Probability in the Engineering and Informational Sciences, **36**, 2022, 824–850.
doi:10.1017/S0269964821000103

SIGNATURES OF MULTI-STATE SYSTEMS BASED ON A SERIES/PARALLEL/RECURRENT STRUCTURE OF MODULES

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In this paper, we consider multi-state coherent systems that can be regarded as a series/parallel/recurrent connection of multi-state modules with binary/multi-state components. The multi-state (survival) signatures of such systems are presented in terms of multi-state (survival) signatures of related modules based on the structures. For a recurrent structure, the multi-state survival signature of the structure is also needed. The results established here are finally illustrated with a number of examples.

Keywords: multi-state coherent system, multi-state module, multi-state (survival) signature, series/parallel/recurrent connection

1. INTRODUCTION

In practice, coherent systems can widely be seen in reliability theory as well as in many practical structures. These are systems in which all components are relevant to the system lifetime and improvement of any component will not lead to a decrease in the system reliability. For binary-state coherent systems with components whose lifetimes are independent and identically distributed (*i.i.d.*) from a common absolutely continuous distribution, Samaniego [31] proposed the system signature as $\boldsymbol{s} = (s_1, \ldots, s_n)$, where s_i is the probability that the ith ordered component failure causes the system to fail and it also equals the proportion of component failure orderings that cause the system to fail at the ith failure, that is

$$s_i = \mathbb{P}(T = X_{i:n}) = \frac{1}{n!} \sum_{\boldsymbol{\pi} \in P} \mathbb{P}(T = X_{i:n} | A_{\boldsymbol{\pi}}).$$

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https://doi.org/10.1017/S0269964821000103 Published online by Cambridge University Press

TABLE 1. Comparisons of signature computation methods for binary-state coherent systems

Computation method	Advantages (A) and Disadvantages (D)
Definition method [21]	A Simple and straightforward, no other theory needed D Time consuming and infeasible for large systems
(Minimal) cut/path set method [4,11]	A Efficient for systems with known (minimal) cut/path sets D The process to get (minimal) cut/path sets may take time
Reliability method [5,23]	A Transform the problem to the computation of system reliability D Computational methods for reliability may have disadvantages
Generating function approach [34]	A Explicit recursive formulas are already given D Only available for consecutive-k type systems
Five-components algorithm [25]	A Results for coherent systems with five components are all given D Unavailable for systems for larger sizes
Markov method [37]	A First provide explicit formulas some well-known systems D The process to get numbers of non-cut sets may take time
Binary-decision diagram algorithm [30]	A No need to compute (minimal) path/cut sets D Large storage is required and unavailable for large systems
Module method	A Can be applied together with other computational methods D Only available for modular systems with given structures, for example, series/parallel/recurrent structure [17], k-out-of-n structure [10], $(\boldsymbol{n}, f, k)$ structure [13] and other structures with joint/disjoint modules [8,15,20,24]

In addition to the system signature, several related concepts have been introduced for different types of binary-state coherent systems, for example, the maximal/minimal signature [27], the dynamic signature [33], the survival signature [6], the ordered signature [3], the joint signature [28] and some others. The signature and all the mentioned related concepts are useful in practice and have a wide array of applications in the field of reliability since they depend only on the system structure and possess a simpler and more interpretable form than the traditional structure function. For example, Kochar *et al.* [21] compared the performance of binary-state coherent systems in terms of their signatures; see Navarro *et al.* [26], Navarro *et al.* [29] and Samaniego [32] for more detailed discussions in this regard. The computation of signature is one of the most significant issues, and several computational methods have already been presented. Each of them has specific advantages and disadvantages, as described in Table 1.

The above discussions on signature are all for binary-state systems, but with the development of multi-state modeling in the fields of reliability and networks, researchers have recently defined and studied multi-state signatures for multi-state systems with binary-state or multi-state components. For multi-state systems with binary-state components, several signature indices have been proposed, such as multi-dimensional D-spectrum [22], (cumulative) first spectrum and second spectrum [16], bivariate signature [9], bivariate tail signature and cumulative bivariate signature [18], multi-state (ordered) signature [35] and some others. More discussions on these can be found in Ashrafi and Asadi [1], Ashrafi and Asadi [2], Zarezadeh *et al.* [39] and Yi *et al.* [36].

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The above cited works are all on multi-state systems with binary-state components. For systems with multi-state components, there are also some results on corresponding signatures. For example, Gertsbakh *et al.* [19] considered a binary-state system with ternary-state components and investigated its ternary D-spectrum. Da Costa Bueno [12] defined a multi-state monotone system signature for multi-state systems whose components have exchangeable lifetimes in each state. Eryilmaz and Tuncel [14] proposed a generalized survival signature for multi-state systems that are in state j if j different rules are satisfied. Yi *et al.* [38] introduced a new method for the computation of multi-state survival signatures based on a finite Markov imbedding chain approach (FMCIA).

Though there exists a lot of results on signatures of multi-state systems, sufficient results are still lacking, especially for multi-state systems with multi-state components. Moreover, the computation of multi-state signatures still remains as a challenging problem. As a natural computational tool, the traditional way of computation based directly on the definition is extremely time-assuming and is usually infeasible for large systems. Take a multi-state system with m *i.i.d.* binary-state components as an example, for the computation of its multi-state signature in [35]. In this case, there are $m!$ component failure orderings to consider, which means the computational efforts increase rapidly with the system size m, especially when $m \geq 12$. Fortunately, for multi-state consecutive type systems, a method based on the FMCIA [38] can be applied for the computation of their multi-state survival signature. For large multi-state systems, this method is much more efficient compared with the traditional method based on the definition, but it is directly available only for consecutive type systems and can only be generalized to multi-state systems whose reliability can easily be obtained by methods like FMCIA.

To provide a new and efficient computational tool for multi-state signatures, we consider here a module method, along the lines of Gertsbakh *et al.* [17]. One of the most important advantages of this method is that it can be used together with other methods, as long as multi-state systems can be regarded as series/parallel/recurrent connection of multi-state modules with binary-state components or multi-state components. Here, a recurrent connection means a connection between several identical modules, implying that the modules have an identical multi-state (survival) signature.

For a coherent system with m components, its structure function can be defined as $y = \phi(x_1, \ldots, x_m)$ to represent the functional relationship between system state $y \in \{0, \ldots, M\}$ and component states $x_1, \ldots, x_m \in \{0, \ldots, M\}$, that is, to describe the structure of the system. In related research, a system with structure function $\phi(x_1, \ldots, x_m)$ is often termed as System ϕ. Usually, State M corresponds to a perfect functioning state, State 0 corresponds to a complete failure state and States $1, \ldots, M-1$ are imperfect functioning states between them, in some increasing form of performance of the system. Specifically, the coherent system is a binary-state system for $M = 1$ and is a multi-state system for $M = 2, 3, \ldots$. In the present work, we assume that all coherent systems and their components start from perfectly functioning states.

Now, for System ϕ_1 with structure function $\phi_1(x_1, \ldots, x_{m_1})$, $x_1, \ldots, x_{m_1} \in \{0, \ldots, M\}$ and System ϕ_2 with structure function $\phi_2(x_{m_1+1}, \ldots, x_{m_1+m_2})$, $x_{m_1+1}, \ldots, x_{m_1+m_2} \in \{0, \ldots, M\}$, a series connection of them is indeed a new system with structure function

$$\phi(x_1, \ldots, x_{m_1+m_2}) = \min\{\phi_1(x_1, \ldots, x_{m_1}), \phi_2(x_{m_1+1}, \ldots, x_{m_1+m_2})\}.$$

Similarly, a parallel connection of them is another new system with structure function

$$\phi(x_1, \ldots, x_{m_1+m_2}) = \max\{\phi_1(x_1, \ldots, x_{m_1}), \phi_2(x_{m_1+1}, \ldots, x_{m_1+m_2})\}.$$

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Finally, a recurrent connection of System ϕ_1 with structure ϕ_2 is a new system with structure function as follows, for $x_1, \ldots, x_{m_1 m_2} \in \{0, \ldots, M\}$:

$$\phi(x_1, \ldots, x_{m_1 m_2}) = \phi_2(\phi_1(x_1, , \ldots, x_{m_1}), \phi_1(x_{m_1+1}, \ldots, x_{2m_1}), \ldots, \times \phi_1(x_{m_1(m_2-1)+1}, \ldots, x_{m_1 m_2})).$$

In this paper, we consider several different types of multi-state module systems and present their system signatures in terms of module signatures and structure signatures. The rest of this paper is organized as follows. In Section 2, we consider a series/parallel connection of two multi-state coherent modules with binary-state components, and in this case, the multi-state signature of the whole system is then derived by multi-state signatures of each module. Next, in Section 3, we generalize the results of Section 2 to systems with multi-state components and develop some results for corresponding multi-state survival signatures. In Section 4, recurrent systems with multi-state coherent modules are considered and then some general results are obtained for systems with binary-state components as well as for systems with multi-state components. In Section 5, we present a number of examples to illustrate all the theoretical results established here. Some concluding remarks are finally made in Section 6.

2. SERIES AND PARALLEL CONNECTION OF TWO MULTI-STATE COHERENT MODULES WITH BINARY-STATE COMPONENTS

The definition of multi-state signature for a multi-state coherent system with binary-state components, as given in Yi *et al.* [35], is as follows.

DEFINITION 2.1: *The multi-state signature of a multi-state coherent system with state space* $\{0, \ldots, M\}$ *and* m *binary-state components whose lifetimes are i.i.d. from an absolutely continuous distribution* $F(x), x \geq 0$, *is*

$$\boldsymbol{s} = (s_{i_1,\ldots,i_M},\ 1 \leq i_1, \ldots, i_M \leq m),$$

where $s_{i_1,\ldots,i_M} = \mathbb{P}\{T_1 = X_{i_1:m}, \ldots, T_M = X_{i_M:m}\}$. *Here, for* $r = 1, \ldots, M$, T_r *is the first time that the system enters state* $M - r$ *or below, and* $X_{i_r:m}$ *is the* i_r*th order statistic among the* m *i.i.d. component lifetimes* $X_1, \ldots, X_m$. *For brevity,* $\boldsymbol{s}$ *can be rewritten as* $\boldsymbol{s} = (s_{i_1,\ldots,i_M}, 1 \leq i_1 \leq \cdots \leq i_M \leq m)$ *by removal of zero elements.*

Let us now consider two independent multi-state coherent systems ϕ_1 and ϕ_2 with state space $\{0, \ldots, M\}$. System ϕ_1 has m_1 binary-state components whose lifetimes are *i.i.d.* from an absolutely continuous distribution $F(x)$ $(x \geq 0)$ and has a multi-state signature $\boldsymbol{s}^1 = (s^1_{i_1,\ldots,i_M}, 1 \leq i_1, \ldots, i_M \leq m_1)$. Similarly, system ϕ_2 has m_2 binary-state components whose lifetimes are *i.i.d.* from the same distribution $F(x)$ $(x \geq 0)$ and has a multi-state signature $\boldsymbol{s}^2 = (s^2_{j_1,\ldots,j_M}, 1 \leq j_1, \ldots, j_M \leq m_2)$. Let us denote subscripts like $i_1, \ldots, i_M$ simply by $\boldsymbol{i}$, and we shall then use this form of notation throughout. Then, we have $\boldsymbol{s}^1 = (s^1_{\boldsymbol{i}}, 1 \leq i_1, \ldots, i_M \leq m_1)$ and $\boldsymbol{s}^2 = (s^2_{\boldsymbol{j}}, 1 \leq j_1, \ldots, j_M \leq m_2)$.

Now, let us consider a series connection between Systems ϕ_1 and ϕ_2. We then obtain a new system with $m_1 + m_2$ *i.i.d.* components whose state is the smaller one among states of Systems ϕ_1 and ϕ_2. To find the multi-state signature $\boldsymbol{s}^{(ser)} = (s_{\boldsymbol{v}}, 1 \leq v_1, \ldots, v_M \leq m_1 + m_2)$ of this new system, multi-state tail signature of the system can be defined, analogous

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to bivariate tail signature in [18] as $\bar{\boldsymbol{S}}^{(ser)} = (\bar{S}_{\boldsymbol{u}}, 0 \le u_1, \ldots, u_M \le m_1 + m_2)$, where

$$\bar{S}_{\boldsymbol{u}} = \sum_{u_1 < v_1 \le m_1+m_2, \ldots, u_M < v_M \le m_1+m_2} s_{\boldsymbol{v}}. \tag{2.1}$$

Here in Eq. (2.1), $\bar{S}_{\boldsymbol{u}}$ is the probability that the new system is in states above $M - r$ right after the u_rth component failure, for all $r = 1, \ldots, M$. (Specifically, we refer to "the 0th component failure" here as the initial time 0.) This concept can be defined for any multi-state signature; for example, we have $\bar{\boldsymbol{S}}^1 = (\bar{S}^1_{\boldsymbol{i}}, 0 \le i_1, \ldots, i_M \le m_1)$ for System ϕ_1 and $\bar{\boldsymbol{S}}^2 = (\bar{S}^2_{\boldsymbol{j}}, 0 \le j_1, \ldots, j_M \le m_2)$ for System ϕ_2. Then, an explicit expression of the multi-state signature $\boldsymbol{s}^{(ser)}$ of the new system can be obtained from the multi-state signatures $\boldsymbol{s}^1$ and $\boldsymbol{s}^2$ of Systems ϕ_1 and ϕ_2, as presented in the following theorem.

THEOREM 2.1: *The elements of* $\boldsymbol{s}^{(ser)} = (s_{\boldsymbol{v}}, 1 \le v_1, \ldots, v_M \le m_1 + m_2)$ *can be computed from* $\boldsymbol{s}^1 = (s^1_{\boldsymbol{i}}, 1 \le i_1, \ldots, i_M \le m_1)$ *and* $\boldsymbol{s}^2 = (s^2_{\boldsymbol{j}}, 1 \le j_1, \ldots, j_M \le m_2)$ *as*

$$s_{\boldsymbol{v}} = \sum_{v_1 - 1 \le u_1 \le v_1, \ldots, v_M - 1 \le u_M \le v_M} (-1)^{(u_1 + \cdots + u_M) - (v_1 + \cdots + v_M) + M} \bar{S}_{\boldsymbol{u}},$$

where for $0 \le u_1 \le \cdots \le u_M \le m_1 + m_2$,

$$\bar{S}_{\boldsymbol{u}} = \sum_{\boldsymbol{i} \in W_{\boldsymbol{u}}} \bar{S}^1_{\boldsymbol{i}} \bar{S}^2_{\boldsymbol{u}-\boldsymbol{i}} \binom{u_1}{i_1} \prod_{r=2}^{M} \binom{u_r - u_{r-1}}{i_r - i_{r-1}} \binom{m_1 + m_2 - u_M}{m_1 - i_M} \binom{m_1 + m_2}{m_1}^{-1}, \tag{2.2}$$

with

$$W_{\boldsymbol{u}} = \{\boldsymbol{i} : \ max(u_r - m_2, i_{r-1} I_{\{r \ne 1\}}) \le i_r \le min(m_1, u_r - (u_{r-1} - i_{r-1}) I_{\{r \ne 1\}}),\ r = 1, \ldots, M\},$$

and, otherwise, $\bar{S}_{\boldsymbol{u}} = \bar{S}_{u_1, max(u_1,u_2), \ldots, max(u_1, \ldots, u_M)}$.

PROOF: Define the first time that the new system/System ϕ_1/System ϕ_2 enters state $M - r$ or below as $T_r / T^1_r / T^2_r$ (for $r = 1, \ldots, M$). According to (2.1), we have

$$\bar{S}_{\boldsymbol{u}} = \mathbb{P}\{T_1 > X_{u_1:m_1+m_2}, \ldots, T_M > X_{u_M:m_1+m_2}\},$$

where $X_{0:m_1+m_2} = 0$ and $X_{i:m_1+m_2}$ $(i = 1, \ldots, m_1 + m_2)$ is the ith order statistic among lifetimes $X_1, \ldots, X_{m_1+m_2}$ of the $m_1 + m_2$ *i.i.d.* components in the new system. Then,

$$\begin{aligned}
s_{\boldsymbol{v}} &= \mathbb{P}\{T_1 = X_{v_1:m_1+m_2}, \ldots, T_M = X_{v_M:m_1+m_2}\} \\
&= \mathbb{P}\{A^1_{v_1 - 1} - A^1_{v_1}, \ldots, A^M_{v_M - 1} - A^M_{v_M})\} \\
&= \sum_{v_1 - 1 \le u_1 \le v_1, \ldots, v_M - 1 \le u_M \le v_M} \left[\prod_{i=1}^{M} (-1)^{u_i - (v_i - 1)} \right] \mathbb{P}\{A^1_{u_1}, \ldots, A^M_{u_M}\} \\
&= \sum_{v_1 - 1 \le u_1 \le v_1, \ldots, v_M - 1 \le u_M \le v_M} (-1)^{(u_1 + \cdots + u_M) - (v_1 + \cdots + v_M) + M} \bar{S}_{\boldsymbol{u}},
\end{aligned}$$

where $A^i_{v_i}$ denotes the event that $T_i > X_{v_i:m_1+m_2}$ with $i = 1, \ldots, M$, and $v_i = 0, \ldots, m_1 + m_2$, that is, $A^i_{v_i - 1} - A^i_{v_i}$ denotes the event that $T_i = X_{v_i:m_1+m_2}$ with $i = 1, \ldots, M$ and $v_i = 1, \ldots, m_1 + m_2$.

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With this relationship, to obtain the multi-state signature $\boldsymbol{s}^{(ser)}$ of this new system, we only need to obtain $\bar{S}_{\boldsymbol{u}}$ for all $0 \le u_1 \le \cdots \le u_M \le m_1 + m_2$ since otherwise $\bar{S}_{\boldsymbol{u}} = \bar{S}_{u_1,\max(u_1,u_2),\ldots,\max(u_1,\ldots,u_M)}$. As in Yi *et al.* [35], with $0 \le t_1 \le \cdots \le t_M < \infty$, for System ϕ_1, we have

$$\mathbb{P}\{T_1^1 > t_1, \ldots, T_M^1 > t_M\} = \sum_{0 \le i_1 \le \cdots \le i_M \le m_1} \bar{S}_{\boldsymbol{i}}^1 \binom{m_1}{i_1, i_2 - i_1, \ldots, i_M - i_{M-1}, m_1 - i_M} \times F^{i_1}(t_1) \prod_{r=1}^{M-1} [F(t_{r+1}) - F(t_r)]^{i_{r+1} - i_r} \times [1 - F(t_M)]^{m_1 - i_M},$$

and similarly, for System ϕ_2, we have

$$\mathbb{P}\{T_1^2 > t_1, \ldots, T_M^2 > t_M\} = \sum_{0 \le j_1 \le \cdots \le j_M \le m_2} \bar{S}_{\boldsymbol{j}}^2 \binom{m_2}{j_1, j_2 - j_1, \ldots, j_M - j_{M-1}, m_2 - j_M} \times F^{j_1}(t_1) \prod_{r=1}^{M-1} [F(t_{r+1}) - F(t_r)]^{j_{r+1} - j_r} \times [1 - F(t_M)]^{m_2 - j_M},$$

which means for the new system, we have

$$\begin{aligned}
&\mathbb{P}\{T_1 > t_1, \ldots, T_M > t_M\} \\
&\quad = \mathbb{P}\{\min(T_1^1, T_1^2) > t_1, \ldots, \min(T_M^1, T_M^2) > t_M\} \\
&\quad = \mathbb{P}\{T_1^1 > t_1, \ldots, T_M^1 > t_M\} \times \mathbb{P}\{T_1^2 > t_1, \ldots, T_M^2 > t_M\} \\
&\quad = \sum_{0 \le i_1 \le \cdots \le i_M \le m_1} \sum_{0 \le j_1 \le \cdots \le j_M \le m_2} \bar{S}_{\boldsymbol{i}}^1 \bar{S}_{\boldsymbol{j}}^2 \binom{m_1}{i_1, i_2 - i_1, \ldots, i_M - i_{M-1}, m_1 - i_M} \\
&\qquad \times \binom{m_2}{j_1, j_2 - j_1, \ldots, j_M - j_{M-1}, m_2 - j_M} F^{i_1 + j_1}(t_1) \\
&\qquad \times \prod_{r=1}^{M-1} [F(t_{r+1}) - F(t_r)]^{i_{r+1} + j_{r+1} - i_r - j_r} [1 - F(t_M)]^{m_1 + m_2 - i_M - j_M}.
\end{aligned}$$

Let $u_r = i_r + j_r$, for $r = 1, \ldots, M$. Then, evidently $0 \le i_1 \le \cdots \le i_M \le m_1$, $0 \le j_1 \le \cdots \le j_M \le m_2$ means that

$$0 \le i_1 \le \cdots \le i_M \le m_1, \quad 0 \le u_1 - i_1 \le \cdots \le u_M - i_M \le m_2,$$

that is, $0 \le u_1 \le \cdots \le u_M \le m_1 + m_2, \boldsymbol{i} \in W_{\boldsymbol{u}}$. Then, we have

$$\begin{aligned}
&\mathbb{P}\{T_1 > t_1, \ldots, T_M > t_M\} \\
&\quad = \sum_{0 \le u_1 \le \cdots \le u_M \le m_1 + m_2} \sum_{\boldsymbol{i} \in W_{\boldsymbol{u}}} \bar{S}_{\boldsymbol{i}}^1 \bar{S}_{\boldsymbol{u} - \boldsymbol{i}}^2 \binom{m_1}{i_1, i_2 - i_1, \ldots, i_M - i_{M-1}, m_1 - i_M} \\
&\qquad \times \binom{m_2}{u_1 - i_1, u_2 - i_2 - u_1 + i_1, \ldots, u_M - i_M - u_{M-1} + i_{M-1}, m_2 - u_M + i_M} \\
&\qquad \times F^{u_1}(t_1) \prod_{r=1}^{M-1} [F(t_{r+1}) - F(t_r)]^{u_{r+1} - u_r} [1 - F(t_M)]^{m_1 + m_2 - u_M}.
\end{aligned}$$

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On the other hand, for the new system, we also have

$$\begin{aligned}&\mathbb{P}\{T_1 > t_1, \ldots, T_M > t_M\}\\ &= \sum_{0\le u_1\le\cdots\le u_M\le m_1+m_2} \bar{S}_{\boldsymbol{u}} \binom{m_1+m_2}{u_1, u_2-u_1, \ldots, u_M-u_{M-1}, m_1+m_2-u_M}\\ &\quad\times F^{u_1}(t_1)\prod_{r=1}^{M-1}[F(t_{r+1})-F(t_r)]^{u_{r+1}-u_r}[1-F(t_M)]^{m_1+m_2-u_M}.\end{aligned}$$

Hence, the theorem proved. ■

REMARK 2.1: *In Eq. (2.2), the subset $W_{\boldsymbol{u}}$ consists of all possible $i_1, \ldots, i_M$, given $\boldsymbol{u}$. The term $\bar{S}^1_{\boldsymbol{i}}\bar{S}^2_{\boldsymbol{u}-\boldsymbol{i}}$ is the probability that System ϕ_1 is in states above $M-r$ right after its i_rth component failure and System ϕ_2 is in states above $M-r$ right after its (u_r-i_r)th component failure, for all $r=1,\ldots,M$, and the term*

$$\binom{u_1}{i_1}\prod_{r=2}^{M}\binom{u_r-u_{r-1}}{i_r-i_{r-1}}\binom{m_1+m_2-u_M}{m_1-i_M}\binom{m_1+m_2}{m_1}^{-1}$$

is the conditional probability that there are $i_1, \ldots, i_M$ component failures in System ϕ_1 and $u_1-i_1, \ldots, u_M-i_M$ component failures in System ϕ_2, given that it is right after the u_1th, …, u_Mth component failures of the new system, respectively.

Similarly, let us now consider a parallel connection between Systems ϕ_1 and ϕ_2. We then obtain another new system with m_1+m_2 *i.i.d.* components whose state is the larger one among states of Systems ϕ_1 and ϕ_2. To find the multi-state signature $\boldsymbol{s}^{(par)} = (s_{\boldsymbol{v}}, 1 \le v_1, \ldots, v_M \le m_1+m_2)$ of this new system, multi-state cumulative signature can be defined, analogous to the bivariate cumulative signature in [18], as $\boldsymbol{S}^{(par)} = (S_{\boldsymbol{u}}, 0 \le u_1, \ldots, u_M \le m_1+m_2)$, where

$$S_{\boldsymbol{u}} = \sum_{1\le v_1\le u_1,\ldots,1\le v_M\le u_M} s_{v_1,\ldots,v_M}. \tag{2.3}$$

In Eq. (2.3), $S_{\boldsymbol{u}}$ is the probability that the new system is in state $M-r$ or below right after the u_rth component failure, for all $r=1,\ldots,M$. This concept can be defined for any multi-state signature; for example, we have $\boldsymbol{S}^1 = (S^1_{\boldsymbol{i}}, 0 \le i_1, \ldots, i_M \le m_1)$ for System ϕ_1 and $\boldsymbol{S}^2 = (S^2_{\boldsymbol{j}}, 0 \le j_1, \ldots, j_M \le m_2)$ for System ϕ_2. Then, an explicit expression of the multi-state signature $\boldsymbol{s}^{(par)}$ of the new system can be obtained from the multi-state signatures $\boldsymbol{s}^1$ and $\boldsymbol{s}^2$ of Systems ϕ_1 and ϕ_2, as presented in the following theorem.

THEOREM 2.2: *The elements of $\boldsymbol{s}^{(par)} = (s_{\boldsymbol{v}}, 1 \le v_1, \ldots, v_M \le m_1+m_2)$ can be computed from $\boldsymbol{s}^1 = (s^1_{\boldsymbol{i}}, 1 \le i_1, \ldots, i_M \le m_1)$ and $\boldsymbol{s}^2 = (s^2_{\boldsymbol{j}}, 1 \le j_1, \ldots, j_M \le m_2)$ as*

$$s_{\boldsymbol{v}} = \sum_{v_1-1\le u_1\le v_1,\ldots,v_M-1\le u_M\le v_M} (-1)^{(v_1+\cdots+v_M)-(u_1+\cdots+u_M)} S_{\boldsymbol{u}},$$

where for $0 \le u_1 \le \cdots \le u_M \le m_1+m_2$,

$$S_{\boldsymbol{u}} = \sum_{\boldsymbol{i}\in W_{\boldsymbol{u}}} S^1_{\boldsymbol{i}} S^2_{\boldsymbol{u}-\boldsymbol{i}} \binom{u_1}{i_1}\prod_{r=2}^{M}\binom{u_r-u_{r-1}}{i_r-i_{r-1}}\binom{m_1+m_2-u_M}{m_1-i_M}\binom{m_1+m_2}{m_1}^{-1}, \tag{2.4}$$

with $W_{\boldsymbol{u}}$ being as defined in Theorem 2.1, and otherwise $S_{\boldsymbol{u}} = S_{min(u_1,\ldots,u_M),\ldots,min(u_{M-1},u_M),u_M}$.

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PROOF: Define the first time that the new system/System ϕ_1/System ϕ_2 enters state $M-r$ or below as $T_r/T_r^1/T_r^2$ (for $r=1,\ldots,M$). According to (2.3), we have

$$S_{\boldsymbol{u}} = \mathbb{P}\{T_1 \le X_{u_1:m_1+m_2}, \ldots, T_M \le X_{u_M:m_1+m_2}\},$$

where $X_{0:m_1+m_2}=0$ and $X_{i:m_1+m_2}$ $(i=1,\ldots,m_1+m_2)$ is the ith order statistic among lifetimes $X_1,\ldots,X_{m_1+m_2}$ of the m_1+m_2 *i.i.d.* components in the new system. Then,

$$\begin{aligned} s_{\boldsymbol{v}} &= \mathbb{P}\{T_1 = X_{v_1:m_1+m_2}, \ldots, T_M = X_{v_M:m_1+m_2}\} \\ &= \mathbb{P}\{\bar{A}^1_{v_1} - \bar{A}^1_{v_1-1}, \ldots, \bar{A}^M_{v_M} - \bar{A}^M_{v_M-1}\} \\ &= \sum_{v_1-1\le u_1\le v_1,\ldots,v_M-1\le u_M\le v_M} \left[\prod_{i=1}^{M}(-1)^{v_i-u_i}\right] \mathbb{P}\{\bar{A}^1_{u_1},\ldots,\bar{A}^M_{u_M}\} \\ &= \sum_{v_1-1\le u_1\le v_1,\ldots,v_M-1\le u_M\le v_M} (-1)^{(v_1+\cdots+v_M)-(u_1+\cdots+u_M)} S_{\boldsymbol{u}}, \end{aligned}$$

where $\bar{A}^i_{v_i}$ denotes the event that $T_i \le X_{v_i:m_1+m_2}$, with $i=1,\ldots,M$ and $v_i=0,\ldots,m_1+m_2$, that is, $\bar{A}^i_{v_i} - \bar{A}^i_{v_i-1}$ denotes the event that $T_i = X_{v_i:m_1+m_2}$, with $i=1,\ldots,M$ and $v_i=1,\ldots,m_1+m_2$.

With this relationship, to obtain the multi-state signature $\boldsymbol{s}^{(par)}$ of this new system, we only need to obtain $S_{\boldsymbol{u}}$ for all $0\le u_1\le\cdots\le u_M\le m_1+m_2$ since otherwise $S_{\boldsymbol{u}} = S_{\min(u_1,\ldots,u_M),\ldots,\min(u_{M-1},u_M),u_M}$. As in Theorem 2.1, with $0\le t_1\le\cdots\le t_M<\infty$, for System ϕ_1, we have

$$\begin{aligned} \mathbb{P}\{T_1^1\le t_1,\ldots,T_M^1\le t_M\} = &\sum_{0\le i_1\le\cdots\le i_M\le m_1} S^1_{\boldsymbol{i}} \begin{pmatrix} m_1 \\ i_1, i_2-i_1, \ldots, i_M-i_{M-1}, m_1-i_M \end{pmatrix} \\ &\times F^{i_1}(t_1) \prod_{r=1}^{M-1} [F(t_{r+1})-F(t_r)]^{i_{r+1}-i_r} \times [1-F(t_M)]^{m_1-i_M}, \end{aligned}$$

and similarly, for System ϕ_2, we have

$$\begin{aligned} \mathbb{P}\{T_1^2\le t_1,\ldots,T_M^2\le t_M\} = &\sum_{0\le j_1\le\cdots\le j_M\le m_2} S^2_{\boldsymbol{j}} \begin{pmatrix} m_2 \\ j_1, j_2-j_1, \ldots, j_M-j_{M-1}, m_2-j_M \end{pmatrix} \\ &\times F^{j_1}(t_1) \prod_{r=1}^{M-1} [F(t_{r+1})-F(t_r)]^{j_{r+1}-j_r} \times [1-F(t_M)]^{m_2-j_M}. \end{aligned}$$

Hence, for the new system, we have

$$\begin{aligned} \mathbb{P}\{T_1\le t_1,\ldots,T_M\le t_M\} &= \mathbb{P}\{\max(T_1^1,T_1^2)\le t_1,\ldots,\max(T_M^1,T_M^2)\le t_M\} \\ &= \mathbb{P}\{T_1^1\le t_1,\ldots,T_M^1\le t_M\} \times \mathbb{P}\{T_1^2\le t_1,\ldots,T_M^2\le t_M\}. \end{aligned}$$

Then, the proof gets completed as done in Theorem 2.1. ■

REMARK 2.2: *In Eq. (2.4), the subset $W_{\boldsymbol{u}}$ consists of all possible $i_1,\ldots,i_M$, given $\boldsymbol{u}$. The term $S^1_{\boldsymbol{i}}S^2_{\boldsymbol{u}-\boldsymbol{i}}$ is the probability that System ϕ_1 is in states below $M-r$ right after its*

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i_rth component failure and System ϕ_2 is in states below $M-r$ right after its $(u_r - i_r)$th component failure, for all $r = 1, \ldots, M$, and the term

$$\begin{pmatrix} u_1 \\ i_1 \end{pmatrix} \prod_{r=2}^{M} \begin{pmatrix} u_r - u_{r-1} \\ i_r - i_{r-1} \end{pmatrix} \begin{pmatrix} m_1 + m_2 - u_M \\ m_1 - i_M \end{pmatrix} \begin{pmatrix} m_1 + m_2 \\ m_1 \end{pmatrix}^{-1}$$

is exactly the same as in Theorem 2.1.

3. SERIES AND PARALLEL CONNECTION OF TWO MULTI-STATE COHERENT MODULES WITH MULTI-STATE COMPONENTS

The definition of multi-state survival signature for a multi-state coherent system with multi-state components, as given in Yi *et al.* [38], is as follows.

DEFINITION 3.1: *For a multi-state coherent system with state space $\{0, \ldots, M\}$ and m multi-state components that enter states below r $(r = 1, \ldots, M)$ at times i.i.d. from an absolutely continuous distribution $F_r(x), x \geq 0$, the multi-state survival signature (at all levels) of the system is defined as $\boldsymbol{D} = (\boldsymbol{D}^{(0)}, \ldots, \boldsymbol{D}^{(M)})$, where for any $a = 0, \ldots, M$, $\boldsymbol{D}^{(a)} = (D^{(a)}_{i_0,\ldots,i_{M-1}}, 0 \leq i_0, \ldots, i_{M-1} \leq m)$, and*

$$D^{(a)}_{i_0,\ldots,i_{M-1}} = \mathbb{P}\{L_a > t \,\big|\, m_0(t) = i_0, \ldots, m_{M-1}(t) = i_{M-1}, m_M(t) = i_M := m - \sum\nolimits_{r=0}^{M-1} i_r\}$$

is the probability that the system is in state a or above at time t if the number $m_r(t)$ of components in state r equals to i_r, for all $r = 0, \ldots, M$. Here, for $a = 0, \ldots, M$, L_a is the first time that the system enters states below a. Specifically, elements of $\boldsymbol{D}^{(0)}$ are all 1 since $L_0 = \infty$. For brevity, $\boldsymbol{D}^{(a)}$ can be rewritten as $\boldsymbol{D}^{(a)} = (D^{(a)}_{i_0,\ldots,i_{M-1}}, 0 \leq i_0 + \cdots + i_{M-1} \leq m)$ by removal of zero elements.

We now consider two independent multi-state coherent Systems ϕ_1 and ϕ_2 with state space $\{0, \ldots, M\}$. System ϕ_1 has m_1 multi-state components that enter states below r $(r = 1, \ldots, M)$ at times *i.i.d.* from an absolutely continuous distribution $F_r(x), x \geq 0$ and has a multi-state survival signature $\boldsymbol{D}^1 = (\boldsymbol{D}^{1(0)}, \ldots, \boldsymbol{D}^{1(M)})$. Similarly, System ϕ_2 has m_2 multi-state components that enter states below r $(r = 1, \ldots, M)$ at times *i.i.d.* from the same distribution $F_r(x), x \geq 0$, and has a multi-state survival signature $\boldsymbol{D}^2 = (\boldsymbol{D}^{2(0)}, \ldots, \boldsymbol{D}^{2(M)})$.

Let us now consider a series connection of Systems ϕ_1 and ϕ_2. We then obtain a new system with $m_1 + m_2$ *i.i.d.* components whose state is the smaller one among states of Systems ϕ_1 and ϕ_2. Let us denote subscripts like $i_0, \ldots, i_{M-1}$ simply by $\boldsymbol{i}$, and we shall use this form of notation throughout. Then, an explicit expression of the multi-state survival signature $\boldsymbol{D}^{(ser)} = (\boldsymbol{D}^{(0)}, \ldots, \boldsymbol{D}^{(M)})$ of the new system can be obtained from the multi-state survival signatures $\boldsymbol{D}^1 = (\boldsymbol{D}^{1(0)}, \ldots, \boldsymbol{D}^{1(M)})$ and $\boldsymbol{D}^2 = (\boldsymbol{D}^{2(0)}, \ldots, \boldsymbol{D}^{2(M)})$ of Systems ϕ_1 and ϕ_2, as presented in the following theorem.

THEOREM 3.1: *The elements of $\boldsymbol{D}^{(a)} = (D^{(a)}_{\boldsymbol{u}}, 0 \leq u_0, \ldots, u_{M-1} \leq m_1 + m_2), a = 0, \ldots, M$, can be computed from $\boldsymbol{D}^{1(a)} = (D^{1(a)}_{\boldsymbol{i}}, 0 \leq i_0, \ldots, i_{M-1} \leq m_1)$ and $\boldsymbol{D}^{2(a)} = (D^{2(a)}_{\boldsymbol{j}}, 1 \leq$*

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$j_0, \ldots, j_{M-1} \leq m_2$) *for* $0 \leq u_1 + \cdots + u_{M-1} \leq m_1 + m_2$, *as*

$$D_{\boldsymbol{u}}^{(a)} = \sum_{\boldsymbol{i} \in V_{\boldsymbol{u}}} D_{\boldsymbol{i}}^{1(a)} D_{\boldsymbol{u}-\boldsymbol{i}}^{2(a)} \prod_{r=0}^{M} \binom{u_r}{i_r} \binom{m_1+m_2}{m_1}^{-1}, \tag{3.1}$$

with $u_M = m_1 + m_2 - \sum_{r=0}^{M-1} u_r$, $i_M = m_1 - \sum_{r=0}^{M-1} i_r$ *and*

$$V_{\boldsymbol{u}} = \left\{ \boldsymbol{i} : max\left(0, \sum_{l=0}^{r} u_l - I_{\{r\neq 0\}} \sum_{l=0}^{r-1} i_l - m_2\right) \leq i_r \right.$$
$$\left. \leq min\left(m_1 - I_{\{r\neq 0\}} \sum_{l=0}^{r-1} i_l, u_r\right),\ r = 0, \ldots, M-1 \right\},$$

and otherwise $D_{\boldsymbol{u}}^{(a)} = 0$.

Proof: Define the first time that the new system/System ϕ_1/System ϕ_2 enters states below a as $L_a/L_a^1/L_a^2$ (for $a = 1, \ldots, M$). Let us also denote $L_0 = L_0^1 = L_0^2 = \infty$. As in Yi *et al.* [38], with $a = 0, \ldots, M$ and $0 \leq t < \infty$, for System ϕ_1, we have

$$\begin{aligned} R^{1(a)}(t) &= \mathbb{P}(L_a^1 > t) \\ &= \sum_{0 \leq i_0 + \cdots + i_{M-1} \leq m_1} D_{\boldsymbol{i}}^{(a)} \binom{m_1}{i_0, \ldots, i_M} [F_1(t)]^{i_0} \\ &\quad \times \prod_{r=1}^{M-1} [F_{r+1}(t) - F_r(t)]^{i_r} [1 - F_M(t)]^{i_M}, \end{aligned}$$

and similarly, for System ϕ_2, we have

$$\begin{aligned} R^{2(a)}(t) &= \mathbb{P}(L_a^2 > t) \\ &= \sum_{0 \leq j_0 + \cdots + j_{M-1} \leq m_2} D_{\boldsymbol{j}}^{(a)} \binom{m_2}{j_0, \ldots, j_M} [F_1(t)]^{j_0} \\ &\quad \times \prod_{r=1}^{M-1} [F_{r+1}(t) - F_r(t)]^{j_r} [1 - F_M(t)]^{j_M}. \end{aligned}$$

Then, with $a = 0, \ldots, M$ and $0 \leq t < \infty$, for the new system, we have

$$\begin{aligned} R^{(a)}(t) &= \mathbb{P}\{L_a > t\} \\ &= \mathbb{P}\{\min(L_a^1, L_a^2) > t\} = \mathbb{P}\{L_a^1 > t\}\mathbb{P}\{L_a^2 > t\} \\ &= \sum_{0 \leq i_0 + \cdots + i_{M-1} \leq m_1} \sum_{0 \leq j_0 + \cdots + j_{M-1} \leq m_2} D_{\boldsymbol{i}}^{1(a)} D_{\boldsymbol{j}}^{2(a)} \binom{m_1}{i_0, \ldots, i_M} \binom{m_2}{j_0, \ldots, j_M} \\ &\quad \times [F_1(t)]^{i_0 + j_0} \prod_{r=1}^{M-1} [F_{r+1}(t) - F_r(t)]^{i_r + j_r} [1 - F_M(t)]^{i_M + j_M} \end{aligned}$$

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$$= \sum_{0\le u_0+\cdots+u_{M-1}\le m_1+m_2} \sum_{\boldsymbol{i}\in V_{\boldsymbol{u}}} D_{\boldsymbol{i}}^{1(a)} D_{\boldsymbol{u}-\boldsymbol{i}}^{2(a)} \binom{m_1}{i_0,\ldots,i_M} \binom{m_2}{u_0-i_0,\ldots,u_M-i_M}$$
$$\times [F_1(t)]^{u_0} \prod_{r=1}^{M-1} [F_{r+1}(t)-F_r(t)]^{u_r} [1-F_M(t)]^{u_M}.$$

On the other hand, for the new system, we have

$$R^{(a)}(t) = \mathbb{P}\{L_a > t\}$$
$$= \sum_{0\le u_0+\cdots+u_{M-1}\le m_1+m_2} D_{\boldsymbol{u}}^{(a)} \binom{m_1+m_2}{u_0,\ldots,u_M} [F_1(t)]^{u_0}$$
$$\times \prod_{r=1}^{M-1} [F_{r+1}(t)-F_r(t)]^{u_r} [1-F_M(t)]^{u_M}.$$

Hence, the theorem proved. ■

REMARK 3.1: *In Eq. (3.1), the subset $V_{\boldsymbol{u}}$ consists of all possible $i_1,\ldots,i_M$, given $\boldsymbol{u}$. The term $D_{\boldsymbol{i}}^{1(a)} D_{\boldsymbol{u}-\boldsymbol{i}}^{2(a)}$ is the conditional probability that the lower state of System ϕ_1 and System ϕ_2 is state a or above, given that System ϕ_1 has i_r components in state r and System ϕ_2 has $(u_r - i_r)$ components in state r, for all $r = 0,\ldots,M$, and the term*

$$\prod_{r=0}^{M} \binom{u_r}{i_r} \binom{m_1+m_2}{m_1}^{-1}$$

is the conditional probability that System ϕ_1 has i_r components in state r and System ϕ_2 has $u_r - i_r$ components in state r, given that the two systems have u_r components in state r in total, for all $r = 0,\ldots,M$.

Next, let us consider a parallel connection between Systems ϕ_1 and ϕ_2. We then obtain another new system with $m_1 + m_2$ *i.i.d.* components whose state is the larger one among states of Systems ϕ_1 and ϕ_2. Then, as in Theorem 3.1, to find the multi-state survival signature $\boldsymbol{D}^{(par)} = (\boldsymbol{D}^{(0)},\ldots,\boldsymbol{D}^{(M)})$ of the new system from the multi-state signatures $\boldsymbol{D}^1 = (\boldsymbol{D}^{1(0)},\ldots,\boldsymbol{D}^{1(M)})$ and $\boldsymbol{D}^2 = (\boldsymbol{D}^{2(0)},\ldots,\boldsymbol{D}^{2(M)})$ of Systems ϕ_1 and ϕ_2, we have the following theorem.

THEOREM 3.2: *The elements of $\boldsymbol{D}^{(a)} = (D_{\boldsymbol{u}}^{(a)}, 0 \le u_0,\ldots,u_{M-1} \le m_1+m_2), a = 0,\ldots,M$, can be computed from $\boldsymbol{D}^{1(a)} = (D_{\boldsymbol{i}}^{1(a)}, 0 \le i_0,\ldots,i_{M-1} \le m_1)$ and $\boldsymbol{D}^{2(a)} = (D_{\boldsymbol{j}}^{2(a)}, 1 \le j_0,\ldots,j_{M-1} \le m_2)$ for $0 \le u_1+\cdots+u_{M-1} \le m_1+m_2$, as*

$$D_{\boldsymbol{u}}^{(a)} = 1 - \sum_{\boldsymbol{i}\in V_{\boldsymbol{u}}} [1-D_{\boldsymbol{i}}^{1(a)}][1-D_{\boldsymbol{u}-\boldsymbol{i}}^{2(a)}] \prod_{r=0}^{M} \binom{u_r}{i_r} \binom{m_1+m_2}{m_1}^{-1}, \tag{3.2}$$

with $u_M, i_M, V_{\boldsymbol{u}}$ being as defined in Theorem 3.1, and otherwise $D_{\boldsymbol{u}}^{(a)} = 0$.

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PROOF: Define the first time that the new system/System ϕ_1/System ϕ_2 enters states below a as $L_a/L_a^1/L_a^2$ $(a = 1, \ldots, M)$. Let us also denote $L_0 = L_0^1 = L_0^2 = \infty$. As in Theorem 3.1, with $a = 0, \ldots, M$ and $0 \leq t < \infty$, for the new system, we have

$$R^{(a)}(t) = \mathbb{P}\{L_a > t\} = \mathbb{P}\{\max(L_a^1, L_a^2) > t\} = 1 - \mathbb{P}\{L_a^1 \leq t\}\mathbb{P}\{L_a^2 \leq t\}.$$

Then, the proof gets completed by proceeding along the lines of Theorem 3.1. ■

REMARK 3.2: *In Eq. (3.2), the subset $V_{\boldsymbol{u}}$ consists of all possible $i_1, \ldots, i_M$, given $\boldsymbol{u}$. The term $[1 - D_{\boldsymbol{i}}^{1(a)}][1 - D_{\boldsymbol{u}-\boldsymbol{i}}^{2(a)}]$ is the conditional probability that the higher state of System ϕ_1 and System ϕ_2 is in states below a, given that System ϕ_1 has i_r components in state r and System ϕ_2 has $(u_r - i_r)$ components in state r, for all $r = 0, \ldots, M$, and the term*

$$\prod_{r=0}^{M} \begin{pmatrix} u_r \\ i_r \end{pmatrix} \begin{pmatrix} m_1 + m_2 \\ m_1 \end{pmatrix}^{-1}$$

is the same as in Theorem 3.1.

4. SIGNATURE OF RECURRENT SYSTEMS

In this section, we study some recurrent systems by making use of the results developed in the last two sections. First of all, let us consider N independent and identical coherent modules, each of which has m binary-state components with *i.i.d.* lifetimes from an absolutely continuous distribution $F(x)$ $(x \geq 0)$ and has a multi-state signature $\boldsymbol{s}^1 = (s_{\boldsymbol{i}}^1, 1 \leq i_1, \ldots, i_M \leq m)$.

Next, let us consider a recurrent system with state space $\{0, \ldots, M\}$ that is composed of N modules. Let us denote the multi-state survival signature of the connection as $\boldsymbol{D}^2 = (\boldsymbol{D}^{2(0)}, \ldots, \boldsymbol{D}^{2(M)})$, where $\boldsymbol{D}^{2(a)} = (D_{\boldsymbol{j}}^{2(a)}, 0 \leq j_0, \ldots, j_{M-1} \leq N)$, $a = 0, \ldots, M$, and $D_{\boldsymbol{j}}^{2(a)}$ is the probability that the recurrent system is in state a or above, given that $j_M = N - \sum_{r=0}^{M-1} j_r$ and there are j_r modules in state r, for all $r = 0, \ldots, M$. To study the multi-state survival signature $\boldsymbol{D}^{(rec)} = (\boldsymbol{D}^{(0)}, \ldots, \boldsymbol{D}^{(M)})$ of the recurrent system, we shall use the following notation:

$$d_u^{(M)} = D_u^{(M)}, \; d_u^{(a)} = D_u^{(a)} - D_u^{(a+1)}, \; a = 0, \ldots, M-1, \tag{4.1}$$

for $u = 0, \ldots, Nm$. Here in Eq. (4.1), $D_u^{(a)}, d_u^{(a)}, a = 0, \ldots, M$ are the probabilities that the new system is in state a or above and in state a right after the uth component failure, respectively. This notation can be applied to any multi-state system with binary-state components; for example, we have $D_k^{1(a)}, d_k^{1(a)}$, $a = 0, \ldots, M$, for each module with $D_k^{1(a)} = \sum_{i_{M-a+1} > k, 1 \leq i_1, \ldots, i_M \leq m} s_{\boldsymbol{i}}^1$, $a = 0, \ldots, M$ and $D_k^{1(0)} = 1$. With this notation, an explicit expression of the multi-state survival signature $\boldsymbol{D}^{(rec)} = (\boldsymbol{D}^{(0)}, \ldots, \boldsymbol{D}^{(M)})$ of the new system can be obtained from the multi-state signature $\boldsymbol{s}^1 = (s_{\boldsymbol{i}}^1, 1 \leq i_1, \ldots, i_M \leq m)$ of each module and the multi-state survival signature $\boldsymbol{D}^2 = (\boldsymbol{D}^{2(0)}, \ldots, \boldsymbol{D}^{2(M)})$ of the connection, as presented in the following theorem.

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THEOREM 4.1: *The elements of* $\boldsymbol{D}^{(a)} = (D_u^{(a)}, 0 \le u \le Nm)$, $a = 0, \ldots, M$, *can be computed from* $\boldsymbol{s}^1 = (s_{\boldsymbol{i}}^1, 1 \le i_1, \ldots, i_M \le m)$ *and* $\boldsymbol{D}^{2(a)} = (D_{\boldsymbol{j}}^{2(a)}, 0 \le j_0, \ldots, j_{M-1} \le N)$, *as*

$$D_u^{(a)} = \left(\begin{array}{c} Nm \\ u \end{array} \right)^{-1} \sum_{0 \le j_0 + \cdots + j_{M-1} \le N} D_{\boldsymbol{j}}^{2(a)} \left(\begin{array}{c} N \\ j_0, \ldots, j_M \end{array} \right) \sum_{\boldsymbol{i} \in \Theta_{u,\boldsymbol{j}}} \prod_{r=0}^{M} \left\{ \left(\begin{array}{c} j_r \\ i_{r0}, \ldots, i_{rm} \end{array} \right) \right.$$
$$\left. \times \prod_{k=0}^{m} \left[d_k^{1(r)} \left(\begin{array}{c} m \\ k \end{array} \right) \right]^{i_{rk}} \right\}, \tag{4.2}$$

where $u = 0, \ldots, Nm$, $j_M = N - \sum_{l=0}^{M-1} j_l$, $i_{rm} = j_r - \sum_{l=0}^{m-1} i_{rl}$ *and*

$$\Theta_{u,\boldsymbol{j}} = \{ \boldsymbol{i} = (\boldsymbol{i}_r, 0 \le i_{r0} + \cdots + i_{r,m-1} \le j_r,\ r = 0, \ldots, M) :$$
$$i_{r0} + \cdots + i_{rm} = j_r,\ r = 0, \ldots, M \text{ and } \sum_{r=0}^{M} \sum_{k=0}^{m} k i_{rk} = u \Big\}.$$

PROOF: Define the first time that the new system/module e ($e = 1, \ldots, N$) enters states below a as L_a / L_a^e (for $a = 1, \ldots, M$). Let us also denote $L_0 = L_0^e = \infty$ and $L_{M+1}^e = 0$. With $0 \le t < \infty$ and $r = 0, \ldots, M+1$, as in Theorem 3.1, for module e, we have

$$R^{1(r)}(t) = \mathbb{P}\{L_r^e > t\} = \sum_{k=0}^{m} D_k^{1(r)} \left(\begin{array}{c} m \\ k \end{array} \right) F^k(t)[1 - F(t)]^{m-k},$$

which means that with $a = 0, \ldots, M$ and $0 \le t < \infty$, for the new system, we have

$$\mathbb{P}\{L_a > t\} = \sum_{0 \le j_0 + \cdots + j_{M-1} \le N} D_{\boldsymbol{j}}^{2(a)} \left(\begin{array}{c} N \\ j_0, \ldots, j_M \end{array} \right) \prod_{r=0}^{M} [R^{1(r)}(t) - R^{1(r+1)}(t)]^{j_r}$$
$$= \sum_{0 \le j_0 + \cdots + j_{M-1} \le N} D_{\boldsymbol{j}}^{2(a)} \left(\begin{array}{c} N \\ j_0, \ldots, j_M \end{array} \right) \prod_{r=0}^{M} \left[\sum_{k=0}^{m} d_k^{1(r)} \left(\begin{array}{c} m \\ k \end{array} \right) F^k(t) \bar{F}^{m-k}(t) \right]^{j_r},$$

where $\bar{F}(t) = 1 - F(t)$. Let us denote $\Theta_{\boldsymbol{j}} = \bigcup_u \Theta_{u,\boldsymbol{j}}$, which means that

$$\Theta_{\boldsymbol{j}} = \{ \boldsymbol{i} = (\boldsymbol{i}_r, 0 \le i_{r0} + \cdots + i_{r,m-1} \le j_r,\ r = 0, \ldots, M) :$$
$$i_{r0} + \cdots + i_{rm} = j_r,\ r = 0, \ldots, M \}.$$

Then, we have

$$\mathbb{P}\{L_a > t\}$$
$$= \sum_{0 \le j_0 + \cdots + j_{M-1} \le N} D_{\boldsymbol{j}}^{2(a)} \left(\begin{array}{c} N \\ j_0, \ldots, j_M \end{array} \right)$$
$$\times \sum_{\boldsymbol{i} \in \Theta_{\boldsymbol{j}}} \prod_{r=0}^{M} \left\{ \left(\begin{array}{c} j_r \\ i_{r0}, \ldots, i_{rm} \end{array} \right) \prod_{k=0}^{m} \left[d_k^{1(r)} \left(\begin{array}{c} m \\ k \end{array} \right) F^k(t) \bar{F}^{m-k}(t) \right]^{i_{rk}} \right\}.$$

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After some algebraic manipulations, we get

$$\begin{aligned}\mathbb{P}\{L_a > t\} &= \sum_{0\le j_0+\cdots+j_{M-1}\le N} D_{\boldsymbol{j}}^{2(a)} \binom{N}{j_0,\ldots,j_M} \\ &\quad\times \sum_{\boldsymbol{i}\in\Theta_{\boldsymbol{j}}} \prod_{r=0}^{M} \left\{ \binom{j_r}{i_{r0},\ldots,i_{rm}} \prod_{k=0}^{m} \left[d_k^{1(r)} \binom{m}{k} \right]^{i_{rk}} \right\} \\ &\quad\times F^{\sum_{r=0}^{M}\sum_{k=0}^{m} k i_{rk}}(t) \bar{F}^{\sum_{r=0}^{M}\sum_{k=0}^{m}(m-k) i_{rk}}(t) \\ &= \sum_{0\le u\le Nm} \sum_{0\le j_0+\cdots+j_{M-1}\le N} D_{\boldsymbol{j}}^{2(a)} \binom{N}{j_0,\ldots,j_M} \\ &\quad\times \sum_{\boldsymbol{i}\in\Theta_{u,\boldsymbol{j}}} \prod_{r=0}^{M} \left\{ \binom{j_r}{i_{r0},\ldots,i_{rm}} \prod_{k=0}^{m} \left[d_k^{1(r)} \binom{m}{k} \right]^{i_{rk}} \right\} \\ &\quad\times F^{u}(t) \bar{F}^{Nm-u}(t).\end{aligned}$$

On the other hand, for the new system, we also have

$$\mathbb{P}\{L_a > t\} = \sum_{0\le u\le Nm} D_u^{(a)} \binom{Nm}{u} F^u(t) \bar{F}^{Nm-u}(t).$$

Hence, the theorem proved. ■

REMARK 4.1: *In Eq. (4.2), all cases are considered for the system to be in state a or above when there are j_r modules in state r and among which there are i_{rk}, $k = 0, 1, \ldots, m$, modules that have k failed components, for all $r = 0, \ldots, M$.*

Similarly, let us now consider N independent and identical coherent modules, each of which has m multi-state components that enter states below r $(r = 1, \ldots, M)$ at times *i.i.d.* from an absolutely continuous distribution $F_r(x), x \ge 0$, and has a multi-state survival signature $(\boldsymbol{D}^{1(0)}, \ldots, \boldsymbol{D}^{1(M)})$.

Then, consider a recurrent system with state space $\{0, \ldots, M\}$ that is composed of N modules and denote the multi-state survival signature of the connection as $\boldsymbol{D}^2 = (\boldsymbol{D}^{2(0)}, \ldots, \boldsymbol{D}^{2(M)})$, where $\boldsymbol{D}^{2(a)} = (D_{\boldsymbol{j}}^{2(a)}, 0 \le j_0, \ldots, j_{M-1} \le N)$, $a = 0, \ldots, M$, and $D_{\boldsymbol{j}}^{2(a)}$ is the probability that the recurrent system is in state a or above, given that $j_M = N - \sum_{r=0}^{M-1} j_r$ and there are j_r modules in state r, for all $r = 0, \ldots, M$. To find the multi-state survival signature $\boldsymbol{D}^{(\mathrm{rec})} = (\boldsymbol{D}^{(0)}, \ldots, \boldsymbol{D}^{(M)})$ of the recurrent system, we shall use the following notation:

$$d_{\boldsymbol{u}}^{(M)} = D_{\boldsymbol{u}}^{(M)},\ d_{\boldsymbol{u}}^{(a)} = D_{\boldsymbol{u}}^{(a)} - D_{\boldsymbol{u}}^{(a+1)},\ a = 0, \ldots, M-1. \tag{4.3}$$

In Eq. (4.3), $d_{\boldsymbol{u}}^{(a)}$, $a = 0, \ldots, M$, is the conditional probability that the new system is in state a, given that there are u_r components in state r, for all $r = 0, \ldots, M-1$. This notation can be applied to any multi-state survival signature; for example, we have $d_{\boldsymbol{i}}^{1(a)}$, $a = 0, \ldots, M$, for each module, and $d_{\boldsymbol{j}}^{2(a)}$, $a = 0, \ldots, M$, for the connection. With this notation, an explicit expression of the multi-state survival signature $\boldsymbol{D}^{(ser)} = (\boldsymbol{D}^{(0)}, \ldots, \boldsymbol{D}^{(M)})$ of the new system can be obtained from the multi-state survival

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signature $\boldsymbol{D}^1 = (\boldsymbol{D}^{1(0)}, \ldots, \boldsymbol{D}^{1(M)})$ of each module and the multi-state survival signature $\boldsymbol{D}^2 = (\boldsymbol{D}^{2(0)}, \ldots, \boldsymbol{D}^{2(M)})$ of the connection, as presented in the following theorem.

THEOREM 4.2: *The elements of* $\boldsymbol{D}^{(a)} = (D^{(a)}_{\boldsymbol{u}}, 0 \le u_0, \ldots, u_{M-1} \le Nm)$, $a = 0, \ldots, M$ *can be computed from* $\boldsymbol{D}^{1(r)} = (D^{1(r)}_{\boldsymbol{i}}, 0 \le i_0, \ldots, i_{M-1} \le m)$, $r = 0, \ldots, M$ *and* $\boldsymbol{D}^{2(a)} = (D^{2(a)}_{\boldsymbol{j}}, 0 \le j_0, \ldots, j_{M-1} \le N)$ *for* $0 \le u_0 + \cdots + u_{M-1} \le Nm$, *as*

$$
\begin{aligned}
D^{(a)}_{\boldsymbol{u}} = {} & \binom{Nm}{u_0, \ldots, u_M}^{-1} \sum_{0 \le j_0 + \cdots + j_{M-1} \le N} D^{2(a)}_{\boldsymbol{j}} \binom{N}{j_0, \ldots, j_M} \sum_{\boldsymbol{l} \in \Omega_{\boldsymbol{u},\boldsymbol{j}}} \prod_{r=0}^{M} \\
& \times \left\{ \binom{j_r}{l_{\boldsymbol{i}_r}, 0 \le i_{r0} + \cdots + i_{r,M-1} \le m} \right. \\
& \left. \times \prod_{0 \le i_{r0} + \cdots + i_{r,M-1} \le m} \left[d^{1(r)}_{\boldsymbol{i}_r} \binom{m}{i_{r0}, \ldots, i_{rM}} \right]^{l_{\boldsymbol{i}_r}} \right\}, \qquad (4.4)
\end{aligned}
$$

where $j_M = N - \sum_{l=0}^{M-1} j_l$, $i_{rM} = m - \sum_{l=0}^{M-1} i_{rl}$ *and*

$$
\begin{aligned}
\Omega_{\boldsymbol{u},\boldsymbol{j}} = \Big\{ \boldsymbol{l} = (l_{\boldsymbol{i}_r}, 0 \le i_{r0} + \cdots + i_{r,M-1} \le m,\ r = 0, \ldots, M) : & \sum_{0 \le i_{r0} + \cdots + i_{r,M-1} \le m} l_{\boldsymbol{i}_r} = j_r \\
\text{for } r = 0, \ldots, M, \ \sum_{r=0}^{M} & \sum_{0 \le i_{r0} + \cdots + i_{r,M-1} \le m} i_{rl} l_{\boldsymbol{i}_r} = u_l \quad \text{for } l = 0, \ldots, M \Big\},
\end{aligned}
$$

and otherwise, $D^{(a)}_{\boldsymbol{u}} = 0$.

PROOF: Define the first time that the new system/module e ($e = 1, \ldots, N$) enters states below a as L_a/L^e_a (for $a = 1, \ldots, M$). Let us also denote $L_0 = L^e_0 = \infty$ and $L^e_{M+1} = 0$. With $0 \le t < \infty$ and $r = 0, \ldots, M+1$, for module e, we have

$$
\begin{aligned}
R^{1(r)}(t) &= \mathbb{P}(L^e_r > t) \\
&= \sum_{0 \le i_{r0} + \cdots + i_{r,M-1} \le m} D^{1(r)}_{\boldsymbol{i}_r} \binom{m}{i_{r0}, \ldots, i_{rM}} [F_1(t)]^{i_{r0}} \\
&\quad \times \prod_{l=1}^{M-1} [F_{l+1}(t) - F_l(t)]^{i_{rl}} [1 - F_M(t)]^{i_{rM}},
\end{aligned}
$$

which means that with $a = 0, \ldots, M$, for the new system, we have

$$
\mathbb{P}\{L_a > t\} = \sum_{0 \le j_0 + \cdots + j_{M-1} \le N} D^{2(a)}_{\boldsymbol{j}} \binom{N}{j_0, \ldots, j_M} \prod_{r=0}^{M} [R^{1(r)}(t) - R^{1(r+1)}(t)]^{j_r}
$$

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$$= \sum_{0\le j_0+\cdots+j_{M-1}\le N} D_{\boldsymbol{j}}^{2(a)} \begin{pmatrix} N \\ j_0,\ldots,j_M \end{pmatrix} \times \prod_{r=0}^{M} \left\{ \sum_{0\le i_{r0}+\cdots+i_{r,M-1}\le m} d_{\boldsymbol{i}_r}^{1(r)} \begin{pmatrix} m \\ i_{r0},\ldots,i_{rM} \end{pmatrix} [F_1(t)]^{i_{r0}} \prod_{l=1}^{M-1} [F_{l+1}(t)-F_l(t)]^{i_{rl}} [1-F_M(t)]^{i_{rM}} \right\}^{j_r}.$$

Let us denote that $\Omega_{\boldsymbol{j}} = \bigcup_{\boldsymbol{u}} \Omega_{\boldsymbol{u},\boldsymbol{j}}$, which means

$$\Omega_{\boldsymbol{j}} = \left\{ \boldsymbol{l} = (l_{\boldsymbol{i}_r}, 0 \le i_{r0}+\cdots+i_{r,M-1}\le m,\ r=0,\ldots,M) : \sum_{0\le i_{r0}+\cdots+i_{r,M-1}\le m} l_{\boldsymbol{i}_r} = j_r,\ r=0,\ldots,M \right\}.$$

We then have

$$\mathbb{P}\{L_a > t\} = \sum_{0\le j_0+\cdots+j_{M-1}\le N} D_{\boldsymbol{j}}^{2(a)} \begin{pmatrix} N \\ j_0,\ldots,j_M \end{pmatrix} \times \sum_{\boldsymbol{l}\in\Omega_{\boldsymbol{j}}} \prod_{r=0}^{M} \left\{ \begin{pmatrix} j_r \\ l_{\boldsymbol{i}_r}, 0\le i_{r0},\ldots,i_{r,M-1}\le m \end{pmatrix} \times \prod_{0\le i_{r0}+\cdots+i_{r,M-1}\le m} \left[d_{\boldsymbol{i}_r}^{1(r)} \begin{pmatrix} m \\ i_{r0},\ldots,i_{rM} \end{pmatrix} [F_1(t)]^{i_{r0}} \prod_{l=1}^{M-1} [F_{l+1}(t)-F_l(t)]^{i_{rl}} [1-F_M(t)]^{i_{rM}} \right]^{l_{\boldsymbol{i}_r}} \right\}.$$

After some algebraic manipulations, we get

$$\mathbb{P}\{L_a > t\} = \sum_{0\le j_0+\cdots+j_{M-1}\le N} D_{\boldsymbol{j}}^{2(a)} \begin{pmatrix} N \\ j_0,\ldots,j_M \end{pmatrix} \sum_{\boldsymbol{l}\in\Omega_{\boldsymbol{j}}} \prod_{r=0}^{M} \left\{ \begin{pmatrix} j_r \\ l_{\boldsymbol{i}_r}, 0\le i_{r0},\ldots,i_{r,M-1}\le m \end{pmatrix} \times \prod_{0\le i_{r0}+\cdots+i_{r,M-1}\le m} \left[d_{\boldsymbol{i}_r}^{1(r)} \begin{pmatrix} m \\ i_{r0},\ldots,i_{rM} \end{pmatrix} \right]^{l_{\boldsymbol{i}_r}} \right\} [F_1(t)]^{\sum_{r=0}^{M}\sum_{0\le i_{r0},\ldots,i_{r,M-1}\le m} i_{r0} l_{\boldsymbol{i}_r}} \times \prod_{l=1}^{M-1} [F_{l+1}(t)-F_l(t)]^{\sum_{r=0}^{M}\sum_{0\le i_{r0},\ldots,i_{r,M-1}\le m} i_{rl} l_{\boldsymbol{i}_r}} [1-F_M(t)]^{\sum_{r=0}^{M}\sum_{0\le i_{r0},\ldots,i_{r,M-1}\le m} i_{rM} l_{\boldsymbol{i}_r}}$$

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$$= \sum_{0\le u_0+\cdots+u_{M-1}\le Nm} \sum_{0\le j_0+\cdots+j_{M-1}\le N} D_{\boldsymbol{j}}^{2(a)} \begin{pmatrix} N \\ j_0,\ldots,j_M \end{pmatrix}$$
$$\times \sum_{\boldsymbol{l}\in\Omega_{\boldsymbol{u},\boldsymbol{j}}} \prod_{r=0}^{M} \left\{ \begin{pmatrix} j_r \\ l_{\boldsymbol{i}_r}, 0\le i_{r0},\ldots,i_{r,M-1}\le m \end{pmatrix} \right.$$
$$\left. \prod_{0\le i_{r0}+\cdots+i_{r,M-1}\le m} \left[d_{\boldsymbol{i}_r}^{1(r)} \begin{pmatrix} m \\ i_{r0},\ldots,i_{rM} \end{pmatrix} \right]^{l_{\boldsymbol{i}_r}} \right\} [F_1(t)]^{u_1}$$
$$\times \prod_{l=1}^{M-1} [F_{l+1}(t)-F_l(t)]^{u_l} [1-F_M(t)]^{u_M}.$$

On the other hand, for the new system, we also have

$$\mathbb{P}\{L_a > t\} = \sum_{0\le u_0+\cdots+u_{M-1}\le Nm} D_{\boldsymbol{u}}^{(a)} \begin{pmatrix} Nm \\ u_0,\ldots,u_M \end{pmatrix} [F_1(t)]^{u_0}$$
$$\times \prod_{l=1}^{M-1} [F_{l+1}(t)-R_l(t)]^{u_l} [1-F_M(t)]^{u_M}.$$

Hence, the theorem proved. ■

REMARK 4.2: *In Eq. (4.4), all cases are considered for the system to be in state a or above when there are j_r modules in state r and among which there are $l_{\boldsymbol{i}_r}$ modules that have i_{rk} components in state k ($k = 0,\ldots,M$), for all $r = 0,\ldots,M$.*

5. ILLUSTRATIVE EXAMPLES

In this section, the theoretical results obtained in the preceding sections are illustrated with some examples. Specifically, Example 5.1 illustrates the results in Section 2, Example 5.2 illustrates the results in Section 3, while Examples 5.3 and 5.4 are for illustrating the results of Section 4.

EXAMPLE 5.1: *Consider two multi-state coherent systems ϕ and $\tilde{\phi}$ with state space $\{0,1,2,3\}$, and suppose each of them has $3N$ ($N = 1,2,\ldots$) i.i.d. binary-state components. With $x_1,x_2,\ldots,x_{3N} \in \{0,1\}$, the structure functions of the two systems are*

$$\phi(x_1,\ldots,x_{3N}) = \min(x_1+x_2+x_3, x_4+x_5+x_6,\ldots,x_{3N-2}+x_{3N-1}+x_{3N}),$$
$$\tilde{\phi}(x_1,\ldots,x_{3N}) = \max(x_1+x_2+x_3, x_4+x_5+x_6,\ldots,x_{3N-2}+x_{3N-1}+x_{3N}).$$

(See Figure 1 for the structure block diagrams of the two systems.) Then, System ϕ can be regarded as a series connection of N multi-state systems with structure functions $\phi_i(x_{3i-2},x_{3i-1},x_{3i}) = x_{3i-2}+x_{3i-1}+x_{3i}$, $i = 1,2,\ldots,N$, while System $\tilde{\phi}$ can be regarded as a parallel connection of them. The multi-state signatures $\boldsymbol{s}^i = (s^i_{i_1,i_2,i_3}, 1\le i_1,i_2,i_3\le 3)$ of System ϕ_i, $i = 1,\ldots,N$, are the same, that is, $s^i_{1,2,3} = 1$ and $s^i_{i_1,i_2,i_3} = 0$ otherwise.

Let us now consider specifically the case $N = 2$. For System ϕ, using Theorem 2.1, we have all nonzero elements in its multi-state tail signature $\bar{\boldsymbol{S}} = (\bar{S}_{u_1,u_2,u_3}, 0\le u_1,u_2,u_3\le 6)$

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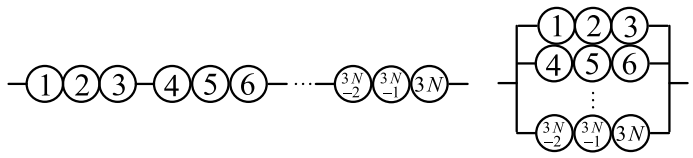


FIGURE **1.** Structure block diagrams for Systems ϕ (left) and $\tilde{\phi}$ (right) in Example 5.1.

to be

$$\bar{S}_{0,0,0} = \bar{S}_{0,1,0} = \bar{S}_{0,0,1} = \bar{S}_{0,1,1} = \bar{S}_{0,0,2} = \bar{S}_{0,1,2} = 1,\ \bar{S}_{0,0,3} = \bar{S}_{0,1,3} = 0.9,$$
$$\bar{S}_{0,2,0} = \bar{S}_{0,2,1} = \bar{S}_{0,2,2} = \bar{S}_{0,2,3} = \bar{S}_{0,0,4} = \bar{S}_{0,1,4} = 0.6,\ \bar{S}_{0,2,4} = 0.4,$$

which gives the multi-state signature of System ϕ *as* $\boldsymbol{s} = (s_{i_1,i_2,i_3}, 1 \le i_1, i_2, i_3 \le 6)$ *with all nonzero elements as follows:*

$$s_{1,2,3} = s_{1,2,4} = 0.1,\ s_{1,3,4} = s_{1,2,5} = 0.2,\ s_{1,3,5} = 0.4.$$

This means that System ϕ *enters state* 2 *at the first component failure for sure, enters state* 1 *at the second/third component failure with probability* 0.4/0.6 *and enters state* 0 *at the third/fourth/fifth component failure with probability* 0.1/0.3/0.6*, which seems reasonable with regard to Figure 1.*

Similarly, for System $\tilde{\phi}$*, using Theorem 2.2, we have all nonzero elements in its multi-state cumulative signature* $\boldsymbol{S} = (S_{u_1,u_2,u_3}, 0 \le u_1, u_2, u_3 \le 6)$ *as follows:*

$$S_{2,4,6} = 0.4, S_{2,5,6} = S_{2,6,6} = S_{3,4,6} = S_{4,4,6} = S_{5,4,6} = S_{6,4,6} = 0.6,$$
$$S_{3,5,6} = S_{3,6,6} = 0.9, S_{4,5,6} = S_{4,6,6} = S_{5,5,6} = S_{5,6,6} = S_{6,5,6} = S_{6,6,6} = 1,$$

which gives the multi-state signature of System $\tilde{\phi}$ *as* $\tilde{\boldsymbol{s}} = (\tilde{s}_{i_1,i_2,i_3}, 1 \le i_1, i_2, i_3 \le 6)$ *with all nonzero elements as follows:*

$$\tilde{s}_{2,4,6} = 0.4, \tilde{s}_{2,5,6} = \tilde{s}_{3,4,6} = 0.2, \tilde{s}_{3,5,6} = \tilde{s}_{4,5,6} = 0.1.$$

This means that System $\tilde{\phi}$ *enters state* 2 *at the second/third/fourth component failure with probability* 0.6/0.3/0.1*, enters state* 1 *at the fourth/fifth component failure with probability* 0.6/0.4 *and enters state* 0 *at the sixth component failure for sure, which seems reasonable with regard to Figure 1.*

For $N = 2$*, comparing the multi-state signatures* $\boldsymbol{s}$ *and* $\tilde{\boldsymbol{s}}$ *of System* ϕ *and System* $\tilde{\phi}$*, we observe the fact that parallel connection is better than series connection for multi-state structures. Besides, to show efficiency of the module method discussed in this paper, we compare its average computational times in ten runs for multi-state signatures of System* ϕ *and System* $\tilde{\phi}$ *for* $N = 2, 3, 4, 5$ *with the traditional method directly based on the definition; see Tables 2 and 3 for results obtained on a PC with Inter Core i5-7200 CPU and 8 GB RAM. It can be seen from these tables that the advantage of the module method becomes more evident when the system size gets larger, which is understandable since the direct method is time-consuming for large* N *(in fact, there are* $(3N)! = 1307674368000$ *cases to deal with for* $N = 5$ *here).*

EXAMPLE 5.2: *Consider two multi-state coherent systems* ϕ *and* $\tilde{\phi}$ *with state space* $\{0, 1, 2\}$*, and suppose each of them has four i.i.d. multi-state components. With* $x_1, x_2, x_3, x_4 \in \{0, 1, 2\}$*, the structure functions of the two systems are as follows:*

$$\phi(x_1, x_2, x_3, x_4) = min(max(x_1, x_2), max(x_3, x_4)),$$
$$\tilde{\phi}(x_1, x_2, x_3, x_4) = max(min(x_1, x_2), min(x_3, x_4)).$$

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TABLE 2. Computational times for multi-state signatures of System ϕ in Example 5.1 for $N = 2, 3, 4, 5$ (s: second, min: minute)

N	Nonzero elements in computational results	Module method	Direct method
2	$s_{1,2,3} = s_{1,2,4} = 0.1, s_{1,3,4} = s_{1,2,5} = 0.2,$ $s_{1,3,5} = 0.4$	0.0124 s	0.0088 s
3	$s_{1,2,3} \approx s_{1,2,4} \approx 0.0357, s_{1,3,4} \approx 0.0714,$ $s_{1,2,5} \approx 0.0500, s_{1,3,5} \approx 0.1000,$ $s_{1,4,5} \approx s_{1,2,6} \approx s_{1,2,7} \approx 0.0643,$ $s_{1,3,6} \approx s_{1,4,6} \approx s_{1,3,7} \approx s_{1,4,7} \approx 0.1286$	0.0456 s	1.4214 s
4	$s_{1,2,3} = s_{1,2,4} \approx 0.0182, s_{1,3,4} \approx 0.0364,$ $s_{1,2,5} \approx 0.0227, s_{1,3,5} \approx 0.0455,$ $s_{1,4,5} \approx 0.0409, s_{1,2,6} \approx 0.0292, s_{1,3,6} \approx 0.0584,$ $s_{1,4,6} \approx 0.0643,$ $s_{1,5,6} = s_{1,2,9} \approx 0.0234, s_{1,2,7} = s_{1,2,8} \approx 0.0351,$ $s_{1,3,7} = s_{1,3,8} \approx 0.0701,$ $s_{1,4,7} \approx 0.0818, s_{1,5,7} = s_{1,3,9} \approx 0.0468,$ $s_{1,4,8} \approx 0.0842,$ $s_{1,5,8} = s_{1,4,9} \approx 0.0561, s_{1,5,9} \approx 0.0374$	0.1838 s	>30 min
5	$s_{1,2,3} = s_{1,2,4} \approx 0.0110, s_{1,3,4} \approx 0.0220,$ $s_{1,2,5} \approx 0.0130, s_{1,3,5} \approx 0.0260,$ $s_{1,4,5} \approx 0.0270, s_{1,2,6} \approx 0.0162, s_{1,3,6} \approx 0.0324,$ $s_{1,4,6} \approx 0.0378,$ $s_{1,5,6} \approx 0.0216, s_{1,2,7} \approx 0.0198, s_{1,3,7} \approx 0.0396,$ $s_{1,4,7} \approx 0.0486,$ $s_{1,5,7} = s_{1,3,10} \approx 0.0360, s_{1,6,7} = s_{1,2,11} \approx 0.0090,$ $s_{1,2,8} = s_{1,2,9} \approx 0.0225,$ $s_{1,3,8} = s_{1,3,9} \approx 0.0450, s_{1,4,8} \approx 0.0556,$ $s_{1,5,8} \approx 0.0468,$ $s_{1,6,8} = s_{1,2,10} = s_{1,3,11} \approx 0.0180,$ $s_{1,4,9} \approx 0.0574,$ $s_{1,5,9} \approx 0.0498, s_{1,6,9} = s_{1,4,11} \approx 0.0231,$ $s_{1,4,10} \approx 0.0462,$ $s_{1,5,10} \approx 0.0411, s_{1,6,10} = s_{1,5,11} \approx 0.0205,$ $s_{1,6,11} \approx 0.0103$	0.3308 s	>30 min

(See Figure 2 for the structure block diagrams of the two systems.) Then, System ϕ can be regarded as a series connection of two systems with structure functions $\phi_{11}(x_1, x_2) = max(x_1, x_2)$ and $\phi_{12}(x_3, x_4) = max(x_3, x_4)$. The multi-state survival signatures of the two parallel systems are the same, namely,

$$(\boldsymbol{D}^{1(0)}, \boldsymbol{D}^{1(1)}, \boldsymbol{D}^{1(2)}) = \begin{pmatrix} D_{2,0}^{1(0)} & D_{1,1}^{1(0)} & D_{0,2}^{1(0)} & D_{1,0}^{1(0)} & D_{0,1}^{1(0)} & D_{0,0}^{1(0)} \\ D_{2,0}^{1(1)} & D_{1,1}^{1(1)} & D_{0,2}^{1(1)} & D_{1,0}^{1(1)} & D_{0,1}^{1(1)} & D_{0,0}^{1(1)} \\ D_{2,0}^{1(2)} & D_{1,1}^{1(2)} & D_{0,2}^{1(2)} & D_{1,0}^{1(2)} & D_{0,1}^{1(2)} & D_{0,0}^{1(2)} \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}^T.$$

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TABLE 3. Computational times for multi-state signatures of System $\tilde{\phi}$ in Example 5.1 for $N = 2, 3, 4, 5$ (s: second, min: minute)

N	Nonzero elements in computational results	Module method	Direct method
2	$\tilde{s}_{2,4,6} = 0.4, \tilde{s}_{2,5,6} = \tilde{s}_{3,4,6} = 0.2, \tilde{s}_{3,5,6} = \tilde{s}_{4,5,6} = 0.1$	0.0124 s	0.0063 s
3	$\tilde{s}_{3,6,9} = \tilde{s}_{3,7,9} = \tilde{s}_{4,6,9} = \tilde{s}_{4,7,9} \approx 0.1286$, $\tilde{s}_{3,8,9} = \tilde{s}_{4,8,9} = \tilde{s}_{5,6,9} \approx 0.0643$, $\tilde{s}_{5,7,9} \approx 0.1000, \tilde{s}_{5,8,9} \approx 0.0500$, $\tilde{s}_{6,7,9} \approx 0.0714, \tilde{s}_{6,8,9} = \tilde{s}_{7,8,9} \approx 0.0357$	0.0457 s	1.4165 s
4	$\tilde{s}_{4,8,12} \approx 0.0374, \tilde{s}_{4,9,12} = \tilde{s}_{5,8,12} \approx 0.0561$, $\tilde{s}_{4,10,12} = \tilde{s}_{6,8,12} \approx 0.0468$, $\tilde{s}_{5,9,12} \approx 0.0842, \tilde{s}_{6,9,12} \approx 0.0818$, $\tilde{s}_{5,10,12} = \tilde{s}_{6,10,12} \approx 0.0701$, $\tilde{s}_{5,11,12} = \tilde{s}_{6,11,12} \approx 0.0351$, $\tilde{s}_{4,11,12} = \tilde{s}_{7,8,12} \approx 0.0234, \tilde{s}_{7,9,12} \approx 0.0643$, $\tilde{s}_{7,10,12} \approx 0.0584, \tilde{s}_{7,11,12} \approx 0.0292$, $\tilde{s}_{8,9,12} \approx 0.0409, \tilde{s}_{8,10,12} \approx 0.0455$, $\tilde{s}_{8,11,12} \approx 0.0227, \tilde{s}_{9,10,12} \approx 0.0364$, $\tilde{s}_{9,11,12} = \tilde{s}_{10,11,12} \approx 0.0182$	0.1847 s	>30 min
5	$\tilde{s}_{5,10,15} \approx 0.0103, \tilde{s}_{5,11,15} = \tilde{s}_{6,10,15} \approx 0.0205$, $\tilde{s}_{5,12,15} = \tilde{s}_{7,10,15} \approx 0.0231$, $\tilde{s}_{5,13,15} = \tilde{s}_{8,10,15} = \tilde{s}_{6,14,15} \approx 0.0180$, $\tilde{s}_{5,14,15} = \tilde{s}_{9,10,15} \approx 0.0090$, $\tilde{s}_{6,11,15} \approx 0.0411, \tilde{s}_{6,12,15} \approx 0.0462$, $\tilde{s}_{6,13,15} = \tilde{s}_{9,11,15} \approx 0.0360$, $\tilde{s}_{7,11,15} \approx 0.0498, \tilde{s}_{7,12,15} \approx 0.0574$, $\tilde{s}_{7,13,15} = \tilde{s}_{8,13,15} \approx 0.0450$, $\tilde{s}_{7,14,15} = \tilde{s}_{8,14,15} \approx 0.0225$, $\tilde{s}_{8,11,15} \approx 0.0468, \tilde{s}_{8,12,15} \approx 0.0566$, $\tilde{s}_{9,12,15} \approx 0.0486, \tilde{s}_{9,13,15} \approx 0.0396$, $\tilde{s}_{9,14,15} \approx 0.0198, \tilde{s}_{10,11,15} \approx 0.0216$, $\tilde{s}_{10,12,15} \approx 0.0378, \tilde{s}_{10,13,15} \approx 0.0324$, $\tilde{s}_{10,14,15} \approx 0.0162$, $\tilde{s}_{11,12,15} \approx 0.0270, \tilde{s}_{11,13,15} \approx 0.0260, \tilde{s}_{11,14,15} \approx 0.0130$, $\tilde{s}_{12,13,15} \approx 0.0220, \tilde{s}_{12,14,15} = \tilde{s}_{13,14,15} \approx 0.0110$	0.4281 s	>30 min



FIGURE **2.** Structure block diagrams for Systems ϕ (left) and $\tilde{\phi}$ (right) in Example 5.2.

Then, using Theorem 3.1, we have the multi-state survival signature of System ϕ as follows:

$$
\begin{aligned}
&(\boldsymbol{D}^{(0)}, \boldsymbol{D}^{(1)}, \boldsymbol{D}^{(2)}) \\
&= \begin{pmatrix}
D_{4,0}^{(0)} & D_{3,1}^{(0)} & D_{2,2}^{(0)} & D_{1,3}^{(0)} & D_{0,4}^{(0)} & D_{3,0}^{(0)} & D_{2,1}^{(0)} & D_{1,2}^{(0)} & D_{0,3}^{(0)} & D_{2,0}^{(0)} & D_{1,1}^{(0)} & D_{0,2}^{(0)} & D_{1,0}^{(0)} & D_{0,1}^{(0)} & D_{0,0}^{(0)} \\
D_{4,0}^{(1)} & D_{3,1}^{(1)} & D_{2,2}^{(1)} & D_{1,3}^{(1)} & D_{0,4}^{(1)} & D_{3,0}^{(1)} & D_{2,1}^{(1)} & D_{1,2}^{(1)} & D_{0,3}^{(1)} & D_{2,0}^{(1)} & D_{1,1}^{(1)} & D_{0,2}^{(1)} & D_{1,0}^{(1)} & D_{0,1}^{(1)} & D_{0,0}^{(1)} \\
D_{4,0}^{(2)} & D_{3,1}^{(2)} & D_{2,2}^{(2)} & D_{1,3}^{(2)} & D_{0,4}^{(2)} & D_{3,0}^{(2)} & D_{2,1}^{(2)} & D_{1,2}^{(2)} & D_{0,3}^{(2)} & D_{2,0}^{(2)} & D_{1,1}^{(2)} & D_{0,2}^{(2)} & D_{1,0}^{(2)} & D_{0,1}^{(2)} & D_{0,0}^{(2)}
\end{pmatrix}^T \\
&= \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 2/3 & 1 & 1 & 0 & 2/3 & 1 & 1 & 2/3 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/3 & 2/3 & 2/3 & 1 & 1 & 1
\end{pmatrix}^T .
\end{aligned}
$$

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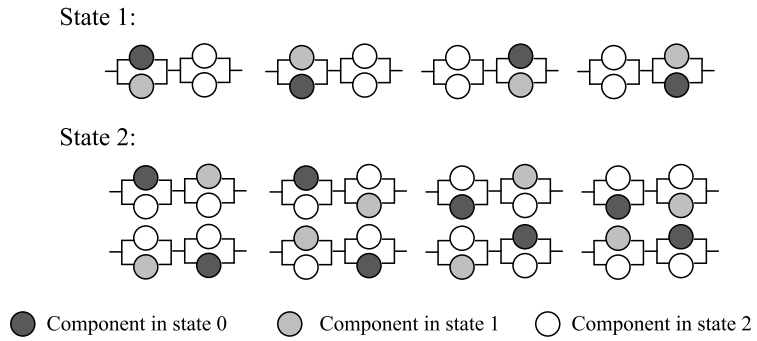


FIGURE **3.** All possible cases for $D_{1,1}^{(1)}, D_{1,1}^{(2)}$ in Example 5.2.

These results can also be verified by the definition of multi-state survival signature. For example, under the condition that System ϕ has one component in state 0, one component in state 1 and two components in state 2, there are four cases that cause System ϕ to be in state 1 and eight cases that cause System ϕ to be in state 2 (see Figure 3), which means that $D_{1,1}^{(1)} = 1$ and $D_{1,1}^{(2)} = 8/12 = 2/3$.

Similarly, System $\tilde{\phi}$ can be regarded as a parallel connection of two systems with structure functions $\phi_{21}(x_1, x_2) = min(x_1, x_2)$ and $\phi_{22}(x_3, x_4) = min(x_3, x_4)$. The multi-state survival signatures of the two series systems are the same, namely,

$$(\boldsymbol{D}^{2(0)}, \boldsymbol{D}^{2(1)}, \boldsymbol{D}^{2(2)}) = \begin{pmatrix} D_{2,0}^{2(0)} & D_{1,1}^{2(0)} & D_{0,2}^{2(0)} & D_{1,0}^{2(0)} & D_{0,1}^{2(0)} & D_{0,0}^{2(0)} \\ D_{2,0}^{2(1)} & D_{1,1}^{2(1)} & D_{0,2}^{2(1)} & D_{1,0}^{2(1)} & D_{0,1}^{2(1)} & D_{0,0}^{2(1)} \\ D_{2,0}^{2(2)} & D_{1,1}^{2(2)} & D_{0,2}^{2(2)} & D_{1,0}^{2(2)} & D_{0,1}^{2(2)} & D_{0,0}^{2(2)} \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Then, using Theorem 3.2, we have the multi-state survival signature of System $\tilde{\phi}$ as follows:

$$(\tilde{\boldsymbol{D}}^{(0)}, \tilde{\boldsymbol{D}}^{(1)}, \tilde{\boldsymbol{D}}^{(2)}) = \begin{pmatrix} \tilde{D}_{4,0}^{(0)} & \tilde{D}_{3,1}^{(0)} & \tilde{D}_{2,2}^{(0)} & \tilde{D}_{1,3}^{(0)} & \tilde{D}_{0,4}^{(0)} & \tilde{D}_{3,0}^{(0)} & \tilde{D}_{2,1}^{(0)} & \tilde{D}_{1,2}^{(0)} & \tilde{D}_{0,3}^{(0)} & \tilde{D}_{2,0}^{(0)} & \tilde{D}_{1,1}^{(0)} & \tilde{D}_{0,2}^{(0)} & \tilde{D}_{1,0}^{(0)} & \tilde{D}_{0,1}^{(0)} & \tilde{D}_{0,0}^{(0)} \\ \tilde{D}_{4,0}^{(1)} & \tilde{D}_{3,1}^{(1)} & \tilde{D}_{2,2}^{(1)} & \tilde{D}_{1,3}^{(1)} & \tilde{D}_{0,4}^{(1)} & \tilde{D}_{3,0}^{(1)} & \tilde{D}_{2,1}^{(1)} & \tilde{D}_{1,2}^{(1)} & \tilde{D}_{0,3}^{(1)} & \tilde{D}_{2,0}^{(1)} & \tilde{D}_{1,1}^{(1)} & \tilde{D}_{0,2}^{(1)} & \tilde{D}_{1,0}^{(1)} & \tilde{D}_{0,1}^{(1)} & \tilde{D}_{0,0}^{(1)} \\ \tilde{D}_{4,0}^{(2)} & \tilde{D}_{3,1}^{(2)} & \tilde{D}_{2,2}^{(2)} & \tilde{D}_{1,3}^{(2)} & \tilde{D}_{0,4}^{(2)} & \tilde{D}_{3,0}^{(2)} & \tilde{D}_{2,1}^{(2)} & \tilde{D}_{1,2}^{(2)} & \tilde{D}_{0,3}^{(2)} & \tilde{D}_{2,0}^{(2)} & \tilde{D}_{1,1}^{(2)} & \tilde{D}_{0,2}^{(2)} & \tilde{D}_{1,0}^{(2)} & \tilde{D}_{0,1}^{(2)} & \tilde{D}_{0,0}^{(2)} \end{pmatrix}^T$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1/3 & 1 & 1 & 0 & 1/3 & 1 & 1 & 1/3 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 1 & 1 & 1 \end{pmatrix}^T.$$

These results can also be verified by the definition of multi-state survival signature. For example, under the condition that System $\tilde{\phi}$ has two components in state 0, one component in state 1 and one component in state 2, there are four cases that cause System $\tilde{\phi}$ to be in state 1 and eight cases that cause System $\tilde{\phi}$ to be in state 0 (see Figure 4), which means that $\tilde{D}_{2,1}^{(1)} = 4/12 = 1/3$ and $\tilde{D}_{2,1}^{(2)} = 0$.

Comparing the multi-state survival signatures of Systems ϕ and $\tilde{\phi}$, we observe that component-level redundancy is better than system-level redundancy for multi-state structures.

EXAMPLE 5.3: *Consider a multi-state coherent system ϕ with state space $\{0, 1, 2\}$ that has three i.i.d. multi-state components. With $x_1, x_2, x_3 \in \{0, 1, 2\}$, the structure function of the*

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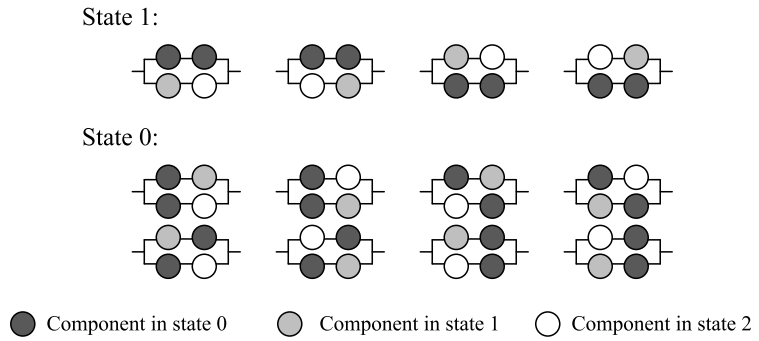


FIGURE **4.** All possible cases for $\tilde{D}_{2,1}^{(1)}, \tilde{D}_{2,1}^{(2)}$ in Example 5.2.

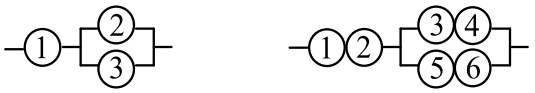


FIGURE **5.** Structure block diagrams for Systems ϕ (left) and $\tilde{\phi}$ (right) in Example 5.3.

system is $\phi(x_1, x_2, x_3) = min(x_1, max(x_2, x_3))$*, which means that the system can be regarded as a series connection of two systems with structure functions* $\phi_1(x_1) = x_1$ *and* $\phi_2(x_2, x_3) = max(x_2, x_3)$*. (See Figure 5 for the structure block diagram of System* ϕ*.) The multi-state survival signatures of the two systems are*

$$(\boldsymbol{D}^{1(0)}, \boldsymbol{D}^{1(1)}, \boldsymbol{D}^{1(2)}) = \begin{pmatrix} D_{1,0}^{1(0)} & D_{0,1}^{1(0)} & D_{0,0}^{1(0)} \\ D_{1,0}^{1(1)} & D_{0,1}^{1(1)} & D_{0,0}^{1(1)} \\ D_{1,0}^{1(2)} & D_{0,1}^{1(2)} & D_{0,0}^{1(2)} \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^T,$$

$$(\boldsymbol{D}^{2(0)}, \boldsymbol{D}^{2(1)}, \boldsymbol{D}^{2(2)}) = \begin{pmatrix} D_{2,0}^{2(0)} & D_{1,1}^{2(0)} & D_{0,2}^{2(0)} & D_{1,0}^{2(0)} & D_{0,1}^{2(0)} & D_{0,0}^{2(0)} \\ D_{2,0}^{2(1)} & D_{1,1}^{2(1)} & D_{0,2}^{2(1)} & D_{1,0}^{2(1)} & D_{0,1}^{2(1)} & D_{0,0}^{2(1)} \\ D_{2,0}^{2(2)} & D_{1,1}^{2(2)} & D_{0,2}^{2(2)} & D_{1,0}^{2(2)} & D_{0,1}^{2(2)} & D_{0,0}^{2(2)} \end{pmatrix}^T$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}^T.$$

Then, using Theorem 3.1, we have the multi-state survival signature of System ϕ *as*

$$(\boldsymbol{D}^{(0)}, \boldsymbol{D}^{(1)}, \boldsymbol{D}^{(2)})$$
$$= \begin{pmatrix} D_{3,0}^{(0)} & D_{2,1}^{(0)} & D_{1,2}^{(0)} & D_{0,3}^{(0)} & D_{2,0}^{(0)} & D_{1,1}^{(0)} & D_{0,2}^{(0)} & D_{1,0}^{(0)} & D_{0,1}^{(0)} & D_{0,0}^{(0)} \\ D_{3,0}^{(1)} & D_{2,1}^{(1)} & D_{1,2}^{(1)} & D_{0,3}^{(1)} & D_{2,0}^{(1)} & D_{1,1}^{(1)} & D_{0,2}^{(1)} & D_{1,0}^{(1)} & D_{0,1}^{(1)} & D_{0,0}^{(1)} \\ D_{3,0}^{(2)} & D_{2,1}^{(2)} & D_{1,2}^{(2)} & D_{0,3}^{(2)} & D_{2,0}^{(2)} & D_{1,1}^{(2)} & D_{0,2}^{(2)} & D_{1,0}^{(2)} & D_{0,1}^{(2)} & D_{0,0}^{(2)} \end{pmatrix}^T$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2/3 & 1 & 0 & 2/3 & 1 & 2/3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/3 & 2/3 & 1 \end{pmatrix}^T.$$

These results can also be verified by the definition of multi-state survival signature. For example, under the condition that System ϕ *has one component in state* 0*, one component in state* 1 *and one component in state* 2*, there are two cases that cause System* ϕ *to be in*

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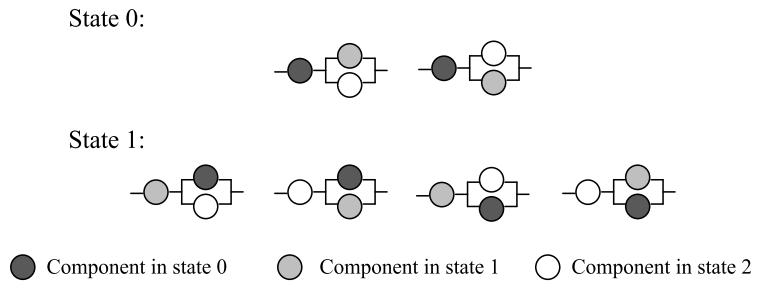


FIGURE **6.** All possible cases for $D_{1,1}^{(1)}, D_{1,1}^{(2)}$ in Example 5.3.

state 0 *and four cases that cause System* ϕ *to be in state* 1 *(see Figure 6), which means that* $D_{1,1}^{(1)} = 4/6 = 2/3$ *and* $D_{1,1}^{(2)} = 0$.

Now, let us further consider a multi-state coherent system $\tilde{\phi}$ *with state space* $\{0,1,2\}$ *that has six i.i.d. binary-state components. With* $x_1, \ldots, x_6 \in \{0,1\}$, *the structure function of the system is* $\tilde{\phi}(x_1, \ldots, x_6) = min(x_1 + x_2, max(x_3 + x_4, x_5 + x_6))$. *(See Figure 5 for the structure block diagram of System* $\tilde{\phi}$.*) Then, System* $\tilde{\phi}$ *can be regarded as a recurrent system composed of three modules: each module has the same multi-state signature* $\boldsymbol{s}^1 = (s^1_{i_1,i_2}, 1 \le i_1, i_2 \le 2)$ *with* $s^1_{12} = 1$ *and* $s^1_{i_1,i_2} = 0$ *otherwise, and the multi-state survival signature of the connection is the same as System* ϕ *in this example.*

Using Theorem 4.1, we then have the multi-state survival signature of System $\tilde{\phi}$ *as*

$$
\begin{aligned}
&(\tilde{\boldsymbol{D}}^{(0)}, \tilde{\boldsymbol{D}}^{(1)}, \tilde{\boldsymbol{D}}^{(2)}) \\
&= \begin{pmatrix} \tilde{D}_0^{(0)} & \tilde{D}_1^{(0)} & \tilde{D}_2^{(0)} & \tilde{D}_3^{(0)} & \tilde{D}_4^{(0)} & \tilde{D}_5^{(0)} & \tilde{D}_6^{(0)} \\ \tilde{D}_0^{(1)} & \tilde{D}_1^{(1)} & \tilde{D}_2^{(1)} & \tilde{D}_3^{(1)} & \tilde{D}_4^{(1)} & \tilde{D}_5^{(1)} & \tilde{D}_6^{(1)} \\ \tilde{D}_0^{(2)} & \tilde{D}_1^{(2)} & \tilde{D}_2^{(2)} & \tilde{D}_3^{(2)} & \tilde{D}_4^{(2)} & \tilde{D}_5^{(2)} & \tilde{D}_6^{(2)} \end{pmatrix}^T \\
&= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 14/15 & 4/5 & 8/15 & 0 & 0 \\ 1 & 2/3 & 2/15 & 0 & 0 & 0 & 0 \end{pmatrix}^T .
\end{aligned}
$$

These results can also be verified by related definitions. For example, under the condition that System $\tilde{\phi}$ *has two components in state* 0 *and four components in state* 1, *there is one case that causes System* $\tilde{\phi}$ *to be in state* 0, *twelve cases that cause System* $\tilde{\phi}$ *to be in state* 1 *and two cases that cause System* $\tilde{\phi}$ *to be in state* 2 *(see Figure 7), which means that* $\tilde{D}_2^{(1)} = 14/15$ *and* $\tilde{D}_2^{(2)} = 2/15$.

EXAMPLE 5.4: *Consider a multi-state coherent system* $\tilde{\phi}$ *with state space* $\{0,1,2\}$ *that has six i.i.d. multi-state components. With* $x_1, \ldots, x_6 \in \{0,1,2\}$, *the structure function of the system is*

$$\tilde{\phi}(x_1, \ldots, x_6) = \min(\min(x_1, x_2), \max(\min(x_3, x_4), \min(x_5, x_6))).$$

(See Figure 8 for the structure block diagram of System $\tilde{\phi}$.*) Then, System* $\tilde{\phi}$ *can be regarded as a recurrent system composed of three modules: each module has the same multi-state survival signature as* $(\boldsymbol{D}^{2(0)}, \boldsymbol{D}^{2(1)}, \boldsymbol{D}^{2(2)})$ *in Example 5.2, and the multi-state survival signature of the connection is the same as* $(\boldsymbol{D}^{(0)}, \boldsymbol{D}^{(1)}, \boldsymbol{D}^{(2)})$ *in Example 5.3. Then, by Theorem 4.2, we have the multi-state survival signature of the recurrent system as* $(\tilde{\boldsymbol{D}}^{(0)}, \tilde{\boldsymbol{D}}^{(1)}, \tilde{\boldsymbol{D}}^{(2)})$,

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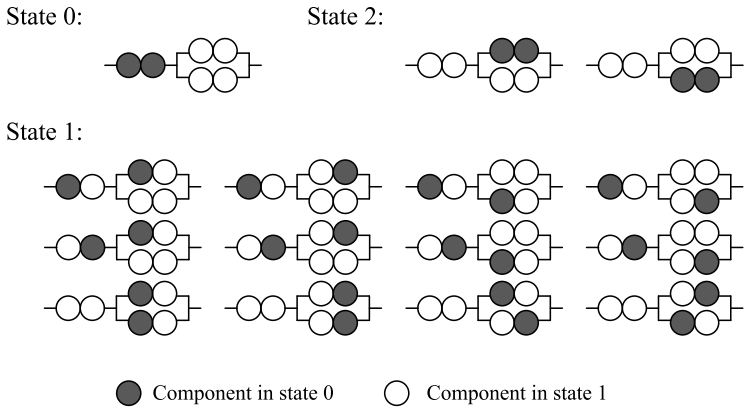


FIGURE **7.** All possible cases for $\tilde{D}_2^{(1)}, \tilde{D}_2^{(2)}$ in Example 5.3.

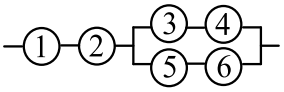


FIGURE **8.** Structure block diagram for System $\tilde{\phi}$ in Example 5.4.

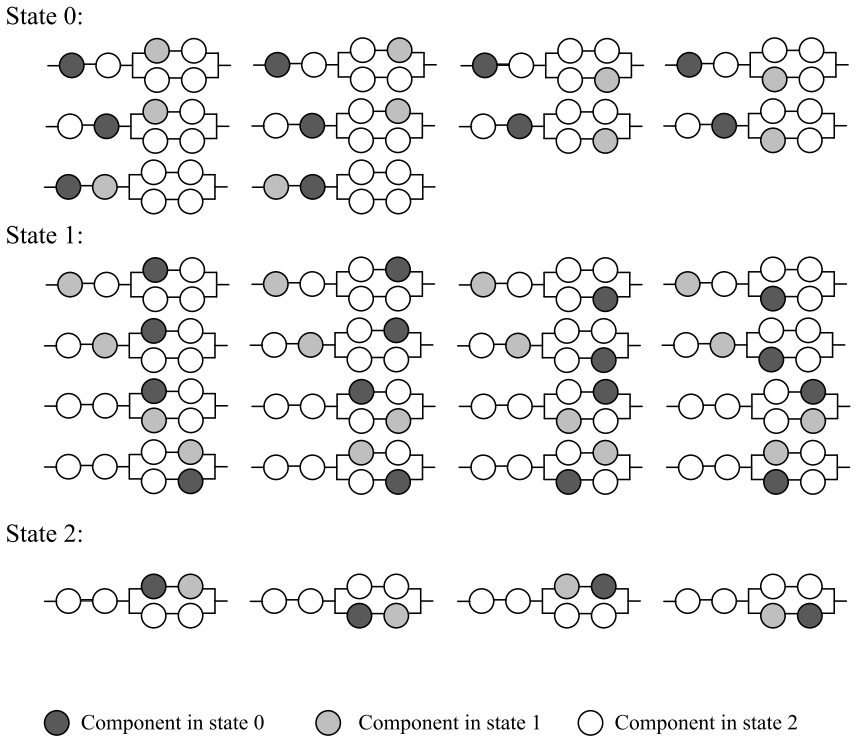


FIGURE **9.** All possible cases for $\tilde{D}_{1,1}^{(1)}, \tilde{D}_{1,1}^{(2)}$ in Example 5.4.

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where the elements of $\tilde{\boldsymbol{D}}^{(0)}$ *are all* 1 *and* $\tilde{\boldsymbol{D}}^{(1)}$ *and* $\tilde{\boldsymbol{D}}^{(2)}$ *have nonzero elements as follows:*

$$\begin{aligned}
&\tilde{D}^{(1)}_{2,4} = \tilde{D}^{(1)}_{2,3} = \tilde{D}^{(1)}_{2,2} = \tilde{D}^{(1)}_{2,1} = \tilde{D}^{(1)}_{2,0} = 2/15,\\
&\tilde{D}^{(1)}_{1,5} = \tilde{D}^{(1)}_{1,4} = \tilde{D}^{(1)}_{1,3} = \tilde{D}^{(1)}_{1,2} = \tilde{D}^{(1)}_{1,1} = \tilde{D}^{(1)}_{1,0} = 2/3,\\
&\tilde{D}^{(1)}_{0,6} = \tilde{D}^{(1)}_{0,5} = \tilde{D}^{(1)}_{0,4} = \tilde{D}^{(1)}_{0,3} = \tilde{D}^{(1)}_{0,2} = \tilde{D}^{(1)}_{0,1} = \tilde{D}^{(1)}_{0,0} = 1,\\
&\tilde{D}^{(2)}_{2,0} = \tilde{D}^{(2)}_{1,1} = \tilde{D}^{(2)}_{0,2} = 2/15, \tilde{D}^{(2)}_{1,0} = \tilde{D}^{(2)}_{0,1} = 2/3, \tilde{D}^{(2)}_{0,0} = 1.
\end{aligned}$$

These results can also be verified by related definitions. For example, under the condition that System $\tilde{\phi}$ *has one component in state* 0*, one component in state* 1 *and four components in state* 2*, there are ten cases that cause System* $\tilde{\phi}$ *to be in state* 0*, sixteen cases that cause System* $\tilde{\phi}$ *to be in state* 1 *and four cases that cause System* $\tilde{\phi}$ *to be in state* 2 *(see Figure 9), which means that* $\tilde{D}^{(1)}_{1,1} = 20/30 = 2/3$ *and* $\tilde{D}^{(2)}_{1,1} = 4/30 = 2/15$*.*

6. CONCLUDING REMARKS

In this paper, we have considered multi-state systems that can be regarded as series/parallel/recurrent connection of multi-state modules. For a multi-state coherent system with binary/multi-state components, the multi-state (survival) signature of the system is expressed by the multi-state (survival) signature of each module (and that of the recurrent connection). A number of examples have also been presented to illustrate the theoretical results established here. In this connection, it will be of interest to consider the following problems for further study:

(1) These results established here may be generalized to multi-state coherent systems with shared modules in which case some results can be presented in terms of multi-state joint signature and multi-state joint survival signature for both systems with binary-state components and multi-state components (see the recent work of Coolen-Maturi *et al.* [7]);
(2) With more and more computational methods proposed for multi-state signatures, it will be of interest to find the best computational strategy for given systems; for example, whether the module method should be applied, with which method it should work together, how many times and in which order they should be used, and so on.

Work on these problems is currently under progress and we hope to report these findings in a future paper. Finally, it is worthwhile to mention that the module method in this paper will become more useful in practice if it can be implemented in software such as R just as some traditional computational methods for binary-state systems have been implemented.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Nos. 72001016, 71931001 and 71722007), the Funds for First-class Discipline Construction (XK18025) and the Fundamental Research Funds for the Central Universities (buctrc202102). The authors also express their sincere thanks to the Editor-in-Chief and the reviewers for many constructive comments and suggestions on the previous version of this manuscript which resulted in this improved version.

Competing interests

The authors declare no conflict of interest.

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