\boldsymbol{\beta}}_2-\boldsymbol{\beta}_2\right)=O_p(1)$ holds, we obtain

$$\frac{1}{\sqrt{k}}\left(\widetilde{\boldsymbol{\beta}}_2-\boldsymbol{\beta}_2\right)'\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}=o_p(1).$$

Under Assumption 5 (many weak instruments case), (A.14) implies $\mathrm{Var}\left[\boldsymbol{H}_{22}^{-1}\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}\right]=o(1)$, which together with (A.13) implies that $\boldsymbol{H}_{22}^{-1}\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}=o_p(1)$. Since $\frac{1}{\sqrt{k}}\boldsymbol{H}_{22}\left(\widetilde{\boldsymbol{\beta}}_2-\boldsymbol{\beta}_2\right)=O_p(1)$ holds, we obtain

$$\frac{1}{\sqrt{k}}\left(\widetilde{\boldsymbol{\beta}}_2-\boldsymbol{\beta}_2\right)'\boldsymbol{X}_2'\boldsymbol{C}\boldsymbol{\varepsilon}=o_p(1).$$

Regarding B_3, from Theorem 1, we have that $B_3\xrightarrow{d}\mathcal{N}(0,1)$. □

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Proof of Theorem 3. Note that

$$\sum_{i\neq j} C_{ij}\varepsilon_i\varepsilon_j = \sum_{i\neq j}\frac{P_{ij}}{2}\left(\frac{1}{1-P_{ii}}+\frac{1}{1-P_{jj}}\right)\varepsilon_i\varepsilon_j = (1+o(1))\sum_{i\neq j} P_{ij}\varepsilon_i\varepsilon_j$$

as $\max_i P_{ii}\to 0$. Furthermore,

$$\sum_{i\neq j} C_{ij}\varepsilon_i\varepsilon_j = (1+o(1))\boldsymbol{\varepsilon}'\boldsymbol{P}\boldsymbol{\varepsilon} - (1+o(1))\sum_{i=1}^{n} P_{ii}\varepsilon_i^2.$$

By assumptions (*ii*) and (*iv*), $\boldsymbol{\varepsilon}'\boldsymbol{P}\boldsymbol{\varepsilon}\to_d \sigma^2\chi_k^2$. Moreover, by independence of ε_i, Assumption 3 and the properties of P_{ii}

$$\mathrm{E}\left[\left(\sum_{i=1}^{n} P_{ii}\varepsilon_i^2 - k\sigma^2\right)^2\right]\to 0,$$

which implies $\sum_{i=1}^n P_{ii}\varepsilon_i^2\to_p k\sigma^2$ (see Chao et al., 2014). Consider now

$$V_n = \frac{2}{k}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}^2\sigma_i^2\sigma_j^2 = \frac{2\sigma^4}{k}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}^2 = \frac{2\sigma^4}{k}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{P_{ij}^2}{4}\left(\frac{1}{1-P_{ii}}+\frac{1}{1-P_{jj}}\right)^2.$$

Since $\sum_{i=1}^n\sum_{j=1}^n P_{ij}^2 = k$ is fixed and $\max_i P_{ii}\to 0$, $V_n\to 2\sigma^4$. Hence, by Lemma A.2, $\widehat{V}(\boldsymbol{\beta}_0)\to_p 2\sigma^4$. Finally,

$$T_1 = \frac{1}{\sqrt{k}}\frac{\boldsymbol{\varepsilon}_0'\boldsymbol{C}\boldsymbol{\varepsilon}_0}{\sqrt{\frac{2}{k}\boldsymbol{\varepsilon}_0^{(2)'}\boldsymbol{C}^{(2)}\boldsymbol{\varepsilon}_0^{(2)}}}\to_d \frac{\chi_k^2-k}{\sqrt{2k}}.$$

Thus, $\sqrt{2k}T_1+k\to_d\chi_k^2$. Let us consider now the T_2 statistic. Notice that $\widetilde{\boldsymbol{\varepsilon}} = \boldsymbol{y}-\boldsymbol{X}_1\boldsymbol{\beta}_{10}-\boldsymbol{X}_2\widetilde{\boldsymbol{\beta}}_2$ where $\widetilde{\boldsymbol{\beta}}_2 = \left(\boldsymbol{X}_2'\boldsymbol{P}\boldsymbol{X}_2\right)^{-1}\boldsymbol{X}_2'\boldsymbol{P}(\boldsymbol{y}-\boldsymbol{X}_1\boldsymbol{\beta}_{10})$, the two-stage least-squares under the null. By standard manipulations, CLT and Slutsky's theorem, we get

$$\widetilde{\boldsymbol{\varepsilon}}'\boldsymbol{P}\widetilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}'\left(\boldsymbol{P}-\boldsymbol{P}\boldsymbol{X}_2(\boldsymbol{X}_2'\boldsymbol{P}\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{P}\right)\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}'\boldsymbol{Z}^*\left(\boldsymbol{I}_k-\boldsymbol{P}_{\boldsymbol{X}_2'\boldsymbol{Z}^*}\right)\boldsymbol{Z}^{*'}\boldsymbol{\varepsilon}\to_d\sigma^2\chi_{k-g_2}^2,$$

where $\boldsymbol{Z}^* = \boldsymbol{Z}(\boldsymbol{Z}'\boldsymbol{Z})^{-1/2}$. Using the results in Lemma A.3, we get $\sum_{i=1}^n P_{ii}\widetilde{\varepsilon}_i^2\to_p k\sigma^2$ and $\widehat{V}(\widetilde{\boldsymbol{\beta}})\to_p 2\sigma^4$. So, by the usual standard arguments $\sqrt{2k}T_2+k\to_d\chi_{k-g_2}^2$. □

Proof of Corollary 1. The proof mimics that of Thm. 1 in Chao et al. (2014). Let $q_\alpha^{\chi_k^2}$ be the generic αth quantile of the chi-square distribution with k degrees of freedom. As $k\to\infty$, $\frac{q_\alpha^{\chi_k^2}-k}{\sqrt{2k}}\to z_\alpha$, where z_α is the generic αth quantile of the standard normal distribution. This proves parts (*i*) and (*ii*). With respect to part (*iii*) and part (*iv*), notice that $\sqrt{\frac{k-g_2}{k}}\frac{q_\alpha^{\chi_{k-g_2}^2}-(k-g_2)}{\sqrt{2(k-g_2)}}-\frac{g_2}{\sqrt{2k}}\to z_\alpha$ as $k\to\infty$. □

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