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FUNCTIONAL LIMIT THEOREMS FOR BRANCHING PROCESSES IN A NEARLY DEGENERATE VARYING ENVIRONMENT

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Abstract

We investigate branching processes in a nearly degenerate varying environment, where the offspring distribution converges to the degenerate distribution at 1. Such processes die out almost surely; therefore, we either condition on non-extinction or add inhomogeneous immigration. Extending our one-dimensional limit results from Kevei and Kubatovics (2024), we derive functional limit theorems. In the former case, the limit process is a time-changed simple birth-and-death process on $(-\infty, \infty)$ conditioned on survival at 0, while in the latter, it is a time-changed stationary continuous-time Markov branching process with immigration.

Keywords: Yaglom-type limit; birth-and-death process; Markov branching process with immigration; quasi-stationary distribution

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1. Introduction

A *branching process in a varying environment* (BPVE) $(X_n)_{n\in\mathbb{N}}$ is defined as

$$X_0 = 1, \qquad X_n = \sum_{j=1}^{X_{n-1}} \xi_{n,j}, \quad n \in \mathbb{N} = \{1, 2, \ldots\}, \tag{1}$$

where $\{\xi_{n,j} : n, j \in \mathbb{N}\}$ are nonnegative integer-valued independent random variables such that, for each n, the variables $\{\xi_{n,j} : j \in \mathbb{N}\}$ are identically distributed; let ξ_n denote a generic copy. Adding immigration leads to a *branching process in a varying environment with immigration* (BPVEI) $(Y_n)_{n\in\mathbb{N}}$, defined as

$$Y_0 = 0, \qquad Y_n = \sum_{j=1}^{Y_{n-1}} \xi_{n,j} + \varepsilon_n, \quad n \in \mathbb{N}, \tag{2}$$

where $\{\xi_{n,j}, \varepsilon_n : n, j \in \mathbb{N}\}$ are nonnegative integer-valued independent random variables such that, for each n, the variables $\{\xi_{n,j} : j \in \mathbb{N}\}$ are identically distributed, and ξ_n is a generic copy.

Put $\bar{f}_n = \mathbb{E}(\xi_n)$ for the offspring mean in generation n. To exclude trivialities, we always assume that $\bar{f}_n > 0$ for all n. Let $f_n(s) = \mathbb{E}(s^{\xi_n})$, $s \in [0, 1]$, denote the generating function (g.f.)

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of the offspring distribution in the nth generation. As in [21], we are interested in *branching processes in a nearly degenerate environment*; that is, we always assume the following conditions:

(C1) $\lim_{n\to\infty} \bar{f}_n = 1$, $\sum_{n=1}^{\infty} (1-\bar{f}_n)_+ = \infty$, and $\sum_{n=1}^{\infty} (\bar{f}_n - 1)_+ < \infty$;

(C2) $\lim_{n\to\infty, \bar{f}_n<1} f_n''(1)/(1-\bar{f}_n) = \nu \in [0,\infty)$, and the sequence $\left(f_n''(1)/|1-\bar{f}_n|\right)_n$ is bounded;

(C3) if $\nu > 0$, then $\lim_{n\to\infty, \bar{f}_n<1} f_n'''(1)/(1-\bar{f}_n) = 0$, and the sequence $\left(f_n'''(1)/|1-\bar{f}_n|\right)_n$ is bounded.

Here and later, $\lim_{n\to\infty, \bar{f}_n<1}$ means that the convergence holds along the subsequence $\{n\colon \bar{f}_n < 1\}$. We allow $\bar{f}_n = 1$, in which case we use the convention $0/0 = 0$. Since by (C1) subcritical regimes dominate in the process, $\mathbb{E}(X_n) = \prod_{j=1}^{n} \bar{f}_j =: \bar{f}_{0,n} \to 0$ as $n \to \infty$, and the process dies out almost surely. Condition (C2) implies that $f_n''(1) \to 0$, thus $f_n(s) \to s$, and the branching mechanism converges to the degenerate branching.

The theory of non-homogeneous branching processes started in [10, 14]. Recent results in [18, 19] have renewed interest in BPVEs. In [18], a necessary and sufficient condition was obtained for almost sure extinction of regular processes; see also [5, 12] for the continuous-time and multitype cases, and [8] for a probabilistic proof. Nearly degenerate BPVEIs were introduced in [16], where it was assumed that the offspring have a Bernoulli distribution. In this case, the resulting process is a non-homogeneous first-order integer-valued autoregressive process (INAR(1)). INAR processes are important in various fields of applied probability; see the survey in [29]. The setting of [16] was extended in [20], and the multitype case was studied in [15].

In [21] we analyzed BPVEs and BPVEIs under conditions (C1)–(C3). We showed that both BPVE processes conditioned on non-extinction and BPVEI processes without normalization converge in distribution to a geometric or a compound Poisson random variable. The results in [21] concern only the asymptotics of the one-dimensional distributions. In this paper, we continue the investigation of the long-term behavior of nearly degenerate BPVEs and BPVEIs and prove functional limit theorems in the Skorokhod space of càdlàg functions.

To derive convergence, it is crucial to find a proper scaling for the process. Condition (C1) implies that, for all nearly degenerate BPVEs, there is a sequence $(A(n))_{n\in\mathbb{N}}$ of natural numbers increasing to infinity such that

$$\bar{f}_{0,A(n)} \sim n^{-1} \quad \text{as } n \to \infty, \tag{3}$$

which turns out to be the proper scaling. In what follows, we are interested in the processes $(X_{A(nt)})_{t>0}$ and $(Y_{A(nt)})_{t>0}$, and to simplify the notation, $A(nt)$ is meant as $A(\lfloor nt \rfloor)$.

The simplest example is given by

$$\bar{f}_n = \begin{cases} 1-\alpha/n, & n \geq \lfloor \alpha \rfloor + 1, \\ 1, & \text{otherwise} \end{cases}$$

for some $\alpha > 0$. Then there exists $c > 0$ such that $\bar{f}_{0,n} \sim cn^{-\alpha}$ as $n \to \infty$. Thus, with $A(n) = \lfloor c^{1/\alpha} n^{1/\alpha} \rfloor$, (3) holds.

In Section 2.1, for the conditioned process we prove that $\mathcal{L}((X_{A(nt)})_{t>0} \mid X_{A(n)} > 0)$, the law of $(X_{A(nt)})_{t>0}$ conditioned on the event $\{X_{A(n)} > 0\}$, converges in distribution in the Skorokhod

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space of càdlàg functions to a time-transformed simple birth-and-death process conditioned on survival. We analyze the limit process and explore its relation to the quasi-stationary distribution. In Section 2.2, for the immigration process we show that $(Y_{A(nt)})_{t>0}$ converges to a time-transformed continuous-time Markov branching process with immigration. In both cases, we only scale in time; therefore, the limit processes are continuous-time Markov chains with state spaces $\mathbb{N}$ and $\mathbb{N}\cup\{0\}$, respectively. Convergence to the Poisson process is a well-studied area; see [6, Section 3.12]. However, we are not aware of results on convergence to non-monotone Markov chains. The proof of the convergence of the finite-dimensional distributions is standard, but the proof of tightness is rather troublesome. We also note that the limit processes in both cases are naturally defined on $(0,\infty)$, which is not common for Markov processes. For the conditioned process (X_n), the limit is non-homogeneous even after a natural time transformation, and its existence can be obtained via the construction of an entrance law. For the immigration process (Y_n), the limit is a time-transformed stationary process. Stationary processes, in discrete and continuous time, are commonly defined on $\mathbb{Z}$ or on $(-\infty,\infty)$, respectively; see, e.g., [4, 7].

Section 2 contains the main results, while proofs and auxiliary results are gathered in Section 3.

2. Main results

2.1. Nearly degenerate BPVEs

Consider a simple continuous-time birth-and-death process $(Z(u))_{u\ge -w}$ defined on the time interval $u\in[-w,\infty)$ for some $w\ge 0$, with birth rate $\nu/2$ and death rate $1+\nu/2$. That is, in state n, the birth and death rates are $n\nu/2$ and $n(1+\nu/2)$, respectively. Note that this is a usual birth-and-death process with a shifted time domain. The transition-generating function of Z is given (see, e.g., [3, III.5], [1, 6.4.3]) as

$$F(s,t)=\mathbb{E}\big[s^{Z(t+u)}\mid Z(u)=1\big]=\frac{(1+\nu/2)(s-1)-\mathrm{e}^{t}(s\nu/2-(1+\nu/2))}{\nu(s-1)/2-\mathrm{e}^{t}(s\nu/2-(1+\nu/2))}$$
$$=1-\mathrm{e}^{-t}\left(\frac{1}{1-s}+\frac{\nu}{2}(1-\mathrm{e}^{-t})\right)^{-1}, \tag{4}$$

where $s\in[0,1]$, $u\ge -w$, and $t\ge 0$.

Let $a,b\in\mathbb{R}$ with $a<b$. The Skorokhod spaces of real-valued càdlàg functions on $[a,b]$ and $[a,\infty)$ with the usual metric are denoted by $D([a,b])$ and $D([a,\infty))$ respectively, and $\xrightarrow{\mathcal{D}}$ represents convergence in distribution in the Skorokhod space. A random variable Y has a *geometric distribution with parameter* $p\in(0,1]$, written $Y\sim\mathrm{Geom}(p)$, if $\mathbb{P}(Y=k)=(1-p)^{k-1}p$ for $k=1,2,\dots$

Theorem 1. *Let $(X_n)_{n\in\mathbb{N}}$ be a BPVE satisfying conditions (C1)–(C3). Then, for any $\varepsilon\in(0,1]$,*

$$\mathcal{L}\big((X_{A(nt)})_{t\ge\varepsilon}\mid X_{A(n)}>0\big)\xrightarrow{\mathcal{D}}\mathcal{L}\big((Z(\log t))_{t\ge\varepsilon}\mid Z(0)>0\big)$$

as $n\to\infty$, where $(Z(u))_{u\ge\log\varepsilon}$ is a simple birth-and-death process with the initial distribution $Z(\log\varepsilon)\sim\mathrm{Geom}(2/(2+\nu))$, birth rate $\nu/2$, and death rate $1+\nu/2$.

Note that if $\nu=0$, $Z(u)$ is a simple death process with death rate 1 and the initial distribution $Z(\log\varepsilon)=1$.

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Since the limit process $(Z(\log t))_{t\geq\varepsilon}$ is conditioned on non-extinction at $t=1$, it is no surprise that the limit behaves differently for $t\in[\varepsilon,1)$ and $t\geq 1$. In fact, for $t\geq 1$, the limit reduces to $(Z(\log t))_{t\geq 1}$ with initial distribution Geom $(2/(2+\nu))$.

Note that Theorem 1 holds for any $\varepsilon>0$; however, we cannot take $\varepsilon=0$ because the limit process is not defined on (0,1]. That is, Z is not defined on $(-\infty,\infty)$. To see the behavior on (0, 1], we first show that the limit process can be extended to (0,1]. To do that, we need some properties of the limit process.

For $a\in[0,1]$, $\nu\geq 0$, we introduce the notation

$$h_a(s)=1-a\left(\frac{1}{1-s}+\frac{\nu}{2}(1-a)\right)^{-1},$$

that is, h_a is the generating function of a linear fractional distribution with mean a. Linear fractional distributions are an important class of offspring distributions because of the possible explicit calculations; see [3, Chapter I.4], with

$$p=\frac{\nu(1-a)}{2+\nu(1-a)},\qquad b=\frac{4a}{(2+\nu(1-a))^2}.$$

We stress that h_a depends on both a and ν; however, since the latter parameter is fixed in our setup, it is suppressed in our notation. Note that we also allow the degenerate case $\nu=0$, when $h_a(s)$ is the g.f. of a Bernoulli-distributed variable with parameter a. Recall the important property of linear fractional distributions that $h_a(h_b(s))=h_{ab}(s)$ for all $a,b\in[0,1]$. With this notation, the function F defined in (4) can be expressed as $F(s,t)=h_{\mathrm{e}^{-t}}(s)$.

Let us define the time-changed process

$$U(t)=Z(\log t),\quad t\geq\varepsilon. \tag{5}$$

Then, for $\varepsilon\leq u<t$ and $x_0\in\{1,2,\ldots\}$,

$$\begin{aligned}\mathbb{E}\left[s^{U(t)}\mid U(u)=x_0\right]&=\mathbb{E}\left[s^{Z(\log t)}\mid Z(\log u)=x_0\right]\\&=\left(\mathbb{E}\left[s^{Z(\log t)}\mid Z(\log u)=1\right]\right)^{x_0}\\&=\left(h_{\mathrm{e}^{-(\log t-\log u)}}(s)\right)^{x_0}=(h_{u/t}(s))^{x_0}.\end{aligned} \tag{6}$$

We analyze the limit process defined in (5). Put $p=2/(2+\nu)$, $q=1-p$. Define the generating function for $t\in(0,1]$ as

$$g_t(s)=\frac{ps}{1-qs}\,\frac{t^{-1}(1-h_t(0))}{1-h_t(0)qs}=\sum_{x=1}^{\infty}pq^{x-1}t^{-1}(1-h_t(0)^x)s^x.$$

Note that $t^{-1}(1-h_t(0))=p/(1-qt)\to p$ as $t\downarrow 0$. Furthermore, g_t is the generating function of $V+W_t$, where $V\sim$ Geom (p), $W_t+1\sim$ Geom $(p/(1-qt))$, and V and W_t are independent. Let $\mathbb{P}_\varepsilon$ denote the law of the process $(U(t)_{t\in[\varepsilon,1]})$ under the initial distribution $U(\varepsilon)\sim$ Geom (p).

The distribution Geom (p) is the extremal quasi-stationary distribution of the birth-and-death process Z; see [11, p. 106]. This means that

$$\mathbb{E}_\varepsilon\left[s^{U(1)}\mid U(1)>0\right]=\frac{ps}{1-qs}=\mathbb{E}_\varepsilon\left(s^{U(\varepsilon)}\right),$$

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i.e., conditioned on non-extinction, the distribution of $U(1)$ is the same as the initial distribution. The second statement below is an extension of the quasi-stationary property, which corresponds to $t = 1$.

The limit process as $\varepsilon \downarrow 0$ is defined on (0,1] (or on $(0, \infty)$), which means that there is no initial distribution for the Markov process. The existence of the process in this case can be handled via *entrance laws*. That is, if we start the process at time $\varepsilon > 0$ with the initial law given by g_ε and run it up to time $t > \varepsilon$ according to the non-homogeneous transition probabilities given by $k_{\varepsilon,t;x_0}$, then the process at time t has the distribution given by g_t.

Lemma 1. *Fix $\varepsilon \in (0, 1)$ arbitrarily. Then, for $\varepsilon \le u < t \le 1$ and $x_0 \in \mathbb{N}$, the generating function of the conditional transition probabilities is given by*

$$\mathbb{E}\big[s^{U(t)} \mid U(u) = x_0,\, U(1) > 0\big] = \frac{(h_{u/t}(s))^{x_0} - (h_{u/t}(h_t(0)s))^{x_0}}{1 - (h_u(0))^{x_0}} =: k_{u,t;x_0}(s). \tag{7}$$

The family of laws with generating functions $(g_t)_{t\in(0,1]}$ is an entrance law for the transition-generating functions $k_{u,t;x}$; that is, for $\varepsilon \in (0, 1]$ and for any $t \in [\varepsilon, 1]$,

$$\mathbb{E}_\varepsilon\big[s^{U(t)} \mid U(1) > 0\big] = g_t(s).$$

Entrance laws for homogeneous Markov chains were treated in [9, Definition I.1.1], [26, III.5.39], and [28, Chapters I.1 and V]. In [26], the existence of a càdlàg Markov process on $(0, \infty)$ with the given transition probabilities and entrance laws was proved. In the general non-homogeneous setup, the problem was already investigated in 1936 in [22] (for the English translation see [23]), where entrance laws are called absolute probabilities. In our setup, the transition probabilities are non-homogeneous even after the time transformation to $(-\infty, 0]$ due to the conditioning at 0.

The existence of an entrance law, together with Kolmogorov's consistency theorem, implies that there exists a Markov process $\widetilde{U}$ on (0, 1] with the given marginals and transition probabilities. For any $\varepsilon > 0$, the process $\widetilde{U}$ on $[\varepsilon, 1]$ is a time-transformed birth-and-death process conditioned on non-extinction; thus, it has a càdlàg version. To show the existence of a càdlàg version on the whole of (0, 1], we have to demonstrate that we can glue the process together backwards in time in a consistent way. This can be done using time reversal, as the reversed process is also Markov, which can be continued forward in time; see the proof for the details.

Corollary 1. *There exists a càdlàg Markov process $(\widetilde{U}(t))_{t\in(0,1]}$ such that its transition probabilities are given by the generating functions $k_{u,t;x}(s) = \mathbb{E}[s^{\widetilde{U}(t)} \mid \widetilde{U}(u) = x]$, and $\mathbb{E}(s^{\widetilde{U}(t)}) = g_t(s)$ for each $t \in (0, 1]$.*

As we noted earlier,

$$\lim_{t\downarrow 0} g_t(s) = \frac{ps}{1 - qs}\,\frac{p}{1 - qs},$$

that is, $\widetilde{U}(t) \xrightarrow{\mathcal{D}} V + W_0$ as $t \downarrow 0$ where V, W_0 are independent, $V, W_0 + 1 \sim \mathrm{Geom}\,(p)$. However, contrary to the homogeneous case, see [26, III.39], $\widetilde{U}(t)$ as $t \downarrow 0$ does not converge in probability. Indeed, a short calculation shows that, for $\alpha \in (0, 1)$, as $t \downarrow 0$,

$$k_{\alpha t,t;x_0}(s) = \frac{(h_\alpha(s))^{x_0} - (h_\alpha(h_t(0)s))^{x_0}}{1 - (h_{\alpha t}(0))^{x_0}} \longrightarrow \frac{s}{\alpha}(h_\alpha(s))^{x_0-1} h'_\alpha(s), \quad s \in [0, 1],$$

that is, the transition probabilities around 0 converge to a nontrivial distribution.

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As a consequence of the existence result above and Theorem 1, we can extend the convergence to (0, 1]. The theory of Skorokhod spaces of càdlàg functions on noncompact domains other than $[0, \infty)$ is not well established. In [30, Section 12.9], convergence on $D(I)$ with I being an interval of $\mathbb{R}$ is defined as the convergence on all compact subintervals $[a, b]$ of I, where both a and b are either continuity points of the limit or boundary points of I. However, the results in [30] are only spelled out for $I = [0, \infty)$ and for the strong M_1 topology. In [17, Section 2.2], the authors consider convergence in $D(\mathbb{R})$. Therefore, instead of claiming convergence on (0, 1], with a simple transformation we prove convergence in the more common function space $D([1, \infty))$.

Corollary 2. *Under the assumptions of Theorem 1,*

$$\mathcal{L}((X_{A(n/t)-})_{t\geq 1} \mid X_{A(n)} > 0) \xrightarrow{\mathcal{D}} \mathcal{L}((\widetilde{U}((1/t)-))_{t\geq 1}) \quad \text{as } n \to \infty.$$

2.2. Nearly degenerate BPVEIs

Consider a BPVEI $(Y_n)_{n\in\mathbb{N}}$ as defined in (2), and introduce the notation

$$m_{n,k} := \mathbb{E}[\varepsilon_n(\varepsilon_n - 1)\cdots(\varepsilon_n - k + 1)], \quad k \in \mathbb{N},$$

for the factorial moments of the immigration in generation n. In what follows, we suppose that $(Y_n)_{n\in\mathbb{N}}$ is a nearly degenerate BPVEI; that is, its offspring distribution satisfies (C1)–(C3). We further assume that one of the following conditions holds:

(C4) $\lim_{n\to\infty, \bar{f}_n<1} m_{n,k}/(k!(1-\bar{f}_n)) = \lambda_k$, $k = 1, 2, \ldots, K$, for some $K \geq 2$, such that $\lambda_K = 0$ and the sequences $(m_{n,k}/(k!|1-\bar{f}_n|))_n$ are bounded for each $k \leq K$;

(C4′) $\lim_{n\to\infty, \bar{f}_n<1} m_{n,k}/(k!(1-\bar{f}_n)) = \lambda_k$, $k = 1, 2, \ldots$, such that $\limsup_{n\to\infty} \lambda_n^{1/n} \leq 1$ and the sequences $(m_{n,k}/(k!|1-\bar{f}_n|))_n$ are bounded for all k.

For BPVEIs, the limit processes are *continuous-time Markov branching processes with immigration*, defined as follows. Each individual has an exponentially distributed lifetime with parameter α, and upon death it leaves a random number of offspring ξ. Furthermore, a random number of immigrants ε arrive at random times determined by a Poisson process of intensity β. All the quantities involved are independent. The offspring- and immigration-generating functions are $f(s) = \sum_{k=0}^{\infty} \mathbb{P}(\xi = k)s^k$ and $h(s) = \sum_{k=1}^{\infty} \mathbb{P}(\varepsilon = k)s^k$, $|s| \leq 1$, where we may and do assume that $\mathbb{P}(\xi = 1) = 0$ and $\mathbb{P}(\varepsilon = 0) = 0$.

When there is no immigration, we obtain the well-known class of continuous-time Markov branching processes; see [3, Chapter III]. In particular, a birth-and-death process is a special continuous-time branching process with $\alpha = 1 + \nu$ and

$$f(s) = \frac{1}{1+\nu}\left(1 + \frac{\nu}{2} + \frac{\nu}{2}s^2\right).$$

Continuous-time Markov branching processes with immigration are less studied, though they were introduced in [27] in 1957. Simple birth-and-death processes with immigration are discussed in detail in [1, Section 6.4.4], where immigrants arrive one by one. For some recent references, see [24, 25].

Let $(W(t))_{t\geq -w}$ denote the resulting continuous-time branching process with immigration on the time interval $t \in [-w, \infty)$, $w \geq 0$, and introduce the notation $G(s, t) = \mathbb{E}(s^{W(t+u)} \mid$

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$W(u) = 0)$, $t \geq 0$, $u \geq -w$, $s \in [0, 1]$. Then $G(s, t)$ satisfies the Kolmogorov forward equation (see [24, (2.5)])

$$\frac{\partial}{\partial t} G(s, t) = a(s) \frac{\partial}{\partial s} G(s, t) + b(s) G(s, t)$$

with boundary condition $G(s, 0) = 1$, where

$$a(s) = \alpha(f(s) - s), \qquad b(s) = \beta(h(s) - 1). \tag{8}$$

Theorem 2. *Assume that the BPVEI $(Y_n)_{n\in\mathbb{N}}$ defined in (2) satisfies conditions (C1)–(C3), and either (C4) or (C4′) holds. Then, for any $0 < \varepsilon \leq 1$,*

$$\mathcal{L}\big((Y_{A(nt)})_{t\geq\varepsilon}\big) \xrightarrow{\mathcal{D}} \mathcal{L}\big((W(\log t))_{t\geq\varepsilon}\big)$$

as $n \to \infty$, where $(W(u))_{u\geq\log\varepsilon}$ is a continuous-time Markov branching process with immigration, with the initial distribution having g.f.

$$\log f_Y(s) = \begin{cases} -\displaystyle\sum_{k=1}^{\kappa} \frac{2^k \lambda_k}{\nu^k} \left(\log\left(1 + \frac{\nu}{2}(1-s)\right) + \sum_{i=1}^{k-1} \frac{\nu^i}{i 2^i} (s-1)^i \right), & \nu > 0, \\ \displaystyle\sum_{k=1}^{\kappa} \frac{\lambda_k}{k} (s-1)^k, & \nu = 0, \end{cases}$$

with parameters $\alpha = 1 + \nu$, $\beta = \sum_{k=1}^{\kappa} (-1)^{k+1} \lambda_k$, and

$$f(s) = (1+\nu)^{-1} \left(1 + \frac{\nu}{2} + \frac{\nu}{2} s^2\right), \qquad h(s) = 1 + \beta^{-1} \sum_{k=1}^{\kappa} \lambda_k (s-1)^k$$

offspring- and immigration-generating functions, where $\kappa = K - 1$ or $\kappa = \infty$ depending on whether condition (C4) or (C4′) holds. The transition-generating function of W is given by

$$G_y(s, t) = \mathbb{E}\big[s^{W(t+u)} \mid W(u) = y\big] = \frac{f_Y(s)}{f_Y(h_{\mathrm{e}^{-t}}(s))} (h_{\mathrm{e}^{-t}}(s))^y.$$

Similarly to Theorem 1, if $\nu = 0$, reproduction occurs according to a simple death process with parameter $\mu = 1$.

If $K = 2$, the limit process is a simple birth-and-death process with immigration, with birth, death, and immigration rates $\nu/2$, $1 + \nu/2$, and λ_1, respectively; see [1, Section 6.4.4].

As in the conditional setup, we cannot choose $\varepsilon = 0$ in the limit theorem. However, the situation is simpler. Contrary to the conditional setup, the limit process is stationary. Indeed,

$$\begin{aligned} \sum_{k=0}^{\infty} \mathbb{E}\big[s^{W(t)} \mid W(\log\varepsilon) = k\big] f_Y[k] &= \sum_{k=0}^{\infty} \frac{f_Y(s)}{f_Y(h_{\mathrm{e}^{-(t-\log\varepsilon)}}(s))} (h_{\mathrm{e}^{-(t-\log\varepsilon)}}(s))^k f_Y[k] \\ &= \frac{f_Y(s)}{f_Y(h_{\mathrm{e}^{-(t-\log\varepsilon)}}(s))} f_Y(h_{\mathrm{e}^{-(t-\log\varepsilon)}}(s)) = f_Y(s). \end{aligned}$$

Therefore, the existence of the process $W(t)$ on $(-\infty, \infty)$ follows directly from Kolmogorov's consistency theorem. The càdlàg version can be constructed as in Corollary 1. For an explicit

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probabilistic construction of the limit process on $\mathbb{R}$, we refer to [17, Section 2.1 and Example 4.1.b].

Corollary 3. *Let $(W(u))_{u\in\mathbb{R}}$ be a stationary continuous-time Markov branching process with immigration, as in Theorem 2. Under the assumptions of Theorem 2,*

$$\mathcal{L}\big((Y_{A(n/t)-})_{t\geq 1}\big) \xrightarrow{\mathcal{D}} \mathcal{L}\big((W(-\log t-))_{t\geq 1}\big) \quad \text{as } n\to\infty.$$

3. Proofs

3.1. Preparation

Recall that $f_n(s)=\mathbb{E}(s^{\xi_n})$ denotes the offspring-generating function in generation n, and $\bar{f}_n=\mathbb{E}(\xi_n)$. For $j<n$, we introduce notation for the composite generating functions and for the corresponding means:

$$f_{n,n}(s)=s, \qquad f_{j,n}(s)=f_{j+1}\circ\cdots\circ f_n(s), \qquad \bar{f}_{n,n}=1, \qquad \bar{f}_{j,n}=\bar{f}_{j+1}\cdots\bar{f}_n.$$

We analyze the g.f. of the underlying processes. For a given g.f. f with mean $\bar{f}$ and $f''(1)<\infty$, define the *shape function* (see, e.g., [18]) as

$$\varphi(s)=\frac{1}{1-f(s)}-\frac{1}{\bar{f}(1-s)}, \quad 0\leq s<1, \qquad \varphi(1)=\frac{f''(1)}{2\bar{f}^2}. \tag{9}$$

Let φ_j be the shape function of f_j. Iteration yields (see [18, Lemma 5] and [19, Proposition 1.3])

$$\frac{1}{1-f_{j,n}(s)}=\frac{1}{\bar{f}_{j,n}(1-s)}+\varphi_{j,n}(s), \qquad \varphi_{j,n}(s):=\sum_{k=j+1}^{n}\frac{\varphi_k(f_{k,n}(s))}{\bar{f}_{j,k-1}}. \tag{10}$$

The following lemmas from [21] are frequently used in our proofs. The first one is a consequence of the Silverman–Toeplitz theorem, while the second one follows from a straightforward calculation.

Lemma 2. ([21, Lemma 2].) *Let $(\bar{f}_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers satisfying (C1), and define $a_{n,j}^{(k)}=(1-\bar{f}_j)\prod_{i=j+1}^{n}\bar{f}_i^{k}=(1-\bar{f}_j)\bar{f}_{j,n}^{k}$, $n,j,k\in\mathbb{N}$, $j\leq n-1$, and $a_{n,n}^{(k)}=1-\bar{f}_n$. If $(x_n)_{n\in\mathbb{N}}$ is bounded and $\lim_{n\to\infty,\bar{f}_n<1}x_n=x\in\mathbb{R}$, then, for all $k\in\mathbb{N}$,*

$$\lim_{n\to\infty}\sum_{j=1}^{n}a_{n,j}^{(k)}x_j=\frac{x}{k}, \qquad \lim_{n\to\infty}\sum_{j=1}^{n}|a_{n,j}^{(k)}|x_j=\frac{x}{k}.$$

Lemma 3. ([21, Lemma 3].) *Let φ_n be the shape function of f_n. Then, under the conditions (C1)–(C3) with $\nu>0$,*

$$\lim_{n\to\infty,\bar{f}_n<1}\sup_{s\in[0,1]}\frac{|\varphi_n(1)-\varphi_n(s)|}{1-\bar{f}_n}=0,$$

and the sequence $\sup_{s\in[0,1]}|\varphi_n(1)-\varphi_n(s)|/|1-\bar{f}_n|$ is bounded.

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3.2. Proofs for subsection 2.1

Lemma 4. *Assume that conditions (C1)–(C3) are satisfied for a BPVE $(X_n)_{n\in\mathbb{N}}$ as defined in (1). Then, for any $0 < u \le t$,*

$$\lim_{n\to\infty} \mathbb{E}\big[s^{X_{A(nt)}} \mid X_{A(nu)} = 1\big] = 1 - \frac{u}{t}\left(\frac{1}{1-s} + \frac{\nu}{2}\left(1 - \frac{u}{t}\right)\right)^{-1} = h_{u/t}(s),$$

that is, $\mathcal{L}(X_{A(nt)} \mid X_{A(nu)} = 1)$ converges in distribution to a linear fractional distribution with generating function $h_{u/t}(s)$.

Proof. By (10), we have

$$\begin{aligned}(1 - f_{j,n}(s))^{-1} &= \frac{1}{\bar{f}_{j,n}}\left(\frac{1}{1-s} + \sum_{k=j+1}^{n} \bar{f}_{k-1,n}\varphi_k(f_{k,n}(s))\right)\\ &= \frac{1}{\bar{f}_{j,n}}\left(\frac{1}{1-s} + \sum_{k=j+1}^{n} \bar{f}_{k-1,n}\varphi_k(1) - \sum_{k=j+1}^{n} \bar{f}_{k-1,n}(\varphi_k(1) - \varphi_k(f_{k,n}(s)))\right).\end{aligned}$$

By Lemmas 2 and 3, for the third term we obtain

$$\left|\sum_{k=j+1}^{n} \bar{f}_k \frac{\varphi_k(1) - \varphi_k(f_{k,n}(s))}{1-\bar{f}_k} a^{(1)}_{n,k}\right| \le \sum_{k=j+1}^{n} \left|\frac{\varphi_k(1) - \varphi_k(f_{k,n}(s))}{1-\bar{f}_k}\right| |a^{(1)}_{n,k}| \to 0.$$

Here and later, all unspecified limit relations are meant as $n \to \infty$. Recalling from (9) that $\varphi_k(1) = f''_k(1)/(2\bar{f}^2_k)$, we get by (C2) that

$$\begin{aligned}\left|\sum_{k=j+1}^{n} \frac{1}{\bar{f}_k}\frac{f''_k(1)}{1-\bar{f}_k} a^{(1)}_{n,k} - \nu \sum_{k=j+1}^{n} a^{(1)}_{n,k}\right| &= \left|\sum_{k=j+1}^{n} a^{(1)}_{n,k}\left(\frac{1}{\bar{f}_k}\frac{f''_k(1)}{1-\bar{f}_k} - \nu\right)\right|\\ &\le \sum_{k=1}^{n} |a^{(1)}_{n,k}|\left|\frac{1}{\bar{f}_k}\frac{f''_k(1)}{1-\bar{f}_k} - \nu\right| \to 0,\end{aligned}$$

where $\sum_{k=j+1}^{n} a^{(1)}_{n,k} = \sum_{k=j+1}^{n} (\bar{f}_{k,n} - \bar{f}_{k-1,n}) = (1 - \bar{f}_{j,n})$. By the definition in (3) of $A(nt)$,

$$\bar{f}_{A(nu),A(nt)} \to \frac{u}{t} \tag{11}$$

as $n \to \infty$. Thus, substituting $j = A(nu)$ and $n = A(nt)$, by (11),

$$\big(1 - f_{A(nu),A(nt)}(s)\big)^{-1} \to \frac{t}{u}\left(\frac{1}{1-s} + \frac{\nu}{2}\left(1 - \frac{u}{t}\right)\right).$$

Since $\mathbb{E}[s^{X_{A(nt)}} \mid X_{A(nu)} = 1] = f_{A(nu),A(nt)}(s)$, the desired result follows after a simple rearrangement. □

Now we prove the properties of the limit process.

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Proof of Lemma 1. First, we calculate the conditional transition-generating functions. Using the Markov property and (6),

$$\begin{aligned}\mathbb{E}\big[s^{U(t)} \mid U(u)=x_0,\, U(1)>0\big] &= \sum_{x=1}^{\infty} s^x \mathbb{P}(U(t)=x \mid U(u)=x_0,\, U(1)>0)\\ &= \sum_{x=1}^{\infty} s^x \frac{\mathbb{P}(U(t)=x,\, U(u)=x_0,\, U(1)>0)}{\mathbb{P}(U(u)=x_0,\, U(1)>0)}\\ &= \sum_{x=1}^{\infty} s^x \frac{\mathbb{P}(U(t)=x \mid U(u)=x_0)\mathbb{P}(U(1)>0 \mid U(t)=x)}{\mathbb{P}(U(1)>0 \mid U(u)=x_0)}\\ &= \sum_{x=1}^{\infty} s^x \frac{\mathbb{P}(U(t)=x \mid U(u)=x_0)[1-(h_t(0))^x]}{1-(h_u(0))^{x_0}}\\ &= \frac{(h_{u/t}(s))^{x_0}-(h_{u/t}(h_t(0)s))^{x_0}}{1-(h_u(0))^{x_0}}.\end{aligned}$$

Recall that $p=2/(2+\nu)=1-q$. In the calculations below, we use the identity

$$\frac{ph_a(s)}{1-qh_a(s)} = 1-\frac{a(1+\nu/2)}{(1-s)^{-1}+\nu/2}, \quad a\in[0,\,1],\ s\in[0,\,1]. \tag{12}$$

The proof of (12) is a straightforward calculation. By the law of total probability, (6), and (12), we have

$$\begin{aligned}\mathbb{P}_\varepsilon(U(1)>0) &= \sum_{x=1}^{\infty} \mathbb{P}(U(\varepsilon)=x)\,\mathbb{P}(U(1)>0 \mid U(\varepsilon)=x)\\ &= \sum_{x=1}^{\infty} pq^{x-1}(1-(h_\varepsilon(0))^x) = 1-\frac{ph_\varepsilon(0)}{1-qh_\varepsilon(0)} = \varepsilon.\end{aligned} \tag{13}$$

Note that this means that the distribution of the extinction time of the birth-and-death process Z is exponential; see, e.g., [11, (3.3)].

Using (13), (7), and (12) again, we obtain

$$\begin{aligned}\mathbb{E}_\varepsilon\big[s^{U(t)} \mid U(1)>0\big] &= \sum_{x=1}^{\infty} \mathbb{P}_\varepsilon(U(\varepsilon)=x \mid U(1)>0)\,\mathbb{E}\big[s^{U(t)} \mid U(\varepsilon)=x,\, U(1)>0\big]\\ &= \sum_{x=1}^{\infty} \frac{\mathbb{P}_\varepsilon(U(\varepsilon)=x,\, U(1)>0)}{\mathbb{P}_\varepsilon(U(1)>0)} k_{\varepsilon,t;x}(s)\\ &= \varepsilon^{-1}\sum_{x=1}^{\infty} pq^{x-1}\big[(h_{\varepsilon/t}(s))^x-(h_{\varepsilon/t}(h_t(0)s))^x\big]\\ &= \varepsilon^{-1}\left[\frac{ph_{\varepsilon/t}(s)}{1-qh_{\varepsilon/t}(s)}-\frac{ph_{\varepsilon/t}(h_t(0)s)}{1-qh_{\varepsilon/t}(h_t(0)s)}\right]\\ &= \frac{ps}{1-qs}\,\frac{t^{-1}(1-h_t(0))}{1-h_t(0)qs} = g_t(s),\end{aligned}$$

proving the statement. □

Next, we prove the existence of a càdlàg version of the limit process on (0, 1].

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Proof of Corollary 1. For each $\varepsilon \in (0, 1)$ the existence of a càdlàg version on $[\varepsilon, 1]$ follows from the representation in Theorem 1. Our aim is to continue the process backwards in a consistent way.

The existence of a Markov process $\widetilde{U}$ on the product space $\mathbb{N}^{(0,1]}$ with the prescribed distribution follows from Kolmogorov's consistency theorem. Using time reversal $\overline{U}(s) = \widetilde{U}(1-s)$, $s \in [0, 1)$, the transition probabilities are

$$\begin{aligned}\mathbb{P}(\overline{U}(t) = y \mid \overline{U}(u) = x) &= \frac{g_{1-t}[y]}{g_{1-u}[x]} \mathbb{P}(\widetilde{U}(1-u) = x \mid \widetilde{U}(1-t) = y) \\ &= \frac{g_{1-t}[y]}{g_{1-u}[x]} k_{1-t,1-u;y}[x] \\ &=: \overline{k}_{u,t;x}[y], \quad x, y \in \mathbb{N},\ 0 \le u < t < 1. \end{aligned} \tag{14}$$

Thus, we have transition probabilities $\overline{k}_{u,t;x}[y]$, and the corresponding Markov chain has a càdlàg version on any $[t_1, t_2] \subset [0, 1)$.

The process $\overline{U}$ has a càdlàg version on the interval $\left[0, \frac{1}{2}\right]$, and can be continued in a càdlàg way to $\left[\frac{1}{2}, \frac{3}{4}\right]$ from the random initial condition $\overline{U}\left(\frac{1}{2}\right)$ according to the transition probabilities $\overline{k}_{u,t;x}[y]$ in (14). Next, we continue to $\left[\frac{3}{4}, \frac{7}{8}\right]$, etc. Concatenating these processes we obtain a càdlàg version on [0,1). From $\overline{U}$ a time reversal provides the required version of $\widetilde{U}$. □

The first step towards the proof of Theorem 1 is to show the convergence of the finite-dimensional distributions.

Proposition 1. *Assume that the BPVE* $(X_n)_{n\in\mathbb{N}}$ *satisfies conditions (C1)–(C3), and for* $k \ge 0$, $\ell \ge 0$, *let* $0 < t_{-k} < t_{-k+1} < \cdots < t_{-1} < t_0 = 1 < t_1 < \cdots < t_\ell < \infty$, *and* $x_i \in \mathbb{N}$, $i = -k, \dots, \ell$. *Then*

$$\lim_{n\to\infty} \mathbb{P}\big(X_{A(nt_i)} = x_i,\ i = -k, \dots, \ell \mid X_{A(n)} > 0\big) = \mathbb{P}_{t_{-k}}\big(U(t_i) = x_i,\ i = -k, \dots, \ell \mid U(1) > 0\big),$$

where $\mathbb{P}_{t_{-k}}$ *denotes the law of the Markov process* $(U(t))_{t\ge t_{-k}}$ *with initial distribution* $U(t_{-k}) \sim$ Geom $(2/(2+\nu))$.

Proof. Recall the notation $p = 2/(2+\nu) = 1 - q$. First, we show that

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{P}\big(X_{A(nt_i)} = x_i,\ i = -k, \dots, \ell \mid X_{A(n)} = x_0\big) \\ &\qquad = q^{x_{-k}-x_0}(t_{-k})^{-1}\, \mathbb{P}\big(U(t_i) = x_i,\ i = -k, \dots, \ell \mid U(t_{-k}) = x_{-k}\big). \end{aligned} \tag{15}$$

By the Markov property,

$$\begin{aligned}&\mathbb{P}\big(X_{A(nt_i)} = x_i,\ i = -k, \dots, \ell \mid X_{A(n)} = x_0\big) \\ &\qquad = \frac{\mathbb{P}(X_{A(nt_{-k})} = x_{-k})}{\mathbb{P}(X_{A(n)} = x_0)} \prod_{i=-k+1}^{\ell} \mathbb{P}(X_{A(nt_i)} = x_i \mid X_{A(nt_{i-1})} = x_{i-1}). \end{aligned}$$

By Lemma 4 and (6),

$$\lim_{n\to\infty} \mathbb{P}\big(X_{A(nt_i)} = x_i \mid X_{A(nt_{i-1})} = x_{i-1}\big) = \mathbb{P}\big(U(t_i) = x_i \mid U(t_{i-1}) = x_{i-1}\big).$$

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Furthermore, by [21, Theorem 1], for all $x \in \mathbb{N}$,

$$\lim_{n\to\infty} \mathbb{P}\big(X_n = x \mid X_n > 0\big) = pq^{x-1}, \tag{16}$$

and by [21, Lemma 4],

$$\frac{\mathbb{P}(X_{A(ns)} > 0)}{\mathbb{P}(X_{A(n)} > 0)} = \frac{1 - f_{0,A(ns)}(0)}{1 - f_{0,A(n)}(0)} \sim \frac{\bar{f}_{0,A(ns)}}{\bar{f}_{0,A(n)}} \sim s^{-1},$$

where the last asymptotic equality follows from (3). Hence,

$$\begin{aligned}\lim_{n\to\infty} \frac{\mathbb{P}(X_{A(nt_{-k})} = x_{-k})}{\mathbb{P}(X_{A(n)} = x_0)} &= \lim_{n\to\infty} \frac{\mathbb{P}(X_{A(nt_{-k})} = x_{-k})}{\mathbb{P}(X_{A(nt_{-k})} > 0)} \cdot \frac{\mathbb{P}(X_{A(nt_{-k})} > 0)}{\mathbb{P}(X_{A(n)} > 0)} \cdot \frac{\mathbb{P}(X_{A(n)} > 0)}{\mathbb{P}(X_{A(n)} = x_0)} \\ &= q^{x_{-k}-x_0}(t_{-k})^{-1},\end{aligned}$$

and thus (15) follows.

Using (15), (16), and (13), we have

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{P}\big(X_{A(nt_i)} = x_i,\, i = -k, \dots, \ell \mid X_{A(n)} > 0\big) \\ &\quad= \lim_{n\to\infty} \frac{\mathbb{P}(X_{A(n)} = x_0)}{\mathbb{P}(X_{A(n)} > 0)} \mathbb{P}\big(X_{A(nt_i)} = x_i,\, i = -k, \dots, \ell \mid X_{A(n)} = x_0\big) \\ &\quad= pq^{x_{-k}-1}(t_{-k})^{-1} \mathbb{P}\big(U(t_i) = x_i,\, i = -k, \dots, \ell \mid U(t_{-k}) = x_{-k}\big) \\ &\quad= \mathbb{P}_{t_{-k}}\big(U(t_i) = x_i,\, i = -k, \dots, \ell \mid U(1) > 0\big) \frac{pq^{x_{-k}-1}\mathbb{P}_{t_{-k}}(U(1) > 0)}{t_{-k}\mathbb{P}_{t_{-k}}(U(t_{-k}) = x_{-k})} \\ &\quad= \mathbb{P}_{t_{-k}}\big(U(t_i) = x_i,\, i = -k, \dots, \ell \mid U(1) > 0\big),\end{aligned}$$

proving the statement. □

Proof of Theorem 1. The convergence of the finite-dimensional distributions follows from Proposition 1. Therefore, by [13, Theorem 3.7.8], it is enough to prove tightness.

We verify the classical tightness criteria given in [6, Theorem 16.8] and in the Corollary following it. First, we recall the standard notation from [6]. For $x \in D[0, \infty)$, $T \subset [0, \infty)$ let $w(x, T) = \sup_{s,t\in T} |x(t) - x(s)|$, and for $m > 0$, $\delta > 0$, $w'_m(x, \delta) = \inf \max_{1\le i\le v} w(x, [t_{i-1}, t_i))$, where the infimum extends over all partitions $[t_{i-1}, t_i)$, $i = 1, \dots, v$, of $[0, m)$ such that $t_i - t_{i-1} > \delta$ for all i.

We have to show that (condition (i) in [6, Corollary of Theorem 16.8])

$$\lim_{a\to\infty} \limsup_{n\to\infty} \mathbb{P}\big(|X_{A(nt)}| \ge a \mid X_{A(n)} > 0\big) = 0 \quad \text{for all } t \ge \varepsilon, \tag{17}$$

and that (condition (ii) in [6, Theorem 16.8]), for all $m > \varepsilon$ and $\eta > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}\big(w'_m((X_{A(nt)})_{t\in[\varepsilon,m]}, \delta) \ge \eta \mid X_{A(n)} > 0\big) = 0. \tag{18}$$

Equation (17) follows immediately from the convergence of the finite-dimensional distributions.

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We prove (18). For $\varepsilon < 1$, we have $\{X_{A(n\varepsilon)} > 0\} \supset \{X_{A(n)} > 0\}$, and by (11) and [21, Lemma 4], $\mathbb{P}(X_{A(n\varepsilon)} > 0)/\mathbb{P}(X_{A(n)} > 0) \to \varepsilon^{-1}$. Thus, for any event F,

$$\limsup_{n\to\infty} \mathbb{P}(F \mid X_{A(n)} > 0) \le \limsup_{n\to\infty} \frac{\mathbb{P}\big(F \cap \{X_{A(n\varepsilon)} > 0\}\big)}{\mathbb{P}(X_{A(n\varepsilon)} > 0)} \frac{\mathbb{P}(X_{A(n\varepsilon)} > 0)}{\mathbb{P}(X_{A(n)} > 0)}$$
$$\le \varepsilon^{-1} \limsup_{n\to\infty} \mathbb{P}(F \mid X_{A(n\varepsilon)} > 0).$$

Therefore, it is enough to prove (18) under the condition $X_{A(n\varepsilon)} > 0$.

Next, we show that we can restrict ourselves to bounded sample paths. [21, Lemma 4] and (11) imply that

$$\lim_{n\to\infty} \mathbb{E}[X_{A(nm)} \mid X_{A(n\varepsilon)} > 0] = \lim_{n\to\infty} \frac{\bar{f}_{0,A(nm)}}{1 - f_{0,A(n\varepsilon)}(0)}$$
$$= \lim_{n\to\infty} \frac{\bar{f}_{0,A(n\varepsilon)}}{1 - f_{0,A(n\varepsilon)}(0)} \frac{\bar{f}_{0,A(nm)}}{\bar{f}_{0,A(n\varepsilon)}} = \frac{2+\nu}{2} \frac{\varepsilon}{m}. \tag{19}$$

Fix $K > 0$ large and define the stopping time (with respect to the natural filtration) $\tau = \min\{k\colon A(n\varepsilon) \le k \le A(nm),\ X_k > K\}$, with the convention $\min \emptyset = \infty$. By the law of total expectation and the branching property, we have

$$\mathbb{E}[X_{A(nm)} \mathbf{1}_{\{A(n\varepsilon) \le \tau \le A(nm)\}} \mid X_{A(n\varepsilon)} > 0]$$
$$= \sum_{k=A(n\varepsilon)}^{A(nm)} \mathbb{E}[X_{A(nm)} \mid \tau = k] \mathbb{P}(\tau = k \mid X_{A(n\varepsilon)} > 0)$$
$$\ge \sum_{k=A(n\varepsilon)}^{A(nm)} K \bar{f}_{k,A(nm)} \mathbb{P}(\tau = k \mid X_{A(n\varepsilon)} > 0)$$
$$\ge \frac{K}{C_0} \bar{f}_{A(n\varepsilon),A(nm)} \mathbb{P}\Big(\max_{A(n\varepsilon) \le k \le A(nm)} X_k > K \mid X_{A(n\varepsilon)} > 0 \Big),$$

where

$$C_0 := \prod_{n=1, \bar{f}_n > 1}^{\infty} \bar{f}_n < \infty. \tag{20}$$

In the last inequality, we used that typically $\bar{f}_n < 1$, and for those n for which the converse inequality holds, the contribution is bounded by C_0. The finiteness of C_0 follows from condition (C1). Thus, by (19) and (11), for any $m > \varepsilon > 0$,

$$\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\max_{A(n\varepsilon) \le k \le A(nm)} X_k > K \mid X_{A(n\varepsilon)} > 0 \Big) = 0. \tag{21}$$

This means that we can indeed consider bounded sample paths.

Since each jump has size at least 1, if $w'_m((X_{A(nt)})_{t\in[\varepsilon,m]}, \delta) > 0$ then the process has either at least $N = \lfloor (m-\varepsilon)/\delta \rfloor + 1$ jumps, where $\lfloor \cdot \rfloor$ denotes the lower integer part, or at least two

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jumps on an interval $[A(nt_0), A(n(t_0+\delta))]$ for some $t_0 \in (\varepsilon, m)$. Fix a large $K > 0$ and introduce the events

$$B_n = \Big\{ \max_{A(n\varepsilon)\le k\le A(nm)} X_k \le K \Big\},$$
$$C_n = \big\{(X_{A(nt)})_{t\in[\varepsilon,m]} \text{ has at least } N \text{ jumps}\big\},$$
$$D_n = \big\{\text{there exists } t_0 \in [\varepsilon, m] \text{ such that } (X_t) \text{ has at least two jumps in } [A(nt_0), A(n(t_0+\delta))]\big\}.$$

Then, for any $\eta \in (0, 1)$,

$$\begin{aligned}\mathbb{P}\big(w'((X_{A(nt)})_{t\in[\varepsilon,m]}, \delta) \ge \eta \mid X_{A(n\varepsilon)} > 0\big)\\ \le \mathbb{P}(C_n \cap B_n \mid X_{A(n\varepsilon)} > 0) + \mathbb{P}(D_n \cap B_n \mid X_{A(n\varepsilon)} > 0) + \mathbb{P}\big(B_n^{\mathrm{c}} \mid X_{A(n\varepsilon)} > 0\big).\end{aligned} \tag{22}$$

For the third term, it follows from (21) that

$$\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{P}\big(B_n^{\mathrm{c}} \mid X_{A(n\varepsilon)} > 0\big) = 0. \tag{23}$$

To handle the first two terms in (22), we have to understand the jump probabilities. For a discrete-time process, a jump means that $X_k \ne X_{k+1}$. Since $f_n''(1)/|1-\bar f_n|$ is bounded by (C2), with $f_n[1] = \mathbb{P}(\xi_n = 1)$, we have, with some $s \in (0, 1)$,

$$1 - f_n[1] = 1 - f_n'(0) = f_n'(1) - f_n'(0) + 1 - f_n'(1) = f_n''(s) + 1 - f_n'(1) \le c|1 - f_n'(1)| \tag{24}$$

for some $c > 0$ and for n large enough. Here and later, c stands for a positive constant whose value does not depend on relevant quantities and can change from line to line. Thus, for $0 < x_k \le K$, by (24) we have

$$\begin{aligned}\mathbb{P}(X_{k+1} \ne X_k \mid X_k = x_k) &\le 1 - (f_{k+1}[1])^{x_k}\\ &= 1 - (1 - (1 - f_{k+1}[1]))^{x_k}\\ &\le 1 - (1 - 2x_k(1 - f_{k+1}[1])) \le cK|1 - \bar f_{k+1}|.\end{aligned} \tag{25}$$

Next, consider the event $C_n \cap B_n$. Note that for any $k_1 < k_2 < \cdots < k_N$,

$$\{X_{k_i} \ne X_{k_i+1}, i = 1, \dots, N\} \subset \{X_{k_i} > 0, i = 1, \dots, N\} = \{X_{k_N} > 0\},$$

and hence

$$\begin{aligned}&\mathbb{P}(B_n \cap C_n \mid X_{A(n\varepsilon)} > 0)\\ &\quad= \mathbb{P}\big(\text{there exist } A(n\varepsilon) \le k_1 < \cdots < k_N < A(nm)\\ &\qquad\qquad : X_{k_i} \ne X_{k_i+1}, i = 1, \dots, N,\ B_n \mid X_{A(n\varepsilon)} > 0\big)\\ &\quad\le \sum_{A(n\varepsilon)\le k_1<\cdots<k_N<A(nm)} \mathbb{P}\big(X_{k_i} \ne X_{k_i+1}, i = 1, \dots, N, B_n \mid X_{A(n\varepsilon)} > 0\big).\end{aligned}$$

For $A(n\varepsilon) \le k_1 < \cdots < k_N < A(nm)$,

$$\begin{aligned}&\mathbb{P}\big(X_{k_i} \ne X_{k_i+1}, i = 1, \dots, N, B_n \mid X_{A(n\varepsilon)} > 0\big)\\ &\quad\le \mathbb{P}\big(X_{k_i} \ne X_{k_i+1}, X_{k_i}, X_{k_i+1} \le K, i = 1, \dots, N \mid X_{A(n\varepsilon)} > 0\big)\\ &\quad= \sum_{\substack{1\le x_{k_1},\dots,x_{k_N}\le K,\\ x_{k_i}\ne x_{k_i+1}}} \mathbb{P}\big(X_{k_i} = x_{k_i}, X_{k_i+1} = x_{k_i+1}, i = 1, \dots, N \mid X_{A(n\varepsilon)} > 0\big).\end{aligned}$$

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Now, let us compute the above expression for a given $1 \le x_{k_i} \le K$ satisfying the condition $x_{k_i} \ne x_{k_i+1}$. By the Markov property and (25), we have

$$\begin{aligned}
&\mathbb{P}\big(X_{k_i} = x_{k_i}, X_{k_i+1} = x_{k_i+1}, i = 1, \dots, N \mid X_{A(n\varepsilon)} > 0\big) \\
&\quad = \mathbb{P}(X_{k_1} = x_{k_1} \mid X_{A(n\varepsilon)} > 0)\mathbb{P}(X_{k_1+1} = x_{k_1+1} \mid X_{k_1} = x_{k_1}) \\
&\qquad \dots \\
&\qquad \times \mathbb{P}(X_{k_N+1} = x_{k_N+1} \mid X_{k_N} = x_{k_N}) \\
&\quad \le 1 \cdot cK|1 - \bar{f}_{k_1}| \cdot 1 \cdot cK|1 - \bar{f}_{k_2}| \cdots cK|1 - \bar{f}_{k_N}|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{P}\big(X_{k_i} \ne X_{k_i+1}, i = 1, \dots, N, \max_{n\varepsilon \le k \le nm} X_k \le K \mid X_{A(n\varepsilon)} > 0\big) \\
&\quad \le \sum_{\substack{1 \le x_{k_1}, \dots, x_{k_N} \le K, \\ x_{k_i} \ne x_{k_i+1}}} \mathbb{P}\big(X_{k_i} = x_{k_i}, X_{k_i+1} = x_{k_i+1}, i = 1, \dots, N \mid X_{A(n\varepsilon)} > 0\big) \\
&\quad \le (cK^2(K-1))^N |1 - \bar{f}_{k_1}| \cdots |1 - \bar{f}_{k_N}|,
\end{aligned} \tag{26}$$

and thus

$$\begin{aligned}
&\mathbb{P}(B_n \cap C_n \mid X_{A(n\varepsilon)} > 0) \\
&\quad \le \sum_{A(n\varepsilon) \le k_1 < \dots < k_N \le A(nm)} \mathbb{P}\big(X_{k_i} \ne X_{k_i+1}, i = 1, \dots, N, B_n \mid X_{A(n\varepsilon)} > 0\big) \\
&\quad \le (cK^3)^N \sum_{A(n\varepsilon) \le k_1 < \dots < k_N \le A(nm)} |1 - \bar{f}_{k_1}| \cdots |1 - \bar{f}_{k_N}| \\
&\quad \le (cK^3)^N \frac{1}{N!} \left(\sum_{k=A(n\varepsilon)}^{A(nm)} |1 - \bar{f}_k| \right)^N \le \frac{C(K)^N}{N!},
\end{aligned}$$

where, here and later, $C(K)$ stands for a positive constant whose actual value is not important and depends only on K, ε, and m. In the last inequality, we used that

$$\begin{aligned}
\sum_{k=A(n\varepsilon)}^{A(nm)} |1 - \bar{f}_k| &= \sum_{k=A(n\varepsilon)}^{A(nm)} (1 - \bar{f}_k) + 2 \sum_{k=A(n\varepsilon)}^{A(nm)} (1 - \bar{f}_k)_- \\
&\le -\log \bar{f}_{A(n\varepsilon)-1, A(nm)} + 2 \sum_{k=A(n\varepsilon)}^{\infty} (1 - \bar{f}_k)_-,
\end{aligned} \tag{27}$$

where the first term tends to $\log m - \log \varepsilon$ by (3) and the second to 0 by (C1) as $n \to \infty$. Since $N = N(\delta) \to \infty$ as $\delta \to 0$, we obtain

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}(C_n \cap B_n \mid X_{A(n\varepsilon)} > 0) = 0. \tag{28}$$

Finally, we bound the probability of two close jumps. Consider a fixed t_0, and bound the probability of having at least two jumps in $[t_0, t_0 + 2\delta]$. Let E_{t_0} denote the latter event; that is,

$$E_n(t_0) = \big\{(X_k)_{A(nt_0) \le k \le A(n(t_0+2\delta))} \text{ has at least two jumps}\big\}.$$

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Similarly to the previous case, by (26) and (27) we have

$$\begin{aligned}
&\mathbb{P}(E_n(t_0) \cap B_n \mid X_{A(n\varepsilon)} > 0) \\
&\quad \le \sum_{A(nt_0) \le k_1 < k_2 \le A(n(t_0+2\delta))} \mathbb{P}\big(X_{k_i} \ne X_{k_i+1},\, i = 1, 2,\, B_n \mid X_{A(n\varepsilon)} > 0\big) \\
&\quad \le (cK^3)^2 \sum_{A(nt_0) \le k_1 < k_2 \le A(n(t_0+2\delta))} |1 - \bar{f}_{k_1}||1 - \bar{f}_{k_2}| \\
&\quad \le \frac{(cK^3)^2}{2} \left(\sum_{k=A(nt_0)}^{A(n(t_0+2\delta))} |1 - \bar{f}_k| \right)^2 \\
&\quad \le \frac{(cK^3)^2}{2} \left(-\log \bar{f}_{A(nt_0)-1, A(n(t_0+2\delta))} + 2 \sum_{k=A(nt_0)}^{A(n(t_0+2\delta))} (1 - \bar{f}_k)_- \right)^2,
\end{aligned}$$

where the first term in the parentheses tends to $\log(1 + 2\delta/t_0) \le 2\delta/t_0$ and the second to 0 as $n \to \infty$ by (C1) and the fact that $A(n) \to \infty$. Therefore, uniformly in t_0,

$$\limsup_{n\to\infty} \mathbb{P}(E_n(t_0) \cap B_n \mid X_{A(n\varepsilon)} > 0) \le C(K)\delta^2. \tag{29}$$

Since $D_n \subset \cup_{\ell=0}^{\lfloor (m-\varepsilon)/\delta \rfloor} E_n(\varepsilon + \ell\delta)$, we have, by a simple union bound combined with (29), that

$$\limsup_{n\to\infty} \mathbb{P}(D_n \cap B_n \mid X_{A(n\varepsilon)} > 0) \le C(K)\delta. \tag{30}$$

By (23), we can choose K large enough to make the limit superior (in n) of the third term in (22) arbitrarily small. Then, with this fixed K, we can apply (28) and (30). Summarizing, we obtain that (18) holds; thus, the proof is complete. □

Proof of Corollary 2. Since, by Theorem 1, we have convergence on $D([\varepsilon, \infty))$ for arbitrary $\varepsilon > 0$ and the limit is almost surely continuous at $t = 1$, [6, Theorem 16.2] implies that the convergence also holds on $D([\varepsilon, 1])$.

Let $\psi_m \colon D([1, m]) \to D([1, m])$ be such that

$$(\psi_m x)(t) = \begin{cases} x(t) & \text{if } 1 \le t \le m-1, \\ (m-t)\, x(t) & \text{if } m-1 \le t \le m, \end{cases}$$

and define $\phi_\varepsilon \colon D([\varepsilon, 1]) \to D([1, 1/\varepsilon])$ such that $(\phi_\varepsilon x)(t) = x((1/t)-)$, where $x(u-)$ represents the left limit in u. It can easily be proved that ψ_m and ϕ_ε are continuous. Thus, the statement follows from [6, Lemma 3 of Section 16] after applying the mapping theorem with $\psi_{1/\varepsilon} \circ \phi_\varepsilon$. □

3.3. Proofs for subsection 2.2

Recall that $h_n(s) = \sum_{k=0}^{\infty} h_n[k]s^k$ denotes the g.f. of the immigration distribution, $g_n(s) = \mathbb{E}s^{Y_n}$ is the g.f. of the population size, and let $g_{j,n}(s) = \mathbb{E}[s^{Y_n} \mid Y_j = 0]$ denote the conditional g.f. of the process $(Y_n)_{n\in\mathbb{N}}$.

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The following result ensures that the limit process is an honest, standard, continuous-time Markov branching process with immigration. For general definitions of continuous-time Markov chains, see [2], and for continuous-time Markov branching processes, see [24].

Lemma 5. *Assume that conditions (C1)–(C3) and either (C4) or (C4′) holds. Then, for the BPVEI* (Y_n) *defined in (2) and for any* $0 < u < t$,

$$\lim_{n\to\infty} g_{A(nu),A(nt)}(s) = \frac{f_Y(s)}{f_Y(h_{u/t}(s))},$$

and for any $y \in \mathbb{N}$,

$$\lim_{n\to\infty} \mathbb{E}\big[s^{Y_{A(nt)}} \mid Y_{A(nu)} = y\big] = \frac{f_Y(s)}{f_Y(h_{u/t}(s))}(h_{u/t}(s))^y.$$

Furthermore, the transition-generating functions

$$G_y(s,t) = \frac{f_Y(s)}{f_Y(h_{\mathrm{e}^{-t}}(s))}(h_{\mathrm{e}^{-t}}(s))^y$$

define a homogeneous standard honest transition function of a Markov branching process with immigration.

Proof. By the branching property (see, e.g., the proof of [21, Theorem 2]),

$$g_{j,n}(s) = \prod_{l=j+1}^{n} h_l(f_{l,n}(s)) = \frac{g_n(s)}{g_j(f_{j,n}(s))}.$$

Therefore, by [21, Theorem 2 or 3] and Lemma 4,

$$\lim_{n\to\infty} g_{A(nu),A(nt)}(s) = \lim_{n\to\infty} \frac{g_{A(nt)}(s)}{g_{A(nu)}(f_{A(nu),A(nt)}(s))} = \frac{f_Y(s)}{f_Y(h_{u/t}(s))}.$$

The second statement follows similarly, upon noting that $\mathbb{E}[s^{Y_k} \mid Y_\ell = y] = g_{\ell,k}(s)(f_{\ell,k}(s))^y$ for $\ell < k$.

Since $G_y(\cdot, t)$ is a limit of generating functions and $G_y(1,t) = 1$, it is clear that $G_y(\cdot, t)$ is a generating function. Thus, it determines the transition functions via

$$G_y(s,t) = \sum_{j=0}^{\infty} p_{y,j}(t)s^j,$$

where $p_{y,j}(t) \in [0,1]$, $\sum_{j=0}^{\infty} p_{y,j}(t) \equiv 1$, $p_{y,j}(0) = \delta_{y,j}$. Furthermore, it satisfies the Chapman–Kolmogorov equation since it is a limit of generating functions of a discrete-time Markov process. The branching property follows from the explicit form of $G_y(s,t)$; see [24, (1.1) and (2.1)]. □

By [24, Theorem 2.1] (see the displayed equation after (2.2) there),

$$a(s) = \lim_{t\downarrow 0} \frac{\partial}{\partial t} h_{\mathrm{e}^{-t}}(s) = (1-s)\left(1 + \frac{\nu}{2}(1-s)\right),$$

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while

$$b(s) = \lim_{t \downarrow 0} \frac{\partial}{\partial t} \frac{f_Y(s)}{f_Y(h_{e^{-t}}(s))} = \sum_{k=1}^{\kappa} \lambda_k (s-1)^k,$$

where the second equality follows from a straightforward calculation. Recall that under (C4) $\kappa = K - 1$ and under (C4′) $\kappa = \infty$. From (8) we find that the offspring and immigration rates and generating functions are given by

$$\alpha = 1 + \nu, \qquad f(s) = \frac{1 + \nu/2}{1 + \nu} + \frac{\nu/2}{1 + \nu} s^2, \qquad \beta = -b(0), \qquad h(s) = 1 + \frac{b(s)}{\beta}.$$

Proof of Theorem 2. The proof is similar to the proof of Theorem 1, so we only sketch it, emphasizing the differences.

The convergence of the finite-dimensional distributions follows from the convergence of one-dimensional marginals and the convergence of the transition probabilities. The first statement is [21, Theorem 2 or 3]; the second is Lemma 5.

We prove the tightness of $(Y_{A(nt)})_{t \geq \varepsilon}$ for all $\varepsilon > 0$ fixed. Similarly to the proof of Theorem 1, it is enough to check that

$$\lim_{K \to \infty} \limsup_{n \to \infty} \mathbb{P}\Big(\max_{A(n\varepsilon) \leq k \leq A(nm)} Y_k > K \Big) = 0 \tag{31}$$

and

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\big(w'((Y_{A(nt)})_{t \in [\varepsilon, m]}, \delta) \geq \eta \big) = 0 \tag{32}$$

for all $\eta > 0$. By Lemma 2,

$$\lim_{n \to \infty} \mathbb{E}(Y_{A(nm)}) = \lim_{n \to \infty} \sum_{j=1}^{A(nm)} \frac{m_{j,1}}{1 - \bar{f}_j} (1 - \bar{f}_j) \bar{f}_{j,A(nm)} = \lambda_1 < \infty.$$

Let $\tau = \min\{k \colon A(n\varepsilon) \leq k \leq A(nm),\ Y_k > K\}$, and with C_0 defined in (20),

$$\begin{aligned} \mathbb{E}(Y_{A(nm)} \mathbf{1}_{\{A(n\varepsilon) \leq \tau \leq A(nm)\}}) &\geq \sum_{k=A(n\varepsilon)}^{A(nm)} \left[K \bar{f}_{k,A(nm)} + \sum_{j=k+1}^{A(nm)} m_{j,1} \bar{f}_{j,A(nm)} \right] \mathbb{P}(\tau = k) \\ &\geq \frac{K}{C_0} \bar{f}_{A(n\varepsilon),A(nm)} \mathbb{P}\Big(\max_{A(n\varepsilon) \leq k \leq A(nm)} Y_k > K \Big), \end{aligned}$$

and thus (31) follows as in the proof of Theorem 1. Again, to prove (32) we need to bound the probability of having many jumps, or at least two close jumps. As in (24) and (25), for any $y_k \leq K$, with $h_{k+1}[0] = \mathbb{P}(\varepsilon_{k+1} = 0)$,

$$\mathbb{P}(Y_{k+1} \neq Y_k \mid Y_k = y_k) \leq 1 - (f_{k+1}[1])^K h_{k+1}[0] \leq cK |1 - \bar{f}_{k+1}|,$$

since, by our assumptions, $h_{k+1}[0] = h_{k+1}(0) \geq 1 - m_{k+1,1} \geq 1 - c|1 - \bar{f}_{k+1}|$. The rest follows exactly as in the proof of Theorem 1. □

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