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On recurrence and ergodicity for geodesic flows on non-compact periodic polygonal surfaces

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In memory of Dan Rudolph

Abstract. We study billiard dynamics on non-compact polygonal surfaces with a free, cocompact action of $\mathbb{Z}$ or $\mathbb{Z}^2$. In the $\mathbb{Z}$-periodic case, we establish criteria for conservativity. In the $\mathbb{Z}^2$-periodic case, we study a particular family of such surfaces, the rectangular Lorenz gas. Assuming that the obstacles are sufficiently small, we obtain the ergodic decomposition of directional billiards for a finite but asymptotically dense set of directions. This is based on our study of ergodicity for $\mathbb{Z}^d$-valued cocycles over irrational rotations.

Contents

1 Introduction 491
2 Setting and preliminaries 492
3 $\mathbb{Z}$-periodic polygonal surfaces 498
4 Z-periodic polygonal surfaces 502
5 Ergodicity of cocycles over irrational rotations 507
Acknowledgements 514
References 514

1. *Introduction*

Since Boltzmann's *ergodic hypothesis*, mathematicians have been studying dynamical systems of physical origin, in particular, the geodesic flows on model spaces in physics. If the space has a boundary, we arrive at a billiard. Proving ergodicity may be very difficult, as the Boltzmann–Sinai model shows, but conservativity is granted by the Poincaré recurrence theorem if the space is compact or has finite volume. The situation changes drastically when the volume is infinite, in particular, if the space is *periodic*, i.e., invariant under the free action of an infinite group: the dynamical system may have a dissipative component.

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Here we study the conservativity, ergodicity, and related asymptotic properties of billiard flows on $\mathbb{Z}$-*periodic and* $\mathbb{Z}^2$-*periodic polygonal surfaces*; these include several popular examples, one of them being the classical *wind-tree model* in statistical physics [**13, 25**].

Theorem 1 and its corollaries yield criteria for conservativity of billiard flows on $\mathbb{Z}$-periodic polygonal surfaces. Proposition 5 is an application of Theorem 1 to a particular class of $\mathbb{Z}$-periodic polygonal surfaces: rational stairways. The case of a $\mathbb{Z}^2$-periodic polygonal surface is more difficult. We study a particular family of these spaces: the *polygonal Lorenz gas*. After making a few remarks on the general situation (Propositions 6 and 7), we concentrate on the *rectangular Lorenz gas* $\tilde{P}(a, b)$†. We investigate billiard flows on $\tilde{P}(a, b)$ in the directions $\arctan(q/p)$ if a, b are sufficiently small and a/b is irrational. Theorem 2 provides an explicit ergodic decomposition. Theorem 2 and Corollaries 4 and 5 imply conservativity of these $\mathbb{Z}^2$-periodic billiard flows. This has been claimed in [**25**] for $p = q = 1$.

Our approach uses billiard maps with respect to the boundary of polygonal surfaces. We identify them with skew products over interval exchanges with the fiber $\mathbb{Z}^2$. In particular, we reduce analysis of the billiard on $\tilde{P}(a, b)$ to a study of cocycles over irrational rotations ρ_α. We carry this out in §5. The methods are related to those of [**3, 5, 7, 14, 16, 32, 34**]. The main results are Theorems 3, 4, and 5. Theorem 3 and Corollary 7 establish the ergodicity of a class of $\mathbb{Z}^d$-valued cocycles over ρ_α under certain genericity assumptions. These results are valid for arbitrary d. Theorem 4 concerns the ergodicity of special $\mathbb{Z}$-valued cocycles over any irrational rotations. It is crucially used in the proof of Theorem 5, which states the ergodicity of particular $\mathbb{Z}^2$-valued ρ_α-cocycles for all irrational α. Theorem 2 is an application of Theorem 5 to the rectangular Lorenz gas.

It is instructive to compare our conclusions with the results on ergodicity for polygonal billiards and related dynamical systems [**5, 17–19, 21, 29, 37**]. Our study and other investigations [**8, 11, 15, 16, 26, 27, 32**] indicate that the billiard analysis for non-compact polygonal surfaces and translation surfaces is more involved than in the compact case. The general picture is not yet clear, and our results might help to unravel it. The preprint [**9**] is a preliminary version of the present work.

2. *Setting and preliminaries*

Let $(X, \mathcal{A})$ be a standard Borel space, endowed with a possibly infinite measure ν. By a *dynamical system* we will mean a measure-preserving automorphism (X, τ, ν) or a measure-preserving flow (X, T^t, ν). For the sake of brevity, we will recall the main concepts only for automorphisms, referring the reader to [**1, 10, 34**] for further material.

The automorphism (X, τ, ν) is *conservative* if for every measurable set $B \subset X$ and for ν-almost every (a.e.) point $x \in B$ there is a sequence $n_k \to \infty$ such that $\tau^{n_k} x \in B$. It is *dissipative* if there is a measurable set $A \subset X$ such that $X = \bigcup_{n\in\mathbb{Z}} \tau^n A$ and $\tau^p A \cap \tau^q A = \emptyset$ for $p \neq q$. It is well known that any dynamical system (X, τ, ν) has a unique (up to sets of measure zero) disjoint invariant decomposition $X = C \cup D$ such that (C, τ, ν) is conservative and (D, τ, ν) is dissipative. Both may be non-trivial if $\nu(X) = \infty$. Let (Y, T^t, μ) be a flow, and let $X \subset Y$ be a cross-section. Then the flow is conservative if and only if the *induced automorphism* (X, τ, ν) is conservative.

† Corresponding to the periodic wind-tree model [**25**]; $a, b > 0$ are the parameters of the rectangle.

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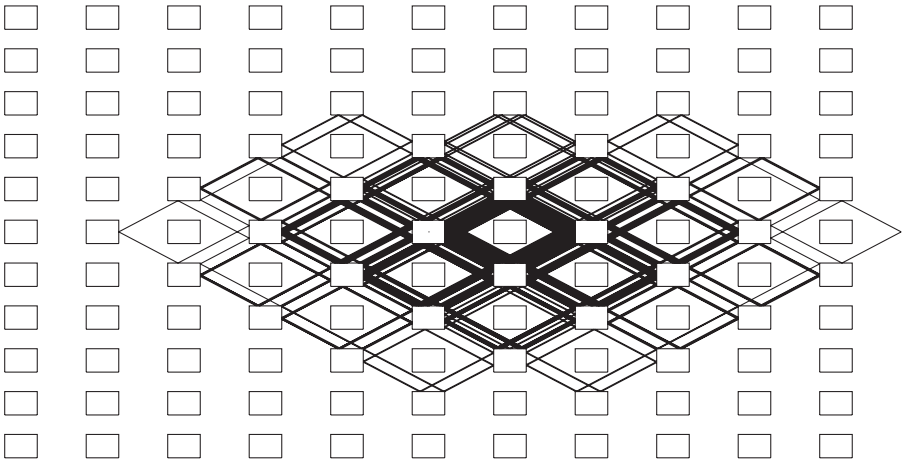


FIGURE 1. An orbit of the billiard flow in direction (1, 1) for the rectangular Lorenz gas.

2.1. *Ergodic theory for skew products.* Let G be a non-compact, locally compact abelian group. We denote by Leb_G a Haar measure on G, suppressing the subscript whenever possible. We will review the material on G-valued cocycles. Since in our applications $G \simeq \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$, we will gear our exposition to these groups. Let $g = (x_1, \ldots, x_{d_1}, x_{d_1+1}, \ldots, x_{d_1+d_2}) \in \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$. Set $|g| = \sup_{1 \le i \le d_1+d_2} |x_i|$. We refer to $|g|$ as a norm on G. For a sequence $g_n \in G$, we write $g_n \to \infty$ if $\lim_n |g_n| = \infty$.

Definition 1. Let (X, τ, ν) be a dynamical system with $\nu(X) < \infty$, and let $\varphi : X \to G$ be a measurable function. The *associated cocycle* is given by $\varphi(0, x) = 0$ and

$$\varphi(n, x) = \sum_{j=0}^{n-1} \varphi(\tau^j x) \quad \text{if } n > 0, \qquad \varphi(n, x) = -\sum_{j=n}^{-1} \varphi(\tau^j x) \quad \text{if } n < 0. \tag{1}$$

We will also use the notation $\varphi_n(x)$ and (φ_n). Note that $\varphi_n(x)$ are the ergodic (or Birkhoff) sums of φ. The cocycle (φ_n) can be viewed as a random walk on G driven by (X, τ, ν). Set $\tilde{X} = X \times G$, $\tilde{\nu} = \nu \times \mathrm{Leb}$ and

$$\tilde{\tau}(x, g) = (\tau x, g + \varphi(x)). \tag{2}$$

The dynamical system $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ (or $(\tilde{X}, \tau_\varphi, \tilde{\nu})$) is the *skew product* over (X, τ, ν) with the *fiber* G and the *displacement function* φ. For $n \in \mathbb{Z}$, we have $\tilde{\tau}^n(x, g) = (\tau^n x, g + \varphi_n(x))$.

Definition 2. Let (X, τ, ν) be a dynamical system with $\nu(X) < \infty$. Let $\varphi : X \to G$ be a measurable function, and let (φ_n) be the corresponding cocycle. A point $x \in X$ is *transient* (respectively *recurrent*) if $\varphi_n(x) \to \infty$ (respectively $\varphi_n(x) \not\to \infty$). The cocycle is transient (respectively recurrent) if a.e. $x \in X$ is transient (respectively recurrent).

Note that there are examples of transient $\mathbb{Z}^2$-valued cocycles with recurrent components [**4**]. The sets of transient and recurrent points for a cocycle are measurable and invariant. Hence, any cocycle over an ergodic dynamical system is either recurrent

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or transient. Let (X, τ, ν) be arbitrary, let (φ_n) be a cocycle, and let $R \subset X$ be its set of recurrent points. Suppose that $\nu(R) > 0$. Let ν_R be the restriction of ν to R, and set $\tilde{R} = R \times G$, $\tilde{\nu}_R = \nu_R \times \text{Leb}$. Then $(\tilde{R}, \tilde{\tau}, \tilde{\nu}_R)$ is conservative [**34**]. If, moreover, X is a separable metric space, then for a.e. $x \in R$ there is a sequence $n_k = n_k(x) \to \infty$ such that $\tau^{n_k} x \to x$ and $\varphi(n_k, x) \to 0$. Therefore, for a.e. point $x \in R$, the sequence $(\varphi_k(x))_{k \geq 0}$ visits arbitrarily close to $0 \in G$. If $G = \mathbb{Z}^d$, then $\varphi_k(x) = 0$ infinitely many times.

From now until the end of this section (except for Lemma 1), we assume that (X, τ, ν) is *ergodic*, $\nu(X) < \infty$, and that $\varphi : X \to G$ is measurable. It is *centered* if $\int_X \varphi \, d\nu = 0$. Recall that φ is a *coboundary* if there exists a measurable function $\psi : X \to G$ such that $\varphi = \psi - \tau\psi$.

PROPOSITION 1. *Let (φ_n) be the cocycle associated with an integrable function $\varphi : X \to \mathbb{R}^d$, and let $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ be the corresponding skew product. Then the following hold.*

(1) *Let $d = 1$. If φ is centered, then the cocycle (φ_n) is recurrent.*

(2) *Let $d \geq 1$. If φ is not a coboundary, then for a.e. $x \in X$ and any compact $K \subset \mathbb{R}^d$,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_K(\varphi(i, x)) = 0. \tag{3}$$

Proof. Claim 1 is well known [**2, 30, 34**]. We will prove claim 2. Set $\tilde{X} = X \times \mathbb{R}^d$ and let $(\tilde{X}, \tau_\varphi, \tilde{\nu})$ be the associated skew product. For a compact $L \subset \mathbb{R}^d$, set $u_L(x, g) = \lim_{n \to \infty} [\sum_{i=0}^{n-1} 1_L(\varphi(i, x) + g)]/n$. By the ergodic theorem, $u_L : \tilde{X} \to \mathbb{R}^d$ is well defined. If $u_L \neq 0$ on a set of $\tilde{\nu}$-positive measure, then φ is a coboundary [**6**], contrary to the assumption. Thus, $u_L(x, g) = 0$ for a.e. $(x, g) \in \tilde{X}$. We have $u_L(x, g) = u_{L-g}(x, 0)$, but for any compact K there exists a compact L such that $K \subset L - g$ for all sufficiently small $g \in \mathbb{R}^d$. □

We will call equation (3) the *zero frequency* property.

PROPOSITION 2. *Let $\varphi : X \to \mathbb{R}$ be integrable and centered. If φ is not a coboundary, then for a.e. $x \in X$,*

$$\sup_{n \geq 0} \varphi(n, x) = +\infty, \quad \inf_{n \geq 0} \varphi(n, x) = -\infty. \tag{4}$$

Proof. If the former part of the claim fails, then, by ergodicity, $\sup_n \varphi(n, x) < \infty$ for a.e. $x \in X$. Set

$$h(x) = \sup_{k \geq 1} \varphi(k, x), \quad g(x) = \sup_{k \geq 2} \varphi(k, x) - h(x). \tag{5}$$

Since $\tau\varphi(k, x) = \varphi(k + 1, x) - \varphi(x)$, we have

$$\varphi(x) = \sup_{k \geq 2} \varphi(k, x) - \tau \sup_{k \geq 1} \varphi(k, x) = h(x) - h(\tau x) + g(x). \tag{6}$$

Iterating equation (6), we obtain

$$\varphi(n, x) = h(x) - h(\tau^n x) + \sum_{0}^{n-1} g(\tau^j x).$$

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Since by Proposition 1, (φ_n) is recurrent, for a.e. x there is an infinite sequence $n_k = n_k(x)$ such that $\varphi(n_k, x)$ and $h(\tau^{n_k}x)$ are bounded. By equation (5), $g \le 0$, and equation (6) implies $|\sum_{j=0}^{\infty} g(\tau^j x)| < \infty$ almost everywhere. Thus, again by recurrence, $g = 0$ almost everywhere, and hence φ is a coboundary. The latter part of the claim is proved the same way. □

We will refer to equation (4) as the *unbounded oscillation* property.

PROPOSITION 3. *Let $\varphi : X \to \mathbb{R}^d$. Then for any strictly increasing sequence $k_n \in \mathbb{N}$, there exists a constant $c \in [0, +\infty]$ (depending on the sequence) such that for a.e. $x \in X$ we have*

$$\limsup_{n\to\infty}(n^{-1/d}|\varphi(k_n, x)|) = c. \tag{7}$$

If for some sequence k_n we have $c = 0$, then (φ_n) is recurrent.

Proof. The function $\limsup_{n\to\infty}(n^{-1/d}|\varphi(k_n, x)|)$ is measurable and invariant. By ergodicity, it is constant, $c \in [0, +\infty]$. By [**4, 35**], $c = 0$ implies recurrence. □

Let the setting be as in Definition 1, and let $H \subset G$ be a closed subgroup. Set $\tilde{X}_H = X \times G/H$, $\tilde{\nu}_H = \nu \times \mathrm{Leb}_{G/H}$. Set $\tilde{\tau}_H(x, g + H) = (\tau x, \varphi(x) + g + H)$. The skew product $(\tilde{X}_H, \tilde{\tau}_H, \tilde{\nu}_H)$ is the *reduction* of $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ *by the subgroup* H.

LEMMA 1. *Let $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ be the skew product over (X, τ, ν) with the fiber G and the displacement function φ. If there is a closed, proper subgroup $H \subset G$ such that $(\tilde{X}_H, \tilde{\tau}_H, \tilde{\nu}_H)$ is ergodic, then φ is not a coboundary.*

Proof. Let $\psi : X \to G$ be such that $\varphi = \psi - \tau\psi$. Then $\Psi(x, g) = (x, \psi(x) + g)$ conjugates $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ and $(X \times G, \tau \times \mathrm{Id}, \tilde{\nu})$. Reducing by H, we obtain the isomorphism Ψ_H conjugating $(\tilde{X}_H, \tilde{\tau}_H, \tilde{\nu}_H)$ and $(X \times G/H, \tau \times \mathrm{Id}_{G/H}, \tilde{\nu}_H)$, contrary to the ergodicity assumption. □

Definition 3. Let $v_n \in \mathbb{R}^d$ be an infinite sequence. A unit vector $u \in \mathbb{R}^d$ is a *projectively asymptotic direction* if there is a subsequence v_{n_k} such that $|v_{n_k}| \to \infty$ and $|v_{n_k}|^{-1}v_{n_k}$ converges to u or $-u$.

We denote by $\langle u, v\rangle$ the scalar product in $\mathbb{R}^2$. Set $|v| = \langle v, v\rangle^{1/2}$ and let $U \subset \mathbb{R}^2$ be the unit circle. If (φ_n) is an $\mathbb{R}^2$-valued cocycle, we denote by $A_\varphi(x) \subset U$ the set of projectively asymptotic directions for the sequence $\varphi_n(x)$.

PROPOSITION 4. *Let $\varphi : X \to \mathbb{R}^2$ be an integrable and centered function, and let (φ_n) be the associated cocycle. If (φ_n) is transient, then $A_\varphi(x) = U$ for a.e. $x \in X$.*

Proof. Let $v \in U$, and let $v^\perp \in U$ be such that $\langle v, v^\perp\rangle = 0$. The function $x \mapsto \langle \varphi(x), v^\perp\rangle$ satisfies the assumptions of Proposition 1, hence the cocycle $\langle(\varphi_n), v^\perp\rangle$ is recurrent. For a.e. $x \in X$, there is a sequence $n_k \to \infty$ and $c > 0$ such that $|\langle \varphi_{n_k}(x), v^\perp\rangle| < c$. If $|\langle \varphi_{n_k}(x), v\rangle| \nrightarrow \infty$, then (φ_n) is recurrent. Otherwise, $\varphi_{m_k}(x)/|\varphi_{m_k}(x)|$ converges to v or $-v$ for a subsequence m_k, hence $v \in A_\varphi(x)$. □

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2.2. *Billiard flows and billiard maps on non-compact, periodic polygonal surfaces.* Compact polygonal surfaces are natural generalizations of planar polygons [**17, 20**]. We will always assume that a polygonal surface P is connected and has a non-empty boundary ∂P. *Regular geodesics* $\gamma(t)$ on P are defined for $-\infty < t < \infty$. *Regular billiard curves* are those regular geodesics that meet ∂P. Let UP be the corresponding set of unit tangent vectors, and let μ be the Liouville measure on UP. The *billiard flow* (UP, T^t, μ) is defined in the same way as the billiard flow on a polygon [**18, 19, 23**]. Let $U_{\partial P}P$ denote the *standard cross-section* for (UP, T^t, μ), i.e., the set of vectors $v \in UP$ with base points in ∂P. Let ν be the induced measure on $U_{\partial P}P$. The Poincaré map $(U_{\partial P}P, \tau, \nu)$ is the *billiard map* of P [**18, 19**].

The group $\Gamma = \Gamma(P) \subset O(2)$ generated by *linear reflections* about the sides of ∂P [**17, 20**] naturally acts on U; it consists of reflections and rotations. We say that P is a *rational polygonal surface* [**17, 20, 38**] if $|\Gamma| < \infty$. Then Γ is the dihedral group R_N composed of N rotations and N reflections. Any reflection in $O(2)$ fixes two vectors $v, -v \in U$. Identifying U with $\mathbb{R}/2\pi\mathbb{Z}$, we obtain two *reflecting directions* $\alpha, \alpha + \pi \mod 2\pi$. We choose Euclidean coordinates so that the set of reflecting directions of R_N is $\{k\pi/N : 0 \le k \le 2N-1\}$. We identify U/R_N with $[0, \pi/N]$.

We associate a *translation surface* $S = S(P)$ with a rational P [**17, 20, 22**]. The group R_N acts on S by isometries, and we have the projection $p : S \to P = S/R_N$. The directions of geodesics in S (respectively P) belong to U (respectively $[0, \pi/N]$). The flow (UP, T^t, μ) decomposes as a one-parameter family of *directional billiard flows* $(UP_\theta, T^t_\theta, \mu_\theta)$, where $\theta \in [0, \pi/N]$. Accordingly, the *linear flows* $(S, L^t_\theta, \text{Leb})$, $0 \le \theta < 2\pi$, decompose the geodesic flow. For $\theta \in (0, \pi/N)$, we have natural identifications $(UP_\theta, T^t_\theta, \mu_\theta) = (S, L^t_\theta, \text{Leb})$. The billiard map decomposes as a one-parameter family of *directional billiard maps* $(U_{\partial P}P_\theta, \tau_\theta, \nu_\theta)$.

Let $\tilde{P}$ be a non-compact polygonal surface, and let G be an infinite, countable group acting freely and cocompactly by isometries on $\tilde{P}$. Then $\tilde{P}$ is a *G-periodic* polygonal surface. The quotient $P = \tilde{P}/G$ is a compact polygonal surface; the projection $p : \tilde{P} \to P$ is a *covering of polygonal surfaces* [**20**]. It induces the covering of billiard flows $q : (U\tilde{P}, \tilde{T}^t, \tilde{\mu}) \to (UP, T^t, \mu)$ and the corresponding billiard maps $qm : (U_{\partial\tilde{P}}\tilde{P}, \tilde{\tau}, \tilde{\nu}) \to (U_{\partial P}P, \tau, \nu)$. There is a function $\varphi : U_{\partial P}P \to G$ such that $(U_{\partial\tilde{P}}\tilde{P}, \tilde{\tau}, \tilde{\nu})$ is the skew product over $(U_{\partial P}P, \tau, \nu)$ with the displacement function φ. We have $\Gamma(\tilde{P}) = \Gamma(P)$, thus $\tilde{P}$ and P are simultaneously rational or not. Suppose they are rational, with $\Gamma = R_N$, $N \ge 1$, and let $\tilde{S}$ and S be the respective translation surfaces. Then $\tilde{S}$ is G-periodic, and we have $\tilde{S}/G = S$, $\tilde{S}/\Gamma = \tilde{P}$, $S/\Gamma = P$, $\tilde{P}/G = P$. The coverings $q : (U\tilde{P}, \tilde{T}^t, \tilde{\mu}) \to (UP, T^t, \mu)$ and $qm : (U_{\partial\tilde{P}}\tilde{P}, \tilde{\tau}, \tilde{\nu}) \to (U_{\partial P}P, \tau, \nu)$ are compatible with the directional decompositions. They induce directional coverings $q_\theta : (U\tilde{P}_\theta, \tilde{T}^t_\theta, \tilde{\mu}_\theta) \to (UP_\theta, T^t_\theta, \mu_\theta)$ and $qm_\theta : (U_{\partial\tilde{P}}\tilde{P}_\theta, \tilde{\tau}_\theta, \tilde{\nu}_\theta) \to (U_{\partial P}P_\theta, \tau_\theta, \nu_\theta)$. Restricting φ to $U_{\partial P}P_\theta \subset U_{\partial P}P$, we obtain $\varphi_\theta : U_{\partial P}P_\theta \to G$ such that $(U_{\partial\tilde{P}}\tilde{P}_\theta, \tilde{\tau}_\theta, \tilde{\nu}_\theta)$ are skew products over $(U_{\partial P}P_\theta, \tau_\theta, \nu_\theta)$ with the displacement functions φ_θ.

We will use the notion of polygons in translation surfaces. A polygon $O \subset S$ may have several connected components; those that have empty interiors are *barriers*. The polygonal surface $P = S \backslash O$ is a *translation surface with polygonal obstacles*.

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LEMMA 2.

(1) *Let $P = S\backslash O$ be a compact translation surface with polygonal obstacles. Let US be the set of unit vectors tangent to regular geodesics on S, and let μ be the Liouville measure on US. Then for almost every $v \in US$ the geodesic in S emanating from v meets O.*

(2) *Let P be a compact, rational polygonal surface, and let R_N be the corresponding group. Let γ be a dense geodesic on P in the direction $\theta \in [0, \pi/N]$. If γ does not meet ∂P, then $N = 1$ and $\theta \in \{0, \pi\}$.*

Proof. Recall that a direction $\theta \in U$ is *minimal* (respectively *uniquely ergodic*) if every regular geodesic on S in the direction θ is dense (respectively uniformly distributed) in S. Let $U_{\min} \subset U$ (respectively $U_{\text{erg}} \subset U$) be the sets of minimal (respectively uniquely ergodic) directions. The set $U\backslash U_{\min}$ (respectively $U\backslash U_{\text{erg}}$) is countable [**28**]† (respectively has Lebesgue measure zero [**29**]). Let $US_{\min} \subset US$ (respectively $US_{\text{erg}} \subset US$) be the corresponding subsets in the unit tangent bundle. Then US_{erg} has full Liouville measure, and $US_{\text{erg}} \subset US_{\min}$. For a typical translation surface the inclusion $US_{\text{erg}} \subset US_{\min}$ is proper.

Denote by $NC \subset US$ the set of vectors tangent to regular geodesics that do not meet O. Let $v \in NC$, and let γ be the corresponding geodesic on S. If interior$(O) \neq \emptyset$, then γ is not dense. Thus, in this case, $NC \subset US\backslash US_{\min}$, and hence $\mu(NC) = 0$. Let interior$(O) = \emptyset$, i.e., O is a union of geodesic segments in S. A geodesic segment yields two directions, say $\alpha, \alpha + \pi \mod 2\pi \in U$. We choose one of the directions and, slightly abusing the language, call it the direction of the segment. A geodesic segment in direction α is a cross-section for all linear flows L_θ^t, $\theta \in U_{\min}$, on S, except, possibly, for $\theta = \alpha, \alpha + \pi$. Thus, $NC \cap US_{\min} = \emptyset$, unless O consists of geodesic segments in a single direction, say α. In the latter case, if α is not a minimal direction, then $NC \cap US_{\min} = \emptyset$ as well. Finally, if α is a minimal direction, then $NC \cap US_{\min}$ consists of geodesics with directions $\alpha, \alpha + \pi$, implying $\mu(NC \cap US_{\min}) = 0$. Thus, in all cases, $\mu(NC \cap US_{\min}) = 0$. Since $\mu(NC \cap US\backslash US_{\min}) = \mu(US\backslash US_{\min}) = 0$, we have established claim 1. Claim 2 follows by lifting the setting to the translation surface of P via the canonical covering $p : S \to P = S/R_N$ [**17**]. □

LEMMA 3. *Let $G \subset \mathbb{R}^n$ be a lattice. Let $\tilde{P}$ be a non-compact, G-periodic polygonal surface, and let $P = \tilde{P}/G$.*

(1) *The displacement function $\varphi : U_{\partial P} P \to G$ for the skew product $(U_{\partial \tilde{P}} \tilde{P}, \tilde{\tau}, \tilde{\nu})$ is centered, i.e.,*

$$\int_{U_{\partial P} P} \varphi \, d\nu = 0. \tag{8}$$

(2) *Let $\tilde{P}$ be rational, and let $\Gamma(P) = R_N$, $N \geq 1$. For $\theta \in [0, \pi/N]$, let φ_θ be the displacement functions for the directional billiard maps $(U_{\partial \tilde{P}} \tilde{P}_\theta, \tilde{\tau}_\theta, \tilde{\mu}_\theta)$. If N is even, then for every $\theta \in [0, \pi/N]$, the function φ_θ is centered. If N is odd, then the function $\varphi_{\pi/2N}$ is centered.*

† This is closely related to the Keane criterion of minimality for interval exchange transformations.

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Proof. (1) For $v \in U_{\partial P} P$, let $\gamma_v(t)$, $0 \le t$, be the forward billiard orbit, and let $t(v) > 0$ be the *first return time*. The tangent vector $\gamma_v'(t)$ is defined for $0 \le t < t(v)$, and $\lim_{t \to t(v)-} \gamma_v'(t)$ exists. The transformation $\sigma(v) = -\lim_{t \to t(v)-} \gamma_v'(t)$ is the *canonical involution* for the billiard map $(U_{\partial P} P, \tau, \nu)$ [**18**]. Let $\tilde{v} = (v, g) \in U_{\partial \tilde{P}} \tilde{P}$, and let $\tilde{\sigma}(v, g) = (\sigma(v), g + \varphi(v))$ be the canonical involution for $(U_{\partial \tilde{P}} \tilde{P}, \tilde{\tau}, \tilde{\nu})$. The identity

$$(v, g) = \tilde{\sigma}^2(v, g) = \tilde{\sigma}(\sigma(v), g + \varphi(v)) = (\sigma^2(v), g + \varphi(v) + \varphi(\sigma(v)))$$

yields $\varphi(\sigma(v)) = -\varphi(v)$. Since σ preserves the Liouville measure, the claim follows.

(2) The symmetry $\theta \mapsto -\theta$ of U and the identification $U/R_N = [0, \pi/N]$ induce a map $\theta \mapsto \theta'$ of $[0, \pi/N]$. If N is even, then $\theta \mapsto -\theta$ belongs to R_N, and hence $\theta' = \theta$. If N is odd, then $\theta' = \pi/N - \theta$. The canonical involution $\sigma : U_{\partial P} P \to U_{\partial P} P$ induces *directional involutions* $\sigma_\theta : U_{\partial P} P_\theta \to U_{\partial P} P_{\theta'}$. The above argument implies that for every $\theta \in [0, \pi/N]$, we have

$$\int_{U_{\partial P} P_{\theta'}} \varphi_{\theta'} \, d\nu_{\theta'} = -\int_{U_{\partial P} P_\theta} \varphi_\theta \, d\nu_\theta. \tag{9}$$

The claims now follow from the preceding remarks. □

3. $\mathbb{Z}$-periodic polygonal surfaces

A compact, rational polygonal surface P is *arithmetic* if its translation surface S admits a covering onto a flat torus whose branch locus is a single point [**21, 22**]†. We will assume, without loss of generality, that S covers $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the branch locus is $\{0\} + \mathbb{Z}^2$. A $\mathbb{Z}$-periodic polygonal surface $\tilde{P}$ is arithmetic if $P = \tilde{P}/\mathbb{Z}$ is arithmetic. A direction $\theta \in U$ is *rational* (respectively *irrational*) if $\tan\theta \in \mathbb{Q}$ (respectively $\tan\theta \notin \mathbb{Q}$) [**17**].

THEOREM 1. *Let $\tilde{P}$ be a $\mathbb{Z}$-periodic polygonal surface and let $P = \tilde{P}/\mathbb{Z}$.*

(1) *If the flow $(U P, T^t, \mu)$ is ergodic, then the billiard flow on $\tilde{P}$ is conservative.*

(2) *Let $\tilde{P}$ be rational, and let R_N be the corresponding group. Suppose that N is even. Then, for a full measure set of directions $\theta \in [0, \pi/N]$, the directional billiard flow $(U \tilde{P}_\theta, \tilde{T}_\theta^t, \tilde{\mu}_\theta)$ is conservative, with zero frequency and unbounded oscillations.*

(3) *Let, moreover, $\tilde{P}$ be arithmetic.*

   (i) *For any irrational direction θ, the flow $(U \tilde{P}_\theta, \tilde{T}_\theta^t, \tilde{\lambda}_\theta)$ has the above properties.*

   (ii) *For any rational direction, the flow $(U \tilde{P}_\theta, \tilde{T}_\theta^t)$ is a disjoint union of periodic and transient orbit bands whose boundaries are concatenations of saddle connections.*

Proof. Claim 1 follows from Lemma 3 and Proposition 1, and we turn to claim 2. The set $\mathcal{E}_{\mathrm{erg}}(P) \subset [0, \pi/N]$ of ergodic directions for P has full measure [**29**]. By Lemmas 2, 3, and Proposition 1, $(U \tilde{P}_\theta, \tilde{T}_\theta^t, \tilde{\lambda}_\theta)$ is conservative for any $\theta \in \mathcal{E}_{\mathrm{erg}}(P)$. For $k > 1$, set $P_k = \tilde{P}/k\mathbb{Z}$. Then P_k is a k-to-1 covering of P; the sets $\mathcal{E}_{\mathrm{erg}}(P_k) \subset \mathcal{E}_{\mathrm{erg}}(P)$ have full measure [**29**]. Let $\theta \in \bigcup_{k>1} \mathcal{E}_{\mathrm{erg}}(P_k)$. Then, by Lemma 1, and Propositions 1 and 2, the flow $\tilde{T}_\theta^t$ has zero frequency and unbounded oscillations. We have proved claim 2.

† These are also known as *square-tiled translation surfaces or origamis* [**36**].

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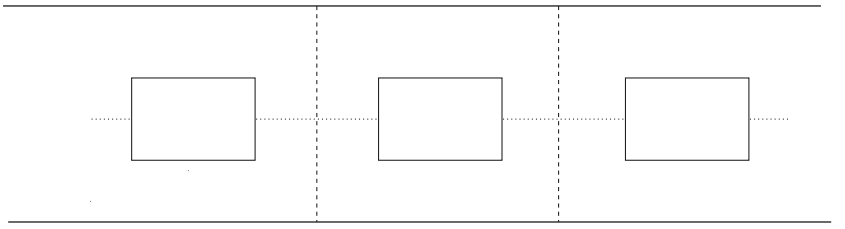


FIGURE 2. A $\mathbb{Z}$-periodic polygonal surface.

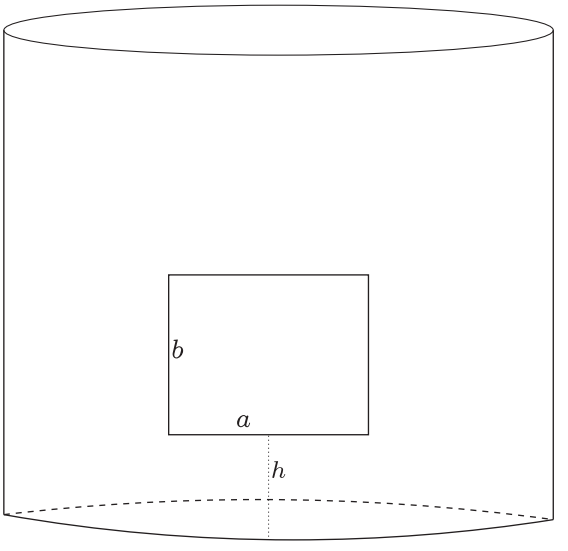


FIGURE 3. Flat cylinder with a rectangular obstacle.

Claim 3(i) follows from the observation that a direction is ergodic if and only if it is irrational [**17**]. To prove 3(ii), let θ be a rational direction. The flow T_θ^t on P decomposes into periodic bands bounded by saddle connections [**17**]. The lift of a periodic band to $\tilde{P}$ is periodic or transient, depending on whether or not the displacement along the orbit vanishes. □

Example 1. Let $B \subset \mathbb{R}^2$ be the strip bounded by the lines $\{y = 0\}$ and $\{y = 1\}$. For $0 \le a, b < 1$ and $b/2 < h < 1 - b/2$, let R_0 be the closed $(a \times b)$-rectangle in the interior of the unit square, centered at $(\frac{1}{2}, h)$. The $\mathbb{Z}$-periodic polygonal surface $\tilde{P}(a, b, h) = B \setminus \bigcup_{k\in\mathbb{Z}}(R_0 + (k, 0))$ is a strip with rectangular obstacles, which become barriers if $a = 0$ or $b = 0$. Let $P(a, b, h)$ be a flat unit cylinder with the obstacle R_0. Then $\tilde{P}(a, b, h)/\mathbb{Z} = P(a, b, h)$ (see Figures 2 and 3). Its translation surface S is made from four copies of $P(a, b, h)$, with appropriate identifications of sides. If $a, b \neq 0$, then S has four cone points with angles 6π, and $g(S) = 5$. If $a = 0$, then S has four cone points with angles 4π, thus $g(S) = 3$. If $b = 0$, then S has two cone points with angles 4π, hence $g(S) = 2$. The surface $\tilde{S}$ is obtained by analogous identifications from four copies

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of $\tilde{P}$. The dihedral group of $P(a, b, h)$ has order 4 if $b \neq 0$, and 2 if $b = 0$. Thus, for $b \neq 0$, the surface $\tilde{P}(a, b, h)$ satisfies the assumptions of claim 2 in Theorem 1. It is arithmetic if and only if $a, b, h \in \mathbb{Q}$ [**21, 22**]. Theorem 1 implies the following.

COROLLARY 1. *Let* $\tilde{P} = \tilde{P}(a, b, h)$ *and* $b \neq 0$. *For a.e.* $\theta \in [0, \pi/2]$, *the flows* $(U\tilde{P}_\theta, \tilde{T}^t_\theta, \tilde{\mu}_\theta)$ *are conservative with zero frequency and unbounded oscillations. If* $a, b, h \in \mathbb{Q}$, *then the claims hold for every* $\theta \in [0, \pi/2]$ *such that* $\tan\theta \notin \mathbb{Q}$.

Remark 1. The surface $\tilde{P}(a, 0, h)$ is a horizontal band with a periodic configuration of horizontal barriers, $N = 1$, and the directions belong to $[0, \pi]$. For $\theta \neq \pi/2$, the flow $\tilde{T}^t_\theta$ is dissipative: every orbit drifts horizontally with the rate $|\cos\theta|$. The flow $\tilde{T}^t_{\pi/2}$ is periodic. This illustrates claim 2 in Lemma 3 and the necessity that N be even in Theorem 1.

The set of planar polygons has a natural topology. The space $\mathcal{P}_n$ of connected n-gons up to scaling is a compact in $\mathbb{R}^{2n}$ [**18, 38**], hence a Baire space [**33**]. We say that a billiard property is *topologically generic* if the set of polygons $P \in \mathcal{P}_n$ with this property is residual, i.e., contains a dense G_δ. Ergodicity is topologically generic [**29**]. It is not known if it is measure theoretically generic [**19**]. Vorobets proved that polygons admitting a certain super-exponentially fast approximation by rational polygons (*Vorobets approximation* in the following) are ergodic [**37**]. Let $\mathcal{O}_n \subset \mathcal{P}_n$ be the topological space of simple n-gons inside the unit square. For $O \in \mathcal{O}_n$, let $\tilde{P}_O$ (respectively P_O) be the strip B with the collection $\{O + (k, 0), k \in \mathbb{Z}\}$ of obstacles (respectively the unit cylinder with obstacle O). Then $\{\tilde{P}_O : O \in \mathcal{P}_n\}$ is a Baire space of $\mathbb{Z}$-periodic polygonal surfaces. The following is immediate from the above material.

COROLLARY 2. *The billiard flow on* $\tilde{P}_O$ *is conservative with zero frequency and unbounded oscillations for:*

(1) *the topologically generic polygon* $O \in \mathcal{O}_n$;
(2) *polygons* $O \in \mathcal{O}_n$ *admitting a Vorobets approximation.*

Specializing to a simple closed subset in $\mathcal{O}_n$, we obtain a 'concrete version' of Corollary 2. For $0 \leq \alpha < \pi/2$, let $R(a, b, h; \alpha)$ be the rectangle in Example 1 rotated by angle α. Let $\tilde{P}(a, b, h; \alpha)$ be the infinite strip with the obstacles $\{R(a, b, h, \alpha) + (k, 0) : k \in \mathbb{Z}\}$.

COROLLARY 3. *Fix* a, b, h. *Let* $0 \leq \alpha < \pi/2$. *The billiard flow on* $\tilde{P}(a, b, h; \alpha)$ *is conservative with zero frequency and unbounded oscillations for:*

(i) *a dense* G_δ *set of angles* α;
(ii) *angles* α *such that* α/π *is a Liouville number.*

As an example, we will discuss a natural class of $\mathbb{Z}$-periodic polygonal surfaces. Let $Q \subset \mathbb{R}^2$ be a simply connected polygon, i.e., ∂Q is connected. Suppose that Q satisfies the following conditions. There is $p \geq 1$, a vector $\vec{v} \neq 0$, and for $1 \leq i \leq p$ there are $2p$ distinct (closed) segments S_i, $S'_i \subset \partial Q$ such that:

(i) $S'_i = S_i + \vec{v}$; and
(ii) $Q \cap (Q + \vec{v}) = \bigcup_{1 \leq i \leq p} S'_i$.

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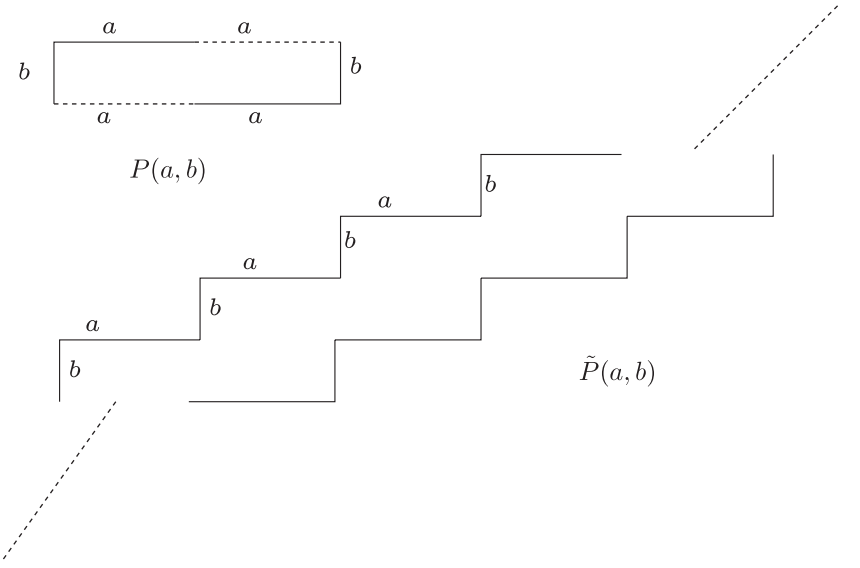


FIGURE 4. A stairway polygonal surface and its quotient.

We claim that $\bigcup_{1\le i\le p}(S_i \cup S_i') \subset \partial Q$ is a proper subset. To prove this, we assume the opposite, and let $C = \bigcup_{1\le i\le p} S_i$. Then $C \subset \mathbb{R}^2$ is a simple (i.e., without self-intersections) curve, and $\partial Q = C \cup (C + \vec{v})$. Suppose first that C is connected. Let $x, y \in C$ be the two endpoints. Then $x' = x + \vec{v}$, $y' = y + \vec{v}$ are the endpoints of the curve $C' = C + \vec{v}$. Since $C \cup C'$ is a simple closed curve, $\{x, y\} = \{x', y'\}$. Hence, up to switching labels, $x' = y$, $y' = x$. Thus, $y = x + \vec{v}$, $x = y + \vec{v}$, and hence $x = x + 2\vec{v}$, implying $\vec{v} = 0$, contrary to the defining assumptions.

We have proved the claim in the special case when $\bigcup_{1\le i\le p} S_i$ is connected. The general case follows by induction on the number of connected components in $\bigcup_{1\le i\le p} S_i$. We leave the details to the reader. Figure 4 shows an example where Q is the rectangle with horizontal and vertical sides of lengths $2a$ and b, respectively. Here $p = 1$; S (respectively S') is the left (respectively right) half of the bottom (respectively top) of Q, and $\vec{v} = (a, b)$.

Let $Q \subset \mathbb{R}^2$ satisfy the above conditions. The *stairway based on* Q is the polygonal surface $\tilde{P}$ obtained by deleting from $\bigcup_{k\in\mathbb{Z}}(Q + k\vec{v})$ the segments $S_i + k\vec{v} : 1 \le i \le p, k \in \mathbb{Z}$. Then $P = \tilde{P}/\mathbb{Z}$ is Q with S_i, S_i' identified for $1 \le i \le p$. We have shown that $\bigcup_{1\le i\le p}(S_i \cup S_i') \ne \partial Q$, hence $\partial P \ne \emptyset$. Let $\Gamma \subset O(2)$ be the group generated by linear reflections about the sides of ∂P. Then $\tilde{P}$ is a *rational stairway* if and only if $|\Gamma| < \infty$.

PROPOSITION 5. *Let $\tilde{P} \subset \mathbb{R}^2$ be a rational stairway. If $|\Gamma|$ is divisible by 4 then the directional billiard flows $\tilde{T}_\theta^t$ on $\tilde{P}$ are conservative with zero frequency and unbounded oscillations for a.e. θ. If the surface P is arithmetic, then the above holds for every irrational direction.*

Example 2. Let $Q = Q(a, b)$ be a $2a \times b$ rectangle. Let S, $S' \subset \partial Q$ be as in Figure 4. The stairway based on Q is $\tilde{P}(a, b)$ in Figure 4. It is arithmetic if and only if $a, b \in \mathbb{Q}$ [**21, 22**].

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4. *$\mathbb{Z}^2$-periodic polygonal surfaces*

Let R_0 be a unit square; let $O \subset \text{interior}(R_0)$ be a polygon. Set $\tilde{P}_O = \mathbb{R}^2 \setminus \bigcup_{(p,q)\in\mathbb{Z}^2}(O + (p, q))$. The $\mathbb{Z}^2$-periodic polygonal surface $\tilde{P}_O$ is a Euclidean plane with a doubly periodic configuration of obstacles. This is said to be a *polygonal Lorenz gas*. The compact polygonal surface $P_O = \tilde{P}_O/\mathbb{Z}^2$ is rational if and only if the angles between the sides of O are π-rational. When O is a rectangle, $\tilde{P}_O$ is the *rectangular Lorenz gas*, or the *wind-tree model* [**13, 25**]. We will study this model in this section. First, we will make a few remarks on the general case, restricting them to the case where O is a triangle. The reader will easily extend these to an arbitrary polygonal Lorenz gas.

Let $z(t) \in \tilde{P}_O$ be a billiard orbit. The functions $|z(t) - z(0)|$, $[z(t) - z(0)]/|z(t) - z(0)|$ have obvious geometric meanings. If there exists $t_k \to \infty$ such that $[z(t_k) - z(0)]/|z(t_k) - z(0)| \to \pm v$, we say that v is a *projective asymptotic direction* for the orbit. If $\limsup_{t\to\infty} |z(t)|/\sqrt{t} > 0$, we say that the *orbit displacement grows asymptotically (at least) at the rate* $\sqrt{t}$. Let $K \subset \tilde{P}_O$. If $\lim_{T\to\infty} \int_0^T 1_K(z(t))\, dt/T = 0$, then $z(t)$ *visits* K *with zero frequency*.

Let $\mathcal{T}$ be the topological space of triangles in the unit square.

PROPOSITION 6. *For the topologically generic $O \in \mathcal{T}$, the billiard on $\tilde{P}_O$ is either:*

(i) *conservative, with zero frequency of visiting every compact domain; or*

(ii) *dissipative, with the full set of projectively asymptotic directions and the asymptotic displacement growth of the order of $\sqrt{t}$ for a.e. orbit.*

Proof. For $(k, \ell) \in \mathbb{Z}^2$, let $P_O(k, \ell) = \tilde{P}_O/(k\mathbb{Z} \oplus \ell\mathbb{Z})$. By [**29**], there is a dense G_δ set $\mathcal{D} \subset \mathcal{T}$ such that for $O \in \mathcal{D}$, the billiard on $P_O(k, \ell)$ is ergodic for all pairs $k \neq 0$, $\ell \neq 0$. For $O \in \mathcal{D}$, the billiard on $\tilde{P}_O$ is either conservative or dissipative. If it is conservative, Lemma 1 and Proposition 1 yield (i). If it is dissipative, (ii) follows likewise from Propositions 4 and 3. □

The following is the counterpart of Proposition 6 for directional billiard flows. We leave the proof to the reader.

PROPOSITION 7. *Let $O \in \mathcal{T}$ be such that P_O is rational, and let R_N be the corresponding dihedral group. Suppose that N is even. Then for a.e. $\theta \in [0, \pi/N]$ the directional billiard flow $\tilde{T}_\theta^t$ is either:*

(i) *conservative, with zero frequency of visiting every compact domain; or*

(ii) *dissipative, with the full set of projectively asymptotic directions and the asymptotic displacement growth of the order of $\sqrt{t}$ for a.e. orbit.*

4.1. *Rectangular Lorenz gas.* Let $0 < a, b < 1$, and let $R(a, b)$ be the upright $a \times b$ rectangle centered at $(0, 0)$. We will study the billiard on the $\mathbb{Z}^2$-periodic rational polygonal surface

$$\tilde{P}(a, b) = \mathbb{R}^2 \Big\backslash \bigcup_{(m,n)\in\mathbb{Z}^2} \{R(a, b) + (m, n)\}. \tag{10}$$

The quotient $P(a, b) = \tilde{P}(a, b)/\mathbb{Z}^2$ is a unit torus with a rectangular obstacle, $\Gamma = R_2$. To simplify the notation, we will suppress a, b whenever possible. We will denote by $(\tilde{Z}_\theta, \tilde{T}_\theta^t, \tilde{\mu}_\theta)$ and $(Z_\theta, T_\theta^t, \mu_\theta)$ (respectively $(\tilde{X}_\theta, \tilde{\tau}_\theta, \tilde{\nu}_\theta)$ and $(X_\theta, \tau_\theta, \nu_\theta)$) the billiard

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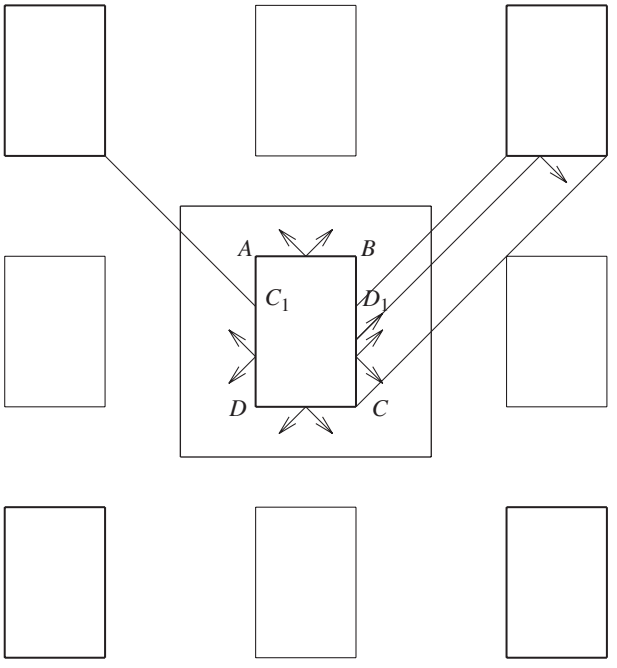


FIGURE 5. The cross-section for a directional billiard flow.

flow (respectively billiard map) in the direction $\theta \in [0, \pi/2]$ for $\tilde{P}(a, b)$ and $P(a, b)$, respectively. Set $R(a, b) = ABCD$. The space X_θ consists of outward pointing unit vectors with base points in $ABCD$ whose directions are $\pm\theta, \pi \pm \theta$ (see Figure 5). The rational directions are $\theta(p, q) = \arctan(q/p)$, where $p, q \in \mathbb{N}$ are relatively prime. We will often use the notation (p, q) for $\theta(p, q)$.

We say that $\tilde{P}(a, b)$ has *small obstacles* with respect to (p, q) if the geodesics in $P(a, b)$ emanating from A or C in the direction (p, q) return to the same point without encountering obstacles.

LEMMA 4. *The polygonal surface $P(a, b)$ satisfies the small obstacles condition if and only if*

$$qa + pb \leq 1. \tag{11}$$

The equality holds if and only if every geodesic in the direction (p, q) is a billiard curve.

Proof. The condition is satisfied if and only if $R(a, b)$ fits in the strip between two parallel lines with slope q/p and vertical distance $1/p$, i.e., $aq/p + b \leq 1/p$. The equality holds if and only if $R(a, b)$ fits in tightly (see Figure 6). □

We fix (p, q) satisfying inequality (11) and suppress p, q, θ whenever possible. We identify X with two copies of the rectangle $ABCD$: the copy X_+ (respectively X_-) carries the outward pointing vectors in the directions $\theta, \pi + \theta$ (respectively $\pi - \theta, 2\pi - \theta$) (see Figures 5 and 7). Set $\tau_\pm = \tau|_{X_\pm}$, $\tau_\pm^2 = \tau^2|_{X_\pm}$, $\tilde{\tau}_\pm = \tilde{\tau}|_{\tilde{X}_\pm}$, and $\tilde{\tau}_\pm^2 = \tilde{\tau}^2|_{\tilde{X}_\pm}$.

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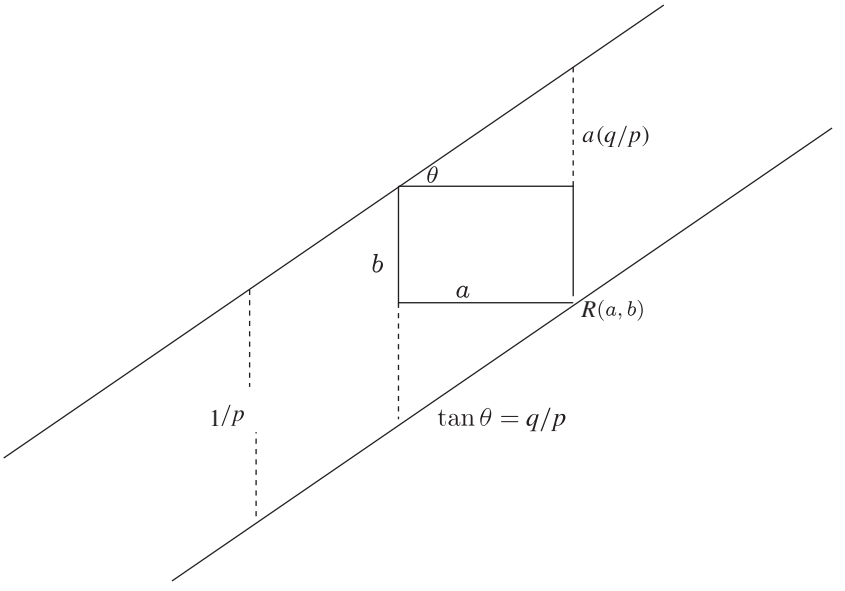


FIGURE 6. The smallness of obstacles condition.

LEMMA 5. *We identify the rectangle $ABCD$ oriented $A \to B \to C \to D$ with the circle $S^1 = \mathbb{R}/\mathbb{Z}$ so that*

$$|AB| = |CD| = qa/2(qa + pb), \quad |BC| = |DA| = pb/2(qa + pb). \tag{12}$$

Then the maps $\tau_\pm : S^1 \to S^1$ are the orthogonal reflections about the axes AC and BD. The maps $\tau_-\tau_+$ and $\tau_+\tau_-$ are the rotations by $qa/(qa + pb)$ and $pb/(qa + pb)$, respectively.

Proof. Vectors emanating from $ABCD$ in direction η assume at the first return the direction $r(\eta)$, where r is a reflection in R_2. Thus, $\tau_\pm(X_\pm) = X_\mp$. From the canonical invariant measure $d\nu = \sin\eta\, ds$ [**23**], we obtain $\nu(AB) = \nu(CD) = qa$, $\nu(BC) = \nu(DA) = pb$. Equation (12) follows from $\nu(S^1) = 2(qa + pb)$. The maps $\tau_\pm : S^1 \to S^1$ preserve the arc length and reverse orientation, hence they are orthogonal reflections. Since τ_+ (respectively τ_-) fixes the points A and C (respectively B and D), these pairs of points yield the axes of the respective reflections. □

We assume that $\alpha < \beta$†, and set

$$\alpha = qa/(qa + pb), \quad \beta = pb/(qa + pb). \tag{13}$$

Let $\psi_\pm : S^1 \to \mathbb{Z}^2$ be the displacement functions so that

$$\tilde{\tau}_+^2(x, g) = (x + \alpha, g + \psi_+(x)), \quad \tilde{\tau}_-^2(x, g) = (x + \beta, g + \psi_-(x)).$$

To calculate $\psi_\pm$, we identify $S^1_\pm$ with $[0, 1]$ so that A, B, C, D (respectively D, A, B, C) go to 0, $\frac{1}{2}\alpha$, $\frac{1}{2}$, $\frac{1}{2} + \frac{1}{2}\alpha$ (respectively 0, $\frac{1}{2} - \frac{1}{2}\alpha$, $\frac{1}{2}$, $1 - \frac{1}{2}\alpha$).

† The other case reduces to this by switching the coordinate axes.

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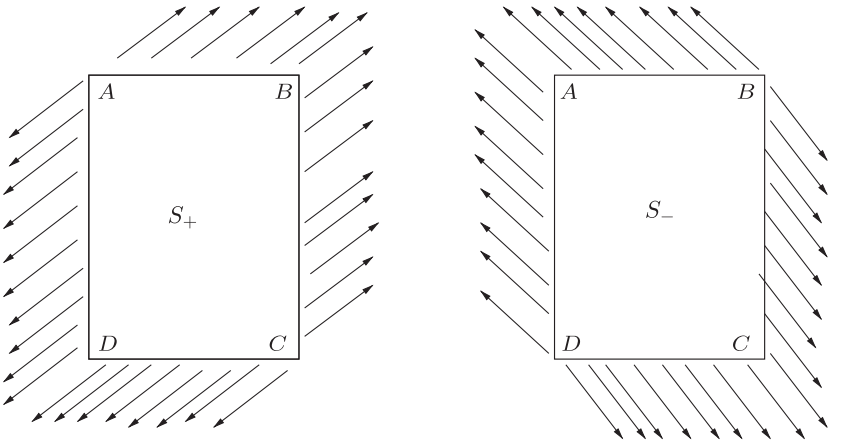


FIGURE 7. The cross-section X.

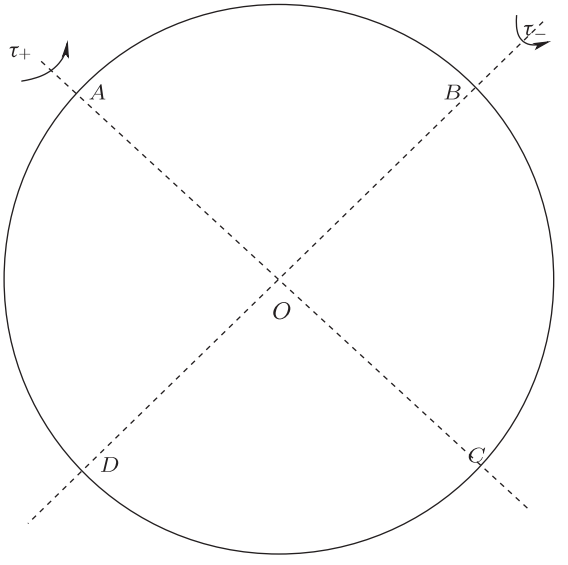


FIGURE 8. The circle and two orthogonal reflections.

LEMMA 6. *The functions* $\psi_\pm$ *take the values* $(2p, 0)$ *on* $]\frac{1}{2} - \frac{1}{2}\alpha, \frac{1}{2}[$ *and* $]\frac{1}{2}, \frac{1}{2} + \frac{1}{2}\alpha[$, *the values* $(-2p, 0)$ *on* $]1 - \frac{1}{2}\alpha, 1[$ *and* $]0, \frac{1}{2}\alpha[$, *the values* $(0, 2q)$ *on* $]0, \frac{1}{2} - \frac{1}{2}\alpha[$ *and* $]\frac{1}{2}\alpha, \frac{1}{2}[$, *and the values* $(0, -2q)$ *on* $]\frac{1}{2}, 1 - \frac{1}{2}\alpha[$ *and* $]\frac{1}{2} + \frac{1}{2}\alpha, 1[$.

Proof. Let $\varphi_\pm : S^1 \to \mathbb{Z}^2$ be the displacement functions for $\tilde{\tau}_\pm$. From Lemma 5 and Figure 8, we obtain $\varphi_+|_{ABC} = (p, q)$, $\varphi_+|_{CDA} = (-p, -q)$, $\varphi_-|_{DAB} = (-p, q)$, *and* $\varphi_-|_{BCD} = (p, -q)$. Using that $\psi_\pm(x) = \varphi_\pm(x) + \varphi_\mp(\tau_\pm(x))$ and the above formulas, we calculate $\psi_\pm$. □

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Since ψ_+, ψ_- take values in the group $2p\mathbb{Z} \times 2q\mathbb{Z} \simeq \mathbb{Z}^2$, we represent them by functions $[0, 1] \to \mathbb{Z}^2$ that do not depend on p, q. In particular, ψ_+ goes to

$$\Psi(x) = \begin{cases} (0, 1) \text{ on }]0, \frac{1}{2} - \frac{1}{2}\alpha[, \ (1, 0) \text{ on }]\frac{1}{2} - \frac{1}{2}\alpha, \frac{1}{2}[, \\ (0, -1) \text{ on }]\frac{1}{2}, 1 - \frac{1}{2}\alpha[, \ (-1, 0) \text{ on }]1 - \frac{1}{2}\alpha, 1[. \end{cases} \tag{14}$$

4.2. *Ergodicity, conservativity, and statistics of collisions.* Let $(2p\mathbb{Z} \times 2q\mathbb{Z})^* \subset \mathbb{Z}^2$ be the group generated by (p, q) and $(p, -q)$. Set

$$G_{(p,q)} = \mathbb{Z}^2/(2p\mathbb{Z} \times 2q\mathbb{Z})^*, \quad H_{(p,q)} = \mathbb{Z}^2/(2p\mathbb{Z} \times 2q\mathbb{Z}).$$

Since $2p\mathbb{Z} \times 2q\mathbb{Z} \subset (2p\mathbb{Z} \times 2q\mathbb{Z})^*$ is a subgroup of index 2, we have a 2-to-1 surjection $\sigma : H_{(p,q)} \to G_{(p,q)}$, and $|G_{(p,q)}| = 2pq$. The group $H_{(p,q)}$ consists of all pairs $(0 \le \bar{k} < 2p, 0 \le \bar{\ell} < 2q)$ of residues with the operation $(\bar{k} + \bar{k_1} \mod 2p, \bar{\ell} + \bar{\ell_1} \mod 2q)$.

Set $h = (\bar{k}, \bar{\ell}) \in H_{(p,q)}$. Each element $g \in G_{(p,q)}$ yields two pairs $h = (\bar{k}, \bar{\ell}), h_1 = (\bar{k} + p \mod 2p, \bar{\ell} + q \mod 2q)$. Let $\mathbb{Z}^2_g \subset \mathbb{Z}^2$ be the corresponding coset. Then

$$\mathbb{Z}^2_g = \{(\bar{k} + 2pi, \bar{\ell} + 2qj) : i, j \in \mathbb{Z}\} \cup \{(\bar{k} + p + 2pi, \bar{\ell} + q + 2qj) : i, j \in \mathbb{Z}\}.$$

Set $\tilde{X}_g = X \times \mathbb{Z}^2_g$.

THEOREM 2. *Let $(p, q) \in \mathbb{N}^2$ with p, q relatively prime. Let $a, b > 0$ satisfy $qa + pb \le 1$, and let $(\tilde{X}, \tilde{\tau}, \tilde{\nu})$ be the billiard map in the direction (p, q). The $2pq$ sets $\tilde{X}_g \subset \tilde{X}$, $g \in G_{(p,q)}$, are $\tilde{\tau}$-invariant. Let a/b be irrational. Then the transformations $(\tilde{X}_g, \tilde{\tau}, \tilde{\nu})$ are ergodic and isomorphic. The partition*

$$\tilde{X} = \bigcup_{g \in G_{(p,q)}} \tilde{X}_g \tag{15}$$

yields the ergodic decomposition of the billiard in the direction (p, q).

Proof. All the claims, except ergodicity, are immediate from Lemmas 5 and 6. The ergodicity of $(\tilde{X}_g, \tilde{\tau}, \tilde{\nu})$ is equivalent to the ergodicity of skew products $\rho_{\alpha,\psi_+}, \rho_{\beta,\psi_-}$, where α, β are given by equation (13). By symmetry, it suffices to prove the ergodicity of ρ_{α,ψ_+}. Let $\Psi : [0, 1] \to \mathbb{Z}^2$ be as in equation (14). By the discussion preceding equation (14), $\rho_{\alpha,\psi_+} = \rho_{\alpha,\Psi}$. By Theorem 5 in §5.3, $\rho_{\alpha,\Psi}$ is ergodic for any irrational α, i.e., if a/b is irrational. □

The ergodic decomposition in equation (15) acquires a simple geometric meaning when we pass from maps to flows. For $g \in G_{(p,q)}$, set

$$\tilde{P}_g(a, b) = \mathbb{R}^2 \setminus \bigcup_{(m,n) \in \mathbb{Z}^2_g} \{R(a, b) + (m, n)\}. \tag{16}$$

Since $\tilde{P}_g(a, b)$ is $(2p\mathbb{Z} \times 2q\mathbb{Z})^*$-invariant, and $(2p\mathbb{Z} \times 2q\mathbb{Z})^* \simeq \mathbb{Z}^2$, it is a $\mathbb{Z}^2$-periodic polygonal surface. Let $(\tilde{Z}_g, \tilde{T}^t_g, \tilde{\mu}_g)$ be the billiard flow in the direction (p, q) on $\tilde{P}_g(a, b)$. The following is immediate from Theorem 2.

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COROLLARY 4. *Let the assumptions be as in Theorem 2. The flows* $(\tilde{Z}_g, \tilde{T}_g^t, \tilde{\mu}_g)$, $g \in G_{(p,q)}$, *are ergodic and isomorphic. The disjoint decomposition* $\tilde{Z} = \bigcup_{g \in G_{(p,q)}} \tilde{Z}_g$ *is invariant under the billiard flow in the direction* (p, q), *and*

$$(\tilde{Z}, \tilde{T}_{(p,q)}^t, \tilde{\mu}) = \bigcup_{g \in G_{(p,q)}} (\tilde{Z}_g, \tilde{T}_g^t, \tilde{\mu}_g) \tag{17}$$

is the ergodic decomposition of the conservative billiard flow on $\tilde{P}(a, b)$ *in the direction* (p, q). *The geodesic flow on* $\tilde{P}(a, b)$ *in the direction* (p, q) *has a dissipative part if and only if* $qa + pb < 1$.

For $\tilde{z} \in \tilde{Z}$, $T > 0$, and $(m, n) \in \mathbb{Z}^2$, let $N(\tilde{z}, T, (m, n))$ be the number of times the billiard curve $\gamma_{\tilde{z}}(t)$, $0 \le t \le T$, emanating from $\tilde{z}$ encounters the obstacle $R(a, b) + (m, n)$. Corollary 4 and the ergodic theorem for infinite invariant measures [**1**] yield the following.

COROLLARY 5. *Let* $g \in G_{(p,q)}$, *and let* (m, n), $(m', n') \in \mathbb{Z}_g^2$. *As* $T \to \infty$, *for* $\tilde{\mu}$*-almost every* $\tilde{z} \in \tilde{Z}_g$, *the counting functions* $N(\tilde{z}, T, (m, n))$ *and* $N(\tilde{z}, T, (m', n'))$ *go to infinity and their ratio converges to* 1.

Remark 2. Figure 1 illustrates Corollary 5. It shows an orbit, $\gamma(t)$, $0 \le t \le T$, of the billiard flow in the direction (1, 1) emanating from a point $\tilde{z} \in R(a, b)$. Set $R_{(m,n)} = R(a, b) + (m, n)$. The orbit encounters only the obstacles $R_{(m,n)}$ with $m + n$ even. The finite curve in Figure 1 covers the region near $R_{(0,0)}$ much more densely than the rest of the space. Corollary 5 asserts that as $T \to \infty$, the orbit will eventually uniformly cover all of the space $\bigcup_{m+n=0 \mod 2} R_{(m,n)}$.

5. *Ergodicity of cocycles over irrational rotations*

Let $X = \mathbb{R}/\mathbb{Z}$, let $\rho_\alpha : X \to X$ be an irrational rotation, let $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$, and let $\Phi : X \to G$ be a piecewise constant function. Let μ be the Lebesgue measure on $X \times G$. We will study the ergodicity of skew products $(X \times G, \rho_{\alpha,\Phi}, \mu)$, i.e., the ergodicity of cocycles (Φ_n) over ρ_α. It is common to assume that the discontinuities of Φ are independent of α. However, this assumption does not hold for $\Psi : X \to \mathbb{Z}^2$ in equation (14). We will first prove in §5.2 the ergodicity of $\mathbb{Z}^d$-valued cocycles (Φ_n), assuming the *(wdd) property*, which holds generically. Then in §5.3 we will establish the ergodicity of the $\mathbb{Z}^2$-valued cocycle (Ψ_n) in equation (14), without any additional assumptions.

5.1. *The Denjoy–Koksma inequality.*

From now on, $0 < \alpha < 1$ is irrational, with the continued fraction expansion $[0; a_1, \ldots, a_n, \ldots]$ and convergents p_n/q_n, where p_n and q_n are the *numerators* and the *denominators* [**31**]. We have $p_{-1} = q_0 = 1$, $q_{-1} = p_0 = 0$, and, for $n \ge 1$,

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}, \quad (-1)^n = p_{n-1} q_n - p_n q_{n-1}. \tag{18}$$

For $u \in \mathbb{R}$, set $\|u\| = \inf_{n \in \mathbb{Z}} |u - n|$. Then $\|q_n \alpha\| = (-1)^n (q_n \alpha - p_n)$, and for $n \ge 0$, $1 \le k < q_n + 1$, we have

$$(q_{n+1} + q_n)^{-1} \le \|q_n \alpha\| \le q_{n+1}^{-1}, \ \|q_n \alpha\| \le \|k\alpha\| \, . \tag{19}$$

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Let BV denote functions of bounded variation, $V(\varphi)$, on X. Let $\varphi \in \mathrm{BV}$ be a centered function. Let p/q be a rational number such that $|\alpha - p/q| < 1/q^2$. By the *Denjoy–Koksma inequality* [**12**],

$$\left|\sum_{\ell=0}^{q-1} \varphi(x + \ell\alpha)\right| \le V(\varphi). \tag{20}$$

The following is immediate from equations (19) and (20).

COROLLARY 6. *Let $\Phi : X \to G$ be a centered function with BV components. Then the cocycle (Φ_n) over any irrational rotation is recurrent.*

For the reader's convenience, we will prove the following useful facts.

LEMMA 7.

(1) *For any pair of consecutive denominators, at least one satisfies $q_n \|q_n\alpha\| < \frac{1}{2}$.*
(2) *Out of any four consecutive denominators, at least one is odd and satisfies $q_n \|q_n\alpha\| < \frac{1}{2}$.*

Proof. (1) For any $n \in \mathbb{N}$, define δ_1 and δ_2 by $q_n\|q_n\alpha\| = \frac{1}{2} - \delta_1$, $q_{n+1}\|q_{n+1}\alpha\| = \frac{1}{2} - \delta_2$. The identity $q_n\|q_{n+1}\alpha\| + q_{n+1}\|q_n\alpha\| = 1$ implies $(q_{n+1} - q_n)^2 = 2\delta_1 q_{n+1}^2 + 2\delta_2 q_n^2$. Hence, δ_1 and δ_2 cannot both be negative.

(2) By equation (18), at least one of any two consecutive numerators (respectively denominators) is odd. If both p_n, q_n are odd, then one of p_{n+1}, q_{n+1} is even. Let $q_{n-1}, \ldots, q_{n+2}$ be four consecutive denominators. Their possible parities are (0, 1, 0, 1), (0, 1, 1, 0), (1, 0, 1, 0), (0, 1, 1, 1), (1, 0, 1, 1), (1, 1, 0, 1), (1, 1, 1, 0), (1, 1, 1, 1). If there are two consecutive odd denominators, then the statement follows from claim 1. In the remaining cases (0, 1, 0, 1) and (1, 0, 1, 0), we have, respectively, q_n is odd, $a_{n+1} \ne 1$, and q_{n+1} is odd, $a_{n+2} \ne 1$. Set $q = q_n$ (respectively $q = q_{n+1}$) in the former (respectively latter) case. Then $q\|q\alpha\| < \frac{1}{2}$. □

5.2. *Ergodicity of generic cocycles.* Let $d(\cdot, \cdot)$ be an invariant distance on G.

Definition 4. Let $a \in G$. Suppose that for $n \ge 1$ there exist $\ell_n \in \mathbb{N}$, $\varepsilon_n > 0$, and $\delta > 0$ such that $\lim_n \varepsilon_n = 0$, $\lim_n \ell_n\alpha \bmod 1 = 0$, and $\mathrm{Leb}(\{x : d(\Phi_{\ell_n}(x), a) < \varepsilon_n\}) \ge \delta$. Then we say that a is a *quasi-period* for the cocycle (Φ_n). We say that a is a *period* if for every $\rho_{\alpha,\Phi}$-invariant measurable function f on $X \times G$ and for a.e. $(x, g) \in X \times G$, we have

$$f(x, g + a) = f(x, g). \tag{21}$$

LEMMA 8. (See [**7**].) *Every quasi-period is a period.*

The set of periods is a closed subgroup of G which coincides with the group of finite essential values of the cocycle [**34**]. A cocycle is ergodic if and only if its group of periods is G. The range $R(\Phi) \subset G$ of a piecewise constant function $\Phi : X \to G$ is the set of a such that $\Phi(x) = a$ on a non-trivial interval. We assume that $\Phi \ne \mathrm{const}$. Let $\mathcal{D} = \{t_i : i = 1, \ldots, d\}$ be its set of discontinuities, where, without loss of generality, $t_1 = 0$. For $N \in \mathbb{N}$, the set of discontinuities of $\Phi_N(t) = \sum_{k=0}^{N-1} \Phi(t + k\alpha)$ is $\mathcal{D}_N = \{t_i - j\alpha \bmod 1 : 1 \le i \le d, 0 \le j < N\}$. (We assume that the points $t_i - j\alpha \bmod 1$, $1 \le i \le d$, $0 \le j < N$ are

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distinct.) Set $\mathcal{D}_N = \{0 = \gamma_{N,1} < \cdots < \gamma_{N,dN} < 1\}$ and $\gamma_{N,dN+1} = \gamma_{N,1}$; for $1 \le \ell \le dN$, the elements $\gamma_{N,\ell}$ run through $\mathcal{D}_N$ in the natural order.

Definition 5.

(1) A cocycle has *well-distributed discontinuities (wdd)*, if there is $c > 0$ and an infinite set W of denominators of α such that

$$\gamma_{q,\ell+1} - \gamma_{q,\ell} \ge c/q \quad \text{for all } q \in W,\ \ell \in \{1, \ldots, dq\}. \tag{22}$$

(2) The number α has *property (D)* if there is $M \in \mathbb{N}$ such that for infinitely many n, either $a_n \in [2, M]$ or $a_n = a_{n+1} = 1$.

LEMMA 9. *Let $t \in \frac{1}{2}(\mathbb{Z}\alpha + \mathbb{Z})\backslash(\mathbb{Z}\alpha + \mathbb{Z})$. Then there exist $c > 0$ and $L \in \mathbb{N}$ such that if $n \ge L$ and either:*

(i) $a_{n+1} \in [2, M]$*; or*

(ii) $a_{n+1} = a_{n+2} = 1$,

then for $1 \le k \le q_n - 1$ we have

$$\|k\alpha - t\| \ge c/q_n. \tag{23}$$

Proof. Let $t = \ell\alpha/2 + r/2$, with $\ell, r \in \mathbb{Z}$, and ℓ or r odd. Let L be such that $|\ell| < q_{n-1}$ for $n \ge L$. Let $n \ge L$ and $k \in [1, q_n - 1]$. If $a_{n+1} \ge 2$, then $|2k - \ell| \le 2k + |\ell| < 2q_n + q_{n-1} \le a_{n+1}q_n + q_{n-1} = q_{n+1}$. If $a_{n+1} \in [2, M]$, then, by equation (19), for all $j \in [1, q_{n+1}[$

$$\|j\alpha\| \ge \|q_n\alpha\| \ge [q_n + q_{n+1}]^{-1} = [(a_{n+1} + 1)q_n + q_{n-1}]^{-1} \ge [(2 + M)q_n]^{-1}. \tag{24}$$

If $2k - \ell \ne 0$, equation (24) implies

$$\|k\alpha - t\| \ge \|(2k - \ell)\alpha\|/2 \ge [2q_n(2 + M)]^{-1}.$$

If $a_{n+1} = a_{n+2} = 1$, then

$$q_{n+1} = q_n + q_{n-1}, \quad q_{n+2} = q_{n+1} + q_n = 2q_n + q_{n-1}$$

and

$$|2k - \ell| < 2q_n + q_{n-1} = q_{n+2}.$$

Thus, if $2k - \ell \ne 0$, then

$$\|(2k - \ell)\alpha\| \ge \|q_{n+1}\alpha\| \ge [q_{n+2} + q_{n+1}]^{-1} = [3q_n + 2q_{n-1}]^{-1} \ge 1/5q_n.$$

If ℓ is even, then r is odd, and for $2k = \ell$, we have $\|k\alpha - t\| = \|(k - \ell/2)\alpha - 1/2\| = 1/2$. It suffices to set $c = \min\{1/[2(2 + M)], 1/10\}$. □

The following is immediate from Lemma 9.

PROPOSITION 8. *If α has property (D) and if $\mathcal{D} \subset \{\frac{1}{2}(\mathbb{Z}\alpha + \mathbb{Z})\backslash(\mathbb{Z}\alpha + \mathbb{Z})\} \mod 1$, with $t - t' \notin (\mathbb{Z}\alpha + \mathbb{Z}) \mod 1$ for $t \ne t'$ in $\mathcal{D}$, then the cocycle (Φ_n) has (wdd).*

THEOREM 3. *Let $\Phi : X \to \mathbb{Z}^r$ be a piecewise constant centered function. Let (Φ_n) be the corresponding cocycle over ρ_α. If (Φ_n) has property (wdd), then its group of periods contains the range $R(\Phi) \subset \mathbb{Z}^r$.*

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Proof. Let $W \subset \mathbb{N}$ be as in Definition 5. We will study the functions $\Phi_q(x) = \sum_{0\le k\le q-1} \Phi(x + k\alpha \mod 1)$, $q \in W$. For $1 \le \ell \le dq$, let $I_{q,\ell} =]\gamma_{q,\ell}, \gamma_{q,\ell+1}[$ be the intervals of continuity. Set $\mathcal{I}_q = \{I_{q,\ell} : 1 \le \ell \le dq\}$. For $t_i \in \mathcal{D}$, let $\sigma_i = \lim_{\varepsilon\to 0^+}[\Phi(t_i + \varepsilon) - \Phi(t_i - \varepsilon)]$ be the jump at t_i. Set $\Sigma(\Phi) = \{\sigma_i : 1 \le i \le d\}$ and $R = \bigcup_{q\in W} R(\Phi_q) \subset \mathbb{Z}^r$. By equation (20), $|R| < \infty$.

Each interval $[k/q, (k+1)/q[$, $0 \le k \le q-1$, contains an element $j\alpha \mod 1$, where $1 \le j \le q$. Thus, for any $t \in X$, the distance on the circle between any consecutive elements of the set $\{t + j\alpha \mod 1 : j = 0, \dots, q-1\}$ is $< 2/q$. Hence, any interval $J \subset X$ of length $\ge 2/q$ contains at least one point of this set. Let c be as in equation (22). Set $L = \lfloor 2/c \rfloor + 1$. Let $I_{q,\ell} \in \mathcal{I}_q$ be arbitrary. Let $J_{q,\ell} \subset X$ be the union of L consecutive intervals in $\mathcal{I}_q$ starting with $I_{q,\ell}$. By equation (22), the interval $J_{q,\ell}$ is longer than or equal to $2/q$. Thus, for any $t_i \in \mathcal{D}$, it contains a point of the set $\{t_i + j\alpha \mod 1 : j = 0, \dots, q-1\}$. Therefore, for any $\sigma \in \Sigma(\Phi)$, there are $v \in R$ and two consecutive intervals $I, I' \in \mathcal{I}_q$ such that $I \cup I' \subset J_{q,\ell}$ and such that Φ_q takes the values v and $v + \sigma$ on I and I', respectively.

Let $\sigma \in \Sigma(\Phi)$, $v \in R$. Let $\mathcal{F}_q(\sigma) \subset \mathcal{I}_q$ be the family of intervals $I \in \mathcal{I}_q$ such that the jump at the right endpoint of I is σ. Let $\mathcal{A}_q(\sigma, v) \subset \mathcal{F}_q(\sigma)$ be the set of intervals $I \in \mathcal{F}_q(\sigma)$ such that the value of Φ_q on I is v; let $\mathcal{A}'_q(\sigma, v) \subset \mathcal{F}_q(\sigma)$ be the set of intervals $I' \in \mathcal{F}_q(\sigma)$ adjacent on the right to the intervals $I \in \mathcal{A}_q(\sigma, v)$. Let $A_q(\sigma, v) \subset X$ (respectively $A'_q(\sigma, v) \subset X$) be the union of intervals $I \in \mathcal{A}_q(\sigma, v)$ (respectively $I' \in \mathcal{A}'_q(\sigma, v)$). Thus, Φ_q takes the value v (respectively $v + \sigma$) on $A_q(\sigma, v)$ (respectively $A'_q(\sigma, v)$).

There is an infinite subset of W (also denoted by W), and $v_0 \in R$ such that for $q \in W$,

$$|\mathcal{A}_q(\sigma, v_0)|, |\mathcal{A}'_q(\sigma, v_0)| \ge |\mathcal{F}_q(\sigma)|/|R| \ge qd/L|R|. \tag{25}$$

By equations (22) and (25), $\mathrm{Leb}(A_q(\sigma, v_0)), \mathrm{Leb}(A'_q(\sigma, v_0)) \ge dc^2/(2 + c)|R|$. Thus, v_0 and $v_0 + \sigma$ are quasi-periods, and by Lemma 8, periods. Hence, σ is a period. Since $\sigma \in \Sigma(\Phi)$ was arbitrary, the group of periods contains $\Sigma(\Phi)$.

We will denote by 'bar' the reduction modulo a closed subgroup $H \subset G$. Then $\overline{\rho_{\alpha,\Phi}} = (X \times G/H, \rho_{\alpha,\bar\Phi}, \bar\mu)$ is the skew product over ρ_α with the fiber G/H and the displacement function $\bar\Phi$.

Let $H \subset G$ (respectively $H' \subset G$) be the group generated by $\Sigma(\Phi)$ (respectively $R(\Phi)$). Then the function $\bar\Phi : X \to G/H$ is constant. Let $a \in H'$ be such that $\bar\Phi = \bar a$. Then $\overline{\rho_{\alpha,\Phi}}(x, \bar g) = (\rho_\alpha(x), \bar g + \bar a)$. Observe that $H'/H \subset G/H$ is the cyclic group generated by $\bar a$. If $|H'/H| = \infty$, then $\rho_{\alpha,\bar\Phi}$ is dissipative, contrary to Corollary 6. Thus, H'/H is the cyclic group of order $n < \infty$. A $\rho_{\alpha,\Phi}$-invariant function f defines a $\overline{\rho_{\alpha,\Phi}}$-invariant function $\bar f$. By the above equation, $\bar f(\rho^n_\alpha(x), \bar g) = \bar f(x, \bar g)$, i.e., $\bar f(x, \bar g)$ depends only on $\bar g$. Thus, $\bar a$ is a period for $\bar f$, and therefore a is a period for f. □

The following is immediate from Proposition 8 and Theorem 3.

COROLLARY 7. *Let $\Phi : X \to G$ be a piecewise constant, centered function whose range generates G. Suppose that α has property (D), and that the set of discontinuities $\mathcal{D}$ of Φ satisfies $\mathcal{D} \subset \{\frac{1}{2}(\mathbb{Z}\alpha + \mathbb{Z})\backslash(\mathbb{Z}\alpha + \mathbb{Z})\} \mod 1$, with $t - t' \notin (\mathbb{Z}\alpha + \mathbb{Z}) \mod 1$ for $t \ne t'$ in $\mathcal{D}$. Then $\rho_{\alpha,\Phi}$ is ergodic.*

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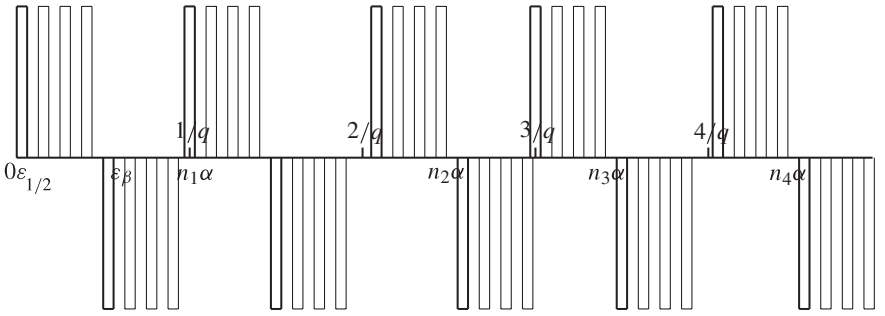


FIGURE 9. Graph of $\zeta(t_n q_n, \cdot)$.

Remark 3. The function $\Psi : X \to \mathbb{Z}^2$ in equation (14) satisfies the assumptions of Corollary 7, and almost every α satisfies (D). Thus, Corollary 7 yields Theorem 2 for generic small obstacles.

5.3. *Removing the genericity assumptions.* For any irrational $0 < \alpha < 1$, we define piecewise constant functions $\gamma, \zeta : X \to \mathbb{Z}$ by

$$\gamma = 1_{[0,\frac{1}{2}[} - 1_{[\frac{1}{2},1[},\ \zeta = 1_{[0,\frac{1}{2}-\frac{1}{2}\alpha[} - 1_{[\frac{1}{2},1-\frac{1}{2}\alpha[}. \tag{26}$$

To explain the heuristics of our proof of the ergodicity of cocycles (γ_n), (ζ_n), assume that α does not satisfy property (D). Passing to a subsequence, if need be, we have $a_n \to \infty$. Suppose that for all sufficiently large n the numbers q_{2n} and p_{2n} are odd, while q_{2n+1} is even and p_{2n+1} is odd. Hence for $n > n_0$, the inequality $\zeta(q_n, \cdot) \neq 0$ holds only on sets of small measure, and we cannot use the method of Theorem 3. Instead, we consider $\zeta(tq, \cdot)$ for such t that $\|tq\alpha\|$ is close to zero but big enough to ensure that $\zeta(tq, x) = \sum_{j=0}^{t-1} \zeta(q, x + jq\alpha) \neq 0$ on a set of measure bounded away from zero. Figure 9 illustrates the idea.

LEMMA 10. *If q is odd and $q\|q\alpha\| < \frac{1}{2}$, then $\sum_{j=0}^{q-1} \gamma(x + j\alpha) = \pm 1$.*

Proof. The discontinuities of γ_q are $t - j\alpha \mod 1$, with $j = 0, \ldots, q-1$, $t = 0, \frac{1}{2}$; the respective jumps are $+2, -2$. Since $q\|q\alpha\| < \frac{1}{2}$, any successive discontinuities are of the form $r/q - j_1(r)\theta$, $r/q - j_2(r)\theta + 1/2q$, with $0 \le j_1(r), j_2(r) < q$ and $j_1(r) - j_2(r) = \pm 1/2q$; hence, the $+2$ and the -2 jumps of γ_q alternate, and the value of γ_q is $u - 2$ or u, for a constant u. Since γ_q is odd and non-constant, $u = 1$. □

THEOREM 4. *The cocycles (γ_n) and (ζ_n) over ρ_α, corresponding to the functions in equation (26) are ergodic.*

Proof. By Lemma 10, 1 is a quasi-period for the cocycle (γ_n). Hence, Lemma 8 yields its ergodicity. Let p_n/q_n be the convergents of α. Let $p'_n, q'_n \in \mathbb{N}$ be such that $q_n = 2q'_n$ or $q_n = 2q'_n + 1$, and $p_n = 2p'_n$ or $p_n = 2p'_n + 1$. Set $\alpha = p_n/q_n + \theta_n$.

The set of discontinuities of ζ is $\{0, \beta = \frac{1}{2} - \frac{1}{2}\alpha, \frac{1}{2}, \beta' = 1 - \frac{1}{2}\alpha\}$, with respective jumps $1, -1, -1, 1$. The set of discontinuities of ζ_q is

$$D_0^{(q)} \cup D_{\frac{1}{2}}^{(q)} \cup D_{1-\frac{1}{2}\alpha}^{(q)} \cup D_{\frac{1}{2}-\frac{1}{2}\alpha}^{(q)},$$

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where $D_t^{(q)} = \{t - j\alpha : 0 \le j \le q-1\}$ for $t \in \{0, \beta, \frac{1}{2}, \beta'\}$. We set $\{0, \beta, \frac{1}{2}, \beta'\} = P_1 \cup P_2$, and consider the following cases.

(1) $P_1 = \{0, \beta'\}$, $P_2 = \{\frac{1}{2}, \beta\}$ if q_n is odd, p_n is even.

(2) $P_1 = \{0, \frac{1}{2}\}$, $P_2 = \{\beta, \beta'\}$ for q_n even, p_n odd.

(3) $P_1 = \{0, \beta\}$, $P_2 = \{\frac{1}{2}, \beta'\}$ when q_n and p_n are odd.

Discontinuities of ζ_{q_n} which come from points in the same (respectively distinct) P_i : $i = 1, 2$ are close to each other (respectively well separated). Since all cases are similar, we will consider in detail only case 2. In what follows, the arithmetic is understood mod 1. To simplify the notation, we will suppress subscripts and write $\alpha = p/q + \theta$, etc. Set $\mathcal{J}_q = \{0, \dots, q-1\}$. For $i = 1, 2, 3, 4$, there are $j_i(r) \in \mathcal{J}_q$, with $(j_2(r) - j_1(r))p \bmod q = (j_4(r) - j_3(r))p \bmod q = \frac{1}{2}q$, such that

$$D_0^{(q)} = \left\{\frac{-jp}{q} - j\theta\right\} = \left\{\frac{r}{q} - j_1(r)\theta\right\},$$

$$D_{\frac{1}{2}}^{(q)} = \left\{\frac{(q'-jp)}{q} - j\theta\right\} = \left\{\frac{r}{q} - j_2(r)\theta\right\},$$

$$D_{1-\frac{1}{2}\alpha}^{(q)} = \left\{\frac{1}{2q} - \frac{p'+1+jp}{q} - \left(j+\frac{1}{2}\right)\theta\right\} = \left\{\frac{r}{q} + \frac{1}{2q} - \left(j_3(r)+\frac{1}{2}\right)\theta\right\},$$

$$D_{\frac{1}{2}-\frac{1}{2}\alpha}^{(q)} = \left\{\frac{1}{2q} - \frac{-q'+p'+1+jp}{q} - \left(j+\frac{1}{2}\right)\theta\right\} = \left\{\frac{r}{q} + \frac{1}{2q} - \left(j_4(r)+\frac{1}{2}\right)\theta\right\},$$

where $j, r \in \mathcal{J}_q$. Observe that $|j_i(r)\theta| \le |q\theta|$, and there is $\varepsilon \in \{+1, -1\}$ such that

$$j_2(r) = j_1(r) + \varepsilon q/2, \quad j_4(r) = j_3(r) + \varepsilon q/2. \tag{27}$$

We will determine the values of the cocycle $\zeta_q(x)$ in a neighborhood of the typical interval $[r/q, (r+1)/q]$, $r \in \mathcal{J}_q$. Let, for concreteness, $\theta < 0$, $j_1 = j_2 + \frac{1}{2}q$, $j_4 = j_3 + \frac{1}{2}q$. (The analysis of other cases is analogous.) Let x start at r/q and move to the right; set $\zeta_q(x) = a$. The value of $\zeta_q(x)$ is constant until x crosses the discontinuity (corresponding to $t = 0$) at $r/q - j_1(r)\theta$, where $\zeta_q(x)$ increases by 1. After that, $\zeta_q(x)$ does not change until x crosses the discontinuity at $r/q - j_2(r)\theta$ (corresponding to $t = \frac{1}{2}$), where it decreases by 1, returning to the value a.

These discontinuities occur before x crosses the two others under the condition $|j_i(r)\theta| < 1/2q$, which holds because $q^2|\theta|$ is small in our case. As x continues moving, the cocycle remains at the value a until, near $r/q + 1/2q$, it increases by 1 at the point $r/q + 1/2q - (j_3(r) + \frac{1}{2})\theta$, a discontinuity corresponding to $t = 1 - \frac{1}{2}\alpha$, and then decreases by 1 at $r/q + 1/2q - (j_4(r) + \frac{1}{2})\theta$, a discontinuity corresponding to $t = \frac{1}{2} - \frac{1}{2}\alpha$. Therefore, we have $\zeta_q = a \pm 1$ on $]r/q - j_1(r)\theta, r/q - j_2(r)\theta[$ and on $]r/q + 1/2q - (j_3(r) + \frac{1}{2})\theta, r/q + 1/2q - (j_4(r) + \frac{1}{2})\theta[$, and $\zeta_q = a$ elsewhere. This analysis is valid for every interval $[r/q, (r+1)/q]$. The order of discontinuities may change, but not the order of the groups of discontinuities. The range of ζ_q is $\{a, a+1, a-1\}$. The identity $\zeta_q(x + 1/2) = -\zeta_q(x)$ implies $a = 0$.

We will now finish the proof. If α satisfies condition (D), then the claim holds, by Corollary 7. Thus, we assume that α does not satisfy condition (D). Then one of the cases (1), (2), or (3) materializes for an infinite sequence (n_k) such that $a_{n_k+1} \to \infty$. We will then say, for brevity, that a case *occurs infinitely often*. If case (1) occurs infinitely often,

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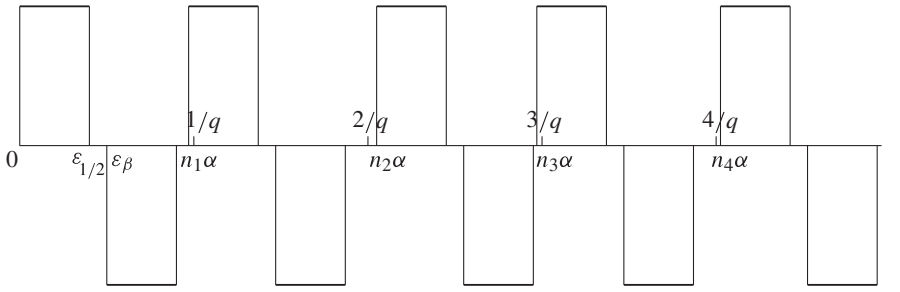


FIGURE 10. The cocycle (ζ_q) for q odd, p even, takes value 1 on a set of measure $\geq \delta > 0$.

then 1 is a quasi-period for (ζ_n) (see Figure 10). Let case (2) occur infinitely often. For $1 \leq r \leq q_{n_k} - 1$, set

$$I_{k,r} = \left]\frac{r}{q_{n_k}} - j_1(r)\theta_{n_k}, \frac{r}{q_{n_k}} - j_2(r)\theta_{n_k}\right[,$$
$$J_{k,r} = \left]\frac{r}{q_{n_k}} + \frac{1}{2q_{n_k}} - \left(j_3(r) + \frac{1}{2}\right)\theta_{n_k}, \frac{r}{q_{n_k}} + \frac{1}{2q_{n_k}} - \left(j_4(r) + \frac{1}{2}\right)\theta_{n_k}\right[.$$

By equation (27), these intervals have length $|\theta_{n_k}|q_{n_k}/2$. At the scale $1/q_{n_k}$, they are close to r/q_{n_k} and to $r/q_{n_k} + 1/2q_{n_k}$, respectively. Outside of these intervals, $\zeta(q_{n_k}, \cdot) = 0$. Let $\delta \in]0, \frac{1}{4}[$. Set $t_k = \lfloor \delta a_{n_k+1} \rfloor$. For $J \subset X$ and $u \in \mathbb{R}$, set $(J + u) = J + u \mod 1$. Let

$$A_k = \bigcup_{j=0}^{q_{n_k}-1} \bigcup_{s=0}^{t_k-1} (I_{k,j} - sq_{n_k}\alpha),\ B_k = \bigcup_{j=0}^{q_{n_k}-1} \bigcup_{s=0}^{t_k-1} (J_{k,j} - sq_{n_k}\alpha).$$

The distance between the intervals $I_{k,r}$ and $J_{k,r}$ is at least $1/2q_{n_k} - q_{n_k}|\theta_{n_k}|$. By our choice of t_k, we have $q_{n_k}|\theta_{n_k}|t_k \leq 1/2q_{n_k} - q_{n_k}|\theta_{n_k}|$, hence the intervals defining A_k and B_k do not overlap.

We have

$$\zeta(t_k q_{n_k}, x) = \sum_{s=0}^{t_k-1} \zeta(q_{n_k}, x + sq_{n_k}\alpha),$$

and by the preceding analysis, $\zeta(t_k q_{n_k}, \cdot) = \pm 1$ on A_k and B_k. Observe that

$$\mathrm{Leb}(A_k) = \frac{1}{2} t_k q_{n_k} q_{n_k} |\theta_{n_k}| \geq \frac{1}{2}\delta a_{n_k+1} \frac{q_{n_k}}{q_{n_k+1}} \geq \frac{1}{2}\delta.$$

Since $t_k q_{n_k}\alpha \bmod 1 \to 0$, and the cocycle takes values ± 1 on A_k, either 1 or -1 is a quasi-period for (ζ_n). If case (3) occurs infinitely often, a similar analysis yields that 1 or -1 is a quasi-period for (ζ_n). The claim follows, by Lemma 8. □

THEOREM 5. *Let $\Psi : X \to \mathbb{Z}^2$ be the function in equation (14). Then the corresponding cocycle over ρ_α is ergodic.*

Proof. Set $\Psi = (\psi_1, \psi_2)$ and $\beta = \frac{1}{2} - \frac{1}{2}\alpha$. Then $\psi_1(x) + \psi_2(x) = \gamma(x)$, $\psi_1(x) - \psi_2(x) = \gamma(x + \beta)$. By Lemmas 7 and 10, $\gamma(q_{n_k}, x)$, $\gamma(q_{n_k}, x + \beta) \in \{\pm 1\}$ for an infinite

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sequence (n_k). Hence, there exist measurable sets $A_k \subset X$ satisfying $\mathrm{Leb}(A_k) \geq \frac{1}{4}$, and such that on A_k, the function $(\gamma(q_{n_k}, x), \gamma(q_{n_k}, x+\beta))$ is constant with the values $(+1, +1)$, $(+1, -1)$, $(-1, +1)$, or $(-1, -1)$.

Thus, on A_k, the function $(\psi_1(q_{n_k}, x), \psi_2(q_{n_k}, x))$ is identically equal to $(1, 0)$, $(0, 1)$, $(-1, 0)$, or $(0, -1)$. Hence, one of the elements $(\pm 1, 0)$, $(0, \pm 1)$ is a quasi-period for the cocycle (Ψ_n). Let, for instance, $(1, 0)$ be a quasi-period. By Lemma 8, $(1, 0)$ is a period. A $\rho_{\alpha,\Psi}$-invariant function f defines a $\rho_{\alpha,\zeta}$-invariant function on $X \times \mathbb{Z}$, where $\zeta : X \to \mathbb{Z}$ is given by equation (26). By Theorem 4, $f = \text{const}$, i.e., the cocycle (Ψ_n) is ergodic. The other cases are disposed of in the same way. □

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