THE REVIEW OF SYMBOLIC LOGIC
Volume 6, Number 4, December 2013

ON THE GENERAL INTERPRETATION OF FIRST-ORDER QUANTIFIERS

G. ALDO ANTONELLI
University of California, Davis

Abstract. While second-order quantifiers have long been known to admit nonstandard, or "general" interpretations, first-order quantifiers (when properly viewed as predicates of predicates) also allow a kind of interpretation that does not presuppose the full power-set of that interpretation's first-order domain. This paper explores some of the consequences of such "general" interpretations for (unary) first-order quantifiers in a general setting, emphasizing the effects of imposing various further constraints that the interpretation is to satisfy.

§1. Introduction. It has been known since Henkin (1950) that second-order quantifiers admit of interpretations on which they are taken to range over some collection of subsets of the domain, a collection that possibly falls short of the full power set. To capture these interpretations formally, Henkin introduced the notion of a *general model* for second-order logic. A general model $\mathfrak{M}$ supplies, besides an interpretation function I for the extra-logical constants, also a sequence $(D_1, D_2, \ldots)$, where D_1 is a nonempty domain of objects, and each D_{n+1} is a collection of n-place relations over D_1 (by identifying subsets of D_1 with the corresponding sets of 1-tuples, we can assume that $D_2 \subseteq \mathscr{P}(D_1)$). On Henkin's account, D_1 is the range for the first-order variables, whereas each D_{n+1} provides the range for second-order relational variables whose values are n-place relations over D_1. In order for the all the instances of the second-order comprehension schema to be satisfied, Henkin also assumed each domain D_{n+1} to be closed under Boolean operations (intersection and complement, say) as well to contain all relations R that can be obtained as projections of relations $S \in D_{n+2}$ (see Henkin, 1996; Enderton, 2009).

There is an asymmetry in Henkin's semantics between the first- and the second-order case, which becomes evident from the more abstract point of view of the theory of generalized quantifiers introduced by Mostowski (1957), Lindström (1966) and Montague (1974). According to the theory, a first-order quantifier Q is a predicate over the power set of the first-order domain D_1: for example, the first-order existential quantifier "there is at least one" denotes the collection of all nonempty subsets of D_1, and the quantifier "there exist exactly k" denotes the collection of all k-membered subsets. A quantifier behaves precisely as such a predicate: when applied to a formula φ, the quantifier Q returns value "true" precisely when the extension of φ in D_1 falls under Q. The asymmetry in Henkin's semantics lies in the fact that whereas the determination of semantic values for the second-order quantifiers requires the specification of various second-order domains, the meaning of the first-order quantifiers is assumed to be fixed by the specification of a single first-order domain. In other words, there is an implicit bias in Henkin's semantics towards the standard interpretations of first-order quantifiers, according to which they are construed as

Received: December 6, 2012.

© Association for Symbolic Logic, 2013

doi:10.1017/S1755020313000270

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

predicates over the full power set of D_1. But it is in fact clear that just specifying a domain of objects *underdetermines* the semantics of first-order quantifiers.

Symmetry can be restored by extending the general semantics to the first-order case. Once first-order quantifiers are conceived as second-order predicates, it becomes apparent that it is indeed possible to develop a general, nonstandard semantics on which they are interpreted as predicates ranging—not over the full power set of the first-order domain—but over *some* collection of subsets. For instance, the first-order existential quantifier $\exists$ can come to denote, on such a semantics, a collection of nonempty subsets that may well be short of the collection of *all* nonempty subsets.

The possibility of such general interpretations is both technically interesting and philosophically relevant. This paper is concerned with the former aspect, by presenting a rigorous formal account of the basic technical notions underpinning the general interpretation of first-order quantifiers, while the philosophical implications are developed in a parallel paper (Antonelli, forthcoming). We will consider a first-order language $\mathcal{L}$ comprising, besides predicate symbols, also any number of first-order quantifiers Q (some of which might or might not be the ordinary quantifiers $\exists$ and $\forall$). Models for such a language will have the form (D_1, D_2, I) providing, in addition to an interpretation I for the predicates, two distinct domains: a first-order domain D_1 of objects and a second-order domain D_2 of subsets of D_1. It is important to keep in mind that both D_1 and D_2 are only needed for the interpretation of first-order notions—a point emphasized by the similar notation adopted for each. The domain D_2 is only "second-order" in the sense that it is comprised of subsets of D_1, but no truly second-order notions are involved.

Furthermore, we restrict ourselves to the case of quantifiers that are both *unary* and *monadic*, that is, quantifiers that, just like $\exists$ and $\forall$, take exactly one formula as argument, binding a unique specific variable in that formula (other variables may occur free, but are treated as parameters). This fact explains why, in contrast to Henkin's semantics for second-order logic, general models need only supply the two domains D_1 and D_2, instead of the multiple domains D_n for $n > 0$. Such domains would be needed were we to extend the framework to the fully general case of quantifiers that are n-ary (applying to n formulas) and k-adic (simultaneously binding k variables), and quantifiers such as the nonlinear quantifiers of Henkin (1961) require more complex still nonstandard models (see López-Escobar, 1991).

Equally crucial is the restriction throughout to a *purely relational* language $\mathcal{L}$, that is, a language with any number of predicate symbols of any number of arguments (including the identity predicate $\doteq$), but no individual constants or function symbols. This allows us to focus on the *modal* properties of the quantifiers, which include the way they interact with Boolean connectives. As an example consider the K schema for the ordinary universal quantifier: $\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$. One way to read the K schema is as stating that the universal quantifier is preserved by supersets: if (commuting the antecedents) $\forall$ applies to φ, and ψ is a superset of φ, then $\forall$ applies to ψ as well. As such, the K schema expresses a form of monotonicity that might in principle pertain to other quantifiers as well (such as, for instance, "there are at least three"). But it is of course accidental that "ψ is a superset of φ" can be expressed using *the same* quantifier $\forall$ that is being claimed to be monotonic, a fact that does not generalize to other monotonic quantifiers (such as Q_1 mentioned below).

It is also intuitively clear why the presence of singular terms would interfere with a proper understanding of general semantics: singular terms are unconstrained in their taking denotations in D_1, thereby giving access to the "dark corners" of the first-order domain

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

where the light of the quantifiers does not shine (i.e., less figuratively, allowing reference to objects that fall outside the range of the quantifiers). Of course a case can be made that some quantifiers are essentially characterized by the way they interact with singular terms, for instance in the schema of universal instantiation, $\forall x\varphi \rightarrow \varphi[^{t}/_{x}]$. In interpreting the schema as asserting that instantiation holds throughout the range of applicability of the quantifiers, we again reveal a bias in favor of the standard interpretation. We will see that perhaps a better way to express this fact—that quantifiers instantiate over their range—is by using the more broadly valid schema $\forall x(\forall y\varphi[^{y}/_{x}] \rightarrow \varphi)$.

The central result of our investigations is that each model $\mathfrak{M}$ is equivalent to its "hull," that is, the model having the same first-order domain and the same interpretation of the predicates as $\mathfrak{M}$, but in which the second-order domain is only comprised of those subsets definable in $\mathfrak{M}$. Philosophically, such "lean" interpretations not only point to the fact the standard interpretations of first-order logic "overshoot the target" in some informal but substantial sense, but also provide a possible criterion of demarcation between genuinely first-order and genuinely second-order notions. They also play a prominent technical role in establishing the correspondence between the "logical" properties of models (the schema therein satisfied) and the "algebraic" properties (the various closure conditions characterizing the second-order domain). For instance, in the case of the existential quantifier, we show that there is a connection between the class of general interpretations satisfying a certain closure condition and the outer-domain models of free logic (thereby showing that the set of validities for the class in question is axiomatizable). Another prominent application will deal with the case of quantifiers expressing *large* properties, that is, those quantifiers that denote a filter over the second-order domain.

It appears that the idea of providing nonstandard interpretations for the first-order quantifiers, even in a generalized setting, first made its appearance with Thomason & Johnson (1969). Arguably, Thomason & Johnson were the first to combine the generalized viewpoint (consideration of quantifiers other than $\exists$ and $\forall$) with the nonstandard approach (consideration of models that in some way or other "liberalize" the interpretation of the quantifiers). Their approach is both more and less general than the present one. It is more general in that they consider ordinary first-order logic augmented with quantifier *variables* (while the present approach deals with quantifier *constants*): accordingly they consider what formulas $Qx\varphi(x)$ are valid under *all* interpretations of the quantifier variable Q. As it turns out, doing so introduces intrinsically higher-order features in the semantics, as shown by Väänänen (1978) (improving upon previous results of Yasuhara (1969)). But the approach of Thomason & Johnson is also less general than the present one in that they restrict themselves to quantifiers that satisfy a form of permutation invariance, and moreover (as mentioned) only in the context of logics extending ordinary first-order logic. The ordinary $\exists$ and $\forall$ are required to express invariance under permutation, but in their absence the logic of quantifier variables becomes much weaker—decidable, in fact (see Anapolitanos & Väänänen, 1981). The notion of permutation invariance is also used by Thomason & Johnson to identify a class of nonstandard models, by allowing models in which the interpretations of the quantifier variables are not required to be invariant under *all* permutations, but just those in a given collection. In contrast, below we will consider a logic with quantifier *constants*, which are not necessarily permutation invariant, and without assuming that the ordinary quantifiers are available in the language. (Although Väänänen (1978) shows how, over first-order logic, quantifier variables are equivalent to an equi-cardinality quantifier constant, this result does not extend to other constants and depends on the availability of $\exists$ and $\forall$.) Nonstandard models for the logic presented below

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

will not be obtained by weakening some invariance condition, but by keeping the meaning of each quantifier fixed while varying the second-order domain to which it is applied.

Another prominent tradition that is relevant to our project is the one originating with Keisler's classic (1970), which employs a nonstandard semantics for the first-order quantifier constant Q_1 ("there exist uncountably many"), in order to show that the set of validities of such a quantifier (now on the standard interpretation) can be recursively axiomatized. In particular, Keisler introduces countable nonstandard structures for the language with the quantifier Q_1: so-called "weak models" in which the quantifier is reinterpreted to apply to countable, definable subsets of the domain (ω_1 iterations of an omitting types construction are then deployed to blow up such subsets to uncountable size while keeping the remaining subsets countable). What is relevant for our purposes is that Keisler's weak-model semantics is quite different from the one offered here both in inspiration and execution. In the first place Keisler presupposes the ordinary quantifiers $\exists$ and $\forall$, which are needed to provide the required axiomatization for Q_1 (to express that Q_1 is monotonic, for instance); whether these quantifiers are present or not can make a substantial difference, as we learn from Anapolitanos & Väänänen (1981). But perhaps most importantly, on Keisler's weak models the meaning of the quantifier Q_1 differs from the meaning it has in the standard case: on such interpretations, "uncountably many" comes to denote collections of subsets of the domain that are, in fact, countable. In contrast, we expend some effort below to make sure that we compare the behavior of *the same* quantifier on the standard vs. the nonstandard interpretations. Such a task is complicated by the fact that we do not have (to the best of our knowledge) a completely general criterion for when a quantifier Q over a domain D_1 is *the same as* a quantifier Q' over D'_1. But we do have answers in some cases: for instance if, say, $\mathsf{Q} = \exists$, then we can say that in each case the quantifier applies to all and only the nonempty subsets of the interpretation's second-order domain. In a slightly more general setting, in our case we can compare quantifiers Q and Q' over (D_1, D_2) and (D'_1, D'_2) if $D_1 \subseteq D'_1$ and similarly $D_2 \subseteq D'_1$ for the corresponding second-order domains.

§2. Basic semantic notions.

DEFINITION 2.1. *Let $\mathfrak{s}$ be a relational and quantificational* signature*, comprised of predicate symbols $P_1, P_2, \ldots$, of varying number of arguments, and unary monadic quantifier symbols $\mathsf{Q}_1, \mathsf{Q}_2, \ldots$ applying to a formula and binding one variable.*

DEFINITION 2.2. *The language determined by the signature $\mathfrak{s}$ is the smallest set $\mathcal{L}(\mathfrak{s})$ of formulas such that:*

1. *If P is a k-place predicate symbol in $\mathfrak{s}$ and $x_1, \ldots, x_k$ are variables then $Px_1 \ldots x_k$ is in $\mathcal{L}(\mathfrak{s})$.*
2. *If x and y are variables, then $x \doteq y$ is in $\mathcal{L}(\mathfrak{s})$.*
3. *if φ and ψ are in $\mathcal{L}(\mathfrak{s})$, then so are $\neg\varphi$ and $\varphi \to \psi$.*
4. *If Q is a unary quantifier symbol in $\mathfrak{s}$, φ is a formula in $\mathcal{L}(\mathfrak{s})$, and x is a variable, then $\mathsf{Q}x\varphi$ is in $\mathcal{L}(\mathfrak{s})$.*

The notions of free and bound occurrences of variables are defined as usual. When we want to make the (finitely many) quantifier symbols of the language explicit, we write $\mathcal{L}(\mathsf{Q}_1, \ldots, \mathsf{Q}_n)$.

Definition 2.2 could, in principle, be easily generalized to arbitrary k-adic, n-ary quantifiers.

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

DEFINITION 2.3. *A* model *for a signature* $\mathfrak{s}$ *is a triple* $\mathfrak{M} = (D_1, D_2, I)$*, where* D_1 *and* D_2 *are both nonempty, with* $D_2 \subseteq \mathscr{P}(D_1)$*, and* I *is a function assigning an interpretation* $I(P) = P^{\mathfrak{M}}$ *to each predicate symbol* P *in the signature.*

DEFINITION 2.4. *A* model $\mathfrak{M} = (D_1, D_2, I)$ *is* standard *if* $D_2 = \mathscr{P}(D)$*, and it is* replete *if* $D_1 \in D_2$.

We will often be comparing models $\mathfrak{M} = (D_1, D_2, I)$ and $\mathfrak{M}' = (D_1, D_2', I)$ that differ at most in their second-order domain, but are otherwise the same.

DEFINITION 2.5. *A monadic, first-order, unary* quantifier *is a function* Q *assigning a predicate* $\mathsf{Q}(D_1)$ *over* $\mathscr{P}(D_1)$ *to each nonempty domain* D_1.

By a slight abuse of language, we use the same symbol Q for both the quantifier *symbol* in the signature $\mathfrak{s}$ and the function assigning the predicate $\mathsf{Q}(D_1)$ to each domain D_1. According to the definition, for example, the quantifier $\exists$ is a function that assigns to each domain D_1 a predicate over $\mathscr{P}(D_1)$, denoted by $\exists(D_1)$, which is true of a subset $X \subseteq D_1$ if and only if $X \neq \varnothing$. Similarly, the quantifier $\forall$ provides a predicate $\forall(D_1)$ which is true of X if and only if $X = D_1$. Many other quantifiers play an important role, for instance $\exists!$ ("there is exactly one") and Q_α ("there exist at least $\aleph_\alpha$"—so that Q_0 is the infinity quantifier, Q_1 is Keisler's quantifier, etc.); see Peters & Westerståhl (2006) for a comprehensive treatment.

Just like $\exists$ and $\forall$ are dual of each other in standard first-order logic, the notion can be extended to any unary quantifier.

DEFINITION 2.6. *For any quantifier* Q*, the* dual *of* Q *is the quantifier* Q^d *such that for any domain* D_1 *and* $X \subseteq D_1$*:* $X \in \mathsf{Q}^d(D_1)$ *if and only if* $D_1 \setminus X \notin \mathsf{Q}(D_1)$.

DEFINITION 2.7. *Given a model* $\mathfrak{M} = (D_1, D_2, I)$ *for the signature* $\mathfrak{s}$*, let* $\mathsf{Q}^{\mathfrak{M}} = \mathsf{Q}(D_1) \cap D_2$ *denote the restriction of* $\mathsf{Q}(D_1)$ *to* D_2*. The predicate* $\mathsf{Q}^{\mathfrak{M}}$ *is the interpretation assigned by the model to the quantifier symbol* Q.

Definition (2.7) provides a way to view quantifiers as constant across domains. We can say that $\mathsf{Q}^{\mathfrak{M}}$ is the same quantifier as $\mathsf{Q}^{\mathfrak{M}'}$ because they are the restrictions of the same quantifier Q to different domains. We now proceed with the notion of satisfaction, which generalizes the ordinary notion of satisfaction for first-order languages.

DEFINITION 2.8. *An* assignment *s of objects from* D_1 *to the variables is a function such that* $s(x) \in D_1$ *for each variable* x*. If* $d \in D_1$*, then* $s[^d/_x]$ *is the assignment such that* $s[^d/_x](x) = d$ *and* $s[^d/_x](y) = s(y)$ *if* y *is a variable other than* x*. We say that* $s[^d/_x]$ *results from* shifting *s at* x.

Notation 2.1 *If* $\boldsymbol{x} = x_1, \ldots, x_n$ *is a sequence of variables and* $\boldsymbol{d} = d_1, \ldots, d_n$ *a sequence members of* D_1*, the multiple shift* $s[^{d_1}/_{x_1}, \ldots, ^{d_n}/_{x_n}] = s[^{d_1}/_{x_1}], \ldots, [^{d_n}/_{x_n}]$ *is sometimes denoted by* $s[^{\boldsymbol{x}}/_{\boldsymbol{d}}]$*, or even* $s[^{x_i}/_{d_i}]$.

DEFINITION 2.9. *We define satisfaction of a formula* φ *in* $\mathcal{L}(\mathfrak{s})$*, by an assignment* s *of objects in* D_1 *to the variables, in a model* $\mathfrak{M} = (D_1, D_2, I)$*, by recursion on* φ*:*

$$\begin{aligned} \mathfrak{M}, s \models Px_1 \ldots x_k \quad &\textit{iff} \quad \langle s(x_1), \ldots, s(x_k)\rangle \in P^{\mathfrak{M}}; \\ \mathfrak{M}, s \models x \doteq y \quad &\textit{iff} \quad s(x) = s(y); \\ \mathfrak{M}, s \models \neg\varphi \quad &\textit{iff} \quad \mathfrak{M}, s \not\models \varphi; \end{aligned}$$

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

$$\mathfrak{M}, s \models \varphi \to \psi \quad \textit{iff} \quad \mathfrak{M}, s \not\models \varphi \ \textit{or}\ \mathfrak{M}, s \models \psi;$$
$$\mathfrak{M}, s \models \mathsf{Q}x\varphi \quad \textit{iff} \quad \{d \in D_1 : \mathfrak{M}, s[^d/_x] \models \varphi\} \in \mathsf{Q}^{\mathfrak{M}}.$$

In ordinary first-order logic it is customary to handle formulas with free variables the same as their universal closure. But of course in the more general case, where quantifiers other than $\exists$ and $\forall$ might or might not be present, special arrangements must be made.

DEFINITION 2.10. *If φ is a formula (with zero or more free variables) then we say that φ is true in $\mathfrak{M}$, written $\mathfrak{M} \models \varphi$, if and only if $\mathfrak{M}, s \models \varphi$ for* every *assignment s. Moreover, two models are $\mathcal{L}(\mathfrak{s})$-equivalent if the same $\mathcal{L}(\mathfrak{s})$-sentences are true in each.*

It is convenient to introduce a notation for the extension of formula in a model, relative to a subset of its free variables and an assignment. We consider the general case of a formula with free variables $x_1, \ldots x_n$ and parameters instantiating the variable $y_1, \ldots y_m$.

DEFINITION 2.11. *Where $\boldsymbol{x} = x_1, \ldots, x_n$ and $\boldsymbol{y} = y_1, \ldots, y_m$ are vectors of variables, $\mathfrak{M}$ is a model, $\varphi(\boldsymbol{x}, \boldsymbol{y})$ a formula, and s an assignment, then:*

$$[\![\varphi]\!]^{\mathfrak{M}}_{s,\boldsymbol{x}} = \{\langle d_1, \ldots d_n\rangle \in D_1^n : \mathfrak{M}, s[^{d_1}/_{x_1}, \ldots, ^{d_n}/_{x_n}] \models \varphi\}.$$

The values $s(y_1), \ldots, s(y_m)$ function as parameters in the previous definition. Using this notation the satisfaction clause for the quantifier in Definition 2.9 becomes:

$$\mathfrak{M}, s \models \mathsf{Q}x\varphi \quad \text{iff} \quad [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \in \mathsf{Q}^{\mathfrak{M}},$$

where we identify a subset $X \subset D_1$ with the set X^1 of its 1-tuples, as already pointed out. If the model is understood, we drop the superscript from $[\![\varphi]\!]^{\mathfrak{M}}_{s,x}$.

REMARK 2.12. *A central theme in the paper is the possibility of imposing certain further restrictions on the second-order domain D_2 of a model. For instance, consider that the inference,*

$$\frac{\mathsf{Q}x(\varphi(x) \wedge \psi(x))}{\mathsf{Q}x\varphi(x)}$$

which is validated by both $\exists$ and $\forall$ on their standard interpretations (but not, for instance, by $\exists!$). Such an inference is not validated by those same quantifiers over general models. But it is of course valid over the class of models $\mathfrak{M} = (D_1, D_2, I)$ in which D_2 is closed under $\subseteq$, in the sense that if $X \in D_2$ and $X \subseteq Y$, then also $Y \in D_2$. This follows since in every $\mathfrak{M}$:

$$[\![\varphi(x) \wedge \psi(x)]\!]_{s,x} \subseteq [\![\varphi(x)]\!]_{s,x}.$$

This observation will play a central role in Section §6. Notice that there are two considerations at work here, that is, whether the quantifier Q is monotonic (which both $\exists$ and $\forall$ are), and whether the second-order domain D_2 of any given model is closed under supersets. And so, accordingly, there are two possible sources of the failure of the inference above.

DEFINITION 2.13. *A general model $\mathfrak{M} = (D_1, D_2, I)$ is* closed under duals *if and only if for any quantifier Q in the signature $\mathfrak{s}$:*

$$\mathfrak{M}, s \models \mathsf{Q}^d x\varphi(x) \Longleftrightarrow \mathfrak{M}, s \models \neg\mathsf{Q}x\neg\varphi(x).$$

REMARK 2.14. *General models are not necessarily closed under duals. Let $\mathfrak{M}$ be a nonreplete general model, that is, such that $D_1 \notin D_2$. Then $\mathfrak{M} \models \neg\exists x\neg(x \doteq x)$ (indeed,*

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

this formula is true in every general model), but $\mathfrak{M} \not\models \forall x(x \doteq x)$. *Conversely, if* $\mathfrak{M}$ *is replete, then* $\mathfrak{M} \models \forall x(x \doteq x)$. *More generally* $\forall x(x \doteq x)$ *is valid in any models in which* D_2 *is nonempty and satisfies the closure under supersets condition mentioned in Remark 2.12. (Notice also that in our counter-example the failure of duality is due to the second-order domain's being nonstandard and is not due to the specific properties of the quantifier* Q.*)*

We close this section by briefly taking up the issue of permutation invariance, which according to a tradition going back to Tarski (1986), is regarded as the proper criterion demarcating logical notions, including quantifiers. Cardinality quantifiers such as "there exists," "exactly three," etc. are all permutation invariant.

DEFINITION 2.15. *A quantifier* Q *over a model* $\mathfrak{M} = (D_1, D_2, I)$ *is* permutation invariant *if and only if, for any permutation* π *of* D_1 *and subset* $A \in D_2$, *it holds that* $A \in \mathsf{Q}^{\mathfrak{M}} \iff \pi[A] \in \mathsf{Q}^{\mathfrak{M}}$, *where* $\pi[A]$ *is the point-wise image of* A *under* π.

Permutation invariance might fail in the context of general models. However, there are, again, two sources of such failures for a quantifier Q over a domain D_1: (*a*) the predicate $\mathsf{Q}(D_1)$ might apply to some subsets but not to their images under permutation; or (*b*) the underlying domain D_2 itself might not be closed under permutation. As to (*a*), some quantifiers lack invariance even on the standard interpretation: consider for instance so-called "Montagovian" quantifiers of the form $\mathsf{Q}_a(D_1) = \{X \subseteq D_1 : a \in X\}$. But it is equally clear that consideration of general interpretations in (*b*) allows for failure of invariance even for those quantifiers that are invariant on the standard interpretation (such as $\exists$). Permutation invariance (or its failure) will not concern us further here, but if one were interested on pursuing such a notion as a criterion of logicality in the Tarskian tradition, then it appears that the relevant test is whether a quantifier fails to be invariant on second-order domains that are themselves closed under permutations (or *p*-closed, in the terminology of Antonelli 2010).

§3. The Hull theorem. One of the main insights that can be gleaned from consideration of general models for the first-order quantifiers is that in some sense, as mentioned, they overshoot their target, in that there are much leaner models, whose second-order domain is only comprised of the definable subsets, to which they are equivalent. Here are the relevant notions.

DEFINITION 3.1. *A* n*-place relation* $R \subseteq D_1^n$ *is* definable *(from parameters) in a model* $\mathfrak{M} = (D_1, D_2, I)$ *if and only if there is a formula* $\varphi(\boldsymbol{x}, \boldsymbol{y})$ *and an assignment* s *such that* $R = [\![\varphi]\!]^{\mathfrak{M}}_{s,\boldsymbol{x}}$ *(where* $\boldsymbol{x}$ *is an* n*-tuple of free variables, and* $\boldsymbol{y}$ *is a sequence of parameters whose denotation is fixed by the assignment).*

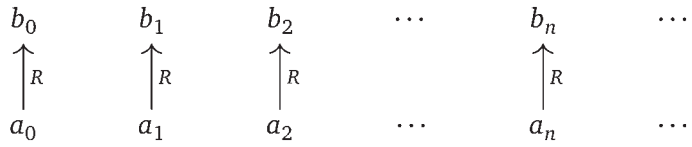
REMARK 3.2. *In Definition 3.1, the notation* $\varphi(\boldsymbol{x}, \boldsymbol{y})$ *indicates that any free variables in* φ *other than the parameters are among* $\boldsymbol{x} = x_1, \ldots, x_n$, *but these variables can occur in* φ *in any particular order. It follows as a result of such a convention that if a binary relation* R *is definable, for instance, so is its converse. This kind of built-in manipulation of the variables will be important in the proof of Theorem 4.8.*

DEFINITION 3.3. *Given a model* $\mathfrak{M} = (D_1, D_2, I)$, *the* hull $H(\mathfrak{M})$ *of* $\mathfrak{M}$ *is the collection of the parameter-definable members of* D_2: $H(\mathfrak{M}) = \{X \in D_2 : X$ *is definable from parameters*$\}$.

If $\mathfrak{M}$ is standard, then $H(\mathfrak{M})$ is just the collection of all parameter-definable subsets of D_1, but if $D_2 \subsetneq \mathscr{P}(D_1)$ then $H(\mathfrak{M})$ might not comprise *all* definable subsets of D_1.

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

Consider for instance the following model $\mathfrak{M}$ for the signature $\mathfrak{s} = \{R, \exists\}$, whose first-order domain is pictured below, where the arrows represent the binary relation R:



Let $A = \{a_i : i \geq 0\}$; the set A is definable if $\mathfrak{M}$ is standard, because then $A = \{x \in D_1 : \exists y Rxy\}$. It remains so definable if $D_2 = \{\{b_i\} : i \geq 0\}$, for then for each $i \geq 0$ the set $\{y : Ra_i y\}$ is a nonempty member of D_2 and so a_i satisfies $\exists y Rxy$. But now suppose that $D_2 = \{\{b_i\} : i > 0\}$ (so that $\{b_0\}$ is missing from the second order domain). Then $\{x \in D_1 : \exists y Rxy\} = A \setminus \{a_0\}$ and A itself ceases to be definable in such a model.

The hull of a model plays a crucial role because in a sense, the sets in the hull are all that is needed in order to fix the meaning of the first-order quantifiers, without any need for the full-fledged power-set of the domain, as borne out in the following theorem.

HULL THEOREM 3.4. *Let $\mathfrak{s}$ be an arbitrary first-order signature, $\mathfrak{M} = (D_1, D_2, I)$ a model for $\mathfrak{s}$; also let $\mathfrak{N} = (D_1, H(\mathfrak{M}), I)$. Then for any formula φ and assignment s:*

$$\mathfrak{M}, s \models \varphi \quad \textit{if and only if} \quad \mathfrak{N}, s \models \varphi.$$

In particular, for any closed formula φ, $\mathfrak{M} \models \varphi$ if and only if $\mathfrak{N} \models \varphi$, and the two models are $\mathscr{L}(\mathfrak{s})$-equivalent.

Proof. By induction on formulas. Since $\mathfrak{M}$ and $\mathfrak{N}$ differ at most on the interpretation of the quantifier symbols, the basis of the induction and the case for Boolean combinations are immediate. For the quantifier case, let Q be any quantifier symbol in the signature. Then $\mathsf{Q}^{\mathfrak{M}}$ and $\mathsf{Q}^{\mathfrak{N}}$ are each the restriction of $\mathsf{Q}(D_1)$ to their respective second-order universes, and since $H(\mathfrak{M}) \subseteq D_2$, we make the observation that $X \in \mathsf{Q}^{\mathfrak{N}}$ implies $X \in \mathsf{Q}^{\mathfrak{M}}$, and conversely, if X is definable in $\mathfrak{M}$ and $X \in \mathsf{Q}^{\mathfrak{M}}$ then $X \in \mathsf{Q}^{\mathfrak{N}}$. The inductive step then is as follows:

$$\begin{aligned}
\mathfrak{M}, s \models \mathsf{Q}x\varphi &\iff [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \in \mathsf{Q}^{\mathfrak{M}}, && \text{def. satisfaction;}\\
&\iff [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \in \mathsf{Q}^{\mathfrak{M}} \cap H(\mathfrak{M}), && \text{def. hull;}\\
&\iff [\![\varphi]\!]^{\mathfrak{N}}_{s,x} \in \mathsf{Q}^{\mathfrak{M}} \cap H(\mathfrak{M}), && \text{ind. hypothesis;}\\
&\iff [\![\varphi]\!]^{\mathfrak{N}}_{s,x} \in \mathsf{Q}^{\mathfrak{N}}, && \text{previous observation;}\\
&\iff \mathfrak{N}, s \models \mathsf{Q}x\varphi, && \text{def. satisfaction.}
\end{aligned}$$

□

Although we have proved the theorem in general for any model $\mathfrak{M}$, it is perhaps most interesting to consider the case where $\mathfrak{M} = (D_1, \mathscr{P}(D), I)$ is standard. Then the model $\mathfrak{N} = (D_1, H(\mathfrak{M}), I)$ is equivalent to it, thereby showing that the full power set is not needed to fix the meaning of the quantifiers. In fact, the hull construction provides a sort of reflective equilibrium, in that its iteration does not produce any new definable sets.

COROLLARY 3.5. *If $\mathfrak{M} = (D_1, D_2, I)$ and $\mathfrak{M}' = (D_1, H(\mathfrak{M}), I)$, then $H(\mathfrak{M}) = H(\mathfrak{M}')$.*

Proof. Suppose $X \in H(\mathfrak{M})$ is definable in $\mathfrak{M}$; then $X \in D_2$ and there are a formula $\varphi(x, \boldsymbol{y})$ (where $\boldsymbol{y}$ is a vector of variables) and an assignment s such that $X = [\![\varphi]\!]^{\mathfrak{M}}_{s,\boldsymbol{x}}$. Since $X \in H(\mathfrak{M})$, and it belongs to the second-order domain of $\mathfrak{M}'$; we need to show that X is

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

also definable in $\mathfrak{M}'$, for then $X \in H(\mathfrak{M}')$. We argue as follows: for every $d \in D_1$:

$$\begin{aligned} d \in X &\iff \mathfrak{M}, s[^d/_x] \models \varphi, && X \text{ definable;} \\ &\iff \mathfrak{M}', s[^d/_x] \models \varphi, && \text{Theorem 3.4.} \end{aligned}$$

For the converse implication, if $X \in H(\mathfrak{M}')$, then also $X \in D_2$, since $H(\mathfrak{M}') \subseteq H(\mathfrak{M}) \subseteq D_2$. Then, by the argument just given, using Theorem 3.4, X is definable in $\mathfrak{M}$, so that $X \in H(\mathfrak{M})$. □

DEFINITION 3.6. *A model $\mathfrak{M} = (D_1, D_2, I)$ is H-closed if and only if $H(\mathfrak{M}) \subseteq D_2$.*

By the previous Corollary 3.5, every model is elementarily equivalent to an H-closed model. The fact that each model is elementarily equivalent to its hull has both philosophical and logical ramifications. Note that the situation is significantly different in the case of second-order logic, where the introduction of first-order definable subsets results in new values for the second-order parameters, so that the construction needs to be iterated transfinitely in order to reach a fixed point. And so for instance, restricting the second-order quantifiers to range over the first-order definable subsets in the standard model $\mathbb{N}$ of arithmetic, results in a model of a theory, ACA_0, which is much weaker than full second-order arithmetic.

REMARK ON DEMARCATION 3.7. *A notion's invariance under restriction to first-order definable subsets, "hull-invariance" (as we might call it) can be used as a* criterion of demarcation *of genuinely first-order notions from second-order ones.*

The Hull theorem also has a converse of sorts.

THEOREM 3.8. *Suppose $\mathfrak{M} = (D_1, D_2, I)$ is H-closed and $\mathfrak{M}^* = (D_1, D_2^*, I)$ is a model such that $D_2 \subseteq D_2^*$ (but is otherwise the same as $\mathfrak{M}$). Then for any assignment s and formula φ:*

$$\mathfrak{M}, s \models \varphi \quad \textit{if and only if} \quad \mathfrak{M}^*, s \models \varphi.$$

Proof. By induction of φ. Since the two models differ only in their second-order domain the basis is immediate and the Boolean cases follow immediately from the inductive hypothesis. For the quantifier case:

$$\begin{aligned} \mathfrak{M}, s \models \mathsf{Q}x\varphi \quad &\Longrightarrow \quad [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \in \mathsf{Q}(D_1) \cap D_2, && \text{def. satisfaction, } \mathsf{Q}^{\mathfrak{M}} \\ &\Longrightarrow \quad [\![\varphi]\!]^{\mathfrak{M}^*}_{s,x} \in \mathsf{Q}(D_1) \cap D_2, && \text{ind. hyp.} \\ &\Longrightarrow \quad [\![\varphi]\!]^{\mathfrak{M}^*}_{s,x} \in \mathsf{Q}(D_1) \cap D_2^*, && D_2 \subseteq D_2^* \\ &\Longrightarrow \quad \mathfrak{M}^*, s \models \mathsf{Q}x\varphi. && \text{def. satisfaction, } \mathsf{Q}^{\mathfrak{M}^*} \end{aligned}$$

Conversely:

$$\begin{aligned} \mathfrak{M}^*, s \models \mathsf{Q}x\varphi \quad &\Longrightarrow \quad [\![\varphi]\!]^{\mathfrak{M}^*}_{s,x} \in \mathsf{Q}(D_1) \cap D_2^*, && \text{def. satisfaction, } \mathsf{Q}^{\mathfrak{M}^*} \\ &\Longrightarrow \quad [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \in \mathsf{Q}(D_1) \cap D_2^*, && \text{ind. hyp.} \\ &\Longrightarrow \quad [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \in \mathsf{Q}(D_1) \cap D_2, && D_2 \text{ is } H\text{-closed} \\ &\Longrightarrow \quad \mathfrak{M}, s \models \mathsf{Q}x\varphi. && \text{def. satisfaction, } \mathsf{Q}^{\mathfrak{M}} \end{aligned}$$ □

In particular, by Theorem 3.8, any H-closed model $\mathfrak{M} = (D_1, D_2, I)$ is $\mathcal{L}(\mathfrak{s})$-equivalent to the corresponding standard model $\mathfrak{M}^* = (D_1, \mathscr{P}(D_1), I)$. Thus we obtain a reduction

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

of the decision problem of $\mathcal{L}(\mathfrak{s})$ on the standard interpretation to that of $\mathcal{L}(\mathfrak{s})$ interpreted over H-closed models.

COROLLARY 3.9. *For each $\mathcal{L}(\mathfrak{s})$-sentence φ, the following are equivalent:*

(a) *φ is valid in the class of standard models for $\mathcal{L}(\mathfrak{s})$;*
(b) *φ is valid in the class of H-closed models for $\mathcal{L}(\mathfrak{s})$;*
(c) *φ is valid in the class of those models $\mathfrak{M}$ for $\mathcal{L}(\mathfrak{s})$ whose second-order domain is precisely $H(\mathfrak{M})$.*

§4. Building hulls from below. As regards the construction of hulls, it might be objected that such a construction still relies on the notion of satisfaction in the original model, so that we cannot convincingly claim to have dispensed with the full power-set of D_1. Consider again the case of a standard $\mathfrak{M}$; in order to define the hull of $\mathfrak{M}$ we still need to use the notion of satisfaction in $\mathfrak{M}$, which makes use of the quantifiers in their standard interpretation. But the objection can be circumvented, by giving a construction of $H(\mathfrak{M})$ "from below," without going through the notion of satisfaction in $\mathfrak{M}$. As in other cases, although we are interested in the case where $\mathfrak{M}$ is standard, the definition is general. We start by identifying a set of Gödel operations on sets of n-tuples.

DEFINITION 4.1 (Gödel operations). *Given a signature $\mathfrak{s}$ and domain D_1, for each n let $[D_1]^n$ be the set of all n-place relations over D_1, that is, $[D_1]^n = \mathscr{P}(D_1^n)$. For any domain D_1, and $D_2 \subseteq \mathscr{P}(D_1)$, define the following* Gödel operations *$F_0, \ldots, F_6$, F_a, F_{Q} (where $a \in D_1$ and Q is a quantifier symbol in the signature), having domain $\bigcup_{n>0}[D_1]^n$:*

$$
\begin{aligned}
F_0(X,Y) &= X \times Y, && \textit{cartesian prod.;}\\
F_1(X,Y) &= X \cap Y, && \textit{intersection;}\\
F_2(X) &= \{\langle x_1,\ldots,x_n\rangle : \langle x_1,\ldots,x_n\rangle \notin X\}, && \textit{complement;}\\
F_3(X) &= \{\langle x_2, x_1 \ldots, x_n\rangle : \langle x_1, x_2 \ldots, x_n\rangle \in X\}, && \textit{permute head;}\\
F_4(X) &= \{\langle x_1, x_1, \ldots, x_n\rangle : \langle x_1,\ldots,x_n\rangle \in X\}, && \textit{duplicate head;}\\
F_5(X) &= \{\langle x_1, \ldots, x_n\rangle : \langle x_1, x_1, \ldots, x_n\rangle \in X\}, && \textit{collapse head;}\\
F_6(X) &= \{\langle x_n, x_1, \ldots, x_{n-1}\rangle : \langle x_1,\ldots,x_n\rangle \in X\}, && \textit{rotate;}\\
F_a(X) &= \{\langle x_1, \ldots, x_n\rangle : \langle a, x_1, \ldots, x_n\rangle \in X\}, && \textit{parametrize;}\\
F_{\mathsf{Q}}(X) &= \{\langle x_1,\ldots,x_n\rangle : \{y : \langle y, x_1, \ldots, x_n\rangle \in X\} \in \mathsf{Q}(D_1) \cap D_2\} && \mathsf{Q}\textit{-project.}
\end{aligned}
$$

Gödel operations as a tool for the analysis of definability have been around since Gödel (1940), where they were used in the context of ZFC. There are important differences between the set-theoretic case and the present setting: obviously the presence of the operations F_a, needed to handle parameters, gives an infinite set of operations, but also the operation F_{Q} generalizes Gödel's projection function, which is used to represent the existential quantifier.

DEFINITION 4.2. *Given a model $\mathfrak{M} = (D_1, D_2, I)$, we say that an n-place relation R over D_1, with $n \geq 1$ is* constructible *in $\mathfrak{M}$ if it can be obtained by repeated application of $F_0, \ldots, F_6$, F_a, F_{Q} to the relations $P^{\mathfrak{M}}$ and D_1 (which we identify with $\{\langle x\rangle : x \in D_1\}$). We say that a set $X \subseteq D_1$ is constructible if its point-wise image under the 1-tuple operation, X^1, is constructible.*

In general, we will, without further notice, identify a set $X \subseteq D_1$ with X^1, including the case where $X = D_1$.

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

REMARK 4.3. *The identity relation* $\boldsymbol{d} = \{\langle x, x\rangle : x \in D_1\}$ *can be obtained as* $F_4(D_1)$.

DEFINITION 4.4. *Given* $\mathfrak{M} = (D_1, D_2, I)$, *let* $L(\mathfrak{M})$ *be the collection of all constructible subsets of* D_1. *We also say that the model* $\mathfrak{M}$ *itself is constructible if* $D_2 = L(\mathfrak{M})$.

Our next task is to show that $H(\mathfrak{M}) = L(\mathfrak{M})$, thereby showing that the set of all definable subsets of D_1 can be obtained from below by means of the Gödel operations. To this end, we begin with a special case.

LEMMA 4.5. *Suppose* $R \subseteq \mathscr{P}(D_1^n)$ *is definable by a quantifier-free, parameter-free formula in* $\mathfrak{M}$. *Then* R *is constructible in* $\mathfrak{M}$.

Proof. Suppose that R is definable by a quantifier-free, parameter-free formula in the model $\mathfrak{M} = (D_1, D_2, I)$. We need to show that R is constructible. We consider a representative example. Let $\varphi(x, y, z)$ be the formula

$$Pxx \wedge \neg Pxy \wedge z \neq x;$$

we show how the relation $\{\langle x, y, z\rangle : \varphi(x, y, z)\} \subseteq D_1^3$ can be obtained by repeated applications of the Gödel operations to D_1 and $P^{\mathfrak{M}}$. Since in this case we have no parameters, the assignment is irrelevant and we declutter notation by dropping any mention of the assignment s altogether. With this understanding, we proceed to define the following relations $S_1, \ldots, S_{10}$ over D_1:

$$\begin{aligned}
S_1 &= \{\langle x, x\rangle : Pxx\} = \boldsymbol{d} \cap P^{\mathfrak{M}};\\
S_2 &= \{\langle x\rangle : Pxx\} = F_5(S_1);\\
S_3 &= \{\langle x, y, z\rangle : Pxx\} = S_2 \times D_1 \times D_1;\\
S_4 &= \{\langle x, y\rangle : \neg Pxy\} = F_2(P^{\mathfrak{M}});\\
S_5 &= \{\langle x, y, z\rangle : \neg Pxy\} = S_4 \times D_1;\\
S_6 &= \{\langle x, y, z\rangle : Pxx \wedge \neg Pxy\} = S_3 \cap S_5;\\
S_7 &= \{\langle z, x\rangle : z \neq x\} = F_2(\boldsymbol{d});\\
S_8 &= \{\langle z, x, y\rangle : z \neq x\} = S_7 \times D_1;\\
S_9 &= \{\langle x, y, z\rangle : z \neq x\} = F_6(F_6(S_8));\\
S_{10} &= \{\langle x, y, z\rangle : Pxx \wedge \neg Pxy \wedge z \neq x\} = S_6 \cap S_9;\\
&= \{\langle x, y, z\rangle : \varphi(x, y, z)\}.
\end{aligned}$$

□

LEMMA 4.6. *Suppose the relation* $R \in [D_1]^{n+m}$ *is definable in* $\mathfrak{M} = (D_1, D_2, I)$ *by a parameter-free formula* $\varphi(y_1, \ldots, y_n, x_1, \ldots, x_m)$. *If* R *is constructible, then so is the relation* S *defined by fixing the first* n *arguments of* φ *with parameters* $a_1, \ldots, a_n \in D$: *where* s *is an assignment such that* $s(y_i) = a_i$,

$$S = \{\langle d_1, \ldots, d_m\rangle : \mathfrak{M}, s[{}^{d_1}/_{x_1} \ldots {}^{d_m}/_{x_m}] \models \varphi\}.$$

Proof. If R is constructible, then so is

$$S = F_{a_n}(\cdots F_{a_1}(R)\cdots) = \{\langle x_1, \ldots, x_m\rangle : \langle a_1, \ldots, a_n, x_1, \ldots, x_m, \rangle \in R\}.$$ □

THEOREM 4.7. *Suppose* $R \subseteq \mathscr{P}(D_1^n)$ *is definable in* $\mathfrak{M}$. *Then* R *is constructible in* $\mathfrak{M}$.

Proof. Fix a model $\mathfrak{M}$ and assignment s. We proceed by induction on the number q of nested quantifiers in a formula $\varphi(x_1, \ldots, x_m)$ defining R. In view of Lemma 4.6 we can

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

assume φ to be parameter-free. If $q = 0$ we invoke Lemma 4.5. So suppose $q > 0$ and without loss of generality assume that φ has the form $\mathsf{Q}y\psi(y, x_1, \ldots, x_n)$ (if not, successively apply the argument to each maximal subformula of φ that does have this form). Notice also that by repeated application of rotation and permutation (F_3 and F_6) we can also assume that the variable of quantification y occurs first. By inductive hypothesis, since ψ has fewer nested quantifiers, there is a constructible relation S over D_1 such that

$$S = \{\langle b, a_1, \ldots, a_n\rangle : \mathfrak{M}, s[{}^{b}/_{y}, {}^{a_i}/_{x_i}] \models \psi(y, x_1, \ldots, x_n)\}.$$

Then we have:

$$\begin{aligned}
F_{\mathsf{Q}}(S) &= \{\langle a_1, \ldots, a_n\rangle : \{a : \langle a, a_1, \ldots, a_n\rangle \in S\} \in \mathsf{Q}(D_1) \cap D_2\}, && \text{def. } F_{\mathsf{Q}};\\
&= \{\langle a_1, \ldots, a_n\rangle : \{a : \mathfrak{M}, s[{}^{b}/_{y}, {}^{a_i}/_{x_i}] \models \psi\} \in \mathsf{Q}^{\mathfrak{M}}\}, && \text{deff. } S, \mathsf{Q}^{\mathfrak{M}};\\
&= \{\langle a_1, \ldots, a_n\rangle : \{a : \mathfrak{M}, s[{}^{a_i}/_{x_i}][{}^{b}/_{y}] \models \psi\} \in \mathsf{Q}^{\mathfrak{M}}\}, && y \text{ distinct from } x_i;\\
&= \{\langle a_1, \ldots, a_n\rangle : \mathfrak{M}, s[{}^{a_i}/_{x_i}] \models \mathsf{Q}y\psi\}, && \text{def. satisfaction.}
\end{aligned}$$

□

THEOREM 4.8. *Suppose $R \subseteq \mathscr{P}(D_1^n)$ is constructible in $\mathfrak{M}$. Then R is definable in $\mathfrak{M}$ (from parameters).*

Proof. By induction on the number of iterations of the Gödel operations: D_1 is definable by $x \doteq x$ and $P^{\mathfrak{M}}$ is definable by $Px_1 \ldots x_n$. Based on the definitions of $F_0 \ldots, F_6, F_a$ and F_{Q} we show that the result of their application to definable relations is again definable.

- Suppose $X, Y \subseteq D_1$ are definable by $\varphi(x)$ and $\psi(y)$; then $F_0(X, Y) = X \times Y$ is definable by $\varphi(x) \wedge \psi(y)$.
- Suppose $X, Y \subseteq D_1^n$ are definable by $\varphi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$; then $F_1(X, Y) = X \cap Y$ is definable by $\varphi(\boldsymbol{x}) \wedge \psi(\boldsymbol{x})$.
- If X is definable by φ then $F_2(X)$ is definable by $\neg\varphi$.
- F_3 is the first of the operations whose definability depends on the details of the definition of $[\![\varphi]\!]_{s,\boldsymbol{x}}$ (see Remark 3.2). Suppose $X \subseteq D_1^n$ is definable by φ, then:

  $$X = \{\langle d_1, \ldots d_n\rangle \in D_1^n : \mathfrak{M}, s[{}^{d_1}/_{x_1}, \ldots, {}^{d_n}/_{x_n}] \models \varphi\};$$

  it follows that $F_3(X)$ is also definable as $\{\langle d_2, d_1, \ldots d_n\rangle \in D_1^n : \mathfrak{M}, s[{}^{d_1}/_{x_1}, \ldots, {}^{d_n}/_{x_n}] \models \varphi\}$.
- Suppose X is definable by $\varphi(x_1, \ldots, x_n)$; then $F_4(X)$ is definable by the formula

  $$x_0 \doteq x_1 \wedge \varphi(x_1, \ldots, x_n).$$

- Suppose $X \subseteq D_1^{n+1}$ is definable by $\varphi(x_0, x_1, \ldots x_n)$; then:

  $$F_5(X) = \{\langle d_1, \ldots, d_n\rangle \in D_1^n : \mathfrak{M}, s[{}^{d_1}/_{x_0}, {}^{d_1}/_{x_1} \ldots, {}^{d_n}/_{x_n}] \models \varphi\}.$$

- Suppose $X \subseteq D_1^n$ is definable by $\varphi(x_1, \ldots, x_n)$, then:

  $$F_6(X) = \{\langle d_n, d_1, \ldots, d_{n-1}\rangle \in D_1^n : \mathfrak{M}, s[{}^{d_1}/_{x_1} \ldots, {}^{d_n}/_{x_n}] \models \varphi\}.$$

- Suppose $X \subseteq D_1^{n+1}$ is definable by $\varphi(x_0, x_1, \ldots x_n)$; then:

  $$F_a(X) = \{\langle d_1, \ldots, d_n\rangle \in D_1^n : \mathfrak{M}, s[{}^{a}/_{x_0}, {}^{d_1}/_{x_1} \ldots, {}^{d_n}/_{x_n}] \models \varphi\}.$$

- Finally, observe that if $X \subseteq D_1^{n+1}$ is defined by $\varphi(y, x_1, \ldots, x_n)$, then $F_{\mathsf{Q}}(X)$ is obviously defined by $\mathsf{Q}y\varphi(y, x_1, \ldots, x_n)$. □

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

REMARK 4.9. *It might seem that in the proof of Theorem 4.8 the details of the definition of* $[\![\varphi]\!]_{s,\boldsymbol{x}}$ *might be avoided by introducing extra bound variables. So for instance, if* $\varphi(x_1, x_2, \ldots, x_n)$ *defines* $X \subseteq D^n$*, then* $F_3(X)$*, permuting the first two elements of each* n*-tuple, is defined as the set of those* n*-tuples* $x_1, x_2, \ldots x_n$ *such that:*

$$\exists y \exists z[y \doteq x_1 \wedge z \doteq x_2 \wedge \varphi(z, y, \ldots, x_n)].$$

But of course this only works not only on the assumption that $\exists$ *is available in the signature, but also that it receives a standard interpretation.*

COROLLARY 4.10. $H(\mathfrak{M}) = L(\mathfrak{M})$ *for each model* $\mathfrak{M}$.

Proof. By Theorems 4.7 and 4.8. □

§5. Outer-domain models for the existential quantifier. In this section we consider general models for $\mathcal{L}(\exists)$, the language whose signature contains the only quantifier symbol $\exists$ denoting the ordinary existential quantifier. A natural family of questions to consider is whether the set of all sentence valid on a given class **C** of models is decidable, where **C** can range from the class of *all* general models at one end, to the class of all *standard* models at the other end. Obviously, validity on all standard models is undecidable, since this is the same as the class of sentences valid on the ordinary semantics for first-order logic; but we also know from Corollary 3.9 that the set of $\mathcal{L}(\exists)$-formulas that are valid over H-closed models is the same as the set of validities over standard models and therefore undecidable. On the other hand, the logic determined by the class of *all* general models appears to be quite weak indeed. While we plan to investigate the issue of decidability of this last class elsewhere, we observe that the situation is in a sense the inverse of that investigated by Andréka *et al.* (1998): Andréka *et al.* introduce *generalized assignment models*, where the interpretation of the quantifiers is standard, but assignments are drawn from a class that might fall short of the whole function space VAR $\rightarrow$ D_1; as a result, validity over such models becomes decidable. In contrast, in this paper assignments to the variables are unconstrained, but the quantifiers receive a nonstandard interpretation; the resulting logic is also likely to be decidable.

In this section, however, we rather turn our attention to closure conditions on the second-order domains of models that fall short of H-closure while still giving an expressive logic. In particular, we will identify a property of general models for $\mathcal{L}(\exists)$ such that each model having such a property turns out to be equivalent to an outer-domain model of the sort developed for the semantics of positive free logic by Leblanc & Thomason (1968), and vice versa. Since the outer-domain models give rise to an axiomatizable set of validities, the same holds for the general models with the property in question. It should be noted that in both cases the quantifier $\exists$ selects, in any model, the nonempty members of the second-order domain, and so it is constant across models of the two classes.

DEFINITION 5.1. *An outer-domain model* $\mathfrak{I} = (D, O, I)$ *provides an interpretation* I *for the predicate symbols, a nonempty outer domain* O*, and a possibly empty inner domain* $D \subseteq O$*. The interpretation* $I(P) = P^{\mathfrak{I}}$ *is a relation over the outer domain* O*, that is, where* P *is a* k*-place predicate symbol then* $P^{\mathfrak{I}} \subseteq O^k$.

Truth in an outer-domain model is defined by restricting the quantifier $\exists$ to range over the inner domain I.

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

DEFINITION 5.2. *Given an outer-domain model $\mathfrak{I} = (D, O, I)$ and an assignment s of objects in O to the variables, satisfaction is defined in the usual way for atomic formulas and Boolean combinations, while for the quantifier case:*

$$\mathfrak{I}, s \models \exists x\varphi \iff \text{for some } d \in D : \mathfrak{I}, s[^d/_x] \models \varphi.$$

REMARK 5.3. *Notice that on the definition, $\neg\exists x\neg\varphi$ is satisfied in $\mathfrak{I}$ by s if and only $[\![\varphi]\!]_{s,x} \supseteq D$, which gives the right truth condition for $\forall$ when the quantifier is construed to range over the inner domain D. In this sense, inner-outer domain models satisfy the duality of $\forall$ and $\exists$ and so we can use $\forall x\varphi$ as an abbreviation for $\neg\exists x\neg\varphi$.*

One connection between outer-domain models and general models is easily uncovered.

THEOREM 5.4. *Let $\mathfrak{I} = (D, O, I)$ be an outer-domain model, and let $\mathfrak{M} = (D_1, D_2, I)$ be defined in such a way that $D_1 = O$ and $D_2 = \{X \subseteq O : X \cap D \neq \varnothing\}$ (retaining the same interpretation function I). Then for any assignment s of objects from $D_1 = O$ to the variables and any formula φ:*

$$\mathfrak{I}, s \models \varphi \iff \mathfrak{M}, s \models \varphi.$$

Proof. By induction on formulas. Since $\mathfrak{I}$ and $\mathfrak{M}$ have the same first-order domain and agree on the interpretation of the predicates, the only nontrivial case is the one for the existential quantifier:

$$\begin{aligned}
\mathfrak{I}, s \models \exists x\varphi &\iff \text{for some } d \in D : \mathfrak{I}, s[^d/_x] \models \varphi, && \text{def. satisfaction;}\\
&\iff \text{for some } d \in D : \mathfrak{M}, s[^d/_x] \models \varphi, && \text{ind. hypothesis;}\\
&\iff [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \cap D \neq \varnothing, && \text{re-writing;}\\
&\iff [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \in D_2, && \text{def. } \mathfrak{M};\\
&\iff \mathfrak{M}, s \models \exists x\varphi, && \text{def. satisfaction.}
\end{aligned}$$

□

The converse of Theorem 5.4 fails in the general case, that is, it is not true that for any general model $\mathfrak{M}$ there is an equivalent outer-domain model $\mathfrak{I}$. This follows from Remark 2.12, since the inference from $\exists x(\varphi(x) \wedge \psi(x))$ to $\exists x\varphi(x)$ is valid over outer-domain models but not over general models. The inference is valid, of course, over models whose second-order domain is closed under supersets, and in fact it only requires closure under supersets of nonempty members of D_2, that is, models such that $\varnothing \neq X \subseteq Y$ and $X \in D_2$ imply $Y \in D_2$. There is, however, a more general condition on general models that gives full equivalence to outer-domain models.

DEFINITION 5.5. *A general model $\mathfrak{M} = (D_1, D_2)$ is* closed under (nonempty) overlaps *if and only if $X \in D_2$ and $Y \cap X \neq \varnothing$ imply $Y \in D_2$.*

THEOREM 5.6. *Let $\mathfrak{M} = (D_1, D_2, I)$ be a general model closed under overlaps. We define an outer-domain model $\mathfrak{I} = (D, O, I)$ by putting $O = D_1$ and letting D be obtained by selecting a (not necessarily distinct) object d from each nonempty $X \in D_2$. As to the interpretation function I, we just set $P^{\mathfrak{I}} = P^{\mathfrak{M}}$ for each predicate P in the signature. Then for any assignment s of objects from the common domain $D_1 = O$ of $\mathfrak{I}$ and $\mathfrak{M}$ to the variables and any formula φ:*

$$\mathfrak{I}, s \models \varphi \iff \mathfrak{M}, s \models \varphi.$$

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

Proof. By induction on formulas. As before, since the two models differ only in the interpretation of the quantifiers, we deal with that case only. We treat the two halves of the biconditional separately, as the argument is somewhat different for each. For the direct implication:

$$\begin{aligned}
\mathfrak{I}, s \models \exists x\varphi &\Longrightarrow \text{for some } d \in D : \mathfrak{I}, s[^{d}/_{x}] \models \varphi, && \text{def. satisfaction;}\\
&\Longrightarrow \text{for some } d \in D : \mathfrak{M}, s[^{d}/_{x}] \models \varphi, && \text{ind. hypothesis;}\\
&\Longrightarrow \text{for some } X \in D_2 : [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \cap X \neq \varnothing, && \text{def. } D;\\
&\Longrightarrow [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \in D_2 \ \&\ [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \neq \varnothing, && \text{overlap-closure;}\\
&\Longrightarrow \mathfrak{M}, s \models \exists x\varphi, && \text{deff. satisfaction, } \exists.
\end{aligned}$$

Conversely:

$$\begin{aligned}
\mathfrak{M}, s \models \exists x\varphi &\Longrightarrow [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \in D_2 \ \&\ [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \neq \varnothing, && \text{def. satisfaction;}\\
&\Longrightarrow [\![\varphi]\!]^{\mathfrak{M}}_{s,x} \cap D \neq \varnothing, && \text{def. } D;\\
&\Longrightarrow \text{for some } d \in D : \mathfrak{M}, s[^{d}/_{x}] \models \varphi, && \text{re-writing;}\\
&\Longrightarrow \text{for some } d \in D : \mathfrak{I}, s[^{d}/_{x}] \models \varphi, && \text{ind. hypothesis;}\\
&\Longrightarrow \mathfrak{I}, s \models \exists x\varphi, && \text{def. satisfaction.}
\end{aligned}$$

□

Thus we see that the class of formulas valid over all overlap-closed general models (in the language with $\exists$ as the only quantifier) coincides with the class of formulas valid over outer-domain models.

THEOREM 5.7 (Leblanc & Thomason, 1968). *The class of formulas valid over outer-domain models can be axiomatized by the (universal closures of the) following schemas (where $\forall$ is defined in terms of $\exists$ and negation):*

- *All propositional tautologies;*
- *Identity axioms: $x \doteq x$ and $x \doteq y \rightarrow (\varphi \rightarrow \varphi[^{x}/_{y}])$;*
- *The K schema: $\forall x(\varphi \rightarrow \psi) \rightarrow (\forall x\varphi \rightarrow \forall x\psi)$;*
- *Restricted universal instantiation: $\forall x(\forall x\varphi \rightarrow \varphi)$;*
- *Vacuous quantification: $\varphi \rightarrow \forall x\varphi$, when x is not free in φ;*
- *The permutation principle: $\forall x\forall y\varphi \rightarrow \forall y\forall x\varphi$.*

It follows that the class of formulas valid in the class of overlap-closed general models is axiomatizable as well.

§6. Filters as generalized universals. While in the previous section we considered the behavior of one quantifier $\exists$ over different kinds of models, here we compare different quantifiers over models essentially of the same kind. In particular, we are interested in comparing the ordinary universal quantifier $\forall$ interpreted over standard models with a quantifier Q interpreted over the class of models $\mathfrak{M}$ such that $\mathsf{Q}^{\mathfrak{M}}$ is a filter. This is essentially different from the results of the previous section, where we compared a single quantifier (constant across models) over different classes of structures. Here, we are re-interpreting the quantifier, with the aim of identifying precisely which algebraic properties of filters give us the well known principles axiomatizing the universal quantifier in first-order logic. It should also be noted that interpretations for the quantifier over filters can be found in the literature, likely making their first appearance with Slomson (1967), and continuing, for example, with Keisler (1970), Bruce (1978) (in the dual form using ideals), and Veloso (2000).

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

A filter F over a set X is standardly defined as a family of nonempty subsets of X that is closed under supersets and binary intersections (occasionally one finds the further condition that F itself may not be empty, but we will not, for the present purposes, rule out this limiting case). Filters can be characterized as representing "large" properties:

- The empty set is not large;
- If A is large and $A \subseteq B$ then B is large;
- if A and B are large then $A \cap B$ is large.

An example we will be using in the proof of Theorem 6.10 is the Fréchet filter of the co-finite subsets of $\mathbb{N}$. In this section we are going to investigate the logical properties of quantifiers that denote filters.

DEFINITION 6.1. *A quantifier* Q *is a filter on a model* $\mathfrak{M} = (D_1, D_2, I)$ *if and only if the following hold:*

F0. $\varnothing \notin \mathsf{Q}^{\mathfrak{M}}$*;*

F1. *If* $A, B \in \mathsf{Q}^{\mathfrak{M}}$ *then* $A \cap B \in \mathsf{Q}^{\mathfrak{M}}$.

F2. *If* $A \in \mathsf{Q}^{\mathfrak{M}}$ *and* $A \subseteq B \in D_2$*, then* $B \in \mathsf{Q}^{\mathfrak{M}}$.

Notice the mention of D_2 *in condition* ***F2****; when* $\mathfrak{M}$ *is standard, this is just the ordinary definition of a filter.*

We will see that the class of models $\mathfrak{M}$ such that Q is a filter on $\mathfrak{M}$ is in fact elementary, *modulo* the taking of hulls. The proof will provide an example of how the Hull Theorem 3.4 can be used to show how certain classes of models are axiomatizable.

DEFINITION 6.2. FIL $= \{\varphi \in \mathcal{L}(\mathsf{Q}) :$ *for all* $\mathfrak{M}[\mathfrak{M}$ *is H-closed* & Q *a filter on* $\mathfrak{M} \Longrightarrow \mathfrak{M} \models \varphi]\}$.

DEFINITION 6.3. *Let* δ_1 *and* δ_2 *be the following sentence and schema:*

δ_1: $\neg \mathsf{Q}x \neg (x \doteq x)$;
δ_2: $\mathsf{Q}x(\varphi \wedge \psi) \leftrightarrow (\mathsf{Q}x\varphi \wedge \mathsf{Q}x\psi)$.

DEFINITION 6.4. $\Delta = \{\varphi \in \mathcal{L}(\mathsf{Q}) :$ *for all* $\mathfrak{M}[\mathfrak{M} \models \delta_1 \wedge \delta_2 \Longrightarrow \mathfrak{M} \models \varphi]\}$*, where of course* $\mathfrak{M} \models \delta_2$ *is construed as a statement about all instances of* δ_2.

Whereas Δ represents a class of formulas that are validated by a logical condition, FIL represents a class of formulas validated by an algebraic condition.

PROPOSITION 6.5. $\Delta \subseteq$ FIL.

Proof. Assume $\theta \in \Delta$ and Q is a filter on the H-closed model $\mathfrak{M}$. We wish to show that $\mathfrak{M} \models \theta$, and so it suffices to prove that $\mathfrak{M} \models \delta_1 \wedge \delta_2$. We take each up in turn:

- If $\mathfrak{M}, s \models \mathsf{Q}x \neg (x \doteq x)$, for arbitrary s, then $\varnothing \in \mathsf{Q}^{\mathfrak{M}}$, against assumption. So $\mathfrak{M}, s \models \delta_1$.
- Suppose $\mathfrak{M}, s \models \mathsf{Q}x(\varphi \wedge \psi)$, for an arbitrary s. Also $[\![\varphi \wedge \psi]\!]_{s,x} \subseteq [\![\varphi]\!]_{s,x}$, and since $\mathfrak{M}$ is H-closed, $[\![\varphi]\!]_{s,x} \in D_2$. Since Q is a filter on $\mathfrak{M}$, $[\![\varphi]\!]_{s,x} \in \mathsf{Q}^{\mathfrak{M}}$, whence $\mathfrak{M}, s \models \mathsf{Q}x\varphi$. Similarly and symmetrically $\mathfrak{M}, s \models \mathsf{Q}x\psi$. For the converse implication, assume $\mathfrak{M}, s \models \mathsf{Q}x\varphi \wedge \mathsf{Q}x\psi$, so that both $[\![\varphi]\!]_{s,x} \in \mathsf{Q}^{\mathfrak{M}}$ and $[\![\psi]\!]_{s,x} \in \mathsf{Q}^{\mathfrak{M}}$. By hypothesis, $[\![\varphi]\!]_{s,x} \cap [\![\psi]\!]_{s,x} = [\![\varphi \wedge \psi]\!]_{s,x} \in \mathsf{Q}^{\mathfrak{M}}$, so that $\mathfrak{M}, s \models \mathsf{Q}x(\varphi \wedge \psi)$. □

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

PROPOSITION 6.6. FIL $\subseteq \Delta$.

Proof. Assume $\theta \in$ FIL. To show $\theta \in \Delta$ suppose $\mathfrak{M} = (D_1, D_2, I)$ is a model of δ_1 and δ_2; we wish to show that $\mathfrak{M} \models \theta$, and this in turn will follow if $\mathfrak{M}$ is equivalent to some H-closed model of θ.

Let $\mathfrak{N} = (D_1, H(\mathfrak{M}), I)$; by Corollary 3.5, $H(\mathfrak{N}) = H(\mathfrak{M})$ so $\mathfrak{N}$ is H-closed. And by the Hull Theorem 3.4, $\mathfrak{N}$ is equivalent to $\mathfrak{M}$, so $\mathfrak{N} \models \delta_1 \wedge \delta_2$ as well. The desired conclusion $\mathfrak{N} \models \theta$ will follow if we can show that Q is a filter on $\mathfrak{N}$, which we do next.

F0. Since for any s we have $\mathfrak{N}, s \models \neg \mathsf{Q}x \neg (x \doteq x)$, which is δ_1, by definition of satisfaction $\varnothing \notin \mathsf{Q}^{\mathfrak{N}}$.

F1. Assume $A, B \in \mathsf{Q}^{\mathfrak{N}}$; since $\mathfrak{N}$ is constructible, there are formulas $\varphi(x, \boldsymbol{y})$ and $\psi(x, \boldsymbol{z})$, possibly involving parameters $\boldsymbol{y}$ and $\boldsymbol{z}$, and an assignment s such that $A = [\![\varphi]\!]_{s,x}$ and $B = [\![\psi]\!]_{s,x}$. By definition of satisfaction again $\mathfrak{N}, s \models \mathsf{Q}x\varphi$ and $\mathfrak{N}, s \models \mathsf{Q}x\psi$, so by δ_2 also $\mathfrak{N}, s \models \mathsf{Q}x(\varphi \wedge \psi)$. It follows that $A \cap B = [\![\varphi \wedge \psi]\!]_{s,x} \in \mathsf{Q}^{\mathfrak{N}}$, as desired.

F2. Suppose $A \in \mathsf{Q}^{\mathfrak{N}}$ and $A \subseteq B \in H(\mathfrak{M}) = H(\mathfrak{N})$. Then there are formulas $\varphi(x, \boldsymbol{y})$ and $\psi(x, \boldsymbol{z})$ and an assignment s such that $[\![\varphi]\!]_{s,x} = A$, and $[\![\psi]\!]_{s,x} = B$. We also have $\mathfrak{N}, s \models \mathsf{Q}x\varphi$ and $[\![\varphi]\!]_{s,x} \subseteq [\![\psi]\!]_{s,x}$. From the latter, $[\![\varphi]\!]_{s,x} = [\![\varphi \wedge \psi]\!]_{s,x}$, so that $\mathfrak{N}, s \models \mathsf{Q}x(\varphi \wedge \psi)$. But now by δ_2 also $\mathfrak{N}, s \models \mathsf{Q}x\psi$, so that $B = [\![\psi]\!]_{s,x} \in \mathsf{Q}^{\mathfrak{N}}$, as desired. □

THEOREM 6.7. FIL $=\Delta$.

Proof. By Propositions 6.5 and 6.6. □

This shows that the class of H-closed models $\mathfrak{M}$ for $\mathcal{L}(\mathsf{Q})$ such that Q is a filter on $\mathfrak{M}$ is elementary, and in fact axiomatized by δ_1 and the single schema δ_2. Of course both δ_1 and δ_2 become theorems of ordinary first-order logic upon a change of notation replacing Q by $\forall$. This is no accident.

DEFINITION 6.8. FOL $= \{\varphi \in \mathcal{L}(\forall) : \textit{for all } \mathfrak{M}[\mathfrak{M} \textit{ is standard} \implies \mathfrak{M} \models \varphi]\}$, *where* $\forall$ *is the quantifier such that* $X \in \forall(D_1)$ *if and only if* $X = D_1$, *for any model* $\mathfrak{M} = (D_1, D_2, I)$.

DEFINITION 6.9. *If* $\varphi \in \mathcal{L}(\mathsf{Q})$ *then* $\varphi^{\forall}$ *is the result of writing* $\forall$ *in place of each occurrence of* Q. *If* $\Theta \subseteq \mathcal{L}(\mathsf{Q})$, *then* $\Theta^{\forall} = \{\varphi^{\forall} : \varphi \in \Theta\}$.

THEOREM 6.10. FIL$^{\forall} \subsetneq$ FOL, *where the inclusion is proper.*

Proof. To see that if $\varphi \in$ FIL then $\varphi^{\forall}$ is a theorem of ordinary first-order logic, suffices to remark that if $\mathfrak{M} = (D_1, \mathscr{P}(D_1), I)$ is standard then it is H-closed, and moreover $\forall(D_1) = \{D_1\}$ is a filter on $\mathfrak{M}$, so $\mathfrak{M} \models \varphi^{\forall}$. To see that the inclusion is proper we need to exhibit a formula $\varphi \notin$ FIL such that $\varphi^{\forall} \in$ FOL. Consider the formula:

$$(*) \qquad \mathsf{Q}x\mathsf{Q}yRxy \rightarrow \mathsf{Q}y\mathsf{Q}xRxy.$$

When re-writing Q as $\forall$ we obtain $\forall x \forall y Rxy \rightarrow \forall y \forall x Rxy$, which is in FOL. But consider the model $\mathfrak{N} = (\mathbb{N}, \mathscr{P}(\mathbb{N}), I)$ for $\mathcal{L}(\mathsf{Q})$ with one 2-place predicate symbol R. Assume that $I(R)$ is the $<$ relation over $\mathbb{N}$, and the quantifier $\mathsf{Q}^{\mathfrak{N}}$ is the Fréchet filter of the co-finite subsets of $\mathbb{N}$. The model $\mathfrak{N}$, being standard, is obviously H-closed. However, $(*)$ is false in $\mathfrak{N}$, since its antecedent "there are co-finitely many x such that there are co-finitely many $y > x$" holds in $\mathfrak{N}$ while the consequent, "there are co-finitely many y's such that there are co-finitely many $x < y$" does not (this counter-example is due to T. Forster and simplifies the author's original proof). □

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

Although FIL$^{\forall}$ is properly contained in first-order logic, several principles of first-order logic are either already in FIL$^{\forall}$, or can be recovered by assuming that the first-order domain D_1 of a model $\mathfrak{M}$ itself is a member of $\mathsf{Q}^{\mathfrak{M}}$. We start with the K axiom of first-order logic, already mentioned at the beginning.

DEFINITION 6.11. *Let* $K(\mathsf{Q})$ *be the schema:* $\mathsf{Q}x(\varphi \to \psi) \to (\mathsf{Q}x\varphi \to \mathsf{Q}x\psi)$.

PROPOSITION 6.12. $K(\mathsf{Q}) \in$ FIL.

Proof. Assume $\mathfrak{M} = (D_1, D_2, I)$ is H-closed and Q is a filter on $\mathfrak{M}$. To show that $K(\mathsf{Q})$ is true in $\mathfrak{M}$, suppose that $\mathfrak{M}, s \models \mathsf{Q}x(\varphi \to \psi)$ and $\mathfrak{M}, s \models \mathsf{Q}x\varphi$ for an arbitrary assignment s. From the latter, since $[\![\varphi]\!]_{s,x} \subseteq [\![\varphi \vee \psi]\!]_{s,x}$ and $[\![\varphi \vee \psi]\!]_{s,x} \in D_2$ by H-closure, we have $\mathfrak{M}, s \models \mathsf{Q}x(\varphi \vee \psi)$. From the former, $\mathfrak{M}, s \models \mathsf{Q}x(\neg\varphi \vee \psi)$ and by the intersection property of filters

$$\mathfrak{M}, s \models \mathsf{Q}x\big((\neg\varphi \vee \psi) \wedge (\varphi \vee \psi)\big),$$

that is, $[\![\big((\neg\varphi \vee \psi) \wedge (\varphi \vee \psi)\big)]\!]_{s,x} \in \mathsf{Q}^{\mathfrak{M}}$. But $[\![\big((\neg\varphi \vee \psi) \wedge (\varphi \vee \psi)\big)]\!]_{s,x} = [\![(\neg\varphi \wedge \varphi) \vee \psi]\!]_{s,x} = [\![\psi]\!]_{s,x}$, so $\mathfrak{M}, s \models \mathsf{Q}x\psi$ as desired. □

A fundamental principle governing $\forall$ is *universal instantiation*: $\forall x\varphi \to \varphi$, (assuming a substitution rule allowing for the further replacement of the free occurrences of x in the consequent by some term t as long as t is free for x in φ). In ordinary first-order logic, open formulas are treated on a par as their universal closures. Accordingly, universal instantiation is in fact equivalent to $\forall x(\forall x\varphi \to \varphi)$, which states that universal quantifiers can be validly instantiated by elements of the domain over which they range. And it is in this form that we show that the corresponding principle for $\mathcal{L}(\mathsf{Q})$ is valid over models $\mathfrak{M}$ whose first-order domain is itself among the members of $\mathsf{Q}^{\mathfrak{M}}$.

PROPOSITION 6.13. *The schema* $\mathsf{Q}x(\mathsf{Q}x\varphi \to \varphi)$ *is true in any model* $\mathfrak{M} = (D_1, D_2, I)$ *such that* $D_1 \in \mathsf{Q}^{\mathfrak{M}}$.

Proof. Let s be an arbitrary assignment in $\mathfrak{M}$; we distinguish two cases:

(*a*) $\mathfrak{M}, s \models \mathsf{Q}x\varphi$. Then $[\![\mathsf{Q}x\varphi \to \varphi]\!]_{s,x} = [\![\varphi]\!]_{s,x}$ by propositional logic, and the hypothesis gives $\mathfrak{M}, s \models \mathsf{Q}x(\mathsf{Q}x\varphi \to \varphi)$, as desired.

(*b*) $\mathfrak{M}, s \not\models \mathsf{Q}x\varphi$. Then $[\![\mathsf{Q}x\varphi \to \varphi]\!]_{s,x} = D_1$ by propositional logic and $D_1 \in \mathsf{Q}^{\mathfrak{M}}$ by assumption, so that again $\mathfrak{M}, s \models \mathsf{Q}x(\mathsf{Q}x\varphi \to \varphi)$. □

PROPOSITION 6.14. *The schema* $\varphi \to \mathsf{Q}x\varphi$, *with* x *not free in* φ, *is true in any model* $\mathfrak{M} = (D_1, D_2, I)$ *such that* $D_1 \in \mathsf{Q}^{\mathfrak{M}}$.

Proof. Assume $\mathfrak{M}, s \models \varphi$.

$$\begin{aligned}
\mathfrak{M}, s \models \mathsf{Q}x\varphi &\iff \{d \in D_1 : \mathfrak{M}, s[^d/_x] \models \varphi\} \in \mathsf{Q}^{\mathfrak{M}}, && \text{Def. 2.9;}\\
&\iff \{d \in D_1 : \mathfrak{M}, s \models \varphi\} \in \mathsf{Q}^{\mathfrak{M}}, && x \text{ not free in } \varphi;\\
&\iff D_1 \in \mathsf{Q}^{\mathfrak{M}} && \text{By the assumption.}
\end{aligned}$$ □

Clearly Propositions 6.13 and 6.14 also follow under the assumption that Q is a *nonempty* filter on a replete model $\mathfrak{M}$, for then $D_1 \in \mathsf{Q}^{\mathfrak{M}}$ by monotonicity. Observe as well that the condition $D_1 \in \mathsf{Q}^{\mathfrak{M}}$ is equivalent to $\mathfrak{M} \models \mathsf{Q}x(x \doteq x)$.

We have seen that FIL is properly contained in first-order logic. Now we show that at least one natural way of strengthening FIL by considering ultrafilters (a suggestion due to

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

Claude LaFlamme) takes us out of FOL: while filters represent large properties, ultrafilters behave more like *truth*. The argument also provides another illustration of how the Hull Theorem 3.4 allows to turn satisfaction of a schema into an algebraic property.

Ultrafilters are maximal filters, that is, filters F on X such that for every $A \subseteq X$, either $A \in F$ or $(X \setminus A) \in F$. An equivalent characterization of ultrafilters, the one we will be using below, states that if $A, B \subseteq X$ and $A \cup B \in F$, then either $A \in F$ or $B \in F$. It's easy to see that this implies maximality, at least in the case of nonempty filters, for then $A \cup (X \setminus A) = X \in F$, whence $A \in F$ or $(X \setminus A) \in F$. Conversely, if $A \cup B \in F$ but neither $A \in F$ nor $B \in F$ then maximality gives $(X \setminus A) \in F$ and $(X \setminus B) \in F$. By the intersection property, $(X \setminus A) \cap (X \setminus B) = (X \setminus (A \cup B)) \in F$ and by the intersection property again $(A \cup B) \cap (X \setminus (A \cup B)) = \varnothing \in F$, a contradiction. We use this alternative characterization below.

DEFINITION 6.15. *A quantifier* Q *is an ultrafilter on a model* $\mathfrak{M} = (D_1, D_2, I)$ *if and only of* Q *is a filter on* $\mathfrak{M}$ *and moreover:*

F3. *If* $A, B \in D_2$ *and* $A \cup B \in \mathsf{Q}^{\mathfrak{M}}$ *then either* $A \in \mathsf{Q}^{\mathfrak{M}}$ *or* $B \in \mathsf{Q}^{\mathfrak{M}}$.

Again, as in Definition 6.1, in the case in which $\mathfrak{M}$ *is standard this gives the ordinary notion of ultrafilter.*

DEFINITION 6.16. $\text{UFIL} = \{\varphi \in \mathcal{L}(\mathsf{Q}) : \textit{for all } \mathfrak{M}[\mathfrak{M} \textit{ is } H\textit{-closed } \& \ \mathsf{Q} \textit{ an ultrafilter on } \mathfrak{M} \Longrightarrow \mathfrak{M} \models \varphi]\}$.

DEFINITION 6.17. *Let* δ_3 *be the following schema:*

δ_3: $\quad \mathsf{Q}x(\varphi \vee \psi) \leftrightarrow (\mathsf{Q}x\varphi \vee \mathsf{Q}x\psi)$.

And define $\Delta^* = \{\varphi \in \mathcal{L}(\mathsf{Q}) : \textit{for all } \mathfrak{M}[\mathfrak{M} \models \delta_1 \wedge \delta_2 \wedge \delta_3 \Longrightarrow \mathfrak{M} \models \varphi]\}$, *where* δ_1 *and* δ_2 *are as in Definition 6.3.*

We wish to show that $\text{UFIL} = \Delta^*$, from which it will follow that $\text{UFIL}^{\forall} \not\subseteq \text{FOL}$, since $\forall$ does not distribute over disjunction.

PROPOSITION 6.18. $\Delta^* \subseteq \text{UFIL}$.

Proof. Suppose $\theta \in \Delta^*$ and $\mathfrak{M} = (D_1, D_2, I)$ is H-closed and Q is an ultrafilter on $\mathfrak{M}$. In order to show that $\mathfrak{M} \models \theta$ it suffices to show that $\mathfrak{M} \models \delta_1 \wedge \delta_2 \wedge \delta_3$.

$\mathfrak{M} \models \delta_1 \wedge \delta_2$ follows just as in the proof of Proposition 6.5. For δ_3 suppose that $\mathfrak{M}, s \models \mathsf{Q}x(\varphi \vee \psi)$. Then $[\![\varphi \vee \psi]\!]_{s,x} \in \mathsf{Q}^{\mathfrak{M}}$ and since $\mathfrak{M}$ is H-closed, both $[\![\varphi]\!]_{s,x} \in D_2$ and $[\![\psi]\!]_{s,x} \in D_2$. Since Q is an ultrafilter, either $[\![\varphi]\!]_{s,x} \in \mathsf{Q}^{\mathfrak{M}}$ or $[\![\psi]\!]_{s,x} \in \mathsf{Q}^{\mathfrak{M}}$, whence either $\mathfrak{M}, s \models \mathsf{Q}x\varphi$ or $\mathfrak{M}, s \models \mathsf{Q}x\psi$.

For the converse, suppose $\mathfrak{M}, s \models \mathsf{Q}x\varphi$; then $[\![\varphi]\!]_{s,x} \in \mathsf{Q}^{\mathfrak{M}}$ and $[\![\varphi]\!]_{s,x} \subseteq [\![\varphi \vee \psi]\!]_{s,x} \in D_2$ by H-closure. By monotonicity $[\![\varphi \vee \psi]\!]_{s,x} \in \mathsf{Q}^{\mathfrak{M}}$, so that $\mathfrak{M}, s \models \mathsf{Q}x(\varphi \vee \psi)$. The case for $\mathsf{Q}x\psi$ is exactly similar. □

PROPOSITION 6.19. $\text{UFIL} \subseteq \Delta^*$.

Proof. Suppose that $\theta \in \text{UFIL}$ and $\mathfrak{M} \models \delta_1 \wedge \delta_2 \wedge \delta_3$ for $\mathfrak{M} = (D_1, D_2, I)$.. In order to show that $\mathfrak{M} \models \theta$ it suffices to show that $\mathfrak{N} \models \theta$ for some H-closed $\mathfrak{N}$ equivalent to $\mathfrak{M}$.

Let $\mathfrak{N} = (D_1, H(\mathfrak{M}), I)$; since $\mathfrak{N}$ is equivalent to $\mathfrak{M}$, also $\mathfrak{N} \models \delta_1 \wedge \delta_2 \wedge \delta_3$. Since $H(\mathfrak{N}) = H(\mathfrak{M})$ by Corollary 3.5, it follows that $\mathfrak{N}$ is H-closed. We show that Q is an ultrafilter on $\mathfrak{N}$, from which it will follow that $\mathfrak{N} \models \theta$, as desired.

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

Since $\mathfrak{N} \models \delta_1 \wedge \delta_2$ it follows that Q is a filter on $\mathfrak{N}$, just as in the proof of Proposition 6.6. To see that $\mathfrak{N}$ satisfies **F3**, suppose $A, B \in H(\mathfrak{M})$ and $A \cup B \in \mathsf{Q}^{\mathfrak{M}}$. Then there are formulas φ and ψ and an assignment s such that $A = [\![\varphi]\!]_{s,x}$ and $B = [\![\psi]\!]_{s,x}$ and $\mathfrak{N}, s \models \mathsf{Q}x(\varphi \vee \psi)$. By δ_3 either $\mathfrak{N}, s \models \mathsf{Q}x\varphi$ or $\mathfrak{N}, s \models \mathsf{Q}x\psi$, whence either $A \in \mathsf{Q}^{\mathfrak{N}}$ or $B \in \mathsf{Q}^{\mathfrak{N}}$, as desired. □

§7. Compactness and saturation. In this brief section we consider another correspondence between logical and algebraic (or, in this case, topological) properties of models.

DEFINITION 7.1. *For a unary quantifier* Q*, a model* $\mathfrak{M} = (D_1, D_2)$ *is called* Q-compact *if the following condition holds: For any collection* $C \subseteq D_2$*, if every finite sub-collection of* C *has intersection in* $\mathsf{Q}^{\mathfrak{M}}$*, then also* $\bigcap C \in \mathsf{Q}^{\mathfrak{M}}$.

If $\mathfrak{M}$ is $\exists$-compact, then $\mathsf{Q}^{\mathfrak{M}}$ is compact in the usual topological sense: if $C \subseteq \mathsf{Q}^{\mathfrak{M}}$ has empty intersection then there is finite sub-collection of C that also has empty intersection.

DEFINITION 7.2. *A model* $\mathfrak{M}$ *is* countably Q-saturated *if and only if for any collection* $\Phi(x; y_1, \ldots, y_n)$ *of formulas in the free variable* x *and the finitely many parameters* $\boldsymbol{y}$*: if for any formulas* $\varphi_1(x, \boldsymbol{y}), \ldots, \varphi_n(x, \boldsymbol{y})$ *in* $\Phi(x, \boldsymbol{y})$ *and assignment* s*,*

$$\mathfrak{M}, s \models \mathsf{Q}x\big(\varphi_1(x, \boldsymbol{y}) \wedge \ldots \wedge \varphi_n(x, \boldsymbol{y})\big),$$

then $\mathfrak{M}, s \models \mathsf{Q}x \bigwedge \Phi(x, \boldsymbol{y})$.

The notion of $\exists$-saturation is just the ordinary model-theoretic notion. Notice the restriction to sets of parameters of cardinality strictly less than $\aleph_0$.

PROPOSITION 7.3. *If* $\mathfrak{M}$ *is* Q*-compact then it is countably* Q*-saturated.*

Proof. The consequent is almost a re-writing of the antecedent. Let $\mathfrak{M}$ be Q-compact and s an assignment. Assume that for any finite subset $\{\varphi_1, \ldots, \varphi_n\}$ of Φ:

$$\mathfrak{M}, s \models \mathsf{Q}x\big(\varphi_1(x, \boldsymbol{y}) \wedge \ldots \wedge \varphi_n(x, \boldsymbol{y})\big),$$

and let $C = \{[\![\varphi]\!]_{s,x} : \varphi \in \Phi\}$. Then any finite subset of C has intersection in $\mathsf{Q}^{\mathfrak{M}}$, when by hypothesis $\bigcap C \in \mathsf{Q}^{\mathfrak{M}}$, that is, $\mathfrak{M}, s \models \mathsf{Q}x \bigwedge \Phi(x, \boldsymbol{y})$. □

Notice that the argument goes through also without the restriction to finite sets of parameters in the definition of saturation. In ordinary first-order model theory, no model of infinite cardinality κ can be saturated with respect to all sets of parameters of cardinality $\leq \kappa$. For instance, for $\kappa = \aleph_0$, all finite sets of formulas $\neg(x \doteq d_n)$ are jointly satisfiable in a model with domain $d_0, d_1, \ldots$, but there is no d that satisfies all of them simultaneously. But the argument just given relies on the model being standard. In the present setting, either some formula $\neg(x \doteq d_n)$ will not have an extension in D_2 (as pointed out by John Baldwin), so the model will not be H-closed, or $\mathfrak{M}$ will not be $\exists$-compact.

DEFINITION 7.4. *A model* $\mathfrak{M} = (D_1, D_2, I)$ *is* Q-saturated *(simpliciter) if it is saturated with respect to sets of parameter of cardinality* $\leq |D_1|$.

This allows us to give a converse of sorts to Proposition 7.3.

PROPOSITION 7.5. *If* $\mathfrak{M}$ *is* Q*-saturated and constructible, the* $\mathfrak{M}$ *is* Q*-compact.*

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

Proof. $\mathfrak{M} = (D_1, D_2, I)$ being constructible means that $D_2 = H(\mathfrak{M})$ (see Definition 4.4 and Corollary 4.10). Therefore, if $C \subseteq \mathsf{Q}^{\mathfrak{M}}$, there is a set $\Phi(x, \boldsymbol{y})$ of formulas in infinitely parameters $\boldsymbol{y} = y_1, y_2, \ldots$ such that $C = \{[\![\varphi(x, \boldsymbol{y})]\!]_{x,s} : \varphi \in \Phi\}$ (of course each $\varphi \in \Phi$ contains only finitely many parameters). If every finite subcollection of C has intersection in $\mathsf{Q}^{\mathfrak{M}}$ then, by saturation, so does C. □

§8. Concluding remark. Just like in the case of second- and higher-order logic, nonstandard interpretations for the quantifiers are quite interesting both from a philosophical and a logico-mathematical point of view in the first-order case as well. Philosophically, first-order quantifiers such as $\exists$ and $\forall$ have been interpreted as carrying the ontological commitments of formalized (or formalizable) theories, as announced in Quine's famous *dictum* that "to be is to be the value of a variable" (see Quine, 1948). But the possibility of nonstandard interpretations reveals that being the value of a variable is at best a sufficient, but not necessary condition for ontological commitment. In fact, it is perhaps the characteristic feature of the kind of interpretations presented here that there can be formulas $\varphi(x)$ that are satisfiable in a model, without the corresponding existential generalization $\exists x \varphi(x)$ being true (because the extension $[\![\varphi]\!]_x$ might not be a member of the second-order domain). In such a case the object $a \in D_1$ satisfying φ can be regarded as "lying beyond the reach" of the first order quantifiers, a fact that, as a consequence of some of our results, might be transparent to users of the language so interpreted. Hence much caution needs to be exercised when attempting to extract ontological commitments of theories from a semantical analysis of the quantifiers. These philosophical issues are further investigated in Antonelli (forthcoming). From a logico-mathematical point of view, nonstandard interpretations reveal the possibility that interesting logical systems might be obtained by appropriately placing restrictions on the second-order domains of models. We have seen examples of this in the case of overlap-closed models and models over which the quantifier is a filter. Especially the latter provide clear instances of how the Hull Theorem can be used to develop a "correspondence theory" relating logical and algebraic properties of models.

§9. Acknowledgments. Many thanks to John Baldwin, Claude Laflamme, and Albert Visser for helpful comments. I am grateful to the referees for their useful criticisms, and especially for pointing me to the question of decidability, as well as to parts of the literature with which I was not previously familiar, notably Thomason & Johnson (1969), Bruce (1978), Väänänen (1978), López-Escobar (1991), and Andréka *et al.* (1998).

BIBLIOGRAPHY

Anapolitanos, D., & Väänänen, J. (1985). Decidability of some logics with free quantifier variables. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, **27**, 17–22.

Andréka, H., Németi, I., & van Benthem, J. (1998). Modal languages and bounded fragments of predicate logic. *Journal of Philosophical Logic*, **27**(3), 217–274.

Antonelli, G. A. (2010). Notions of invariance for abstraction principles. *Philosophia Mathematica*, **18**(3), 276–292.

Antonelli, G. A. (forthcoming). Life on the range: Quine's thesis and semantic indeterminacy. In Torza, A., editor. *Quantifiers, Quantifiers, Quantifiers*. Berlin and New York: Springer.

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press

Bruce, K. B. (1978). Ideal models and some not so ideal problems in the model theory of $L(Q)$. *The Journal of Symbolic Logic*, **43**(2), 304–321.

Enderton, H. (2009). Second-order and higher-order logic. In Zalta, E., editor. *Stanford Encyclopedia of Philosophy*. Stanford, CA.

Gödel, K. (1940). *The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the Axioms of Set Theory*, Vol. 3. Annals of Mathematics Studies. Stanford, CA: Princeton University Press, .

Henkin, L. (1950). Completeness in the theory of types. *Journal of Symbolic Logic*, **15**(2), 81–91.

Henkin, L. (1961). Some remarks on infinitely long formulas. In *Infinistic Methods. Proccedings of the Symposium on the Foundations of Mathematics, Warsaw*, 2–9 September 1959. Pergamon Press, Oxford UK, pp. 167–183.

Henkin, L. (1996). The discovery of my completeness proofs. *Bulletin of Symbolic Logic*, **2**(2), 127–158.

Keisler, H. J. (1970). Logic with quantifier "there exist uncountably many". *Annals of Mathematical Logic*, **1**, 1–93.

Leblanc, H. & Thomason, R. H. (1968). Completeness theorems for some presupposition-free logics. *Fundamenta Mathematicæ*, **62**, 125–154.

Lindström, P. (1966). First-order predicate logic with generalized quantifiers. *Theoria*, **32**, 186–195.

López-Escobar, E. G. K. (1991). Formalizing a non-linear Henkin quantifier. *Fundamenta Mathematicae*, **138**(2), 83–101.

Montague, R. (1974). English as a formal language. In Thomason, R. H., editor. *Formal Philosophy* (originally published in 1969). Yale University Press.

Mostowski, A. (1957). On a generalization of quantifiers. *Fundamenta Mathematicæ*, **44**, 12–36.

Peters, S., & Westerståhl, D. (2006). *Quantifiers in Logic and Language*. Oxford UK, 560 pp.

Quine, W. V. O. (1948). On what there is. *Review of Metaphysics*, **2**(5), 21–38.

Slomson, A. B. (1967). Some problems in mathematical logic. PhD Thesis, Oxford University. Available from: http://books.google.com/books?id=-VaQtgAACAAJ.

Tarski, A. (1986). What are logical notions? *History and Philosophy of Logic*, **7**, 143–154. Original text 1966; edited (with an introduction) by John Corcoran.

Thomason, R. H., & Johnson, D. R. (1969). Predicate calculus with free quantifier variables. *The Journal of Symbolic Logic*, **34**(1), 1–7.

Veloso, P. A. S. (2000). On the power of ultrafilter logic. *Bulletin of the Section of Logic*, **29**, 89–97.

Väänänen, J. (1978). Remarks on free quantifier variables. In Hintikka, J., Niiniluoto, I., & Saarinen, E., editors. *Essays on Mathematical and Philosophical Logic*, Vol. 122. Synthese Library. Springer: Berlin and New York, pp. 267–272.

Yasuhara, M. (1969). Incompleteness of L_p languages. *Fundamenta Mathematicae*, **66**, 147–152.

DEPARTMENT OF PHILOSOPHY
UNIVERSITY OF CALIFORNIA, DAVIS
DAVIS, CA 95616 (USA)
E-mail: antonelli@ucdavis.edu

https://doi.org/10.1017/S1755020313000270 Published online by Cambridge University Press