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Lyapunov optimization for non-generic one-dimensional expanding Markov maps

MAO SHINODA and HIROKI TAKAHASI

Department of Mathematics, Keio University, Yokohama, 223-8522, Japan
(e-mail: shinoda-mao@z3.keio.jp, hiroki@math.keio.ac.jp)

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To the memory of Kenichi Shiraiwa (1928–2017)

Abstract. For a non-generic, yet dense subset of C^1 expanding Markov maps of the interval we prove the existence of uncountably many Lyapunov optimizing measures which are ergodic, fully supported and have positive entropy. These measures are equilibrium states for some Hölder continuous potentials. We also prove the existence of another non-generic dense subset for which the optimizing measure is unique and supported on a periodic orbit. A key ingredient is a new C^1 perturbation theorem which allows us to interpolate between expanding Markov maps and the shift map on a finite number of symbols.

Key words: expanding Markov map, Lyapunov optimizing measure, non-generic property
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1. *Introduction*

Ergodic optimization aims to describe invariant probability measures of a dynamical system which optimize the integral of a given 'performance' function. In its basic form, given a continuous self-map T of a compact metric space M and a real-valued continuous function ϕ on M, one seeks *ϕ-minimizing measures*, i.e., measures in $\mathcal{M}(T)$ which attain the infimum

$$\inf\left\{\int \phi\, d\mu \colon \mu \in \mathcal{M}(T)\right\},$$

where $\mathcal{M}(T)$ denotes the set of T-invariant Borel probability measures. A root of this theory is found in the works of Mather [**20**] and Mañé [**18**, **19**] on the dynamics of the Euler–Lagrange flow: orbits with prescribed properties can be obtained by considering ergodic invariant probability measures which minimize the integral of the Lagrangian, and the orbits are obtained as typical points for these measures (called *action minimizing measures*, see Sorrentino [**26**]).

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A general belief is that the minimizing measure is unique and supported on a periodic orbit for 'most' performance functions. The meaning of 'most' is the genericity in the sense of Baire's category theorem, the typicality in the sense of the Lebesgue measure in parameter space and so on. For instance, see Mañé's conjecture [**19**] and that of Yuan and Hunt [**28**] after numerical observations by Hunt and Ott [**12**, **13**]. For the doubling map $x \mapsto 2x \pmod 1$ and a parametrized performance function $\cos(2\pi(x-\theta))$ $(\theta \in \mathbb{R})$, Bousch [**2**] proved that for all θ the minimizing measure is unique and is supported on a subset of a semi-circle. Further, he proved that for Lebesgue almost every θ the corresponding minimizing measure is supported on a periodic orbit. Contreras, Lopes and Thieullen [**10**] studied C^1 expanding maps of the circle and performance functions in the Banach space $C^{0,\alpha}$ of α-Hölder continuous functions. They considered its subspace consisting of those $\phi \in C^{0,\alpha}$ for which $|\phi(x)-\phi(y)| = o(|x-y|^{\alpha})$ and showed that for an open dense subset of this subspace the minimizing measure is unique and supported on a periodic orbit. For transitive and 'expanding' dynamical systems, the uniqueness of the minimizing measure was established for open dense subsets of suitable separable Banach spaces; see Bousch [**3**] and Contreras [**9**]. Quas and Siefken [**22**] proved the uniqueness of the minimizing measure for the one-sided full shift and an open dense subset of a certain non-separable Banach space. In all of these three results, the unique minimizing measure is supported on a periodic orbit. For generic functions in the space C^0 of continuous functions the uniqueness of the minimizing measure still holds, but the measure is fully supported; see Bousch and Jenkinson [**4**] and Jenkinson [**15**]. Brémont [**7**] proved that for generic C^0 functions, any minimizing measure has zero entropy.

In these developments, the non-uniqueness of the minimizing measure was considered somewhat irrelevant, as it is considered to be a non-generic property. However, the non-uniqueness of the action minimizing measure is a universal phenomenon (the action minimizing measure is, in a sense, a generalization of the KAM tori [**26**]). In the context of the thermodynamic formalism, the study of the non-uniqueness of the minimizing measure has a connection with the asymptotic behavior of Gibbs measures as the temperature drops to zero; see Baraviera, Leplaideur and Lopes [**1**]. Despite its importance, the non-uniqueness of the minimizing measure has not yet received an adequate deal of attention.

In this paper we treat expanding Markov maps of the interval and show that the uniqueness of the Lyapunov optimizing measure fails in a severe way. Let $p \geq 2$ be an integer and $\{\alpha_i\}_{i=0}^{p-1}$, $\{\beta_i\}_{i=0}^{p-1}$ sequences in $[0, 1]$ such that

$$0 = \alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_{p-1} = 1.$$

Put $X = \bigcup_{i=0}^{p-1}[\alpha_i, \beta_i]$. Denote by $\mathscr{E}$ the set of C^1 maps on X such that the following holds for every $f \in \mathscr{E}$:

(E1) f maps each interval $[\alpha_i, \beta_i]$, $i \in \{0, \ldots, p-1\}$, diffeomorphically onto $[0, 1]$;

(E2) there exist constants $c > 0$, $\lambda > 1$ such that for every $n \geq 1$ and every $x \in \bigcap_{k=0}^{n-1} f^{-k}(X)$, $|Df^n(x)| \geq c\lambda^n$.

We endow $\mathscr{E}$ with the C^1 topology given by the norm

$$\|f\|_{C^1} = \sup_X |f| + \sup_X |Df|.$$

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The space $\mathscr{E}$ is an open subset of C^1 functions on X (see Lemma 2.2 to check the openness of (E2)) and hence becomes a (non-complete) Baire space.

Define

$$\Lambda(f) = \bigcap_{n\geq 0} f^{-n}(X).$$

Restricting f to $\Lambda(f)$, we obtain a dynamical system which is also denoted by f with a slight abuse of notation. Then $\Lambda(f)$ is a Cantor set with a Markov partition given by the collection $\{[\alpha_i, \beta_i]\}_{i=0}^{p-1}$ of intervals which topologically conjugates f to the left shift on p symbols.

Put

$$\chi_{\inf}(f) = \inf\left\{\int \log|Df|\, d\mu \colon \mu \in \mathcal{M}(f)\right\}$$

and

$$\chi_{\sup}(f) = \sup\left\{\int \log|Df|\, d\mu \colon \mu \in \mathcal{M}(f)\right\}.$$

Since $\mathcal{M}(f)$ is compact and $\nu \in \mathcal{M}(f) \mapsto \int \log|Df|\, d\nu$ is continuous, the infimum and supremum are attained. A measure $\mu \in \mathcal{M}(f)$ is *Lyapunov minimizing* if

$$\int \log|Df|\, d\mu = \chi_{\inf}.$$

Lyapunov maximizing measures are defined similarly, with $\chi_{\inf}$ replaced by $\chi_{\sup}$.

The notion of Lyapunov optimizing measures was introduced by Contreras, Lopes and Thieullen [**10**]. They showed that for an open dense subset of the space $\bigcup_{\beta>\alpha} C^{1+\beta}$ of expanding maps of the circle in the $C^{1+\alpha}$ topology, the Lyapunov minimizing measure is unique and supported on a periodic orbit. For a generic C^1 expanding map of the circle, Jenkinson and Morris [**16**] proved that the Lyapunov minimizing measure is unique and has zero entropy. See Morita and Tokunaga [**21**] and Tokunaga [**27**] for extensions of the results of [**16**] to higher dimension. With the method of [**16**] one can show that for generic maps in $\mathscr{E}$ the Lyapunov minimizing measure is unique, is fully supported and has zero entropy. In the realm of non-genericity the structure of Lyapunov minimizing measures is in contrast.

THEOREM A. *There exists a dense subset $\mathscr{A}$ of $\mathscr{E}$ such that the following holds for every $f \in \mathscr{A}$:*

- $\chi_{\inf}(f) \neq \chi_{\sup}(f)$*;*
- *there exist uncountably many Lyapunov minimizing measures of f which are ergodic, fully supported and have positive entropy;*
- $\log|Df|$ *is not Hölder continuous.*

Theorem A has been inspired by the following result of the first-named author [**24**] on the ergodic optimization for the subshift of finite type. For an integer $p \geq 2$, let $\Sigma_p = \{0, \ldots, p-1\}^{\mathbb{Z}_{\geq 0}}$ denote the one-sided shift space on p symbols, endowed with the product topology of the discrete topology on $\{0, \ldots, p-1\}$. Let $\sigma : \Sigma_p \circlearrowleft$ denote the left shift: if $\underline{a}, \underline{b} \in \Sigma_p$, $\underline{a} = \{a_n\}_{n\geq 0}$, $\underline{b} = \{b_n\}_{n\geq 0}$ and $\sigma(\underline{a}) = \underline{b}$, then $b_n = a_{n+1}$ holds for every $n \geq 0$. For a $p \times p$ matrix $A = (A_{ij})$ whose every entry is 0 or 1, define

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$\Sigma_A = \{\{a_n\}_{n\geq 0} \in \Sigma_p \colon A_{a_n a_{n+1}} = 1 \ \forall n \geq 0\}$. *The subshift of finite type* is the dynamical system $\sigma_A \colon \Sigma_A \circlearrowleft$ given by $\sigma_A = \sigma|_{\Sigma_A}$. We say that σ_A is *topologically mixing* if there exists an integer $M > 0$ such that all the entries of A^M are positive. Let $C(\Sigma_A)$ denote the space of real-valued continuous functions on Σ_A endowed with the supremum norm.

THEOREM 1. [**24**, Theorem A] *Let $\sigma_A \colon \Sigma_A \circlearrowleft$ be a topologically mixing subshift of finite type. There exists a dense subset $\mathscr{C}_{\sigma_A}$ of $C(\Sigma_A)$ such that for every $\phi \in \mathscr{C}_{\sigma_A}$ there exist uncountably many ϕ-minimizing measures which are ergodic, fully supported and have positive entropy.*

The argument in [**24**] is to view elements of $\mathcal{M}(\sigma_A)$ as duals of $C(\Sigma_A)$ and use the Bishop–Phelps theorem [**14**, Theorem V.1.1] for a continuous path of ergodic Markovian measures. Our strategy for proving Theorem A is to use the Markov partition formed by the intervals $\{[\alpha_i, \beta_i]\}_{i=0}^{p-1}$ to translate the Lyapunov optimization to the ergodic optimization for σ and then appeal to Theorem 1. As a result, for every $f \in \mathscr{A}$ the uncountably many minimizing measures in the statement lie on a continuous path of ergodic measures which are equilibrium states of f for some Hölder continuous potentials.

We have suppressed small generalizations of Theorem A for brevity of presentation. An extension is possible to the case where f is topologically conjugate to the topologically mixing subshift of finite type. Moreover, we can drop the conditions $0 = \alpha_0$ and $\beta_{p-1} = 1$. Statements analogous to Theorem A hold for Lyapunov maximizing measures. Since both proofs are identical, we restrict ourselves to Lyapunov minimizing ones.

For $f \in \mathscr{E}$, define a coding map $\pi_f \colon \Sigma_p \to \Lambda(f)$ by $\pi_f(\underline{a}) \in \bigcap_{n\geq 0} f^{-n}([\alpha_{a_n}, \beta_{a_n}])$, where $\underline{a} = \{a_n\}_{n\geq 0}$. Then π_f is a homeomorphism satisfying $\pi_f \circ \sigma = f \circ \pi_f$ and, for each fixed $\phi \in C(\Sigma_p)$, there is a one-to-one correspondence between ϕ-minimizing measures in $\mathcal{M}(\sigma)$ and $\phi \circ \pi_f^{-1}$-minimizing ones in $\mathcal{M}(f)$. Theorem A is proved by combining Theorem 1 and the next lemma, which allows one to realize a perturbation in $C(\Sigma_p)$ as a perturbation in $\mathscr{E}$.

THEOREM (The realization theorem). *Let $f_0 \in \mathscr{E}$ be of class C^2. For every $\varepsilon > 0$, there exists a neighborhood $\mathscr{U}$ of $\log|Df_0| \circ \pi_{f_0}$ in $C(\Sigma_p)$ such that for every $\phi \in \mathscr{U}$ there exists $f_\infty \in \mathscr{E}$ such that*

$$\|f_0 - f_\infty\|_{C^1} \leq \varepsilon \quad \textit{and} \quad \log|Df_\infty| \circ \pi_{f_\infty} = \phi.$$

We finish the proof of Theorem A assuming the realization theorem.

Proof of Theorem A. For the shift map $\sigma_p \colon \Sigma_p \circlearrowleft$, consider the dense subset $\mathscr{C}_{\sigma_p}$ of $C(\Sigma_p)$ in Theorem 1. Since C^2 maps are dense in $\mathscr{E}$, the realization theorem implies that the set $\widehat{\mathscr{A}} = \{f \in \mathscr{E} \colon \log|Df| \circ \pi_f \in \mathscr{C}_{\sigma_p}\}$ is dense in $\mathscr{E}$. From Theorem 1, if $f \in \widehat{\mathscr{A}}$, then there exist uncountably many Lyapunov minimizing measures which are ergodic, fully supported and have positive entropy.

Let $\widehat{\mathscr{E}}$ denote the set of elements of $\mathscr{E}$ for which there exist two periodic measures with different Lyapunov exponents. Clearly, if $f \in \widehat{\mathscr{E}}$, then $\chi_{\inf}(f) \neq \chi_{\sup}(f)$. Set $\mathscr{A} = \widehat{\mathscr{A}} \cap \widehat{\mathscr{E}}$. The set $\mathscr{A}$ satisfies the desired properties. Indeed, since $\widehat{\mathscr{E}}$ is an open dense subset of $\mathscr{E}$ and $\widehat{\mathscr{A}}$ is a dense subset of $\mathscr{E}$, $\mathscr{A}$ is a dense subset of $\mathscr{E}$.

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Let $f \in \mathscr{A}$ and suppose that $\log |Df|$ is Hölder continuous. We use the result of Bousch [**3**, Théorème 1] and derive a contradiction. The Hölder continuity of $\log |Df|$ and (E2) together imply that $\log |Df|$ satisfies *la condition de Walters* [**3**, Définition 1]. Although (E2) does not imply *la propriété dilatation faible* [**3**, Définition 3], (E2) implies that some iterate of f has this property: there exist an integer $n \geq 1$ and $\lambda > 1$ such that $|f^n(x) - f^n(y)| \geq \lambda |x - y|$ for all x, y in the same branch of f^n. Since any f^n-invariant Borel probability measure is weak*-approximated by periodic ones [**25**], we have

$$\inf \left\{ \int \log |Df| \, d\mu : \mu \text{ is } f^n\text{-invariant} \right\} \geq \chi_{\inf}.$$

By [**3**, Théorème 1] applied to the map f^n and the performance function $-\log |Df|$, there exists a continuous function u such that

$$\begin{aligned} -\log |Df| &\leq u \circ f^n - u - \inf \left\{ \int \log |Df| \, d\mu : \mu \text{ is } f^n\text{-invariant} \right\} \\ &\leq u \circ f^n - u - \chi_{\inf}. \end{aligned}$$

Hence, it is possible to write $-\log |Df| = u \circ f^n - u - \chi_{\inf} - r$, where r is a continuous function on $\Lambda(f)$ satisfying $r \geq 0$. We then obtain $-\int \log |Df| \, d\nu = -\chi_{\inf} - \int r d\nu$ for every $\nu \in \mathcal{M}(f)$ and so ν is Lyapunov minimizing if and only if $\int r d\nu = 0$. Since $r(x) \geq 0$ for all $x \in \Lambda(f)$, r is continuous and there is a Lyapunov minimizing measure fully supported on $\Lambda(f)$, we conclude that $r \equiv 0$ and $-\log |Df| = u \circ f - u - \chi_{\inf}$. Hence, $\chi_{\inf} = \chi_{\sup}$ holds and $f \notin \widehat{\mathscr{E}}$, which is a contradiction. □

For a proof of the realization theorem, we construct a Cauchy sequence $\{f_n\}_{n \geq 0}$ in $\mathscr{E}$ and obtain f_∞ as a limit of this sequence. This construction has two main steps, which are carried out in §3. First, we construct by induction a sequence of continuous piecewise linear maps $h_n : X \to [0, 1]$ $(n = 0, 1, \ldots)$. Then we perturb each h_n and obtain the desired sequence $\{f_n\}_{n \geq 0}$. Although the constructions of $\{h_n\}_{n \geq 0}$ and $\{f_n\}_{n \geq 0}$ are rather intuitive, the difficulty is to ensure that the induction does not halt on the way. We do this by showing that the sizes of gaps at each step of the induction have a definite proportion (see Lemma 3.1). This is the reason for the assumption of C^2 smoothness in the realization theorem. The realization theorem is clearly false for expanding circle maps: there is no space to absorb differences which stem from the perturbation. Indeed, our proof exploits the total disconnectedness of the Cantor set.

The realization theorem implies that any property of minimizing measures which holds for a dense subset of functions in $C(\Sigma_p)$ transmits to a dense subset of $\mathscr{E}$. By the result of Brémont [**7**, Proposition 2.1], for a dense subset of $C(\Sigma_p)$ the minimizing measure is unique and it is supported on a periodic orbit. It follows that for a dense subset of $\mathscr{E}$ the Lyapunov minimizing measure is unique and it is supported on a periodic orbit. Below we give a stronger statement, which in particular indicates that a version of Theorem A does not hold in the $C^{1+\mathrm{Lip}}$ topology.

Let $\mathscr{E}_{\mathrm{Lip}}$ denote the space of maps in $\mathscr{E}$ with Lipschitz continuous derivative endowed with the topology given by the norm

$$\|f\|_{C^{1+\mathrm{Lip}}} = \|f\|_{C^1} + \sup_{\substack{x,y \in X \\ x \neq y}} \frac{|Df(x) - Df(y)|}{|x - y|}.$$

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THEOREM B. *There exists an open subset $\mathcal{O}$ of $\mathcal{E}_{\mathrm{Lip}}$ such that for every $f \in \mathcal{O}$ there exists a unique Lyapunov minimizing measure and it is supported on a periodic orbit. In addition, $\mathcal{O}$ is a dense subset of $\mathcal{E}$.*

The first statement of Theorem B is a consequence of the result of Contreras [**9**]. A proof of the last statement of Theorem B is briefly outlined as follows. The total disconnectedness of the phase space implies that maps with locally constant derivative are dense in $\mathcal{E}$ (see Lemma 4.1). If Df is a locally constant function on $\Lambda(f)$, then $\log|Df| \circ \pi_f$ becomes Lipschitz continuous with respect to the standard distance on Σ_p of *any scale*. By the realization theorem and the result of Contreras [**9**], f is approximated by another for which the Lyapunov minimizing measure is unique and supported on a periodic orbit. Choosing a distance of sufficiently small scale relative to the expansion rate of f in (E2), we obtain the Lipschitz continuity of Df (see §4.3).

Theorem B has one important consequence on *the zero-temperature limit* in the thermodynamic formalism (see e.g. [**1**] and the references therein). For $f \in \mathcal{E}$, define a geometric pressure function $t \in \mathbb{R} \mapsto \mathcal{P}(t)$ by

$$\mathcal{P}(t) = \sup\left\{h_\mu(f) - t\int \log|Df|\,d\mu \colon \mu \in \mathcal{M}(f)\right\},$$

where $h_\mu(f)$ denotes the Kolmogorov–Sinaĭ entropy of (f, μ). *An equilibrium state for the potential* $-t\log|Df|$ is a measure in $\mathcal{M}(f)$ which attains this supremum. If $\log|Df|$ is Hölder continuous, then $t \in \mathbb{R} \mapsto \mathcal{P}(t)$ is real analytic and, for every $t \in \mathbb{R}$, there exists a unique equilibrium state for the potential $-t\log|Df|$ [**5**, **23**], which we denote by μ_t. Lyapunov minimizing measures are obtained by freezing the system: any accumulation point of $\{\mu_t\}_{t\in\mathbb{R}}$ as $t \to +\infty$ is a Lyapunov minimizing measure. By *the zero-temperature limit* we mean the weak*-limit of $\{\mu_t\}_{t\in\mathbb{R}}$ as $t \to +\infty$. The uniqueness of the Lyapunov minimizing measure implies the existence of the zero-temperature limit because there is a unique possible accumulation point.

COROLLARY 1. *For every $f \in \mathcal{O}$, the zero-temperature limit $\lim_{t\to+\infty}\mu_t$ exists and is supported on a periodic orbit.*

If $\log|Df|$ is a locally constant function on $\Lambda(f)$, then the zero-temperature limit exists; see Brémont [**6**] and Leplaideur [**17**]. Such maps are dense in $\mathcal{E}$ (see Lemma 4.1). For maps in $\mathcal{A}$ in Theorem A, it is not clear whether or not the zero-temperature limit exists. For the full shift, a Lipschitz continuous function without zero-temperature limit was constructed by Chazottes and Hochman [**8**], and subsequently by Coronel and Rivera-Letelier [**11**].

The rest of this paper consists of three sections. Sections 2 and 3 are entirely dedicated to a proof of the realization theorem. In §4 we prove Theorem B.

2. *Preliminaries*

In this section we develop fundamental estimates needed for the proofs of the main results.

2.1. *Control of variations.*

For an integer $n \geq 1$, by *a word of length n* we mean an n-string of integers in $\{0, \ldots, p-1\}$. For each integer $n \geq 1$, let W_n denote the set of

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words of length n. For $a_0 \cdots a_{n-1} \in W_n$, define

$$[a_0 \cdots a_{n-1}] = \{\{b_k\}_{k\geq 0} \in \Sigma_p : a_k = b_k \text{ for every } k \in \{0, \ldots, n-1\}\}.$$

For $\phi \in C(\Sigma_p)$ and $a_0 \cdots a_{n-1} \in W_n$, define

$$E_{a_0\cdots a_{n-1}}(\phi) = \exp\left(\min_{[a_0\cdots a_{n-1}]} \phi\right),$$
$$F_{a_0\cdots a_{n-1}}(\phi) = \exp\left(\max_{[a_0\cdots a_{n-1}]} \phi\right)$$

and

$$V_{a_0\cdots a_{n-1}}(\phi) = F_{a_0\cdots a_{n-1}}(\phi) - E_{a_0\cdots a_{n-1}}(\phi).$$

Set

$$M(\phi) = \sup_{\Sigma_p} |\phi|.$$

LEMMA 2.1. *For every $\phi, \phi' \in C(\Sigma_p)$, every $n \geq 1$ and every $a_0 \cdots a_{n-1} \in W_n$, the following holds:*

$$|E_{a_0\cdots a_{n-1}}(\phi) - E_{a_0\cdots a_{n-1}}(\phi')| \leq V_{a_0\cdots a_{n-1}}(\phi) + M(e^{\phi} - e^{\phi'}),$$
$$|F_{a_0\cdots a_{n-1}}(\phi) - F_{a_0\cdots a_{n-1}}(\phi')| \leq V_{a_0\cdots a_{n-1}}(\phi) + M(e^{\phi} - e^{\phi'}).$$

Proof. Let $\underline{a}, \underline{a'} \in [a_0 \cdots a_{n-1}]$ be such that $E_{a_0\cdots a_{n-1}}(\phi) = e^{\phi(\underline{a})}$ and $E_{a_0\cdots a_{n-1}}(\phi') = e^{\phi'(\underline{a'})}$. Then

$$\begin{aligned}|E_{a_0\cdots a_{n-1}}(\phi) - E_{a_0\cdots a_{n-1}}(\phi')| &\leq |e^{\phi(\underline{a})} - e^{\phi(\underline{a'})}| + |e^{\phi(\underline{a'})} - e^{\phi'(\underline{a'})}| \\ &\leq V_{a_0\cdots a_{n-1}}(\phi) + M(e^{\phi} - e^{\phi'}).\end{aligned}$$

A proof of the second inequality is analogous. □

2.2. *Uniform expansion for nearby maps.* The next lemma ensures that $\mathscr{E}$ is an open subset of C^1 functions on X.

LEMMA 2.2. *Let $f_0 \in \mathscr{E}$. There exist constants $\varepsilon_0 > 0$, $c_0 > 0$, $\lambda_0 > 1$ such that the following holds: if $f \in \mathscr{E}$ and $\|f_0 - f\|_{C^1} \leq \varepsilon_0$, then, for every $x \in X$ and every $n \geq 1$ such that $x, f(x), \ldots, f^{n-1}(x) \in X$, $|Df^n(x)| \geq c_0\lambda_0^n$.*

Proof. By (E2), there exist constants $c > 0$, $\lambda > 1$ such that if $x \in X$ and $n \geq 1$ are such that $x, f_0(x), \ldots, f_0^{n-1}(x) \in X$, then $|Df_0^n(x)| \geq c\lambda^n$. Choose an integer $N_0 \geq 1$ such that $c\lambda^{N_0} \geq 3$. Choose $\varepsilon_0 > 0$ such that $|Df^{N_0}(x)| \geq 2$ holds for every $f \in \mathscr{E}$ satisfying $\|f_0 - f\|_{C^1} \leq \varepsilon_0$ and $x \in X$ such that $x, f(x), \ldots, f^{N_0-1}(x) \in X$. For such an f, let $x \in X$ and $n \geq 1$ be such that $x, f(x), \ldots, f^{n-1}(x) \in X$. Write $n = N_0k + l$, where k, l are non-negative integers with $0 \leq l \leq N_0 - 1$. Put

$$c_0 = \frac{1}{2} \min_{\substack{y \in \bigcap_{k=0}^{l-1} f_0^{-k}(X) \\ 1\leq l\leq N_0-1}} |Df_0^l(y)| \quad \text{and} \quad \lambda_0 = 2^{1/N_0}.$$

Splitting the orbit of x into a concatenation of segments of length N_0 and then using the chain rule gives $|Df^n(x)| \geq c_0\lambda_0^n$. □

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2.3. *Consequence of bounded distortion.* Let $a_0 \cdots a_{n-1} \in W_n$ and $f\colon X \to [0, 1]$ a continuous map. Define

$$X_{a_0 \cdots a_{n-1}}(f) = \{x \in X\colon f^k(x) \in [\alpha_{a_k}, \beta_{a_k}] \text{ for every } k \in \{0, \ldots, n-1\}\}.$$

For each $i \in \{1, \ldots, p-1\}$, define

$$X_{a_0 \cdots a_{n-1} \square_i}(f) = \{x \in X_{a_0 \cdots a_{n-1}}(f)\colon f^n(x) \in (\beta_{i-1}, \alpha_i)\}$$

and

$$X_{a_0 \cdots a_{n-1} \widehat{i}}(f) = \{x \in X_{a_0 \cdots a_{n-1}}(f)\colon f^n(x) \in [\alpha_{i-1}, \beta_i]\}.$$

Denote by $|I|$ the length of a bounded interval I. Note that

$$|X_{a_0 \cdots a_{n-1} \widehat{i}}(f)| = |X_{a_0 \cdots a_{n-1}(i-1)}(f)| + |X_{a_0 \cdots a_{n-1} \square_i}(f)| + |X_{a_0 \cdots a_{n-1} i}(f)|.$$

Put

$$\Delta^{(i)}_{a_0 \cdots a_{n-1}}(f) = \frac{|X_{a_0 \cdots a_{n-1} \widehat{i}}(f)|}{|X_{a_0 \cdots a_{n-1} \square_i}(f)|} - 1.$$

LEMMA 2.3. *Let $f_0 \in \mathscr{E}$ be of class C^2. There exists a constant $K(f_0) > 0$ such that for every $i \in \{1, \ldots, p-1\}$, every $n \geq 1$ and every $a_0 \cdots a_{n-1} \in W_n$,*

$$\Delta^{(i)}_{a_0 \cdots a_{n-1}}(f_0) \leq K(f_0).$$

Proof. Since $f_0 \in \mathscr{E}$ is of class C^2, the bounded distortion holds: there exists $M_0 > 1$ such that for every $n \geq 1$, every $a_0 \cdots a_{n-1} \in W_n$ and every $x, y \in X_{a_0 \cdots a_{n-1}}(f_0)$, we have

$$\frac{|Df_0^n(x)|}{|Df_0^n(y)|} \leq M_0.$$

Let $i \in \{1, \ldots, p-1\}$. For every $a_0 \cdots a_{n-1} \in W_n$, we have

$$\frac{|X_{a_0 \cdots a_{n-1} \square_i}(f_0)|}{|X_{a_0 \cdots a_{n-1}}(f_0)|} \geq M_0^{-1} \frac{|f_0^n(X_{a_0 \cdots a_{n-1} \square_i}(f_0))|}{|f_0^n(X_{a_0 \cdots a_{n-1}}(f_0))|} = M_0^{-1}(\alpha_i - \beta_{i-1})$$

and therefore

$$\begin{aligned} \frac{|X_{a_0 \cdots a_{n-1}(i-1)}(f_0)| + |X_{a_0 \cdots a_{n-1} i}(f_0)|}{|X_{a_0 \cdots a_{n-1}}(f_0)|} &\leq \frac{|X_{a_0 \cdots a_{n-1}}(f_0)| - |X_{a_0 \cdots a_{n-1} \square_i}(f_0)|}{|X_{a_0 \cdots a_{n-1}}(f_0)|} \\ &\leq 1 - M_0^{-1}(\alpha_i - \beta_{i-1}). \end{aligned}$$

Put $K(f_0) = \max_{i \in \{1, \ldots, p-1\}} (1 - M_0^{-1}(\alpha_i - \beta_{i-1}))/(M_0^{-1}(\alpha_i - \beta_{i-1}))$. These two inequalities yield the desired one. □

3. *On the proof of the realization theorem*

In this section we complete the proof of the realization theorem. Throughout this section, let $f_0 \in \mathscr{E}$ be of class C^2 and put $\phi_0 = \log |Df_0| \circ \pi_{f_0}$.

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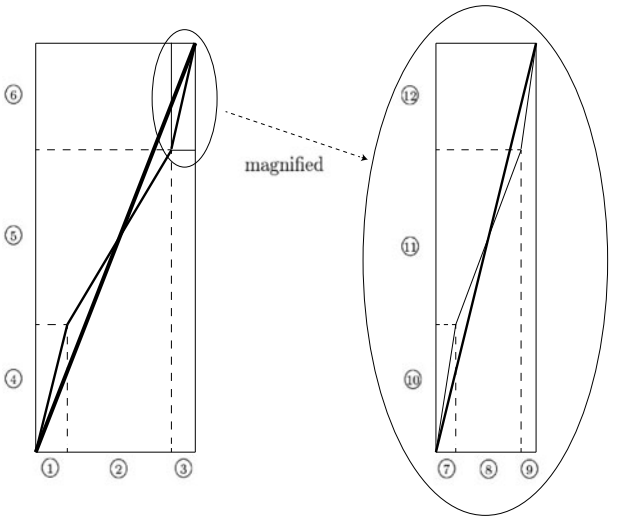


FIGURE 1. The case $p = 2$ and $n \geq N + 1$ with $a_0 \cdots a_{n-1} \in W_n$: the construction of $h_n|_{X_{a_0\cdots a_{n-1}}(h_{n-1})}$ from $h_{n-1}|_{X_{a_0\cdots a_{n-1}}(h_{n-1})}$ and that of $h_{n+1}|_{X_{a_0\cdots a_{n-1}1}(h_n)}$ from $h_n|_{X_{a_0\cdots a_{n-1}1}(h_n)}$ (in the ellipses). The left-hand rectangle is $X_{a_0\cdots a_{n-1}}(h_n) \times X_{a_1\cdots a_{n-1}}(h_{n-1}) \subset X \times [0, 1]$. Its diagonal is the graph of $h_{n-1}|_{X_{a_0\cdots a_{n-1}}(h_n)}$ and the polygonal segment is the graph of $h_n|_{X_{a_0\cdots a_{n-1}}(h_n)}$. The rectangle in the magnified ellipse is $X_{a_0\cdots a_{n-1}1}(h_{n+1}) \times X_{a_1\cdots a_{n-1}1}(h_n)$. Its diagonal is the graph of $h_n|_{X_{a_0\cdots a_{n-1}1}(h_{n+1})}$ and the polygonal segment is the graph of $h_{n+1}|_{X_{a_0\cdots a_{n-1}1}(h_{n+1})}$. The labelled segments are as follows: ① $= X_{a_0\cdots a_{n-1}0}(h_n)$; ② $= X_{a_0\cdots a_{n-1}\square_1}(h_n)$; ③ $= X_{a_0\cdots a_{n-1}1}(h_n)$; ④ $= X_{a_1\cdots a_{n-1}0}(h_{n-1})$; ⑤ $= X_{a_1\cdots a_{n-1}\square_1}(h_{n-1})$; ⑥ $= X_{a_1\cdots a_{n-1}1}(h_{n-1})$; ⑦ $= X_{a_0\cdots a_{n-1}10}(h_{n+1})$; ⑧ $= X_{a_0\cdots a_{n-1}1\square_1}(h_{n+1})$; ⑨ $= X_{a_0\cdots a_{n-1}11}(h_{n+1})$; ⑩ $= X_{a_1\cdots a_{n-1}10}(h_n)$; ⑪ $= X_{a_1\cdots a_{n-1}1\square_1}(h_n)$; ⑫ $= X_{a_1\cdots a_{n-1}11}(h_n)$.

3.1. *Construction of a sequence of continuous piecewise linear maps.* Let $\phi \in C(\Sigma_p)$ be sufficiently close to ϕ_0 and $N \geq 2$ an integer. We construct by induction a sequence $\{h_n(\phi, N)\}_{n\geq 0}$ of continuous piecewise linear maps on X which maps each connected component of X bijectively onto $[0, 1]$. In what follows, we will write h_n for $h_n(\phi, N)$.

Start with $h_n = f_0$ for every $n \in \{0, \ldots, N - 1\}$. Let $n \geq N$ and suppose that h_{n-1}, h_{n-2} has been defined so that the following holds:

(P)$_{n-1}$: $h_{n-1}(X_{a_0\cdots a_{n-1}}(h_{n-1})) = X_{a_1\cdots a_{n-1}}(h_{n-2})$ for every $a_0 \cdots a_{n-1} \in W_n$.

Put

$$G_{n-1} = \bigcup_{k=1}^{n-1} \bigcup_{a_0\cdots a_{k-1}\in W_k} \bigcup_{i=1}^{p-1} X_{a_0\cdots a_{k-1}\square_i}(h_{n-1}).$$

The open intervals $X_{a_0\cdots a_{k-1}\square_i}(h_{n-1})$ in the union are called *a gap of order* k. The total number of gaps of order k is $p^k(p - 1)$. Note that

$$X = G_{n-1} \bigcup \Bigg(\bigcup_{a_0\cdots a_{n-1}\in W_n} X_{a_0\cdots a_{n-1}}(h_{n-1}) \Bigg),$$

where all unions are disjoint. Define h_n so that the following holds (see Figure 1):

(i) $h_n \equiv h_{n-1}$ on G_{n-1};

For each $a_0 \cdots a_{n-1} \in W_n$, $h_n|_{X_{a_0\cdots a_{n-1}}(h_{n-1})}$ is defined as follows:

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(ii) for every $i \in \{0, \dots, p-1\}$,

$$h_n^{-1}(X_{a_1 \cdots a_{n-1} i}(h_{n-1})) \cap X_{a_0 \cdots a_{n-1}}(h_{n-1}) = X_{a_0 \cdots a_{n-1} i}(h_n)$$

and

$$Dh_n|_{X_{a_0 \cdots a_{n-1} i}(h_n)} \equiv \pm E_{a_0 \cdots a_{n-1} i}(\phi),$$

where the sign is + if $Df_0 > 0$ on $[\alpha_{a_0}, \beta_{a_0}]$ and – otherwise;

(iii) for every $i \in \{1, \dots, p-1\}$, $Dh_n|_{X_{a_0 \cdots a_{n-1} \square_i}(h_n)}$ is a constant function.

Since h_n is required to be continuous, there is no ambiguity in this definition. Note that $(\mathrm{P})_n$ holds, which recovers the assumption of the induction.

There is a difference between the transition from $N-1$ to N and that from $n-1$ to n, $n \geq N+1$. Since $h_{N-1} = f_0$, an 'overhang' may happen, in which case the definition of $h_N|_{X_{a_0 \cdots a_{N-1}}(f_0)}$ does not make sense. As developed in the proof of Lemma 3.1 below, the overhang does not happen for an appropriately chosen ϕ and N.

3.2. *Analytic estimates on the sequence of continuous piecewise linear maps.* In this subsection we develop three estimates on the sequence $\{h_n(\phi, N)\}_{n \geq 0}$. The next lemma states that $\{h_n\}_{n \geq 0}$ respects the proportions of gaps.

LEMMA 3.1. *There exist a neighborhood $\hat{\mathscr{U}}$ of ϕ_0 and $\hat{N} \geq 1$ such that the following holds for every $\phi \in \hat{\mathscr{U}}$ and every integer $N > \hat{N}$: the sequence $\{h_n(\phi, N)\}_{n \geq N}$ is well defined and, for every $i \in \{1, \dots, p-1\}$, every $n \geq 1$ and every $a_0 \cdots a_{n-1} \in W_n$,*

$$\Delta^{(i)}_{a_0 \cdots a_{n-1}}(h_n(\phi, N)) \leq 2K(f_0).$$

Proof. We argue in two steps.

Step 1. Well-definedness of $\{h_n(\phi, N)\}_{n \geq N}$. If $n \geq N+1$, then the transition from $n-1$ to n makes sense. It suffices to show that the transition from $N-1$ to N makes sense.

Fix $\delta \in (0, 1)$ such that $K(f_0)(1+\delta)^2/(1+K(f_0)) < 1$. Choose a neighborhood $\hat{\mathscr{U}}$ of ϕ_0 and $\hat{N} \geq 1$ such that for every integer $N \geq \hat{N}$, every $\phi \in \hat{\mathscr{U}}$, every $a_0 \cdots a_{N-1} \in W_N$, every $i \in \{1, \dots, p-1\}$ and every $j \in \{i-1, i\}$, the following holds:

$$\sup_{X_{a_0 \cdots a_{N-1} j}(f_0)} |Df_0| \leq (1+\delta) E_{a_0 \cdots a_{N-1} j}(\phi_0), \tag{1}$$

$$E_{a_0 \cdots a_{N-1} j}(\phi_0) \leq (1+\delta) E_{a_0 \cdots a_{N-1} j}(\phi). \tag{2}$$

The second condition follows from Lemma 2.1. Lemma 2.3 implies that

$$|X_{a_0 \cdots a_{N-1} \widehat{i}}(f_0)|^{-1} \sum_{j \in \{i-1, i\}} |X_{a_0 \cdots a_{N-1} j}(f_0)| \leq \frac{K(f_0)}{1+K(f_0)}.$$

By the mean value theorem, for each $j \in \{i-1, i\}$ there exists $\xi_j \in X_{a_0 \cdots a_{N-1} j}(f_0)$ such that $|Df_0(\xi)||X_{a_0 \cdots a_{N-1} j}(f_0)| = |X_{a_1 \cdots a_{N-1} j}(f_0)|$. Hence,

$$|X_{a_0 \cdots a_{N-1} \widehat{i}}(f_0)|^{-1} \sum_{j \in \{i-1, i\}} \frac{|X_{a_1 \cdots a_{N-1} j}(f_0)|}{|Df_0(\xi_j)|} \leq \frac{K(f_0)}{1+K(f_0)}.$$

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Equations (1) and (2) yield

$$|X_{a_0\cdots a_{N-1}\widehat{i}}(f_0)|^{-1} \sum_{j\in\{i-1,i\}} \frac{|X_{a_1\cdots a_{N-1}j}(f_0)|}{E_{a_0\cdots a_{N-1}j}(\phi)} \le (1+\delta)^2 \frac{K(f_0)}{1+K(f_0)} < 1.$$

This condition prevents the overhang mentioned in the last paragraph of §3.1. Hence, the transition from $N-1$ to N makes sense.

Step 2. Proof of the inequality. Let $N \ge \hat{N}$ be an integer and $\phi \in \hat{\mathscr{U}}$. If $n \in \{0, \dots, N-1\}$, then $h_n = f_0$ and Lemma 2.3 gives $\Delta^{(i)}_{a_0\cdots a_{n-1}}(h_n) \le K(f_0)$. Suppose that $n = N$. The definition of h_N gives

$$\begin{aligned}\Delta^{(i)}_{a_0\cdots a_{N-1}}(h_N) = &\left\{ \sum_{j\in\{i-1,i\}} \frac{|X_{a_1\cdots a_{N-1}j}(f_0)|}{E_{a_0\cdots a_{N-1}j}(\phi)} \right\} \\ &\times \left\{ \int_{X_{a_1\cdots a_{N-1}\widehat{i}}(f_0)} |Df_0(f_0^{-1}(y))|^{-1} dy - \sum_{j\in\{i-1,i\}} \frac{|X_{a_1\cdots a_{N-1}j}(f_0)|}{E_{a_0\cdots a_{N-1}j}(\phi)} \right\}^{-1}. \end{aligned} \tag{3}$$

To estimate the denominator of this fraction, put

$$s = \sup_{X_{a_0\cdots a_{N-1}}(f_0)} |Df_0| \quad \text{and} \quad \eta = 2V_{a_0\cdots a_{N-1}}(\phi_0) + M(e^{\phi_0} - e^{\phi}).$$

Shrink $\hat{\mathscr{U}}$ and enlarge $\hat{N}$ if necessary so that

$$s \ge 2\eta(1 + K(f_0)). \tag{4}$$

We have $s^{-1}|X_{a_1\cdots a_{N-1}\widehat{i}}(f_0)| \le \int_{X_{a_1\cdots a_{N-1}\widehat{i}}(f_0)} |Df_0(f_0^{-1}(y))|^{-1}\, dy$ and Lemma 2.1 gives $E_{a_0\cdots a_{N-1}j}(\phi) \ge s - \eta$ for every $j \in \{i-1, i\}$. Hence, the denominator of the fraction in (3) is bounded from below by

$$\begin{aligned} &s^{-1}|X_{a_1\cdots a_{N-1}\widehat{i}}(f_0)| - \frac{1}{s-\eta} \sum_{j\in\{i-1,i\}} |X_{a_1\cdots a_{N-1}j}(f_0)| \\ &\quad = \frac{s|X_{a_1\cdots a_{N-1}\square_i}(f_0)| - \eta|X_{a_1\cdots a_{N-1}\widehat{i}}(f_0)|}{s(s-\eta)} \\ &\quad \ge |X_{a_1\cdots a_{N-1}\square_i}(f_0)| \cdot \frac{s - \eta(1+K(f_0))}{s(s-\eta)} > 0, \end{aligned}$$

where the first inequality follows from Lemma 2.3 and the last from (4). From (3), we have

$$\begin{aligned}\Delta^{(i)}_{a_0\cdots a_{N-1}}(h_N) \le &\left\{ \sum_{j\in\{i-1,i\}} \frac{|X_{a_1\cdots a_{N-1}j}(f_0)|}{E_{a_0\cdots a_{N-1}j}(\phi)} \right\} \\ &\times \left\{ s^{-1}|X_{a_1\cdots a_{N-1}\widehat{i}}(f_0)| - \sum_{j\in\{i-1,i\}} \frac{|X_{a_1\cdots a_{N-1}j}(f_0)|}{E_{a_0\cdots a_{N-1}j}(\phi)} \right\}^{-1}. \end{aligned} \tag{5}$$

From Lemma 2.1 and the definition of η, for every $j \in \{i-1, i\}$ we have

$$\begin{aligned} s - E_{a_0\cdots a_{N-1}j}(\phi) &\le F_{a_0\cdots a_{N-1}}(\phi_0) - E_{a_0\cdots a_{N-1}}(\phi) \\ &= V_{a_0\cdots a_{N-1}}(\phi_0) + E_{a_0\cdots a_{N-1}}(\phi_0) - E_{a_0\cdots a_{N-1}}(\phi) \\ &\le \eta. \end{aligned}$$

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Substituting $E_{a_0\cdots a_{N-1}j}(\phi) \geq s - \eta$ into (5) yields

$$\begin{aligned}\Delta^{(i)}_{a_0\cdots a_{N-1}}(h_N) &\leq \frac{\sum_{j\in\{i-1,i\}} |X_{a_1\cdots a_{N-1}j}(f_0)|}{|X_{a_1\cdots a_{N-1}\square_i}(f_0)| - (\eta/s)|X_{a_1\cdots a_{N-1}\widehat{i}}(f_0)|}\\ &= \left(1 - \frac{\eta}{s}\frac{|X_{a_1\cdots a_{N-1}\widehat{i}}(f_0)|}{|X_{a_1\cdots a_{N-1}\square_i}(f_0)|}\right)^{-1} \frac{\sum_{j\in\{i-1,i\}} |X_{a_1\cdots a_{N-1}j}(f_0)|}{|X_{a_1\cdots a_{N-1}\square_i}(f_0)|}\\ &\leq \left(1 - \frac{\eta}{s}(1 + K(f_0))\right)^{-1} \Delta^{(i)}_{a_1\cdots a_{N-1}}(f_0)\\ &\leq 2\Delta^{(i)}_{a_1\cdots a_{N-1}}(f_0) \leq 2K(f_0).\end{aligned}$$

It is left to treat the case $n \geq N + 1$. The construction of h_n from h_{n-1} in §3.1 implies that

$$\Delta^{(i)}_{a_0\cdots a_{n-1}}(h_n) \leq \Delta^{(i)}_{a_1\cdots a_{n-1}}(h_{n-1}).$$

Using this inductively yields $\Delta^{(i)}_{a_0\cdots a_{n-1}}(h_n) \leq \Delta^{(i)}_{a_{n-N}\cdots a_{n-1}}(h_N) \leq 2K(f_0)$. □

In what follows, let $\hat{\mathscr{U}}$ be the neighborhood of ϕ_0 in $C(\Sigma_p)$ and $\hat{N} \geq 1$ the number in the statement of Lemma 3.1. For an integer $n \geq 1$ and $a_0 \cdots a_{n-1} \in W_n$, define $\tau_{a_0\cdots a_{n-1}\square_i}$ to be the constant value of $|Dh_n|$ on $X_{a_0\cdots a_{n-1}\square_i}(h_n)$.

LEMMA 3.2. *The following holds for every $\phi \in \hat{\mathscr{U}}$, every integer $N > \hat{N}$ and $\{h_n(\phi, N)\}_{n\geq 0}$.*

(a) *For every $i \in \{1, \ldots, p-1\}$, every $n \geq N+1$ and every $a_0 \cdots a_{n-1} \in W_n$,*

$$\tau_{a_0\cdots a_{n-1}\square_i} > E_{a_0\cdots a_{n-1}}(\phi) - 2M(e^{-2\phi})V_{a_0\cdots a_{n-1}}(\phi)E^2_{a_0\cdots a_{n-1}}(\phi)K(f_0).$$

Moreover, for every $i \in \{1, \ldots, p-1\}$ and every $a_0 \cdots a_{N-1} \in W_N$,

$$\begin{aligned}\tau_{a_0\cdots a_{N-1}\square_i} > E_{a_0\cdots a_{N-1}}(\phi_0) &- 2M(e^{-\phi-\phi_0})(2V_{a_0\cdots a_{N-1}}(\phi_0) + M(e^{\phi} - e^{\phi_0}))\\ &\times E^2_{a_0\cdots a_{N-1}}(\phi_0)K(f_0).\end{aligned}$$

(b) *For every integers m, n with $m > n \geq N$, every $a_0 \cdots a_{m-1} \in W_m$ and every i, $j \in \{1, \ldots, p-1\}$,*

$$|\tau_{a_0\cdots a_{m-1}\square_j} - \tau_{a_0\cdots a_{n-1}\square_i}| \leq (1 + M(e^{2\phi})K(f_0))V_{a_0\cdots a_{n-1}}(\phi).$$

Proof. As for (a), let $i \in \{1, \ldots, p-1\}$ and $a_0 \cdots a_{n-1} \in W_n$. We first consider the case $n \geq N + 1$. From the definition of h_n,

$$\begin{aligned}\tau^{-1}_{a_0\cdots a_{n-1}\square_i} &= |X_{a_1\cdots a_{n-1}\square_i}(h_{n-1})|^{-1}\left(\frac{|X_{a_1\cdots a_{n-1}\widehat{i}}(h_{n-1})|}{E_{a_0\cdots a_{n-1}}(\phi)} - \sum_{j\in\{i-1,i\}} \frac{|X_{a_1\cdots a_{n-1}j}(h_{n-1})|}{E_{a_0\cdots a_{n-1}j}(\phi)}\right)\\ &= E^{-1}_{a_0\cdots a_{n-1}}(\phi) + \left\{\sum_{j\in\{i-1,i\}} \frac{E_{a_0\cdots a_{n-1}j}(\phi) - E_{a_0\cdots a_{n-1}}(\phi)}{E_{a_0\cdots a_{n-1}}(\phi)E_{a_0\cdots a_{n-1}j}(\phi)}|X_{a_1\cdots a_{n-1}j}(h_{n-1})|\right\}\\ &\quad \times \{|X_{a_1\cdots a_{n-1}\square_i}(h_{n-1})|\}^{-1}.\end{aligned}$$

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Hence,

$$\begin{aligned}\tau^{-1}_{a_0\cdots a_{n-1}\square_i} - E^{-1}_{a_0\cdots a_{n-1}}(\phi) &= \left\{ \sum_{j\in\{i-1,i\}} \frac{E_{a_0\cdots a_{n-1}j}(\phi) - E_{a_0\cdots a_{n-1}}(\phi)}{E_{a_0\cdots a_{n-1}}(\phi)E_{a_0\cdots a_{n-1}j}(\phi)} |X_{a_1\cdots a_{n-1}j}(h_{n-1})| \right\} \\ &\quad \times \{|X_{a_1\cdots a_{n-1}\square_i}(h_{n-1})|\}^{-1} \\ &\le M(e^{-2\phi}) \frac{\sum_{j\in\{i-1,i\}} V_{a_0\cdots a_{n-1}}(\phi)|X_{a_1\cdots a_{n-1}j}(h_{n-1})|}{|X_{a_1\cdots a_{n-1}\square_i}(h_{n-1})|} \\ &= M(e^{-2\phi}) V_{a_0\cdots a_{n-1}}(\phi)\Delta^{(i)}_{a_1\cdots a_{n-1}}(h_{n-1}).\end{aligned}$$

This yields

$$\begin{aligned}\tau_{a_0\cdots a_{n-1}\square_i} &\ge E_{a_0\cdots a_{n-1}}(\phi) - \frac{V_{a_0\cdots a_{n-1}}(\phi)E^2_{a_0\cdots a_{n-1}}(\phi)\Delta^{(i)}_{a_1\cdots a_{n-1}}(h_{n-1})}{V_{a_0\cdots a_{n-1}}(\phi)E_{a_0\cdots a_{n-1}}(\phi)\Delta^{(i)}_{a_1\cdots a_{n-1}}(h_{n-1}) + 1/M(e^{-2\phi})} \\ &> E_{a_0\cdots a_{n-1}}(\phi) - M(e^{-2\phi})V_{a_0\cdots a_{n-1}}(\phi)E^2_{a_0\cdots a_{n-1}}(\phi)\Delta^{(i)}_{a_1\cdots a_{n-1}}(h_{n-1}) \\ &\ge E_{a_0\cdots a_{n-1}}(\phi) - 2M(e^{-2\phi})V_{a_0\cdots a_{n-1}}(\phi)E^2_{a_0\cdots a_{n-1}}(\phi)K(f_0).\end{aligned}$$

The last inequality follows from Lemma 3.1.

A proof for the case $n = N$ is analogous to the above argument modulo minor differences. We simply replace $E_{a_0\cdots a_{N-1}}(\phi)$ by $\sup_{X_{a_0\cdots a_{N-1}}(f_0)} |Df_0|$ and argue in the same way. On the fraction in the summand, for every $j \in \{i-1, i\}$,

$$\begin{aligned}E_{a_0\cdots a_{N-1}j}(\phi) - \sup_{X_{a_0\cdots a_{N-1}}(f_0)} |Df_0| &\le E_{a_0\cdots a_{N-1}j}(\phi) - E_{a_0\cdots a_{N-1}}(\phi_0) \\ &\le F_{a_0\cdots a_{N-1}}(\phi) - E_{a_0\cdots a_{N-1}}(\phi_0) \\ &\le 2V_{a_0\cdots a_{N-1}}(\phi_0) + M(e^{\phi} - e^{\phi_0}),\end{aligned}$$

where the last inequality follows from the second estimate in Lemma 2.1. This completes the proof of Lemma 3.2(a).

From the construction in §3.1,

$$\max\{\tau_{a_0\cdots a_{m-1}\square_j}, \tau_{a_0\cdots a_{n-1}\square_i}\} \le F_{a_0\cdots a_{n-1}}(\phi).$$

Lemma 3.2(a) implies that

$$\tau_{a_0\cdots a_{m-1}\square_j} > E_{a_0\cdots a_{m-1}}(\phi) - V_{a_0\cdots a_{m-1}}(\phi)E^2_{a_0\cdots a_{m-1}}(\phi)K(f_0)$$

and

$$\tau_{a_0\cdots a_{n-1}\square_i} > E_{a_0\cdots a_{n-1}}(\phi) - V_{a_0\cdots a_{n-1}}(\phi)E^2_{a_0\cdots a_{n-1}}(\phi)K(f_0).$$

If $\tau_{a_0\cdots a_{n-1}\square_i} > \tau_{a_0\cdots a_{m-1}\square_j}$, then

$$\tau_{a_0\cdots a_{n-1}\square_i} - \tau_{a_0\cdots a_{m-1}\square_j} \le (1 + E^2_{a_0\cdots a_{n-1}}(\phi)K(f_0))V_{a_0\cdots a_{n-1}}(\phi).$$

In the case $\tau_{a_0\cdots a_{n-1}\square_i} < \tau_{a_0\cdots a_{m-1}\square_j}$, we get the same inequality. This completes the proof of Lemma 3.2(b). □

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3.3. *Perturbation to C^1 maps.* We have constructed a sequence $\{h_n(\phi, N)\}_{n\geq 0}$ of continuous piecewise linear maps. For each $n \geq N$, we define a C^1 map $f_n\colon X \to [0, 1]$ by perturbing h_n on each gap of order $k \in \{N, \ldots, n\}$.

Start with $f_n = h_n$ on G_N. Let $k \geq N$ and $a_0 \cdots a_{k-1} \in W_k$. Define $f_n|_{X_{a_0\cdots a_{k-1}}(h_{n-1})}$ as follows. Recall that

$$X_{a_0\cdots a_{k-1}}(h_{n-1}) = \bigcup_{i=1}^{p-1} X_{a_0\cdots a_{k-1}\widehat{i}}(h_{n-1})$$

and, for every $i \in \{1, \ldots, p-1\}$,

$$X_{a_0\cdots a_{k-1}\widehat{i}}(h_{n-1}) = X_{a_0\cdots a_{k-1}(i-1)}(h_{n-1}) \bigcup X_{a_0\cdots a_{k-1}\square_i}(h_{n-1}) \bigcup X_{a_0\cdots a_{k-1}i}(h_{n-1}).$$

For every $x \in X_{a_0\cdots a_{k-1}(i-1)}(h_{n-1}) \bigcup X_{a_0\cdots a_{k-1}i}(h_{n-1})$, set $f_n(x) = h_n(x)$. In order to define f_n on $X_{a_0\cdots a_{k-1}\square_i}(h_{n-1})$, we need the next lemma.

LEMMA 3.3. *Let $E > 0$, $E' > 0$, $\tau > 0$ be such that $2\tau - (E + E')/2 > 0$ and $I = [\alpha, \beta]$ a compact interval. There exists a C^1 diffeomorphism $\varphi\colon I \to \mathbb{R}$ such that $D\varphi(\alpha) = E$, $D\varphi(\beta) = E'$, $|\varphi(I)| = \tau|I|$ and, for every $x \in I$,*

$$|D\varphi(x) - \tau| \leq \max\{E, E'\} - \tau.$$

Proof. Define a continuous function $g\colon I \to \mathbb{R}$ by the following set of conditions: $g(\alpha) = E$, $g(\beta) = E'$, $g((\alpha+\beta)/2) = 2\tau - (E + E')/2$, g is linear on $[\alpha, (\alpha+\beta)/2]$ and $[(\alpha + \beta/2), \beta]$. Define $\varphi\colon I \to \mathbb{R}$ by $\varphi(x) = \int_0^x g(y)dy$. It is easy to check that f satisfies the desired properties apart from the last one. To show the last property, note that $2\tau - (E + E')/2 \leq D\varphi(x) \leq \max\{E, E'\}$ for every $x \in I$. Let $x \in I$ and suppose that $D\varphi(x) - \tau < 0$. Then

$$|D\varphi(x) - \tau| = -D\varphi(x) + \tau \leq (E + E')/2 - \tau \leq \max\{E, E'\} - \tau.$$

If $D\varphi(x) - \tau \geq 0$, then

$$|D\varphi(x) - \tau| = D\varphi(x) - \tau \leq \max\{E, E'\} - \tau.$$

This completes the proof of the lemma. □

Let $\psi\colon \overline{X_{a_0\cdots a_{k-1}\square_i}(h_{n-1})} \to \mathbb{R}$ be a C^1 diffeomorphism for which the conclusion of Lemma 3.3 holds with $E = E_{a_0\cdots a_{k-1}(i-1)}(\phi)$, $E' = E_{a_0\cdots a_{k-1}i}(\phi)$, $\tau = \tau_{a_0\cdots a_{k-1}\square_i}$ and $I = \overline{X_{a_0\cdots a_{k-1}\square_i}(h_{n-1})}$. Lemma 3.2 ensures the condition $2\tau - (E + E')/2 > 0$ in Lemma 3.3. Define

$$f_n(x) = h_n(l) \pm \psi(x) \quad \text{for every } x \in X_{a_0\cdots a_{k-1}\square_i}(h_{n-1}),$$

where l denotes the left boundary point of $X_{a_0\cdots a_{k-1}\square_i}(h_{n-1})$. The sign is $+$ if $Df_0 > 0$ on (r, s) and $-$ otherwise. This finishes the definition of f_n. Lemma 3.3 implies that $f_n\colon X \to [0, 1]$ is C^1 and satisfies (E1).

LEMMA 3.4. *The following holds for every $\phi \in \hat{\mathscr{U}}$, every integer $N > \hat{N}$ and the sequences $\{h_n(\phi, N)\}_{n\geq 0}$ and $\{f_n(\phi, N)\}_{n\geq 0}$: let $n \geq N + 1$. For every $k \in \{N, \ldots, n\}$, every $a_0 \cdots a_{k-1} \in W_k$, every $i \in \{1, \ldots, p-1\}$ and every $x \in X_{a_0\cdots a_{k-1}\widehat{i}}(h_{n-1})$, we have*

$$||Df_n(x)| - \tau_{a_0\cdots a_{k-1}\square_i}| \leq M(e^{-2\phi})V_{a_0\cdots a_{k-1}}(\phi)E^2_{a_0\cdots a_{k-1}}(\phi)K(f_0).$$

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Moreover, for every $a_0 \cdots a_{N-1} \in W_N$, *every* $i \in \{1, \ldots, p-1\}$ *and every* $x \in X_{a_0 \cdots a_{N-1}\widehat{i}}(f_0)$, *we have*

$$||Df_N(x)| - \tau_{a_0 \cdots a_{N-1}\square_i}| \le 2M(e^{-\phi-\phi_0})V_{a_0 \cdots a_{N-1}}(\phi_0)E^2_{a_0 \cdots a_{N-1}}(\phi_0)K(f_0).$$

Proof. From Lemma 3.3 and Lemma 3.2(a), we have

$$\begin{aligned} ||Df_n(x)| - \tau_{a_0 \cdots a_{k-1}\square_i}| &\le \max\{E_{a_0 \cdots a_{k-1}(i-1)}(\phi),\, E_{a_0 \cdots a_{k-1}i}(\phi)\} - \tau_{a_0 \cdots a_{k-1}\square_i} \\ &\le E_{a_0 \cdots a_{k-1}}(\phi) - \tau_{a_0 \cdots a_{k-1}\square_i} \\ &\le M(e^{-2\phi})V_{a_0 \cdots a_{k-1}}(\phi)E^2_{a_0 \cdots a_{k-1}}(\phi)K(f_0). \end{aligned}$$

A proof of the second inequality in the lemma is analogous and hence we omit it. □

3.4. *Cauchy property.* Starting from a C^2 map $f_0 \in \mathscr{E}$, we have constructed a sequence $\{f_n(\phi, N)\}_{n\ge 0}$ of C^1 maps on X. We show that $\{f_n\}_{n\ge 0}$ is a Cauchy sequence in $\mathscr{E}$ which is contained in a C^1 neighborhood of f_0.

LEMMA 3.5. *For every* $\varepsilon > 0$, *there exist a neighborhood* $\mathscr{U}$ *of* $\phi_0 = \log|Df_0| \circ \pi_{f_0}$ *in* $C(\Sigma_p)$ *and* $N_0 \ge 1$ *such that the following holds for every* $\phi \in \mathscr{U}$, *every integer* $N > N_0$ *and the sequence* $\{f_n(\phi, N)\}_{n\ge 0}$:

(a) *for every* $n \ge N$,

$$\|f_0 - f_n\|_{C^1} \le \varepsilon;$$

(b) $\{f_n\}_{n\ge 0}$ *is a Cauchy sequence in* $\mathscr{E}$ *with respect to the norm* $\|\cdot\|_{C^1}$.

Proof. Let $\varepsilon > 0$. Let $\phi \in \hat{\mathscr{U}}$ and $N > \hat{N}$ be an integer. Depending on ε, we will choose ϕ that is sufficiently close to ϕ_0 and then choose a sufficiently large N.

We first estimate $\|f_0 - f_N\|_{C^1}$. Let $a_0 \cdots a_{N-1} \in W_N$ and $i \in \{1, \ldots, p-1\}$. From the construction, $|f_0(x) - f_N(x)| \le |X_{a_1 \cdots a_{N-1}\widehat{i}}(f_0)|$ holds for every $x \in X_{a_0 \cdots a_{N-1}\widehat{i}}(f_0)$. The result (E2) for f_0 implies that $|X_{a_1 \cdots a_{N-1}\widehat{i}}(f_0)| \le c\lambda^{-N+1}$. Hence,

$$\sup_X |f_0 - f_N| \le c\lambda^{-N+1} \le \frac{\varepsilon}{4}, \tag{6}$$

where the last inequality holds for sufficiently large N.

Let $a_0 \cdots a_{N-1} \in W_N$ and $i \in \{1, \ldots, p-1\}$. For every $x \in X_{a_0 \cdots a_{N-1}\widehat{i}}(f_0)$, we have

$$|Df_0(x) - Df_N(x)| \le ||Df_0(x)| - \tau_{a_0 \cdots a_{N-1}\square_i}| + |\tau_{a_0 \cdots a_{N-1}\square_i} - |Df_N(x)||.$$

The second term is bounded by Lemma 3.4. We estimate the first term. If $|Df_0(x)| \le \tau_{a_0 \cdots a_{N-1}\square_i}$, then from $\tau_{a_0 \cdots a_{N-1}\square_i} \le E_{a_0 \cdots a_{N-1}}(\phi)$ in Lemma 3.2(a) and from Lemma 2.1 we have

$$\begin{aligned} ||Df_0(x)| - \tau_{a_0 \cdots a_{N-1}\square_i}| &\le E_{a_0 \cdots a_{N-1}}(\phi) - E_{a_0 \cdots a_{N-1}}(\phi_0) \\ &\le V_{a_0 \cdots a_{N-1}}(\phi) + M(e^{\phi} - e^{\phi_0}). \end{aligned}$$

If $|Df_0(x)| > \tau_{a_0 \cdots a_{N-1}\square_i}$, then from $|Df_0(x)| \le F_{a_0 \cdots a_{N-1}}(\phi_0)$ and the second inequality in Lemma 3.2(a) we have

$$\begin{aligned} ||Df_0(x)| - \tau_{a_0 \cdots a_{N-1}\square_i}| \le{}& V_{a_0 \cdots a_{N-1}}(\phi_0) + 2M(e^{-\phi-\phi_0})V_{a_0 \cdots a_{N-1}}(\phi_0) \\ &+ M(e^{-\phi-\phi_0})M(e^{\phi_0} - e^{\phi})E^2_{a_0 \cdots a_{N-1}}(\phi_0)K(f_0). \end{aligned}$$

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It follows that

$$\sup_X |Df_0 - Df_N| \le \frac{\varepsilon}{4}, \tag{7}$$

provided ϕ is sufficiently close to ϕ_0 and N is sufficiently large. From (6) and (7), we obtain

$$\|f_0 - f_N\|_{C^1} \le \frac{\varepsilon}{2}. \tag{8}$$

Let $\varepsilon_0, c_0, \lambda_0$ be the constants for which the conclusion of Lemma 2.2 holds with respect to f_0. Let $\varepsilon \in (0, \varepsilon_0)$. Let m, n be integers with $m > n \ge N$. We estimate $\|f_n - f_m\|_{C^1}$. If $x \in X$ is contained in a gap of order $\le n$, we have $h_m(x) = h_n(x)$ and thus $f_m(x) = f_n(x)$. Suppose that $x \in X$ is not contained in any gap of order $\le n$. Then there exist $a_0 \cdots a_{n-1} \in W_n$ and $i \in \{1, \dots, p-1\}$ such that $x \in X_{a_0 \cdots a_{n-1}\widehat{i}}(h_{n-1})$. The construction of $\{h_n\}_{n \ge 0}$ in §3.1 implies that

$$\begin{aligned} |f_n(x) - f_m(x)| &\le \sup_{X_{a_0 \cdots a_{n-1}\widehat{i}}(h_{n-1})} |h_n - h_m| \\ &= |X_{a_1 \cdots a_{n-1}\widehat{i}}(h_{n-1})| \\ &= |X_{a_1 \cdots a_{n-1}\widehat{i}}(f_{n-1})|. \end{aligned}$$

Since $x \in X_{a_0 \cdots a_{n-1}\widehat{i}}(h_{n-1})$ and $a_0 \cdots a_{n-1} \in W_n$ are arbitrary, we obtain

$$\sup_X |f_n - f_m| \le \sup_{\substack{a_1 \cdots a_{n-1} \in W_{n-1} \\ i \in \{1, \dots, p-1\}}} |X_{a_1 \cdots a_{n-1}\widehat{i}}(f_{n-1})|.$$

Proceeding to the estimate of derivatives, again let $x \in X$ and let $a_0 \cdots a_{n-1} \in W_n$, $i \in \{1, \dots, p-1\}$ be such that $x \in X_{a_0 \cdots a_{n-1}\widehat{i}}(h_{n-1})$. We treat two cases separately.

Case I. x is not contained in a gap of order $\le m$. We have $|Df_n(x)| = E_{a_0 \cdots a_{n-1}(i-1)}(\phi)$ or $|Df_n(x)| = E_{a_0 \cdots a_{n-1} i}(\phi)$, and $|Df_m(x)| = E_{a_0 \cdots a_{m-1}}(\phi)$. Hence,

$$|Df_n(x) - Df_m(x)| \le V_{a_0 \cdots a_{n-1}}(\phi).$$

Case II. x is contained in a gap of order $k \in [n, m]$. Let $a_0 \cdots a_{k-1} \in W_k$ and $j \in \{1, \dots, p-1\}$ be such that $x \in X_{a_0 \cdots a_{k-1}\square_j}(h_{m-1})$. We have

$$\begin{aligned} |Df_n(x) - Df_m(x)| \le\ & ||Df_n(x)| - \tau_{a_0 \cdots a_{n-1}\square_i}| + |\tau_{a_0 \cdots a_{n-1}\square_i} - \tau_{a_0 \cdots a_{k-1}\square_j}| \\ & + |\tau_{a_0 \cdots a_{k-1}\square_j} - |Df_m(x)||. \end{aligned}$$

The first and the third terms are bounded by Lemma 3.4. For the second term, Lemma 3.2(b) gives

$$|\tau_{a_0 \cdots a_{n-1}\square_i} - \tau_{a_0 \cdots a_{k-1}\square_j}| \le (1 + M(e^{2\phi})K(f_0))V_{a_0 \cdots a_{n-1}}(\phi).$$

Hence, we obtain

$$\begin{aligned} |Df_n(x) - Df_m(x)| \le\ & 2M(e^{-2\phi}) \sum_{\ell \in \{k,n\}} V_{a_0 \cdots a_{\ell-1}}(\phi) E^2_{a_0 \cdots a_{\ell-1}}(\phi) K(f_0) \\ & + (1 + M(e^{2\phi})K(f_0))V_{a_0 \cdots a_{n-1}}(\phi). \end{aligned}$$

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Since $x \in X_{a_0 \cdots a_{n-1}\widehat{i}}(h_{n-1})$ and $a_0 \cdots a_{n-1} \in W_n$ are arbitrary, we obtain

$$\sup_X |Df_n - Df_m| \le \text{const} \cdot \sup_{a_0 \cdots a_{n-1} \in W_n} V_{a_0 \cdots a_{n-1}}(\phi),$$

where the multiplicative constant only depends on ϕ and f_0. Overall, for every integers m, n with $m > n \ge N$,

$$\begin{aligned} \|f_n - f_m\|_{C^1} \le & \sup_{\substack{a_1 \cdots a_{n-1} \in W_{n-1} \\ i \in \{1,\ldots,p-1\}}} |X_{a_1 \cdots a_{n-1}\widehat{i}}(f_{n-1})| \\ & + \text{const} \cdot \sup_{a_0 \cdots a_{n-1} \in W_n} V_{a_0 \cdots a_{n-1}}(\phi). \end{aligned} \tag{9}$$

Since $f_{N-1} = f_0$, for every $n \ge N+1$ we have

$$\begin{aligned} \|f_N - f_n\|_{C^1} \le & \sup_{\substack{a_1 \cdots a_{N-1} \in W_{N-1} \\ i \in \{1,\ldots,p-1\}}} |X_{a_1 \cdots a_{N-1}\widehat{i}}(f_0)| \\ & + \text{const} \cdot \sup_{a_0 \cdots a_{N-1} \in W_N} V_{a_0 \cdots a_{N-1}}(\phi) \le \frac{\varepsilon}{2}, \end{aligned}$$

where the last inequality holds provided ϕ is sufficiently close to ϕ_0 and N is sufficiently large depending on ϕ. From this and (8), we obtain $\|f_0 - f_n\|_{C^1} \le \varepsilon$ for every $n \ge N$. Lemma 2.2 gives $|X_{a_1 \cdots a_{n-1}\widehat{i}}(f_{n-1})| \le c_0^{-1}\lambda_0^{-n}$ and so the first term of (9) converges to zero as $n \to \infty$. The convergence of the second term follows from the uniform continuity of ϕ. Equation (9) implies that $\{f_n\}_{n\ge 0}$ is a Cauchy sequence. □

3.5. *End of the proof of the realization theorem.* We complete the proof of the realization theorem.

Proof of the realization theorem. Let f_∞ denote the limit of the Cauchy sequence $\{f_n\}_{n\ge 0}$. Then f_∞ satisfies (E1). By Lemma 3.5, $\|f_0 - f_\infty\|_{C^1} \le \varepsilon$ holds. Lemma 2.2 implies that f_∞ satisfies (E2). If $x \in \Lambda(f_\infty)$ and $\pi_{f_\infty}^{-1}(x) = \{a_n\}_{n\ge 0}$, then $x \in \bigcap_{n\ge 0} X_{a_0 \cdots a_n}(f_\infty)$. By construction, $|Df_n(x)| \equiv E_{a_0 \cdots a_n}(\phi)$ holds for every $n \ge N$ and therefore

$$\log |Df_\infty(x)| = \lim_{n\to\infty} \log |Df_n(x)| = \lim_{n\to\infty} \log E_{a_0 \cdots a_n}(\phi) = \phi(\pi_{f_\infty}^{-1}(x)),$$

which yields $\log |Df_\infty| \circ \pi_{f_\infty} = \phi$. □

4. *On the proof of Theorem B*

In this last section we prove Theorem B. In §4.1 we recall the result of Contreras [**9**] on ergodic optimization for expanding maps. In §4.2 we show that any map in $\mathscr{E}$ is approximated by another whose derivative is locally constant. In §4.3 we refine the construction in §3.1 and prove a Lipschitz version of the realization theorem. In §4.4 we complete the proof of Theorem B.

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4.1. *The result of Contreras on optimization by periodic measures.* Let (Y, d) be a compact metric space and ϕ a real-valued Lipschitz continuous function on Y. The Lipschitz norm of ϕ is given by

$$\|\phi\|_{\mathrm{Lip}} = \max_Y |\phi| + \sup_{\substack{y,z\in Y \\ y\neq z}} \frac{|\phi(y)-\phi(z)|}{d(y,z)}.$$

Let $\mathrm{Lip}(Y, d)$ denote the space of Lipschitz continuous functions on Y endowed with the topology given by the norm $\|\cdot\|_{\mathrm{Lip}}$. A Lipschitz continuous map $T: Y \circlearrowleft$ is *expanding* if there exist an integer $m \geq 1$ and a constant $\lambda > 1$ such that for every $y \in Y$ there exist $\ell_y \in \{1, \ldots, m\}$, a neighborhood U_y of y in Y and continuous maps $S_i: U_y \to Y$, $i \in \{1, \ldots, \ell_y\}$, such that the following holds:

- $S_i(U_y) \cap S_j(U_y) = \emptyset$ if $i \neq j$;
- $T^{-1}(U_y) = \bigcup_{i=1}^{\ell_y} S_i(U_y)$;
- $T(S_i(z)) = z$ for every $i \in \{1, \ldots, \ell_y\}$ and every $z \in U_y$;
- $d(z, w) \geq \lambda d(S_i(z), S_i(w))$ for every $i \in \{1, \ldots, \ell_y\}$ and every $z, w \in U_y$.

THEOREM 2. [**9**, Theorem A] *Let (Y, d) be a compact metric space and $T: Y \circlearrowleft$ an expanding map. There exists an open dense subset $\mathscr{O}_{(Y,d)}$ of $\mathrm{Lip}(Y, d)$ such that for every $\phi \in \mathscr{O}_{(Y,d)}$ there exists a unique ϕ-minimizing measure and it is supported on a periodic orbit.*

We will apply Theorem 2 to the left shift $\sigma: \Sigma_p \circlearrowleft$. The topology of Σ_p is equivalent to the topology generated by the distance d_θ given by

$$d_\theta(\underline{a}, \underline{b}) = \begin{cases} 0 & \text{if } \underline{a} = \underline{b}, \\ \theta^{s(\underline{a},\underline{b})} & \text{otherwise,} \end{cases}$$

where $\theta \in (0, 1)$, $\underline{a} = \{a_n\}_{n\geq 0}$, $\underline{b} = \{b_n\}_{n\geq 0}$ and $s(\underline{a}, \underline{b}) = \min\{n \geq 0: a_n \neq b_n\}$. Note that σ is expanding in the above sense.

4.2. *Approximation by maps with locally constant derivatives.* For $f \in \mathscr{E}$ we introduce the following additional condition:

(E3) there exists an integer $n \geq 1$ such that the value of Df is constant on each connected component of $\bigcap_{k=0}^{n-1} f^{-k}(X)$.

LEMMA 4.1. *Let $f_0 \in \mathscr{E}$. For any $\varepsilon > 0$, there exists $f \in \mathscr{E}$ which satisfies $\|f_0 - f\|_{C^1} \leq \varepsilon$ and (E3).*

Proof. Let $f_0 \in \mathscr{E}$ and $\varepsilon > 0$. From (E2), there exists an integer $n_0 \geq 1$ such that for every $n \geq n_0$ and every $a_0 \cdots a_{n-1} \in W_n$,

$$|X_{a_0\cdots a_{n-1}}(f_0)| \leq \frac{\varepsilon}{2}. \tag{10}$$

Since Df_0 is uniformly continuous, there exists an integer $n_1 \geq 1$ such that for every $n \geq n_1$ and every $a_0 \cdots a_{n-1} \in W_n$,

$$\sup_{x,y\in X_{a_0\cdots a_{n-1}}(f_0)} |Df_0(x) - Df_0(y)| \leq \frac{\varepsilon}{4}. \tag{11}$$

Put $n_2 = \max\{n_0, n_1\}$. Let $n \geq n_2$. Define $f \in \mathscr{E}$ so that the following holds.

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- (on the complement of the union of gaps of order $\le n-1$) Let $a_0 \cdots a_{n-1} \in W_n$. Then $f(X_{a_0 \cdots a_{n-1}}(f_0)) = X_{a_1 \cdots a_{n-1}}(f_0)$ and, for every $x \in X_{a_0 \cdots a_{n-1}}(f_0)$,
$$Df(x) = \pm \frac{|X_{a_1 \cdots a_{n-1}}(f_0)|}{|X_{a_0 \cdots a_{n-1}}(f_0)|}.$$
The sign is + if $Df_0(x) > 0$ and – otherwise.
- (on gaps of low order) Let $k \in \{1, \ldots, n_2 - 1\}$, $i \in \{1, \ldots, p-1\}$ and let G be a gap of order k. Then
$$\sup_G |f_0 - f| + \sup_G |Df_0 - Df| \le \varepsilon. \tag{12}$$
- (on gaps of high order) Let $k \in \{n_2, \ldots, n-1\}$, $i \in \{1, \ldots, p-1\}$ and consider the gap $(l, r) = X_{a_0 \cdots a_{k-1} \square_i}(h_{n-1})$ of order k. Let $\psi : [l, r] \to \mathbb{R}$ be a C^1 diffeomorphism for which the conclusion of Lemma 3.3 holds with $E = Df(l)$, $E' = Df(r)$, $\tau = |f_0(l) - f_0(r)|/(l - r)$ and $I = [l, r]$. Then $f(x) = f_0(l) + \psi(x)$ holds for every $x \in (l, r)$.

Let $a_0 \cdots a_{n-1} \in W_n$. From the definition of f and (10), we have
$$\sup_{X_{a_0 \cdots a_{n-1}}(f_0)} |f_0 - f| \le |X_{a_1 \cdots a_{n-1}}(f_0)| \le \frac{\varepsilon}{2}. \tag{13}$$
From the definition of f and (11), we have
$$\sup_{X_{a_0 \cdots a_{n-1}}(f_0)} |Df_0 - Df| \le \frac{\varepsilon}{4}. \tag{14}$$
Let $k \in \{n_2, \ldots, n-1\}$, $i \in \{1, \ldots, p-1\}$ and consider the gap $X_{a_0 \cdots a_{k-1} \square_i}(h_{n-1})$ of order k. Let $x \in X_{a_0 \cdots a_{k-1} \square_i}(h_{n-1})$. From (10),
$$|f_0(x) - f(x)| \le |f_0(X_{a_0 \cdots a_{k-1} \square_i}(h_{n-1}))| \le \frac{\varepsilon}{2}. \tag{15}$$
From Lemma 3.3 and (11),
$$|Df_0(x) - Df(x)| \le ||Df_0(x)| - \tau| + |\tau - |Df(x)|| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4}. \tag{16}$$
From (12)–(16), $\|f_0 - f\|_{C^1} \le \varepsilon$ follows. By Lemma 2.2, (E2) holds for f. Hence, $f \in \mathscr{E}$. Condition (E3) follows from the definition of f. □

4.3. *Lipschitz continuity of Df_∞.* The next lemma is a version of the realization theorem.

LEMMA 4.2. *Let $f_0 \in \mathscr{E}$ satisfy (E3) and let $\theta \in (0, 1/\sup_X |Df_0|)$. For every $\varepsilon > 0$, there exists a neighborhood $\mathscr{U}$ of $\log |Df_0| \circ \pi_{f_0}$ in $C(\Sigma_p)$ such that for every $\phi \in \mathscr{U} \cap \mathrm{Lip}(\Sigma_p, d_\theta)$ there exists $f_\infty \in \mathscr{E}_{\mathrm{Lip}}$ such that*
$$\|f_0 - f_\infty\|_{C^1} \le \varepsilon \quad \textit{and} \quad \log |Df_\infty| \circ \pi_{f_\infty} = \phi.$$

Proof. Let $\varepsilon > 0$ be such that $\theta \le 1/(\sup_X |Df_0| + \varepsilon)$. Although f_0 may not be of class C^2, from (E3) the estimate as in Lemma 2.3 continues to hold and thus the construction in §3.1 continues to work: there exists a neighborhood $\mathscr{U}$ of $\log |Df_0| \circ \pi_{f_0}$ in $C(\Sigma_p)$ such that for every $\phi \in \mathscr{U}$ there exists a Cauchy sequence $\{f_n\}_{n \ge 0}$ in $\mathscr{E}$ such that $\|f_0 - f_n\|_{C^1} \le$

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ε for every $n \geq 1$ and its C^1 limit f_∞ belongs to $\mathscr{E}$ and satisfies $\log |Df_\infty| \circ \pi_{f_\infty} = \phi$. We have $\sup_X |Df_n| \leq \sup_X |Df_0| + \varepsilon$. Let $a_0 \cdots a_{n-1} \in W_n$ and $i \in \{1, \ldots, p-1\}$. From $X_{a_0 \cdots a_{n-1}\widehat{i}}(h_n) = X_{a_0 \cdots a_{n-1}\widehat{i}}(f_n)$, $f^n(X_{a_0 \cdots a_{n-1}\widehat{i}}(f_n)) = [\alpha_{i-1}, \beta_i]$ and the mean value theorem, we have

$$|X_{a_0 \cdots a_{n-1}\widehat{i}}(h_n)| \geq \frac{|f^n(X_{a_0 \cdots a_{n-1}\widehat{i}}(f_n))|}{(\sup_X |Df_n|)^n} \geq \frac{\beta_i - \alpha_{i-1}}{(\sup_X |Df_0| + \varepsilon)^n}.$$

Let $\phi \in \mathrm{Lip}(\Sigma_p, d_\theta)$. Let $L(\phi)$ denote the Lipschitz constant of ϕ. Then

$$V_{a_0 \cdots a_{n-1}}(\phi) \leq M(e^\phi) L(\phi) \theta^n$$

and therefore

$$\frac{V_{a_0 \cdots a_{n-1}}(\phi)}{|X_{a_0 \cdots a_{n-1}\widehat{i}}(h_n)|} \leq \frac{M(e^\phi)L(\phi)}{\beta_i - \alpha_{i-1}} \theta^n (\sup_X |Df_0| + \varepsilon)^n \leq \frac{M(e^\phi)L(\phi)}{\beta_i - \alpha_{i-1}}.$$

Lemma 3.1 implies that

$$|X_{a_0 \cdots a_{n-1}\square_i}(h_n)| \geq \frac{1}{2K(f_0)+1} |X_{a_0 \cdots a_{n-1}\widehat{i}}(h_n)|.$$

Hence,

$$\frac{V_{a_0 \cdots a_{n-1}}(\phi)}{|X_{a_0 \cdots a_{n-1}\square_i}(h_n)|} \leq (2K(f_0)+1) \frac{V_{a_0 \cdots a_{n-1}}(\phi)}{|X_{a_0 \cdots a_{n-1}\widehat{i}}(h_n)|} \leq (2K(f_0)+1) \frac{M(e^\phi)L(\phi)}{\beta_i - \alpha_{i-1}}.$$

From this estimate and the construction of $\{f_n\}_{n \geq 0}$ in §3.3, it follows that the Lipschitz constant of Df_n is uniformly bounded over all n. The Lipschitz continuity of Df_∞ is a direct consequence of the uniform convergence of Df_n to Df_∞. □

4.4. *End of proof of Theorem B.* To finish, we need an auxiliary lemma.

LEMMA 4.3. *Let $f_0 \in \mathscr{E}$. There exist $\theta \in (0, 1)$ and a neighborhood $\mathscr{W}$ of f_0 in $\mathscr{E}$ such that for every $f \in \mathscr{W}$, $\pi_f \colon \Sigma_p \to \Lambda(f)$ is Lipschitz continuous with respect to the distance d_θ.*

Proof. By Lemma 2.2, there exist constants $c > 0$, $\lambda > 1$ and a neighborhood $\mathscr{W}$ of f_0 in $\mathscr{E}$ such that the following holds for every $f \in \mathscr{W}$: if $n \geq 1$ and $x \in \bigcap_{k=0}^{n-1} f^{-k}(X)$, then $|Df^n(x)| \geq c\lambda^n$. Let $\theta \in [\lambda^{-1}, 1)$. For every $\underline{a}, \underline{b} \in \Sigma_p$, $\underline{a} \neq \underline{b}$ we have $|\pi_f(\underline{a}) - \pi_f(\underline{b})| \leq c\lambda^{-s(\underline{a},\underline{b})} \leq c\theta^{s(\underline{a},\underline{b})} = c d_\theta(\underline{a}, \underline{b})$. □

Proof of Theorem B. Set

$$\mathscr{O} = \left\{ f \in \mathscr{E}_{\mathrm{Lip}} \colon \log |Df| \circ \pi_f \in \bigcup_{0<\theta<1} \mathscr{O}_{(\Sigma_p, d_\theta)} \right\}.$$

From Theorem 2, for maps in $\mathscr{O}$ the Lyapunov minimizing measure is unique and it is supported on a periodic orbit. From Lemma 4.3, $\mathscr{O}$ is an open subset of $\mathscr{E}_{\mathrm{Lip}}$.

It remains to show that $\mathscr{O}$ is dense in $\mathscr{E}$. Let $f_0 \in \mathscr{E}$ satisfy (E3). By virtue of Lemma 4.1, it suffices to show that f_0 is approximated by elements of $\mathscr{O}$. Let $\theta \in (0, 1/\sup_X |Df_0|)$. Since $\phi_0 = \log |Df_0| \circ \pi_{f_0}$ is a locally constant function,

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$\phi_0 \in \mathrm{Lip}(\Sigma_p, d_\theta)$ holds. By Lemma 4.2, for every $\varepsilon > 0$ there exists a neighborhood $\mathscr{U}$ of ϕ_0 in $C(\Sigma_p)$ such that for every $\phi \in \mathscr{U} \cap \mathrm{Lip}(\Sigma_p, d_\theta)$ there exists $f_\infty \in \mathscr{E}_{\mathrm{Lip}}$ such that $\|f_0 - f_\infty\|_{C^1} \leq \varepsilon$ and $\log |Df_\infty| \circ \pi_{f_\infty} = \phi$. Since $\mathscr{O}_{(\Sigma_p, d_\theta)}$ is a dense subset of $\mathrm{Lip}(\Sigma_p, d_\theta)$ from Theorem 2, $\mathscr{U} \cap \mathscr{O}_{(\Sigma_p, d_\theta)} \neq \emptyset$ holds. If $\phi \in \mathscr{U} \cap \mathscr{O}_{(\Sigma_p, d_\theta)}$, then $f_\infty \in \mathscr{O}$. □

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