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Detachments of Complete Graphs

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A detachment of a graph G is formed by splitting each vertex into one or more subvertices, and sharing the incident edges arbitrarily among the subvertices. In this paper we consider the question of whether a graph H is a detachment of some complete graph K_n. When H is large and restricted to belong to certain classes of graphs, for example bounded degree planar triangle-free graphs, we obtain necessary and sufficient conditions which give a complete characterization.

A *harmonious colouring* of a simple graph G is a proper vertex colouring such that each pair of colours appears together on at most one edge. The *harmonious chromatic number* $h(G)$ is the least number of colours in such a colouring. The results on detachments of complete graphs give exact results on harmonious chromatic number for many classes of graphs, as well as algorithmic results.

1. Introduction

A detachment of a graph G is formed by splitting each vertex into one or more subvertices, and sharing the incident edges arbitrarily among the subvertices. Thus a detachment of G has at least as many vertices as G, and the same number of edges as G. The question of whether or not a graph H is a detachment of another graph G is, in a sense, one of the oldest in graph theory; Euler's Königsberg bridges problem can be stated this way since it is easy to see that a graph with m edges has an eulerian path if and only if the path with m edges is a detachment of it. A survey of detachments was given by Nash-Williams [19].

A *harmonious colouring* of a simple graph G is a proper vertex colouring such that each pair of colours appears together on at most one edge. An *exact colouring* is a proper vertex colouring such that each pair of colours appears on *exactly* one edge. A survey of harmonious colourings can be found in [6].

Below we give more formal definitions of these concepts, and observe that a simple graph G has an exact colouring with n colours if and only if G is a detachment of the complete graph K_n. Also, G has a harmonious colouring with n colours if and only if the

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graph obtained from G by adding $\binom{n}{2} - |E(G)|$ isolated edges to G has an exact colouring (with n colours).

In this paper we study the general question of whether a graph H is a detachment of a complete graph, or, equivalently, has an exact colouring. Clearly this is only possible if the number of edges of H is $\binom{n}{2}$ for some n; in that case H may be a detachment of K_n, and equivalently have an exact colouring with n colours.

The more general concept of harmonious colouring applies to any simple graph. In particular, the *harmonious chromatic number* $h(G)$ is the least number of colours in a harmonious colouring of G. As indicated above, this concept is very closely related to the other two.

In general, the problem of deciding if a graph H is a detachment of a complete graph is NP-complete, even when H is a tree. This follows from the results of [9]. However, when the degree of the graph H is bounded by a constant, the problem is somewhat more tractable. As shown in [5], it can be solved in polynomial time for H a bounded degree tree. In this paper we extend this result to many other classes of graphs, by giving necessary and sufficient conditions, checkable in polynomial time, for a graph in these classes to be a detachment of a complete graph.

The classes of graphs will be described later, but include for example bounded-degree planar triangle-free graphs. In many cases, for example when the graphs are regular, the necessary and sufficient conditions are trivial to check, and we obtain simple conditions. As an example of the kind of results obtained, we will show the following.

Theorem. *For all sufficiently large n, every cubic planar triangle-free graph on $n(3n+1)$ vertices is a detachment of K_{3n+1}.*

It must be emphasized that the results in this paper are valid only for graphs which are 'sufficiently large', and that in each case sufficiently large means very large indeed. This is unavoidable given the methods used, which involve the application of theorems such as those of Gustavsson [13, 14], which already involve very large constants. Therefore we have made little attempt to optimize the constants, particularly when doing so would have resulted in a more complex argument.

2. Notation and terminology

In this section we will give formal definitions of the terms detachment, harmonious colouring and exact colouring, and introduce some further notation and terminology which will be used.

Detachment. Let G be a simple graph with vertex set $v_1, \ldots, v_n$. Then a graph H is a *detachment* of G if the vertex set $V(H)$ can be partitioned into n sets $V_1, \ldots, V_n$ such that (i) for each i, V_i is an independent set, and (ii) for each i, j, $i \neq j$,

$$|E(V_i, V_j)| = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

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In the case that G is the complete graph K_n, condition (ii) simply becomes $|E(V_i, V_j)| = 1$ for each i, j, $i \neq j$.

Following Nash-Williams [20], we say that G is an *amalgamation* of H if and only if H is a detachment of G.

Harmonious colouring. A harmonious colouring is a function c from a colour set C to the set $V(G)$ of vertices of G such that, for any edge e of G, with endpoints x, y say, $c(x) \neq c(y)$, and for any pair of distinct edges e, e', with endpoints x, y and x', y' respectively, then $\{c(x), c(y)\} \neq \{c(x'), c(y')\}$. The *harmonious chromatic number* $h(G)$ is the least number of colours in such a colouring.

Exact colouring. An exact colouring of G with n colours is a harmonious colouring of G with n colours in which, for each pair of colours c, c', there is exactly one edge (x, y) such that x has colour c and y has colour c'.

Observation 2.1. *A simple graph H has an exact colouring with n colours if and only if it is a detachment of K_n.*

This is easily established; the colour classes correspond to the vertex sets $V_1, \ldots, V_n$. □

Similarly we have the following.

Observation 2.2. *A simple graph H has a harmonious colouring with n colours if and only if it is a detachment of a simple graph on n vertices.* □

We also note the following.

Observation 2.3. *If H is a detachment of G and G has a harmonious (respectively exact) colouring with n colours, then so does H.* □

2.1. Terminology

We now introduce a few pieces of terminology which we will use.

Partial colourings. A partial harmonious colouring of a graph G is a colouring of a subset of the vertices of G which is a harmonious colouring of the graph induced by the coloured vertices, and is such that no uncoloured vertex has two or more coloured neighbours with the same colour.

We will say that a pair s, s' of colours *occurs on* or is *used on* an edge e if the endpoints of e have colours s and s'.

We say that an edge is coloured if both of its endpoints are coloured, otherwise we say it is uncoloured.

Degree sum. Suppose that we have a (possibly partial) harmonious colouring C of a graph $G = (V, E)$, and let S_c be the set of vertices with colour c. Then we define the degree sum $\mathrm{ds}_C(c)$ of colour c to be $\sum_{v \in S_c} d(v)$, where $d(v)$ denotes the degree of v in G. If X is a subset of V, denote by $d(v, X)$ the number of neighbours of v in X, and define the degree sum of c in X, denoted $\mathrm{ds}_C(c, X)$, to be $\sum_{v \in S_c} d(v, X)$.

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Fragmentable graphs. Informally, a graph is fragmentable if the removal of a small proportion of the vertices can break it up into bounded sized pieces.

Formally, we will say that a graph $G = (V, E)$ is (C, ε)-fragmentable if there is a set $X \subseteq V$ such that (i) $|X| \leqslant \varepsilon |V|$, and (ii) every component of $G \setminus X$ has at most C vertices.

We say that a class Γ of graphs is fragmentable if for any $\varepsilon > 0$, there is a constant C_ε such that every graph in Γ is $(C_\varepsilon, \varepsilon)$-fragmentable.

Exact graph. As we wish to study detachments of a complete graph, we shall mostly be concerned with graphs that have exactly $\binom{r}{2}$ edges for some integer r. We shall call such a graph an *exact graph*.

3. Outline of paper

In this section we outline the structure of the rest of the paper, and give a sketch of the main ideas.

Section 4 is devoted to proving the main technical result of the paper, concerning exact colourings of graphs which are nearly regular, fragmentable and having a proportion of vertices whose neighbourhood sets are independent. We prove that given certain obviously necessary conditions on the degrees, such a graph is a detachment of a complete graph provided it is large enough.

The proof is based on the proof of a similar result for trees with many leaves [4], but is considerably more complicated. The main idea of the earlier proof can be stated as follows. We wish to find an exact r-colouring of a tree with $\binom{r}{2}$ edges. We first choose a fragmenting set, *i.e.*, a small proportion of the vertices, whose removal breaks up the tree into bounded sized pieces, called fragments. We then identify in these fragments a large (compared to the fragmenting set) set of leaves which we also remove for the moment. We then use a simple construction to colour the vertices of the fragments, using most of the r colours, in such a way that the set of unused pairs of colours is highly symmetrical. We then colour the fragmenting set using the rest of the colours. This unavoidably perturbs the symmetry of the colouring, in particular of the set of unused colour pairs, but the asymmetry is limited because the fragmenting set is small. We then have to colour the removed leaves. Because each of these has only one neighbour, there is a considerable degree of flexibility available in colouring them, and this is sufficient to overcome the asymmetry in the structure of the set of unused colour pairs.

In attempting to extend this result to, say, cubic planar graphs, we immediately face the problem that these graphs have no leaves. This can be overcome by using vertices with independent neighbourhood sets to create what we call superleaves, which we eventually colour in the same way as the leaves of the trees. The construction of these superleaves involves identifying vertices with independent neighbourhoods, and other vertices, to create a set of vertices which have a particular graph induced by the vertex and its first and second neighbourhoods. The reason that vertices with independent neighbourhoods are so important is that two or more such vertices can have the same set of colours on their neighbours, without this resulting in colour pairs being repeated.

The simple construction used to colour the fragments in the tree theorem is also unusable here. Instead we use an argument based on the theorems of Wilson and Gustavsson on

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packing copies of a fixed graph into complete or nearly complete graphs. Gustavsson's theorems are also crucial at a later stage for colouring some of the superleaves. Most other parts of the tree proof are also inadequate, but can be replaced by more complicated arguments which suffice for the more complex graphs.

In Section 5 we extend the result to the case of bounded degree, rather than regular, graphs of the same classes. We give conditions, akin to bin-packing, which prove to be both necessary and sufficient for large enough graphs in these classes. Since these conditions can be checked, for bounded degree graphs, in polynomial time, we are able to give, in Section 6, a polynomial time algorithm for determining if certain graphs are detachments of a complete graph.

In Section 7 we turn to harmonious chromatic number, and give a number of results for bounded degree triangle-free fragmentable classes of graphs. These include a polynomial time algorithm, an upper bound, and exact results for D-dimensional lattice graphs and for graphs of maximum degree 2.

4. Almost-regular graphs

Recall that an exact graph is one which has $\binom{r}{2}$ edges for some r. We first consider exact graphs which are nearly regular.

Definition. An exact graph G with $\binom{r}{2}$ edges is (d, d')-almost-regular if it has r vertices of degree d', and the remainder are of degree d.

As we shall see in Section 5, the problem for more general classes of bounded degree graphs can be reduced to that for almost-regular graphs. Note that in an exact colouring with r colours, the degree sum of each colour must be $r-1$, which may not be divisible by d; however by assigning the r distinct colours to the degree d' vertices, we have some chance of extending this to an exact colouring.

Suppose that $G=(V,E)$ is a graph satisfying the following properties:

(1) G is an exact graph with $\binom{r}{2}$ edges,
(2) G is (d, d')-almost-regular,
(3) G is (C, ε)-fragmentable,
(4) at least $\varepsilon_1 |V|$ vertices of G have an independent neighbourhood set,
(5) d divides $r-1-d'$.

We will show that, provided ε is small enough and r is large enough, then G is a detachment of K_r.

We will need a number of technical lemmas. We first give a lemma on orienting the edges of a certain graph. This will be used in the final step of the colouring process.

Lemma 4.1. ([7]) *Let $G=(V,E)$ be a graph with n vertices cyclically arranged such that each vertex is joined to at least $2k-t$ of the $2k$ vertices nearest to it ($n \geqslant 2k+1$) and the total number of edges is m. For each vertex v let k_v be a positive integer. Then the edges can be oriented so that for each vertex v, the outdegree of v is at least k_v, provided that* (i) *$m \geqslant \sum_{v \in V} k_v$, and* (ii) *for each $v \in V$, $k/2 \leqslant k_v \leqslant k'$ where $k' = k - t + k^2/4(n-k)$.* □

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The next lemma allows us to extend partial harmonious colourings while controlling the degree sums of the colours. It is used a number of times in this paper.

Lemma 4.2. *Let $G=(V,E)$ be a graph with maximum degree at most d. Let X,Y be disjoint subsets of V, and let C_1 and $C_2=\{1,\dots,k\}$ be disjoint sets of colours. Suppose that $c_0 : X \to C_1 \cup C_2$ is a partial harmonious colouring of G, and let $X_1=c_0^{-1}(C_1), X_2=c_0^{-1}(C_2)$. Let $N(Y)$ be the set of vertices adjacent to a vertex of Y, and let $C_3=c_0(N(Y)\cap X_1)\subseteq C_1$. Suppose also that for each colour $z\in C_3$, the degree sum of z in $X_2\cup Y$, $\mathrm{ds}_{c_0}(z, X_2\cup Y)$, is at most θ, and for each $z\in C_2$, the degree sum of z, $\mathrm{ds}_{c_0}(z)$, is at most θ. Let $d_Y=\max\{d(v)|v\in Y\}$.*

Then there is a partial harmonious colouring $c : X\cup Y \to C_1\cup C_2$ such that

(i) $c(v)\in C_2$ for all $v\in Y$,
(ii) $c|_X = c_0$,
(iii) for each $z\in C_2$, $\mathrm{ds}_c(z)\leqslant\theta$,

provided that $k > d^2+d\theta+|X_2\cup Y|d/(\theta-d)$.

If, in addition, $r_1,\dots,r_k$ are integers such that $r_i d_Y + \mathrm{ds}_{c_0}(i)\leqslant\theta$ for each $i=1,\dots,k$ and $\sum_{i=1}^k r_i = |Y|$, then c can be chosen to satisfy (i), (ii), (iii) and

(iv) $|Y\cap c^{-1}(i)| = r_i$ for each $i=1,\dots,k$,

provided that $|Y| > (d+1+\theta)\theta + (d\theta+d^2+1)(\max_i r_i)$.

Proof. First suppose that

$$k > d^2+d\theta+|X_2\cup Y|d/(\theta-d)$$

and let $Y'\subseteq Y$ be a maximal set such that there is partial harmonious colouring $c : X\cup Y'\to C_1\cup C_2$ satisfying (i), (ii) and (iii). We will show that $Y'=Y$. For suppose not; then there is an uncoloured vertex v in Y. Then a given colour $i\in C_2$ can be used on vertex v unless one of the following holds.

(1) Colour i is used on a neighbour of v. There are at most d such colours.
(2) Vertex v has a neighbour with colour z, and colour i appears adjacent to some vertex coloured z. Then $z\in C_2\cup C_3$. There are at most d choices for z, and each has degree sum at most θ in $X_2\cup Y$, so this forbids at most $d\theta$ colours.
(3) Colour i is used adjacent to an uncoloured neighbour of v. There are at most d such neighbours, and each has at most $d-1$ neighbours apart from v, so this forbids at most $d(d-1)$ colours.
(4) Using colour i on v would cause $\mathrm{ds}_c(i)$ to exceed θ. This cannot happen if $\mathrm{ds}_c(i)\leqslant\theta-d$, so let a be the number of colours j in C_2 for which $\mathrm{ds}_c(j)>\theta-d$. Then if b is the total degree of the vertices in $X_2\cup Y$, we have

$$a(\theta-d) < b \leqslant |X_2\cup Y|d.$$

Hence the number of colours which cannot be used for v is at most

$$d+d\theta+d(d-1)+|X_2\cup Y|d/(\theta-d) = d^2+d\theta+|X_2\cup Y|d/(\theta-d).$$

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Since this is less than k, the result follows.

Now suppose that

$$|Y| > (d+1+\theta)\theta + (d\theta + d^2 + 1)(\max_i r_i)$$

and let $Y' \subseteq Y$ be a maximal set such that there is partial harmonious colouring $c : X \cup Y' \to C_1 \cup C_2$ satisfying (i), (ii), (iii) and

(iv′) $|Y' \cap c^{-1}(i)| \leqslant r_i$ for each $i = 1, \ldots, k$.

We will show that $Y' = Y$. For suppose not; then for some i, we have $|Y' \cap c^{-1}(i)| < r_i$. Let v be an uncoloured vertex in Y.

Now consider the set S_1 of vertices w in Y which cannot be coloured or recoloured i. This can occur for the following reasons.

(1) Vertex w is adjacent to a vertex of colour i. There are at most θ such vertices w.
(2) Some neighbour of a vertex coloured i is coloured z, and w is adjacent to a vertex coloured z. There are at most θ choices for z, since $\mathrm{ds}_c(i) \leqslant \theta$, and at most θ vertices in Y adjacent to a vertex of colour z, since if $z \in C_2$, then $\mathrm{ds}_c(z) \leqslant \theta$, while if $z \in C_1$, then either $z \notin C_3$, so that no vertices in Y are adjacent to colour z, or $\mathrm{ds}_c(z, X_2 \cup Y) \leqslant \theta$. Hence there are at most θ^2 of these in all.
(3) Some neighbour u of a vertex coloured i is uncoloured, and is adjacent to w. There are at most θ choices for u, since $\mathrm{ds}_c(i) \leqslant \theta$, and at most d for w, so there are at most $d\theta$ of these in all.

Hence $|S_1| \leqslant (d+1+\theta)\theta$.

On the other hand, let S_2 be the set of vertices w in Y' whose colour cannot be transferred to v. Now if w is coloured j, we could uncolour w and colour v with j unless:

(1) Colour j is used on a neighbour (possibly w itself) of v. There are at most d such colours.
(2) Vertex v has a neighbour with colour z, and colour j appears adjacent to some vertex coloured z. There are at most d choices for z, and each has degree sum at most θ in Y, so this forbids at most $d\theta$ colours.
(3) Colour j is used adjacent to an uncoloured neighbour of v. There are at most d such neighbours, and each has at most $d-1$ neighbours apart from v, so this forbids at most $d(d-1)+1$ colours (including i itself).

Hence at most $d^2 + 1 + d\theta$ colours are forbidden in this way. Therefore there are at most $(d^2 + 1 + d\theta)\max_i r_i$ vertices in Y' whose colour cannot be transferred to v, *i.e.*, $|S_2| \leqslant (d^2 + 1 + d\theta)\max_i r_i$.

Now since

$$|Y| > (d+1+\theta)\theta + (d\theta + d^2 + 1)(\max_i r_i)$$

there exists a vertex w in Y which is not in S_1 or S_2. If w is uncoloured, then since it is not in S_1 it can be coloured i. Otherwise, $w \in Y'$, and is coloured j, say. Then w is not adjacent to v, and $j \neq i$. Since $w \notin S_1$, it can be recoloured i, and since $w \notin S_2$, v can be

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coloured j. But in either case this new colouring satisfies (i), (ii), (iii) and (iv′) and colours one more vertex, contradicting the maximality of c. Hence $Y' = Y$ as claimed. □

For the case when only previously unused colours are used to extend the colouring we obtain the following useful corollary.

Corollary 4.3. *Let $G = (V, E)$ be a graph with maximum degree at most d. Let C_1 and $C_2 = \{1, \ldots, k\}$ be disjoint sets of colours. Suppose that $c_0 : X \to C_1$ is a partial harmonious colouring of G, such that, for each colour z, $\mathrm{ds}_{c_0}(z)$ is at most θ. Let $Y \subseteq V \setminus X$, and $r_1, \ldots, r_k$ be integers such that $r_i d_Y \leqslant \theta$ for each $i = 1, \ldots, k$ and $\sum_{i=1}^{k} r_i = |Y|$.*

Then there is a partial harmonious colouring $c : X \cup Y \to C_1 \cup C_2$ such that

(i) *$c(v) \in C_2$ for all $v \in Y$,*
(ii) *$c|_X = c_0$,*
(iii) *$|Y \cap c^{-1}(i)| = r_i$ for each $i = 1, \ldots, k$,*

provided that $|Y| > (d + 1 + \theta)\theta + (d\theta + d^2 + 1)(\max_i r_i)$. □

We will be concerned with graphs which are induced by a vertex together with its first and second neighbourhoods, where the first neighbourhood is an independent set. We will call a graph with this structure an N2-graph.

Definition. An N2-graph is a simple graph $G = (V, E)$ where V and E have the following structure: $V(G)$ can be partitioned into three sets $V_0(G)$, $V_1(G)$, $V_2(G)$, where $V_0(G) = \{v\}$. $V_1(G)$ is an independent set, each element of $V_1(G)$ is joined to v, and there is at least one edge between $V_1(G)$ and $V_2(G)$. We will call v the *home vertex* of G.

Definition. We will say that an N2-graph H decomposes an N2-graph G if there are edge-disjoint subgraphs $H_1, \ldots, H_k$ of G such that (i) each of $H_1, \ldots, H_k$ is isomorphic to H, (ii) each edge of G is contained in exactly one of $H_1, \ldots, H_k$, (iii) $V_i(H_j) \subseteq V_i(G)$ for each $i = 0, \ldots, 2$ and $j = 1, \ldots, k$.

Thus we have a standard decomposition into edge-disjoint copies of H with the added requirement that the decomposition respect the partition into the three parts. Note that conditions (ii) and (iii) imply that: (iv) $V_0(H_i) = V_0(G) = \{v\}$ for each $i = 1, \ldots, k$, (v) the sets $V_1(H_1), \ldots, V_1(H_k)$ partition $V_1(G)$.

Lemma 4.4. *Let H be an* N2*-graph. Then there is an* N2*-graph G such that* (i) *H decomposes G,* (ii) *the set $V_1(G)$ can be partitioned into subsets $X_1, \ldots, X_k$ of equal size, such that each element of $V_2(G)$ is joined to exactly one element from each X_i, and* (iii) *the chromatic number $\chi(G)$ is bounded by $f(|V(H)|)$, where f is a function independent of H. Furthermore, for any fixed integer M, we can ensure that $|V_1(G)| \geqslant M$.*

To prove this we use a theorem of Häggkvist [15].

Theorem 4.5. (Häggkvist [15]) *Every k regular bipartite graph on $2n$ vertices decomposes $K_{k^2!n, k^2!n}$.*

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Remark. The proof [15] shows that the decomposition respects the bipartition.

Proof of Lemma 4.4. We first consider only the bipartite subgraph B of H induced by the vertices $V_1(H)$ and $V_2(H)$ (ignoring for the moment any edges joining vertices of $V_2(H)$). Let $a = |V_1(H)|$ and $b = |V_2(H)|$. Let $V_1(H) = \{v_1, \ldots, v_a\}$ and $V_2(H) = \{w_1, \ldots, w_b\}$. We form a regular bipartite graph B' as follows. Take b copies $A^{(1)}, \ldots, A^{(b)}$ of $V_1(H)$ for one part and a copies $B^{(1)}, \ldots, B^{(a)}$ of $V_2(H)$ for the other part, where $A^{(i)} = \{v_1^{(i)}, \ldots, v_a^{(i)}\}$, $B^{(j)} = \{w_1^{(j)}, \ldots, w_b^{(j)}\}$. Then, for each $i = 1, \ldots, b$, $j = 1, \ldots, a$, join the vertex $v_{x+j}^{(i)}$ to the vertex $w_{y+i}^{(j)}$ if and only if $(v_x, w_y) \in E(H)$ (with sums taken modulo a or b as appropriate). It is clear that the graph induced by the vertices $A^{(i)}, B^{(j)}$ is a copy of B, for each i, j, and that the graph B' is regular of degree $|E(B)|$, since each edge of B occurs once at each vertex of B'. Note also that each vertex $v_j^{(i)}$ meets the same number of copies of B.

We now apply Häggkvist's theorem to obtain a complete bipartite graph $K_{r,r}$ such that B', and therefore B, decomposes $K_{r,r}$. It is clear that r is bounded by a function of $|V(H)|$.

Now consider the subgraph B^+ of H induced by $V_1(H) \cup V_2(H)$, *i.e.*, B together with the edges of H joining vertices of $V_2(H)$. We form a new (multi)graph H' by replacing each copy of B in $K_{r,r}$ by a copy of B^+. Partition $V(H')$ as $V_1(H') \cup V_2(H')$ in the obvious way. Note that H' has no loops because we have never identified two vertices from the same copy of B^+.

At this point we can increase the size of $V_1(H')$ arbitrarily by taking a number of copies of H' and identifying all corresponding vertices in $V_2(H')$.

We now have to eliminate the multiple edges. Orient arbitrarily the edges of B^+ joining vertices of $V_2(H)$. Now consider all the vertices of $V_2(H)$ in all the copies of B^+ which form H'. Label all these vertices with natural numbers so that the differences of the labels on all the directed edges are distinct. If the maximum label is t, we form a new graph H'' as follows. The vertex set consists of $V_1(H'')$, which equals $V_1(H')$, and $V_2(H'')$, which consists of t copies of $V_2(H')$. Now for each copy X of B^+ in H', we add t copies $X_1, \ldots, X_t$ of B^+ to H''. To do this, suppose a vertex in X has label α, and in H' it is identified with vertex $w \in V_2(H')$. Then in copy X_i we will identify this vertex with vertex w in copy $\alpha + i$ of $V_2(H')$. Rearranging the vertices in this way makes no difference to the edges joining $V_1(H'')$ to $V_2(H'')$, so every possible edge is still present. However there is now at most one edge joining each pair of vertices in $V_2(H'')$. Note that every such edge joins a vertex w in one copy with a vertex w' in another copy, where $w \neq w'$. Now recall that $|V_2(H')| = r$, and $V_2(H'')$ consists of t copies of $V_2(H')$. If we colour the vertices in each copy with colours $1, \ldots, r$, then no adjacent vertices get the same colour.

Finally we form the graph G by splitting each vertex v of $V_1(H'')$ into its set S_v of constituent vertices from all the copies of B^+ making up H''. The union of the sets S_v forms the set $V_1(G)$, while $V_2(G) = V_2(H'')$. To obtain $V_0(G)$, we add a new home vertex v_0 joined to each vertex of $V_1(G)$. Thus G can be decomposed into edge-disjoint copies of H as required. Furthermore, each set S_v is the same size, and for each v, each element of $V_2(G)$ is joined to exactly one element of S_v.

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The vertices of $V_2(G)$ can be coloured with r colours, and a further two colours are required for $V_1(G)$ and $V_0(G)$, hence $\chi(G) \leqslant r+2$, which is bounded by a function of $|V(H)|$.

This completes the proof. □

We now prove a lemma about packing copies of a graph of bounded chromatic number into a graph formed from a complete graph by removing all 'short' edges.

Definition. Let t be a positive real number, and assume that $n \geqslant 2t+1$. Define the graph $A_n(t)$ to be the graph on vertex set $\{v_1, \ldots, v_n\}$ such that (v_i, v_j) is an edge if and only if

$$t < |i-j| < n-t.$$

Informally, $A_n(t)$ is the complete graph K_n minus any edge which joins two vertices at most t apart in the cyclic ordering.

For the lemma we will need two theorems due to Wilson and Gustavsson, respectively. Before quoting Wilson's theorem we need to describe the notation used. Let q be a prime power, e an integer dividing $q-1$ and $r \geqslant 2$ an integer. Let H^e be the subgroup consisting of eth powers in the multiplicative group of nonzero elements of $GF(q)$. Let P_r be the set of ordered pairs $\{(i,j) | 1 \leqslant i < j \leqslant r\}$. Define a choice to be any map $C : P_r \to H^e$, assigning to each pair $(i,j) \in P_r$ a coset $C(i,j)$ of H^e. An r-tuple $(a_1, \ldots, a_r)$ of elements of $GF(q)$ is consistent with the choice C if and only if $a_j - a_i \in C(i,j)$ for all $1 \leqslant i < j \leqslant r$.

Theorem 4.6. (Wilson [21]) *If $q \equiv 1 (\mathrm{mod}\, e)$ is a prime power and $q > e^{r(r-1)}$, then for any choice $C : P_r \to H^e$, there exists an r-tuple $(a_1, \ldots, a_r)$ of elements of $GF(q)$ consistent with C.* □

Theorem 4.7. (Gustavsson [14]) *For every integer $m \geqslant 3$ there exists a constant $\varepsilon_m > 0$ and an integer N_m, such that if G is a simple graph fulfilling all of the following conditions:*

(i) *$|E(G)|$ is divisible by $\binom{m}{2}$,*
(ii) *for every vertex v, $d(v)$ is divisible by $m-1$,*
(iii) *$|V(G)| > N_m$,*
(iv) *$\delta(G) > (1-\varepsilon_m)\,|V(G)|$,*

then G admits a K_m decomposition.

Lemma 4.8. *Suppose λ is a positive integer and let Γ be the class of graphs with chromatic number at most λ. Let $\varepsilon_\Gamma = \varepsilon_\lambda / 2$, and let ε be a constant such that $0 < \varepsilon < \varepsilon_\Gamma$. Let G be a graph in Γ with m edges. Then for sufficiently large prime numbers q satisfying $q \equiv 1 (\mathrm{mod}\, m\lambda!)$, and sufficient large integers n satisfying $n \equiv q (\mathrm{mod}\, 2q\binom{\lambda}{2})$, $A_n(\varepsilon n)$ contains*

$$(1 - 1/q - O(q/n)) \left(\frac{|E(A_n(\varepsilon n))|}{|E(G)|} \right)$$

edge-disjoint copies of G, such that, for each vertex v in G, the number of copies of v in each vertex w of $A_n(\varepsilon n)$ is independent of v and w.

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Proof. Let G be a λ-partite graph. By taking all possible permutations of the λ parts, we obtain $\lambda!$ graphs. Let G' consist of the union of these graphs. Then G' is λ-partite, with the same number of vertices in each part, and the same number e of edges between each pair of parts. Let $r = |V(G')|$, and suppose that q is a prime such that $q \equiv 1 (\mathrm{mod}\, e)$ and $q > e^{r(r-1)}$ (note that by Dirichlet's theorem on the existence of primes in an arithmetic progression, there certainly are such primes). Define a choice C so that for each pair of parts in G', the e edges joining the two parts are assigned each of the e different cosets, in some order. Then by Wilson's theorem, there is an assignment of labels $(a_1, \ldots, a_r)$ from $GF(q)$ to the vertices of G' which is consistent with C. Let M be the λ-partite graph with q vertices $\{v_0^{(i)}, \ldots, v_{q-1}^{(i)}\}$ in part i for each i, and all edges $(v_k^{(i)}, v_l^{(j)})$ with $k \neq l$ and $i \neq j$. We can identify G' with a subgraph of M in the obvious way; if $c: V(G') \to \{1, \ldots, \lambda\}$ is a colouring of G', map $u \in V(G')$ to $v_{a_u}^{(c(u))}$. Furthermore, if we apply each of the permutations $x \mapsto cx + d$, $c \in H^e$, $d \in GF(q)$ to the labels of G', we obtain a new copy of G' (with $u \in V(G')$ mapped to $v_{ca_u+d}^{(c(u))}$). Together these $q(q-1)/e$ copies of G' form an edge-decomposition of M. Note that the number of copies of a given vertex v of G in a vertex w of M is independent of v and w; in fact it is $(\lambda - 1)!(q-1)/e$.

Now consider the graph $A_n(t)$ defined above, with t an integer. This graph has common degree $n - 1 - 2t$ and has $n(n-1-2t)/2$ edges. Hence, by Gustavsson's theorem, if $n > N_\lambda$, and $n - 1 - 2t > (1 - \varepsilon_\lambda)n$, then $A_n(t)$ admits a K_λ decomposition if, for example, $2\binom{\lambda}{2}$ divides $n - 1 - 2t$.

Now suppose that n is a multiple of q, say $n = qn'$. Choose t to be the least multiple of $\binom{\lambda}{2}$ which is at least $\varepsilon n'$. Then if n' is large enough, $A_{n'}(t)$ admits a K_λ decomposition if $n' \equiv 1 (\mathrm{mod}\, 2\binom{\lambda}{2})$, *i.e.*, if $n \equiv q (\mathrm{mod}\, 2q\binom{\lambda}{2})$. Note that $A_{n'}(\varepsilon n')$ contains every edge of $A_{n'}(t)$, and hence contains all of the edge-disjoint copies of K_λ.

If we replace every vertex of $A_{n'}(\varepsilon n')$ with q vertices and every edge with a complete bipartite graph $K_{q,q}$, we obtain a subgraph of $A_n(\varepsilon n)$. Furthermore this graph contains $|E(A_{n'}(t))|/\binom{\lambda}{2}$ copies of M. Hence the number of edge-disjoint copies of G in $A_n(\varepsilon n)$ is

$$\left(\frac{|E(A_{n'}(t))|}{\binom{\lambda}{2}}\right)\left(\frac{\lambda!q(q-1)}{e}\right).$$

Now $\varepsilon n' \leqslant t < \varepsilon n' + \binom{\lambda}{2}$, and

$$\begin{aligned}
\left(\frac{q^2\,|E(A_{n'}(t))|}{|E(A_n(\varepsilon n))|}\right) &= \left(\frac{q^2 n'(n'-1-2t)}{n(n-1-2\lfloor \varepsilon n \rfloor)}\right) \\
&= \left(\frac{n(n-q-2tq)}{n(n-1-2\lfloor \varepsilon n \rfloor)}\right) \\
&= \left(\frac{n(n-2\varepsilon n - O(\binom{\lambda}{2}q))}{n(n-2\varepsilon n - O(1))}\right) \\
&= \left(\frac{1-2\varepsilon - O(q/n)}{1-2\varepsilon - O(1/n)}\right) \\
&= 1 - O(q/n).
\end{aligned}$$

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Hence we have

$$|E(A_{n'}(t))| = (1 - O(q/n))\,|E(A_n(\varepsilon n))|/q^2.$$

Also $\binom{\lambda}{2}e/\lambda! = |E(G)|$, hence the number of edge-disjoint copies of G in $A_n(\varepsilon n)$ is

$$\begin{aligned}(1 - O(q/n))\left(\frac{|E(A_n(\varepsilon n))|}{q^2}\right)\left(\frac{q(q-1)}{|E(G)|}\right) &= (1 - 1/q)\left(\frac{|E(A_n(\varepsilon n))|}{|E(G)|}\right)(1 - O(q/n)) \\ &= (1 - 1/q - O(q/n))\left(\frac{|E(A_n(\varepsilon n))|}{|E(G)|}\right),\end{aligned}$$

as required. Since the number of copies of a given vertex v of G in a vertex w of M is independent of v and w, the same is true of the number of copies of a given vertex v of G in a vertex w of $A_n(\varepsilon n)$. □

Note that if Γ is any class of graphs for which such a constant ε_Γ exists, then since $A_n(\varepsilon_\Gamma n)$ has chromatic number at most $\lceil 1/\varepsilon_\Gamma \rceil$, the graphs in the class Γ all have chromatic number at most the same constant.

Next we quote Gustavsson's main theorem, on the decomposition of nearly complete digraphs.

Theorem 4.9. (Gustavsson [14]) *For any digraph D there exists an $\varepsilon_D > 0$ and an integer N_D, such that if G is a digraph satisfying:*

(i) *$|E(G)|$ is divisible by $|E(D)|$;*

(ii) *there exist nonnegative integers a_{ij} such that*

$$\sum_{v_i \in V(D)} a_{ij} d^+(v_i) = d^+(u_j)$$

$$\sum_{v_i \in V(D)} a_{ij} d^-(v_i) = d^-(u_j),$$

for every $u_j \in V(G)$;

(iii) *if there exist vertices $u_1, u_2 \in V(G)$ such that $u_1u_2 \in E(G)$, $u_2u_1 \notin E(G)$, then there also exist vertices $v_1, v_2 \in V(D)$ such that $v_1v_2 \in E(D)$, $v_2v_1 \notin E(D)$;*

(iv) *$|V(G)| > N_D$;*

(v) *$\delta^+(G), \delta^-(G) > (1 - \varepsilon_D)\,|V(G)|$;*

then G admits a D-decomposition. □

For undirected graphs, we can obtain digraphs by replacing each undirected edge with a pair of directed edges in opposite directions. Then provided G, and therefore the vertex degrees, are large enough, condition (ii) can be replaced by (ii) $d(u_j)$ is divisible by the greatest common divisor of the vertex degrees of D, for every $u_j \in V(G)$. Also condition (iii) is irrelevant. Hence we obtain the following.

Theorem 4.10. (Gustavsson) *For any graph H there exists an $\varepsilon_H > 0$ and an integer N_H, such that if G is a graph satisfying:*

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(i) $|E(G)|$ *is divisible by* $|E(H)|$;
(ii) *if* m *is the greatest common divisor of the vertex degrees of* H, *then* m *divides* $d(u_j)$ *for every* $u_j \in V(G)$;
(iii) $|V(G)| > N_H$;
(iv) $\delta(G) > (1 - \varepsilon_H)|V(G)|$;

then G *admits a* H*-decomposition.* □

We now prove the main theorem of this section.

Theorem 4.11. *Let* d, d' *be positive integers, and* $\varepsilon_1 > 0$ *be a real number. There exists a real number* $\varepsilon_0 > 0$ *such that for any positive integer* C *there is an integer* N *such that for any graph* $G = (V, E)$ *satisfying the following properties:*
(1) G *is an exact graph with* $\binom{r}{2}$ *edges,*
(2) G *is* (d, d')*-almost-regular,*
(3) G *is* (C, ε_0)*-fragmentable,*
(4) *at least* $\varepsilon_1|V|$ *vertices of* G *have an independent neighbourhood set,*
(5) d *divides* $r - 1 - d'$,
(6) $r \geqslant N$,

then G *is a detachment of* K_r.

Proof. Let $G = (V, E)$ be a graph with n vertices and m edges, and suppose that $m = \binom{r}{2}$, G is (d, d')-almost-regular and (C, ε_0)-fragmentable, and at least $\varepsilon_1 n$ vertices of G have an independent neighbourhood set. Also assume that d divides $r - 1 - d'$. We wish to find an exact r-colouring of G. If $d = 1$ the result is trivial, so henceforth assume that $d \geqslant 2$.

Note that

$$2m = rd' + (n - r)d = nd + r(d' - d).$$

Construction of fragment graph. The first step is to fragment G. Since G is (C, ε_0)-fragmentable, there is a set $J \subseteq V$ (the *junk* set) such that $|J| \leqslant \varepsilon_0 n$ whose removal from G leaves a subgraph in which every component (fragment) has at most C vertices. Note that any superset of J will also fragment G into components of size at most C. We will in fact add a number of vertices to J to obtain the final fragmentation.

Firstly, we will add all vertices of degree d' to J. Provided n is large enough, there are less than $\varepsilon_0 n$ of these.

Secondly, we require that no two vertices in distinct fragments are too close together in G. Therefore we find all vertices which are at distance at most 2 from J, and add them to J. This adds at most $d \max(d, d') 2\varepsilon_0 n$ vertices to J.

Thirdly, for ease of explanation it helps if all the fragments are the same. This can be achieved as follows. We say that two fragments are equivalent if they are isomorphic as graphs, and the set of vertices adjacent to a vertex of J is the same for both fragments.

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Since the fragments all have size at most C, the number of equivalence classes is finite, say K. Fix an integer $M_f = \lfloor \frac{\varepsilon_0 n}{CK} \rfloor$, and discard at most M_f copies from each equivalence class so M_f divides the number of copies in each equivalence class. Add the vertices of the discarded copies to J (note that this has no effect on the equivalence or otherwise of the remaining fragments); there are at most $CKM_f \leqslant \varepsilon_0 n$ of them. Now the remaining fragments can be arranged into M_f new fragments which are all isomorphic to a graph F (and have the same vertices adjacent to J). Also F has C' vertices where $C' < n/M_f \leqslant 2\frac{CK}{\varepsilon_0}$ provided n is large enough.

Define ε_5 by $|J| = \varepsilon_5 n$, so that $\varepsilon_5 \leqslant (2d\max(d, d') + 3)\varepsilon_0$. Thus the number of fragments $M_f = (1 - \varepsilon_5)n/C'$.

We now select a set of vertices which play an important role in the colouring procedure. Recall that by hypothesis there are at least $\varepsilon_1 n$ vertices of G with independent neighbourhood sets. Now consider the N2-graph induced by such a vertex and its first and second neighbourhoods (the latter is non-empty since we are assuming that $d \geqslant 2$). We consider isomorphisms between N2-graphs which preserve the partition V_0, V_1, V_2 of the vertex set. Since G has maximum degree d, the number of isomorphism classes is a constant I_d. Hence for at least $(\varepsilon_1/I_d)n$ vertices v of G, the subgraph $N_2(v)$ induced by the vertex and its first and second neighbourhood is isomorphic to a fixed N2-graph N. We next pick a subset of these vertices v so that the graphs $N_2(v)$ are at distance at least two from each other and from vertices which are adjacent to the set J. We can ensure this by requiring that the vertices v are at distance at least 4 from vertices which are adjacent to the set J and distance at least 6 from each other. The first requirement disallows at most $d^5\varepsilon_5 n$ vertices. Provided ε_5 is small enough, this leaves at least $(\varepsilon_1/2I_d)n$ vertices. By constructing a graph on these vertices by joining any pair at distance at most 6 in G, and choosing a maximal independent set, we obtain a set L of vertices with the desired properties, of size at least $(\varepsilon_1/2I_d d^6)n$. The graphs $N_2(v)$ for vertices $v \in L$ will be called L-graphs. Let ℓ be the number of L-graphs in a fragment F. Then clearly $\ell \geqslant 1$.

Now remove $\lfloor \frac{\varepsilon_0 n}{C'} \rfloor$ fragments and add their vertices to J. There are at most $\varepsilon_0 n$ such vertices. This ensures that at least $\lfloor \frac{\varepsilon_0 n}{C'} \rfloor$ vertices of J have independent neighbourhood sets. Also now $|J| \leqslant \varepsilon_5 n + \varepsilon_0 n$. Further, provided ε_0 is small we have certainly removed at most half of the fragments, so we still have $|L| \geqslant (\varepsilon_1/4I_d d^6)n$. Finally note that we have added to J at least $\varepsilon_0 n - C'$ vertices all of which are distance at least 4 from any vertex of J which is adjacent to any remaining fragment.

We now form larger fragments by taking sets of fragments and identifying vertices. Consider the set L of vertices each with its first and second neighbourhood graph isomorphic to N. Note that N has at most $d^2 + 1$ vertices. By Lemma 4.4, we find that there is an N2-graph N' such that (i) N decomposes N', (ii) the set $V_1(N')$ can be partitioned into subsets $X_1, \ldots, X_b$ each of size a say, such that each element of $V_2(N')$ is joined to exactly one element from each X_i, and (iii) the chromatic number $\chi(N')$ is bounded by the function f of $|V(N)|$, and hence by a function of d. Let $D = |V_1(N')|$. By the lemma, we can also choose N' so that D is as large as we wish. Let $\lambda = f(|V(N)|) + d + 1$. We choose D just large enough so that the number of mutually orthogonal latin squares of order D is at least $\lambda - 1$. This is true if D is at least λ^{15}, say; see for example [3]. By Proposition 1.1 of Häggkvist [15], we then have that DK_λ (*i.e.*, D disjoint copies of K_λ)

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decomposes

$$K_\lambda(D) = K_{\underbrace{D,\ldots,D}_{\lambda}},$$

the complete λ-partite graph with each part having D vertices.

Let T be the number of copies of N in N'. Now let ε_λ be the constant from Theorem 4.7. We need to ensure that $M_f\ell = |L| < T\varepsilon_\lambda m/2D$. If $\ell = 1$, then we have $M_f < \varepsilon_0 n < T\varepsilon_\lambda m/2D$ provided that ε_0 is small enough (bounded by a function of d). Otherwise we can reduce ℓ by removing from L a certain number of elements in each fragment. In this case we will eventually have $T\varepsilon_\lambda m/4D \leqslant |L| < T\varepsilon_\lambda m/2D$. Thus we now have $|L| < T\varepsilon_\lambda m/2D$ and either $|L| \geqslant T\varepsilon_\lambda m/4D$ or $|L| \geqslant \varepsilon_1 n/4I_d d^6$.

Now by discarding at most D^2T fragments if necessary (and adding their vertices to J), we ensure that the number of fragments M_f is a multiple of D^2T. Let $M' = M_f/D^2T$. Then we take sets of T fragments, and for each L-graph N in a fragment F, we identify vertices in the T copies of F to obtain a copy of N', as above using Lemma 4.4. Thus we obtain a set of $M'D^2$ new fragments, each isomorphic to F', say. Denote by FG (the *fragment graph*) the graph consisting of these $M'D^2$ fragments. Partition FG into M' subgraphs $\mathrm{FG}(1),\ldots,\mathrm{FG}(M')$, each consisting of D^2 fragments isomorphic to F'. Let L' be the set of copies of N' obtained from the L-graphs, and define ε_8 by $|L'| = \varepsilon_8 m$. Note that $|L'| = |L|/T$, so $\varepsilon_8 < \varepsilon_\lambda/2D$. Also we have either $\varepsilon_8 \geqslant \varepsilon_\lambda/4D$ or $\varepsilon_8 \geqslant \varepsilon_1 n/4I_d d^6 Tm$. In either case, ε_8 is bounded below by a constant depending only on d, d' and ε_1.

The number of vertices added to J is bounded by a constant, so is certainly at most $\varepsilon_0 n$ if n is large enough. So now $|J| \leqslant \varepsilon_5 n + 2\varepsilon_0 n$. Let $\varepsilon_{10} = |J|/n$, so that $\varepsilon_{10} \leqslant \varepsilon_5 + 2\varepsilon_0 \leqslant (2d\max(d,d') + 5)\varepsilon_0$.

We note that the chromatic number of F' is at most λ. For F has maximum degree d so that $\chi(F) \leqslant d+1$, hence also $\chi(TF) \leqslant d+1$. But since the copies of N' are separated by distance at least two in F', it is clear that $\chi(F') \leqslant \chi(TF) + \chi(N') \leqslant d + 1 + \chi(N') \leqslant d + 1 + f(|V(N)|) = \lambda$.

Super-fragment graph. First we colour all the fragments, apart from the home vertices. Firstly, from the graph F' we form graph F'' as follows. Recall that $V_1(N')$ consists of D vertices arranged in sets $X_1,\ldots,X_b$ each of size a. To form N'' we delete $V_0(N')$, and identify the vertices in each set X_i to form b new vertices, each joined to every vertex of $V_2(N')$. Now to form F'', we replace each copy of N' in F' by a copy of N''. Note that $\chi(N'')$ is also bounded by $f(|V(N)|)$, so $\chi(F'') \leqslant d + 1 + f(|V(N)|) = \lambda$. Note also that $|E(N'')| = |E(N')| - D$.

For the colouring of the fragments, we construct a new graph SFG (the *super-fragment graph*) from FG as follows. Replace each $\mathrm{FG}(i)$ with a graph $\mathrm{SFG}(i)$ isomorphic to F''. Then

$$\begin{aligned} |E(\mathrm{SFG})| &= (|E(\mathrm{FG})| - D\,|L'|)/D^2 \\ &= (|E(\mathrm{FG})|/D^2) - \varepsilon_8 m/D. \end{aligned}$$

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But

$$\begin{aligned} m &\geqslant |E(\mathrm{FG})| \geqslant m - \max(d, d')\,|J| \\ &= m - \max(d, d')\varepsilon_{10} n \\ &\geqslant m - 2\max(d, d')\varepsilon_{10} m/\min(d, d'), \end{aligned}$$

hence

$$(1 - D\varepsilon_8)m/D^2 \geqslant |E(\mathrm{SFG})| \geqslant (1 - D\varepsilon_8 - 2\max(d, d')\varepsilon_{10}/\min(d, d'))m/D^2.$$

Now let Γ be the class of graphs with chromatic number at most λ, and let $\varepsilon_\Gamma = \varepsilon_\lambda/2$ as in Lemma 4.8. Let e be the number of edges in F''.

Colours, supercolours and shades. We now consider the colours to be used in the exact colouring. There are r colours in all, in three sets. We first designate $q' = r \mod D$ of them as residue colours. Next choose a prime q_0, so that $q_0 \equiv 1 (\mathrm{mod}\, e\lambda!)$ and q_0 is about $n^{1/4}$ (say $n^{1/4} < q_0 < 2n^{1/4}$; this will be possible if n is large enough). The remaining colours are divided into p standard colours and q special colours. These will be chosen so that D divides p and q, and such that $p/D \equiv q_0 (\mathrm{mod}\, 2q_0\binom{\lambda}{2})$. The standard colours are partitioned into $P = p/D$ sets called (standard) *supercolours*; similarly we have $Q = q/D$ special supercolours. We define ε_4 by $q + q' = \varepsilon_4 r$, so that $p = (1 - \varepsilon_4)r$. Note that we will choose q and p so that ε_4 is small compared to ε_8, but can still be bounded below in terms of ε_1, d and d' (so q is much larger than its minimum possible value if n is large). We will denote the set of colours by S, standard colours by S', special colours by S'' and residue colours by S'''. Similarly we denote the set of supercolours by Σ, standard supercolours by Σ' and special supercolours by Σ''. A supercolour σ contains D colours $\sigma_1, \ldots, \sigma_D$. We will call these the *shades* of σ.

Colouring the super-fragment graph. First define ε_3 by

$$(1 - \varepsilon_4)^2(1 - 2\varepsilon_3) = \frac{D^2\,|E(\mathrm{SFG})|}{m}.$$

As noted above, the number of edges in SFG satisfies

$$(1 - D\varepsilon_8)m/D^2 \geqslant |E(\mathrm{SFG})| \geqslant (1 - D\varepsilon_8 - 2\max(d, d')\varepsilon_{10}/\min(d, d'))m/D^2,$$

so that

$$(1 - D\varepsilon_8) \geqslant (1 - \varepsilon_4)^2(1 - 2\varepsilon_3) \geqslant (1 - D\varepsilon_8 - 2\max(d, d')\varepsilon_{10}/\min(d, d')).$$

Hence, provided $2\max(d, d')\varepsilon_{10}/\min(d, d') < D\varepsilon_8/2$, we have

$$D\varepsilon_8 \leqslant \varepsilon_3\big(2 - 2\varepsilon_4 + \varepsilon_4^2\big) + 2\varepsilon_4 - \varepsilon_4^2 \leqslant 3D\varepsilon_8/2.$$

So, provided ε_4 is small compared with $D\varepsilon_8$, we can conclude that $\varepsilon_3 < D\varepsilon_8 < \varepsilon_\lambda/2 = \varepsilon_\Gamma$.

Now number the standard supercolours Σ' as $1, \ldots, P$, and consider the graph $A_P(\varepsilon_3 P)$ with these colours as vertices. By Lemma 4.8, if P is large enough, then $A_P(\varepsilon_3 P)$ can be

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decomposed into

$$(1 - 1/q_0 - O(q_0/P))\left(\frac{|E(A_P(\varepsilon_3 P))|}{|E(F'')|}\right)$$

edge-disjoint copies of F''. This gives a harmonious colouring of these copies of F'' with P (super) colours. The number of (super)-coloured edges is therefore at least

$$\begin{aligned}(1 - 1/q_0 - O(q_0/P))\,|E(A_P(\varepsilon_3 P))| &= (1 - O(1/P^{1/2}))\,|E(A_P(\varepsilon_3 P))| \\ &= |E(A_P(\varepsilon_3 P))| - O(P^{3/2}) \\ &= P(P - 1 - 2\lfloor \varepsilon_3 P \rfloor)/2 - O(P^{3/2}) \\ &= P^2(1 - 2\varepsilon_3)/2 - O(P) - O(P^{3/2}) \\ &= p^2(1 - 2\varepsilon_3)/2D^2 - O(P^{3/2}) \\ &= (1 - \varepsilon_4)^2(1 - 2\varepsilon_3)r^2/2D^2 - O(P^{3/2}) \\ &= (1 - \varepsilon_4)^2(1 - 2\varepsilon_3)m/D^2 + O(r) - O(P^{3/2}) \\ &= (1 - \varepsilon_4)^2(1 - 2\varepsilon_3)m/D^2 - O(P^{3/2}) \\ &= |E(\mathrm{SFG})| - O(P^{3/2}).\end{aligned}$$

Furthermore, for each vertex v of F'', the copies of v receive each supercolour the same number of times. For any uncoloured graphs SFG(i) in SFG, we add the vertices of FG(i) to J. There will be at most $O(P^{3/2})$ such vertices, hence certainly at most $\varepsilon_0 n$ if n is large enough.

No further vertices will be added to J. Let $\varepsilon_{12} = |J|/n$, so that $\varepsilon_{12} \leqslant (2d \max(d, d') + 6)\varepsilon_0$.

Colouring the vertices of degree d'. We next colour some vertices in the set J. Note that in any exact colouring with r colours, each colour must have degree sum $r - 1 \equiv d' (\mathrm{mod}\, d)$. Recall that there are r vertices of degree d', all in the set J. We use each colour for one of the vertices of degree d', so that the set of vertices coloured with a special or residue colour is independent. (This is easily seen to be possible provided that $\varepsilon_4 \leqslant 1/(d' + 1)$.)

(Σ', Σ')-, (S''', S')- and (S''', S''')-pairs. We then have to ensure (i) that if we have used a pair (σ_i, τ_j) when colouring the degree d' vertices, for some standard supercolours σ, τ, then we use all D^2 such pairs. We also need to use (ii) all unused pairs from a single standard supercolour, *i.e.*, $(\sigma_i, \sigma_j), i \neq j, \sigma \in \Sigma'$, and (iii) all pairs (s, s') where s, s' are distinct residue colours. Let UP be the set of all these pairs. Note that we have at most $D^2 d' r$ pairs from (i), plus $D^2 r$ from (ii) and D^2 from (iii), so that $|\mathrm{UP}| = O(r)$. To use the pairs in UP we find a collection of $|\mathrm{UP}|$ edges in $G[J]$ which are at distance at least 4 from vertices of J adjacent to fragments, from vertices of degree d', and from each other (this is obviously possible if n is large enough). We colour the endpoints of these edges so that each pair from UP occurs on an edge.

Also, for each residue colour s, let UP_s be the set of all standard colours s' such that (s, s') has not been used on $G[J]$. We find a set of $\sum_s \lceil |\mathrm{UP}_s|/d \rceil$ vertices in J with independent neighbourhoods, which are all at distance at least 4 in G from any vertex already coloured, from each other, and from the fragments. Then for each residue colour s

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we colour $\lceil |\mathrm{UP}_s|/d \rceil$ vertices with colour s and colour $|\mathrm{UP}_s|$ of the neighbours of these vertices with the elements of UP_s. As above this is easily achieved if n is large enough. Note that $\sum_s |\mathrm{UP}_s| < Dr = O(r)$, so the total number of coloured vertices is still $O(r)$.

Now, using special colours, we colour all those uncoloured vertices in J which are adjacent to a vertex coloured with a standard or residue colour, ensuring that we do not use any colour pair already used on $G[J]$ and that each special colour has restricted degree sum. To do this we apply Lemma 4.2, with C_1 the set of standard and residue colours, and C_2 the special colours. Let X be the set of coloured vertices and Y the set of uncoloured vertices adjacent to some member of X_1, where X_1 and X_2 are as defined in Lemma 4.2. Let $s = \sigma_i$ be some standard colour. Then the number of pairs used so far involving s can be estimated as follows.

(a) In colouring the vertices of degree d', at most $d'D$ pairs involving a shade of σ and a shade of another standard supercolour are used. Hence we now have at most $d'D^2$ such pairs.
(b) There are $D-1$ pairs consisting of s and another shade of σ.
(c) There are at most $D-1$ pairs consisting of s and a residue colour.

Together with one vertex of degree d' this means that the number of vertices coloured s is at most $d'D^2 + 2D - 1$. Hence the degree sum of s in $X_2 \cup Y$ is at most $\beta = d(d'D^2 + 2D - 2) + d'$. It is easily seen that for a residue colour s, $\mathrm{ds}_{c_0}(s, X_2 \cup Y)$ is at most $(D-1)d + d'$. Similarly for any special colour s, $\mathrm{ds}_{c_0}(s) \leqslant d'$. Hence $|X_2 \cup Y| \leqslant r\beta$. Then if we take $\theta = \lceil \frac{2\beta d}{\varepsilon_4} \rceil + d$ (so that $\theta \geqslant \beta$), it follows easily that provided n (and hence r) is big enough, then we will have

$$\varepsilon_4 r - q' > d^2 + d\theta + r\beta d/(\theta - d),$$

as required by Lemma 4.2 to allow us to colour the vertices of Y so that each special colour has degree sum at most θ.

(Σ', S'')- and (S''', S'')-pairs. Now, similarly, we must ensure that if we have used a pair (s, s'), for some standard colour s which is a shade of supercolour σ, and some special colour s', then we use all the pairs (σ_i, s'). For each standard supercolour σ, let adj_σ be the set of special colours which occur adjacent to some shade of σ. Then $|adj_\sigma| \leqslant D\beta \leqslant D\theta$, and each special colour occurs in at most θ sets adj_σ. By adding up to d elements to each adj_σ as necessary, we can ensure that for each σ, d divides $|adj_\sigma|$, and that each special colour occurs in at most 2θ sets adj_σ. This is easy to achieve. Let NA_s be the set of all special colours s' such that (i) $s' \in adj_\sigma$, where σ is the supercolour containing s, and (ii) (s, s') has not been used on $G[J]$. Then d divides $|\mathrm{NA}_s|$. For suppose there are γ_s vertices coloured s in J. Each of them has every neighbour coloured, and their total degree is $(\gamma_s - 1)d + d'$. The colour of a neighbour is one of the following.

(a) A shade of another standard supercolour. The number of these is divisible by D.
(b) Another shade of σ. There are $D-1$ of these.
(c) A special colour. Let the number of these be γ'_s.
(d) A residue colour. There are q' of these.

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Hence since d divides D, we find that $\gamma'_s + q' - 1 \equiv d' (\mathrm{mod}\, d)$. Since also $q' - 1 \equiv r - 1 \equiv d' (\mathrm{mod}\, d)$, then d divides γ'_s. But then d divides $|\mathrm{NA}_s| = |adj_\sigma| - \gamma'_s$, as required. Note also that $|\mathrm{NA}_s| \leqslant D\theta$.

Also, for each residue colour s, let NA_s now be the set of all special colours s' such that (s, s') has not been used on $G[J]$. Then, since s has been used adjacent to every standard colour and residue colour (except s itself), and since all vertices adjacent to a residue colour are themselves coloured, we see that

$$p + (q - |\mathrm{NA}_s|) + q' - 1 \equiv d' (\mathrm{mod}\, d),$$

from which it follows easily that d divides $|\mathrm{NA}_s|$. Also, for a residue colour s, $|\mathrm{NA}_s| \leqslant q$.

We find a set of $\sum |\mathrm{NA}_s|/d$ vertices in J with independent neighbourhoods, which are all at distance at least 4 in G from any vertex already coloured, from each other, and from the fragments. Then for each standard or residue colour s we colour $|\mathrm{NA}_s|/d$ with colour s and colour the neighbours of these vertices with the elements of NA_s. Note that $\sum |\mathrm{NA}_s| \leqslant D\theta p + Dq = O(r)$, so the number of coloured vertices is still $O(r)$.

We have now used all colour pairs involving residue colours. The degree sums of all the other colours are bounded by constants depending only on d, d' and ε_4.

Adjusting the colouring of the super-fragment graph. We then recolour a certain set R of vertices in SFG. First place in R any vertex which is adjacent to a vertex of J. Also, suppose that an edge has endpoints coloured with supercolours σ, τ, where some pair (σ_i, τ_j) of shades of these supercolours has already been used on J. Then we remove the colour from one of the endvertices of any such edge, so that the pair (σ, τ) is no longer used. If one of the endvertices is in a copy of N'', we remove the colour from the other end vertex. If both endvertices are in a copy of N'', we also replace it with a copy of N', and add the home vertex of this N' to R. For any standard supercolour σ, any shade of σ has been used adjacent to some shade of at most d' other standard supercolours. Hence, for any supercolour σ, at most Dd' copies of N'' with a vertex coloured σ are replaced in this way.

For the vertices in R we use the special supercolours Σ''. We do this colouring in such a way that the colouring remains harmonious and we do not use any pair (σ, τ) if some pair (σ_i, τ_j) has already been used. To ensure this we apply Lemma 4.2 to a graph consisting of SFG, together with a temporary collection of isolated edges which use all pairs (σ, τ) such that some pair (σ_i, τ_j) is used on J. Take c_0 to be the partial colouring of SFG and these temporary edges, C_1 to be the standard supercolours, C_2 the special supercolours, X the coloured vertices, and $Y = R$. The number of vertices in R adjacent to a vertex of J is at most $d\varepsilon_{12} n$, and the number of extra temporary vertices of each supercolour is bounded by a constant depending on d, d' and ε_4. Hence $|R| \leqslant (d+1)\varepsilon_{12} n$ if n is large enough. Also the total degree sum of all the standard supercolours in $X_2 \cup Y$ is at most $d^2\varepsilon_{12} n$ plus r times a constant depending on d, d' and ε_4. Because each vertex of F'' receives each supercolour an equal number of times, the degree sum of each standard supercolour is at most $d^2\varepsilon_{12} n/P$ plus a constant. We choose θ to be $\varepsilon_{11} Q$, where ε_{11} is a constant. Then provided that ε_{11} is small enough compared to $1/d$, and ε_{12} is small enough compared

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with ε_{11}, by Lemma 4.2 we can colour the vertices of R with special supercolours so that each special supercolour has degree sum at most $\varepsilon_{11}Q$. Discard the temporary edges.

Colouring the fragments. Now we use the supercolouring of each SFG(i) to obtain a colouring of FG(i), except for the home vertices. Recall that D was chosen large enough so that DK_λ decomposes

$$K_\lambda(D) = K_{\underbrace{D,\ldots,D}_{\lambda}}.$$

Let $S_D = \{1,\ldots,D\}$. Label the vertices of $K_\lambda(D)$ as $v(i,j)$, $i = 1,\ldots,D, j = 1,\ldots,\lambda$ in the obvious way. Similarly label the vertices of DK_λ as $w(i,j)$, $i = 1,\ldots,D, j = 1,\ldots,\lambda$. There are D copies $Y^{(1)},\ldots,Y^{(D)}$ of DK_λ in the decomposition. From each copy $Y^{(k)}$, we obtain λ permutations $\rho_1^{(k)},\ldots,\rho_\lambda^{(k)}$ of S_D such that vertex $w(i,j)$ of $Y^{(k)}$ is equal to vertex $v(\rho_j^{(k)}(i),j)$ of $K_\lambda(D)$.

We note that for any distinct j,j', the map $\mu_{j,j'} : S_D \times S_D \to S_D \times S_D$ given by $(i,k) \mapsto (\rho_j^{(k)}(i),\rho_{j'}^{(k)}(i))$ is a permutation. For otherwise there exist $(i,k) \neq (i',k')$ such that $\mu_{j,j'}(i,k) = \mu_{j,j'}(i',k')$. But this means that vertices $w(i,j),w(i,j')$ of $Y^{(k)}$ and vertices $w(i',j),w(i',j')$ of $Y^{(k')}$ are mapped to the same pair of vertices, which is impossible since the copies of DK_λ are edge-disjoint.

We will regard the D^2 copies of F' as being D copies $Z^{(1)},\ldots,Z^{(D)}$ of DF' where $Z^{(k)}$ consists of D copies $Z_1^{(k)},\ldots,Z_D^{(k)}$ of F'.

Choose a proper vertex colouring c of F' with colours $1,\ldots,\lambda$. We use c to continue the construction of an exact colouring. First we will colour the vertices which are not home vertices or their neighbours. To colour a vertex v in copy $Z_i^{(k)}$ of F', we find the supercolour σ of the corresponding vertex v in F'', and let $j = c(v)$. Then let $s = \rho_j^{(k)}(i)$, and colour v with colour σ_s.

Now consider the D copies of vertex v in $Z^{(k)}$. Since $\rho_j^{(k)}$ is a permutation of $1,\ldots,D$, these vertices are given the D shades of σ once each.

Secondly, consider two adjacent vertices v,v' of F'', and suppose that these are given supercolours σ,σ', and colours $c(v) = j, c(v') = j'$. Then over the D^2 copies of F', vertices v and v' are given shades σ_s, $\sigma'_{s'}$ respectively, where $(s,s') = (\rho_j^{(k)}(i),\rho_{j'}^{(k)}(i))$. Then, as noted above, as the pair (i,k) ranges over the set $S_D \times S_D$, so does the pair (s,s'). Hence in D^2F' each shade of σ occurs adjacent to each shade of σ' exactly once.

Now we have to colour the neighbours of home vertices. Consider a particular home vertex v and its N2-graph N'_v. Recall that $V_1(N'_v)$ consists of D vertices arranged in sets $X_1,\ldots,X_b$ each of size a. In F'' each N'_v is replaced by N''_v, in which the vertices in each set X_i to form b new vertices, each joined to every vertex of $V_2(N'_v)$. These vertices receive b distinct supercolours (they must be distinct because the vertices have common neighbours).

We have to use these b supercolours to provide colours for the vertices of $V_1(N'_v)$ in the $D^2 = Dab$ copies of F' which make up FG(i). Hence we use each of these supercolours σ to provide colours for the vertices of $V_1(N'_v)$ in a copies of DF', as follows. Consider one copy $Z^{(k)}$. For this copy we assign the D shades of σ to the vertices of $V_1(N'_v)$ arbitrarily, but in the same way for each of $Z_1^{(k)},\ldots,Z_D^{(k)}$. Note that the colour of a particular vertex of $V_2(N'_v)$ ranges in $Z^{(k)}$ over all shades of some supercolour. Now for the other copies of

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DF', we cyclically rotate the colours within each set $X_1, \ldots, X_b$. Since the vertices of each X_i are collectively adjacent to each element of $V_2(N'_v)$ once each, then each shade of σ will occur exactly once adjacent to each shade of each of the supercolours used on $V_2(N'_v)$.

Superleaf graphs. Now each FG(i), $i = 1, \ldots, M'$, consists of b copies FG(i, j), $j = 1, \ldots, b$ of aDF' coloured as described above. From each copy FG(i, j) we form an amalgamation $F^*(i, j)$ as follows: For each home vertex v, consider the aD copies of the N2-graph N'_v, and identify all the vertices in these copies which have the same colour. In fact no colour can occur both on $V_1(N'_v)$ and $V_2(N'_v)$. Let the graph formed from the aD copies of N'_v be N^*_v. We will call N^*_v a superleaf graph. The graph N^*_v has the following properties.

(i) The vertices are in three sets $V_0(N^*_v)$, $V_1(N^*_v)$ and $V_2(N^*_v)$, with $|V_0(N^*_v)| = aD$ and $|V_1(N^*_v)| = D$.
(ii) Every vertex of $V_1(N^*_v)$ is is joined to every vertex of $V_2(N^*_v)$.
(iii) The graph induced by $V_0(N^*_v)$ and $V_1(N^*_v)$ is the complete bipartite graph $K_{aD,D}$.

Now the fragment graph consists of the graphs $F^*(i, j)$, $i = 1, \ldots, M'$, $j = 1, \ldots, b$.

The colouring of the fragments achieves several things.

(1) For any two supercolours σ, τ, either no pairs (σ_i, τ_j) occur on any edge, or all D^2 pairs do exactly once each.
(2) In each superleaf graph N^*_v, the D vertices of $V_1(N^*_v)$ are coloured with the D shades of some standard supercolour, and the vertices of $V_2(N^*_v)$ are coloured with the shades of some set of supercolours.

N^*_v will be called a σ-superleaf graph if $V_1(N^*_v)$ is coloured with the shades of σ. The number of σ-superleaf graphs is almost independent of the standard supercolour σ, differing by at most a constant. By considering the number of copies of N', it is easy to see that the number of σ-superleaf graphs is $(\varepsilon_8 m/DPa) - O(1)$.

We know that

$$D\varepsilon_8 = 1 - (1 - \varepsilon_4)^2(1 - 2\varepsilon_3) - O(\varepsilon_{10}),$$

from which

$$D\varepsilon_8 = 2\varepsilon_3 + 2\varepsilon_4 - O\left(\varepsilon_4^2\right) - O(\varepsilon_3\varepsilon_4) - O(\varepsilon_0).$$

Now

$$\begin{aligned}
\frac{\varepsilon_8 m}{DPa} - O(1) &= \frac{\varepsilon_8 r(r-1)}{2DPa} - O(1) \\
&= \frac{\varepsilon_8 (r-1)}{2(1-\varepsilon_4)a} - O(1) \\
&= \frac{\varepsilon_8 r}{2(1-\varepsilon_4)a} - O(1) \\
&= \frac{\varepsilon_8 PD}{2(1-\varepsilon_4)^2 a} - O(1) \\
&= \frac{P}{a}\left(\frac{\varepsilon_3 + \varepsilon_4 - O\left(\varepsilon_4^2\right) - O(\varepsilon_3\varepsilon_4) - O(\varepsilon_0)}{(1-\varepsilon_4)^2}\right) - O(1) \\
&= \frac{P}{a}\left(\varepsilon_3 + \varepsilon_4 - O\left(\varepsilon_4^2\right) - O(\varepsilon_3\varepsilon_4) - O(\varepsilon_0)\right) - O(1).
\end{aligned}$$

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Hence the number of σ-superleaf graphs is

$$\frac{P}{a}\left(\varepsilon_3+\varepsilon_4-O\left(\varepsilon_4^2\right)-O(\varepsilon_3\varepsilon_4)-O(\varepsilon_0)\right)-O(1).$$

Completion of colouring of fragmenting set. We must now complete the colouring of the set J. Firstly, for each standard supercolour σ, there is a set U_σ of colours s such that for each shade σ_i of σ, the set of those colours which have not been used adjacent to σ_i is precisely U_σ. Note that $|U_\sigma|$ is a multiple of d. We wish it to be a multiple of D^3. This can be achieved by finding (for each σ) at most D^3 special colours $s' \notin U_\sigma$. Then, by finding a set of vertices in J suitably isolated from coloured vertices and each other, we can easily use up all these colour pairs (σ_i, s'). Now the number of unused colour pairs involving a standard colour is a multiple of D^3, say k_1D^3. Note that all uncoloured vertices of J are adjacent only to special colours.

We now complete the colouring of the set J using only special colours for the remaining uncoloured vertices. In doing this, we must ensure that the degree sum of each special colour is at most $\varepsilon_6 q$. Also, for each special colour s, let α_s be the number of special colours $s' \neq s$ for which the colour pair (s, s') has not yet been used for any edge. We must ensure that after colouring J, each α_s is divisible by D^3.

We do this as follows. Let c_0 be the partial colouring so far, and let Y be the set of uncoloured vertices of J. For each special colour s, let t_s be the number of vertices in Y adjacent to a vertex coloured s. Now, for each $s \in S'$, we have used the following colours adjacent to s. (a) q' residue colours, (b) every shade of, say, k_3 standard supercolours, *i.e.*, k_3D standard colours, (c) $q-1-\alpha_s$ special colours. Hence

$$d_{c_0}(s) = t_s + q' + k_3D + q - 1 - \alpha_s.$$

Then

$$\alpha_s - t_s \equiv (r-1) - d_{c_0}(s) \equiv d' - d' = 0 (\mathrm{mod}\, d)$$

Hence $\alpha_s - t_s$ is divisible by d. We wish to make it divisible by D^3. To do this pick, for each s, a set of $(\alpha_s - t_s)/d \mod (D^3/d)$ vertices of Y and extend c_0 by colouring them with colour s. We choose these vertices so that they are all at distance at least 3 from each other and any previously coloured vertex. This is easily done provided n is large enough. Now if we again set Y to be the set of uncoloured vertices and t_s to be the number of vertices in Y adjacent to a vertex coloured s, then it is easily calculated that $\alpha_s - t_s$ is divisible by D^3.

Now the uncoloured edges in the graph are the following.

(a) Those in the superleaf graphs. The number of these is a multiple of D^3, say k_2D^3.
(b) Uncoloured edges in $G[J]$. The number of these is $\frac{1}{2}(\sum_{s\in S'} t_s + d\,|Y|)$.

On the other hand, the number of unused colour pairs in the graph is $k_1D^3 + \frac{1}{2}\sum_{s\in S'} \alpha_s$. Since the number of edges equals the number of colour pairs, we have

$$k_2D^3 + \tfrac{1}{2}(\textstyle\sum_{s\in S'} t_s + d\,|Y|) = k_1D^3 + \tfrac{1}{2}\sum_{s\in S'}\alpha_s,$$

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hence

$$\sum_{s\in S'}(\alpha_s - t_s) = d\,|Y| + 2(k_2 - k_1)D^3.$$

It follows that $|Y|$ is a multiple of D^3/d.

Now we can apply Lemma 4.2 to colour Y with new temporary colours so that each new colour is used on exactly D^3/d vertices. By Lemma 4.2 this is easily done provided that Y is large enough, *i.e.*, provided n is large enough. Now identify vertices of Y with the same colour, and discard the temporary colours, to obtain a new set Y of vertices of degree D^3. Note that identifying vertices in this way creates an amalgamation of the graph, and the partial harmonious colouring c_0 is still valid.

Now $d_{c_0}(s)$ is at most $D\varepsilon_{11}q + k_4$ where k_4 is a constant depending on d, d' and ε_4. Set $\theta = \varepsilon_6 q$. Set $C_1 = S' \cup S'''$, and $C_2 = S''$, so that X_1 consists of the vertices with standard or residue colours, and X_2 the vertices with special colours. Then we use Lemma 4.2 to colour Y and so complete the colouring of J. This is possible provided that

$$q > D^6 + D^3\theta + |X_2 \cup Y|D^3/(\theta - D^3).$$

This is true provided that ε_6 is small, and ε_{11} and ε_0 are small enough compared with ε_6.

Now, for each s, the number β_s of special colours which have not been used adjacent to s is $\alpha_s - t_s$ minus a multiple of D^3. Hence D^3 divides β_s. Also the degree sum of each special colour is at most $\varepsilon_6 q$. Thus, for each s, $\beta_s \geqslant q - 1 - \varepsilon_6 q > q(1 - 2\varepsilon_6)$ for large enough q.

Remaining (S'', S'') pairs. Now construct a graph U of unused special colour pairs. This graph has vertex set S'', and an edge joining s and s' if and only if the colour pair (s, s') has not yet been used for any edge. Thus the degree $d_U(s)$ of s in U is equal to β_s. Then by Theorem 4.10 we can find an decomposition of U into edge-disjoint copies of $H = K_{aD,D}$, provided that ε_6 is at most $\frac{1}{2}\varepsilon_H$. This gives a colouring of at most $\binom{q}{2}/aD^2$ copies of $K_{aD,D}$ such that each unused pair of special colours occurs on an edge. Assign to each copy a standard supercolour, so that each supercolour is assigned to the same number of copies (± 1).

Now consider the copies of $K_{D,aD}$ which have been assigned the supercolour σ. For each copy let the vertex set be $B_1 \cup B_2$, where $|B_1| = D$, $|B_2| = aD$. Now choose a σ-superleaf graph N^*, recolour the vertices $V_1(N^*)$ with the colours used on B_1, and colour the vertices $V_0(N^*)$ with the colours used on B_2.

To see that this can be done, note from above that the number of σ-superleaf graphs is at least $\frac{\varepsilon_8 m}{aDP} - O(1)$. The number of copies of $K_{D,aD}$ is at most $\binom{q}{2}/aD^2 \leqslant \varepsilon_4^2 r^2/2aD^2$. Hence the number assigned each supercolour is at most $\lceil \varepsilon_4^2 r^2 D/2aD^2 p\rceil = \lceil \varepsilon_4^2 r/2aD(1-\varepsilon_4)\rceil$. Also, the degree sum of each of the colours on B_1 is at most $\varepsilon_6 q$, hence the number of supercolours adjacent to one of these colours is also at most $\varepsilon_6 q \leqslant \varepsilon_6\varepsilon_4 r$. Hence the number of σ-superleaf graphs where the colours of B_1 cannot be used is at most $\varepsilon_6\varepsilon_4 r$. Provided ε_4 is small enough compared with ε_1, and hence compared with ε_8, it is clear that we can choose the required copies of N^*.

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We have now used all colour pairs consisting of two special colours.

Note that the standard colour pairs which are removed by this recolouring are whole sets of pairs of shades of two standard supercolours. The new pairs also occur in sets; if a special colour s is used next to a shade σ_i then in fact D special colours are used next to every shade of σ. Thus the size of each set U_σ is still a multiple of D.

Remaining pairs involving a special colour. Now for each standard supercolour σ we have that $|U_\sigma|$ is divisible by D. Let U'_σ be the set of special colours in U_σ. Since the number of standard colours in U_σ is a multiple of D, then $|U'_\sigma|$ is a multiple of D also.

The only uncoloured vertices are now those in the sets $V_0(N^*)$ of superleaf graphs N^*. In each σ-superleaf graph there is a set of aD uncoloured vertices in $V_0(N^*)$. We divide these into a sets of size D; each of these sets is called a σ-superleaf. A σ-superleaf together with the vertices of $V_1(N^*)$ induces a graph $K_{D,D}$. For each D colours in U'_σ, choose a σ-superleaf and use them to colour its vertices.

We have now used all colour pairs involving special colours.

Colouring superleaves with remaining pairs. The only unused colour pairs are now supercolour pairs (σ, τ), *i.e.*, sets consisting of all D^2 pairs (σ_i, τ_j). Note that colouring a σ-superleaf with the shades of τ will use up the supercolour pair (σ, τ).

Now, to complete the colouring, construct a graph U_Σ on vertex set Σ' with edges the unused supercolour pairs. Thus U_Σ has P vertices. Let k_σ be the number of uncoloured σ-superleaves. From above, the number of σ-superleaf graphs is

$$\frac{P}{a}\left(\varepsilon_3 + \varepsilon_4 - O\left(\varepsilon_4^2\right) - O(\varepsilon_3\varepsilon_4) - O(\varepsilon_0)\right) - O(1).$$

Hence the number of σ-superleaves k_σ equals

$$P\left(\varepsilon_3 + \varepsilon_4 - O\left(\varepsilon_4^2\right) - O(\varepsilon_3\varepsilon_4) - O(\varepsilon_0)\right) - O(1).$$

Now, for each standard supercolour σ, the number of pairs (σ, τ) which have been used, apart from those used in the fragments, is bounded by a constant t. Let $k = \varepsilon_3 P$. Then in U_Σ, each vertex is joined to at least $2k - t$ of the $2k$ vertices nearest to it.

Then, by Lemma 4.1, we can orient the edges of U_Σ so that for each σ, the outdegree of σ is at least k_σ, provided that (i) $|E(U_\Sigma)| \geqslant \sum_{\sigma\in\Sigma'} k_\sigma$, and (ii) for each $\sigma \in \Sigma'$, $k/2 \leqslant k_\sigma \leqslant k'$ where $k' = k - t + k^2/4(P - k)$.

Condition (i) is automatic as this simply states that the number of unused colour pairs is at least as great as the number of uncoloured edges. Condition (ii) is true provided

$$\varepsilon_3 P/2 \leqslant P\left(\varepsilon_3 + \varepsilon_4 - O\left(\varepsilon_4^2\right) - O(\varepsilon_3\varepsilon_4) - O(\varepsilon_0)\right) - O(1)$$

and

$$P\left(\varepsilon_3 + \varepsilon_4 - O\left(\varepsilon_4^2\right) - O(\varepsilon_3\varepsilon_4) - O(\varepsilon_0)\right) - O(1) \leqslant \varepsilon_3 P - t + \varepsilon_3^2 P^2/4P(1 - \varepsilon_3).$$

Recalling that ε_3 is bounded below by a constant multiple of ε_1, it is clear that condition (ii) holds provided that ε_4 and ε_0 are small enough compared to ε_1.

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To complete the colouring we colour the uncoloured σ-superleaves with the out-neighbours of σ in U_Σ. We now have an exact colouring of an amalgamation of G with r colours, which by Observation 2.3 immediately induces an exact colouring of G itself.

The various conditions required in the proof can be summed up as follows. Write $x \prec y$ to mean 'x must be sufficiently small relative to y'. Then we have $\varepsilon_8 \prec \varepsilon_\lambda$, $\varepsilon_{11} \prec \varepsilon_6 \prec \varepsilon_H$, $\varepsilon_4 \prec \varepsilon_1$, and finally $\varepsilon_0 \prec \varepsilon_6\varepsilon_4^2$, $\varepsilon_0 \prec \varepsilon_1$, $\varepsilon_0 \prec \varepsilon_\lambda$ and $\varepsilon_0 \prec \varepsilon_{11}$. Hence it is clear that all of these constants can be chosen to satisfy the necessary conditions. □

We can now prove the result mentioned in the introduction. This is a special case of the more general results proved in the next section.

Corollary 4.12. *For all sufficiently large n, every cubic planar triangle-free graph on $n(3n+1)$ vertices is a detachment of K_{3n+1}.*

Proof. Let G be a cubic planar triangle-free graph on $n(3n+1)$ vertices, and let $r = 3n+1$. We take $d = d' = 3$ and $\varepsilon_1 = 1$ in Theorem 4.11. Then, as noted in [8], the class of planar graphs is fragmentable, hence for the ε_0 given by Theorem 4.11, there is a C such that all planar graphs are (C, ε_0)-fragmentable. Then conditions (i)–(v) of Theorem 4.11 are satisfied; hence, if n (and therefore r), is large enough, then G is a detachment of K_r, as required. □

Remark. Corollary 4.12 is not true for all values of n, as there are cubic planar triangle-free graphs on 14 vertices which are not detachments of K_7, for example the graph formed by joining corresponding vertices in two 7-cycles.

5. Bounded degree graphs

In this section we extend the results of Section 4 to graphs which have a degree bound. We first consider some necessary conditions for a graph to be a detachment of a complete graph.

Degree partition condition. Let G be a graph with $\binom{r}{2}$ edges. Then G satisfies the *degree partition condition* if and only if the vertices can be partitioned into r sets $V_1, \ldots, V_r$ such that, for each $i = 1, \ldots, r$, the sum of the degrees of the vertices in V_i is $r-1$.

It is immediate that this condition is a necessary condition for a graph to be a detachment of K_r. For graphs of degree at most d say, it is not difficult to see that this condition can be checked in polynomial time (see Lemma 6.1). Unfortunately, even for graphs such as bounded degree trees, it is not quite sufficient, even for large r. We therefore need a slight refinement.

Rare vertex degrees. Let G be a graph with maximum degree d, and i be a positive integer. Let n_i be the number of vertices of degree i in G, *i.e.*, $n_i = |\{v : d(v) = i\}|$. Let $M_d = \operatorname{lcm}\{1, \ldots, d\}$, and let $R_d = (d^3 + d^2)M_d^2 + (d^3 + d^2 + 2d)M_d$. We say that i is a *rare vertex degree* if and only if $0 < n_i \leqslant R_d$. Let R_G be the set of vertices whose degree is rare.

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Remark. If G is small it may be that every vertex has rare degree. However, we shall be interested only in large graphs where there must be some degree which is not rare. Note that $|R_G| \leqslant dR_d$.

Degree partition extension condition. Let G be a graph with $\binom{r}{2}$ edges, and maximum degree d. Then G satisfies the *degree partition extension condition* if and only if there is a partition Π of the vertices into r sets $V_1, \ldots, V_r$ such that, for each $i = 1, \ldots, r$, the sum of the degrees of the vertices in V_i is $r-1$, and such that the function $c_\Pi : R_G \to \{1, \ldots, r\}$ given by

$$c_\Pi(v) = i \iff v \in V_i$$

is a partial harmonious colouring of G.

Remark. It is perhaps more intuitive to think of this condition the other way round, *i.e.*, there is a partial harmonious colouring of the rare degree vertices R_G, which, viewed simply as a partition of R_G, can be extended to a partition of V satisfying the degree partition condition.

We can now state and prove the main theorem of the paper.

Theorem 5.1. *Let d be a positive integer, and $\varepsilon_1 > 0$ be a real number. There exists a real number $\varepsilon_0 > 0$ such that for any positive integer C there is an integer N such that for any graph $G = (V, E)$ with maximum degree d satisfying the following properties:*
(1) *G is an exact graph with $\binom{r}{2}$ edges, where $r \geqslant N$,*
(2) *G is (C, ε_0)-fragmentable,*
(3) *at least $\varepsilon_1 |V|$ vertices of G have an independent neighbourhood set,*

then G is a detachment of K_r if and only if G satisfies the degree partition extension condition.

Proof. Let $G = (V, E)$ be an exact graph with $\binom{r}{2}$ edges and maximum degree d, and suppose that G is (C, ε_0)-fragmentable, and at least $\varepsilon_1 |V|$ vertices of G have an independent neighbourhood set.

First note that if G is a detachment of K_r, then the partition of the vertices of G given by the detachment shows immediately that G satisfies the degree partition extension condition.

So suppose that G satisfies the degree partition extension condition. Then there is a partition Π of the vertex set into r sets $V_1, \ldots, V_r$ such that, for each $i = 1, \ldots, r$, the sum of the degrees of the vertices in V_i is $r-1$, and such that the function $c_\Pi : R_G \to \{1, \ldots, r\}$ given by

$$c_\Pi(v) = i \iff v \in V_i$$

is a partial harmonious colouring of G.

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Now, as above, let $M_d = \mathrm{lcm}\{1,\ldots,d\}$. An i-block will be a set of M_d/i vertices, all of degree i, which all lie in the same part of the partition. We will refer to a block to mean an i-block for some i.

Now consider the degree partition $V_1,\ldots,V_r$. We can assume that each V_i contains at most M_d/j vertices of any rare degree j, since the number of rare degree vertices is bounded, so that if r is large enough, and V_i contains more than M_d/j vertices of some rare degree j, we can find another part of the partition containing no vertices of rare degree and swap a block from this part with a block of degree j from V_i. Note that this just has the effect of subdividing some colour classes in c_Π, so it remains a partial harmonious colouring. Now for each i, the total degree of the vertices in V_i which are rare, or are not in a block is at most dM_d. Hence there is an integer M' satisfying $(d-1)M_d < M' \leqslant dM_d$ and $M' \equiv r-1 (\mathrm{mod}\, M_d)$, such that for each $i = 1,\ldots,r$, there is a subset S_i of V_i of total degree M' which includes all rare degree vertices, and such that the remaining vertices of V_i are all in blocks. Let $S = S_1 \cup \cdots \cup S_r$.

Now as in Theorem 4.11 we find a set $J \subseteq V$, satisfying $|J| \leqslant \varepsilon_0 n$, whose removal from G leaves a subgraph in which every component (fragment) has at most C vertices. Add the vertices of S to J.

Next, if i is a vertex degree which is not rare, *i.e.*, there are more than R_d vertices of degree i in G, then we wish to ensure that there are more than R_d vertices of degree i in J. We can do this by adding at most $(R_d+1)d$ vertices to J.

Finally we ensure that the number of fragments in each isomorphism class is a multiple of R_d (where isomorphism of fragments respects the number of neighbours which each fragment vertex has in J). We can do this by discarding at most R_d fragments of each class, and adding their vertices to J. Now, provided n is large enough, it is easy to see that $|J| \leqslant 2\varepsilon_0 n$. Note that R_d is a multiple of M_d.

Now we can partition the fragments into sets $F_1,\ldots,F_N$ say, where each F_j consists of R_d isomorphic fragments. Now, for each j, if each element of F_j is isomorphic to some graph F say, and v is a vertex of F, then let $F_j(v)$ be the set of copies of v in F_j. Then $|F_j(v)| = R_d$ for each j, v.

Now recall that, for each $i = 1,\ldots,r$, there is a subset S_i of $J \cap V_i$ of total degree M' which includes all rare degree vertices in V_i, and that the vertices of $V \setminus S$ are all in blocks. Then, since the vertices in the fragments occur in sets $F_j(v)$ with common degree and total degree a multiple of M_d, it follows that the vertices of $J \setminus S$ can be grouped into sets of common degree and total degree M_d. Let b be the number of such sets, and let $k = b + r$. Number these sets $S_{r+1},\ldots,S_k$. We will extend the harmonious colouring with b further colours $r+1,\ldots,k$, so that these b colours have degree sum M_d, while colours $1,\ldots,r$ have degree sum M'.

For $z = 1,\ldots,d$, let J_z be the set of uncoloured vertices of J of degree z (J_z is empty if z is a rare degree). Let n_{zi} be the number of vertices of degree z in S_i. Now suppose that we have extended c_Π to $J_1,\ldots,J_{z-1}$ so that the number of vertices of degree x coloured with colour i is n_{xi}, for each $x = 1,\ldots,z-1$. We then extend the colouring to J_z. If J_z is empty there is nothing to do, otherwise we use Lemma 4.2. Let c_0 be the partial colouring so far, and let $C_1 = \emptyset$ and $C_2 = 1,\ldots,k$. Then every colour has degree sum at most M'. Then $X_1 = \emptyset$ while $X_2 = R_G \cup J_1 \cup \ldots \cup J_{z-1}$. Let $Y = J_z$, so that $d_Y = z$, and

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let $r_i = n_{zi}$. Let $\theta = M'$. Then it is automatic that $r_i d_Y + \mathrm{ds}_{c_0}(i) \leqslant \theta$ for each $i = 1, \ldots, k$, and that $\sum_{i=1}^{k} r_i = |Y|$, hence all the conditions of Lemma 4.2 are satisfied. Hence J_z can be coloured with colours from C_2 so that the number of vertices coloured with colour i is n_{zi}, provided that $|Y| > (d+1+\theta)\theta + (d\theta + d^2 + 1)(\max_i r_i)$. Now each r_i is at most M', while $|Y| > R_d$. Hence

$$\begin{aligned}(d+1+\theta)\theta + (d\theta + d^2 + 1)(\max_i r_i) &\leqslant (d + 1 + dM_d)dM_d + (d^2 M_d + d^2 + 1)dM_d \\ &= (d^3 + d^2)M_d^2 + (d^3 + d^2 + 2d)M_d \\ &= R_d < |Y|,\end{aligned}$$

as required. Hence, by induction on z we can extend the harmonious colouring to the whole of J with r colours of degree sum M' and the rest of degree sum M_d.

Now to colour the vertices in the sets $F_j(v)$, we repeatedly apply Corollary 4.3, once for each set. We take $Y = F_j(v)$, and use $R_d d(v)/M_d$ new colours. Take $\theta = M'$ and take each r_i to be $M_d/d(v)$ so that $r_i d_Y = M_d \leqslant \theta$. Then, since $|F_j(v)| = R_d$, it follows from Corollary 4.3 that we can colour the elements of $F_j(v)$ so that each new colour has degree sum M_d.

Finally, by identifying vertices with the same colour, we obtain a graph G'' which is (M_d, M') almost regular. G'' has $n' \geqslant n/M'$ vertices, and is fragmented into fragments of size at most $R_d C$ by the removal of at most $|J| \leqslant 2\varepsilon_0 n \leqslant 2\varepsilon_0 M' n'$ vertices.

Now recall that G has at least $\varepsilon_1 n$ vertices with an independent neighbourhood set. Since at most $2\varepsilon_0 nd$ vertices in the fragments have a neighbour in J, then, provided ε_0 is small enough, there are at least $\varepsilon_1 n/2$ vertices v in the fragments whose neighbourhood sets are both independent and disjoint from J. But such vertices v, and their neighbours, are identified only with corresponding vertices in isomorphic fragments, hence the vertices formed by identifying copies of v also have independent neighbourhood sets. It follows that G'' has at least $(\varepsilon_1/2M_d)n'$ vertices with an independent neighbourhood set. Finally G'' has $\binom{r}{2}$ edges and by construction M_d divides $r - 1 - M'$. Hence by Theorem 4.11, if ε_0 is small enough, then if r is large enough, G'' is a detachment of K_r. Hence since G is a detachment of G'', then G is a detachment of K_r also, which completes the proof. □

Note by the remark at the end of Section 4 that this result cannot be extended to all values of r.

The following corollary applies to fragmentable classes of graphs.

Theorem 5.2. *Let Γ be a fragmentable class of triangle-free graphs, and d be a positive integer. Then there is an integer N, such that if $G \in \Gamma$ is an exact graph with $\binom{r}{2}$ edges and maximum degree at most d, and $r \geqslant N$, then G is a detachment of K_r if and only if G satisfies the degree partition extension condition.* □

As noted in [8], any class of graphs with a suitable separator theorem is fragmentable. These include any class of graphs of bounded genus, indeed any class with a fixed excluded minor [1]. Also fragmentable are classes such as lattice graphs of bounded dimension, *e.g.*, square and cubic lattices.

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6. Algorithms

We first establish that for bounded degree graphs the degree partition extension condition can be tested in polynomial time. We see that what is required is bin-packing with bounded sized objects.

Lemma 6.1. *Let d be a fixed positive integer. Let $U = \{u_1, \ldots, u_n\}$ be a finite set of items, such that each $u \in U$ has a positive integer size $s(u) \leqslant d$. Let K and $B_1, \ldots, B_K$ be positive integers (K is the number of bins, and $B_1, \ldots, B_K$ are the bin capacities). Then it can be determined in time polynomial in n if the objects can be packed into the bins, i.e., if U can be partitioned into disjoint sets $U_1, \ldots, U_K$ such that, for each $k = 1, \ldots, K$, $\sum_{u \in U_k} s(u) = B_k$.*

Proof. First we may assume that $\sum_{u \in U} s(u) = \sum_k B_k$, for otherwise there can be no solution. Hence each $B_k \leqslant nd$, and also $K \leqslant nd$. Let n_i be the number of objects of size i, $i = 1, \ldots, d$. Now consider the set

$$S = \{(r_1, \ldots, r_d) \mid 0 \leqslant r_i \leqslant n_i, i = 1, \ldots, d\}.$$

Then $|S| \leqslant (n+1)^d$. Now for k satisfying $1 \leqslant k \leqslant K$, define a subset $S_k \subseteq S$ as follows: $(r_1, \ldots, r_d) \in S_k$ if and only if there are integers $r_{i,j}$, $i = 1, \ldots, d$, $j = 1, \ldots, k$ such that (i) for each $i = 1, \ldots, d$, $\sum_j r_{i,j} = r_i$, and (ii) for each $j = 1, \ldots, k$, $\sum_i r_{i,j} i \leqslant B_j$. Thus $(r_1, \ldots, r_d) \in S_k$ means that a collection made up of r_i objects of size i for $i = 1, \ldots, d$, will fit into the first k bins.

Now it is easy to see that S_{k+1} can be calculated in polynomial time from S_k. For the vector $(r_1, \ldots, r_d) \in S_{k+1}$ if and only if there is a collection of integers $r_{i,k+1}$, $i = 1, \ldots, d$, such that (i) $(r_1 - r_{1,k+1}, \ldots, r_d - r_{d,k+1}) \in S_k$, and (ii) $\sum_i r_{i,k+1} i \leqslant B_{k+1}$. But the collection can only be chosen in polynomially many ways to satisfy (ii), and for each valid collection it is easy to determine from S_k if (i) holds.

Hence we can calculate S_K in polynomial time. It is clear that a solution exists if and only if $(n_1, \ldots, n_d) \in S_K$. □

Lemma 6.2. *Let d be a fixed positive integer. Then if G is a graph with $\binom{r}{2}$ edges and maximum degree d, it can be tested in polynomial time whether or not G satisfies the degree partition extension condition.*

Proof. Recall that G satisfies the condition if and only if there is a partition Π of the vertices into r sets $V_1, \ldots, V_r$ such that for each $i = 1, \ldots, r$, the sum of the degrees of the vertices in V_i is $r - 1$, and such that the function $c_\Pi : R_G \to \{1, \ldots, r\}$ given by

$$c_\Pi(v) = i \iff v \in V_i$$

is a partial harmonious colouring. Let $T = |R_G|$, and set $U = V \setminus R_G$. Then we may assume that every vertex in R_G is in one of $V_1, \ldots, V_T$. Hence the number of possible partial harmonious colourings c_Π is bounded by a function of T and hence of d. G satisfies the condition if, for at least one of these colourings, we can partition the vertices of U into r sets $U_1, \ldots, U_r$ such that, for each U_k, $k = 1, \ldots, r$, the sum $\sum_{v \in U_k} d(v)$ is at

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most B_k, where $B_k = r - 1 - \sum_{v:c_\Pi(v)=k} d(v)$. But since each degree $d(v)$ is bounded by d, then as shown in Lemma 6.1 this can be determined in polynomial time, as required. □

We now have the following result.

Theorem 6.3. *Let d be a fixed positive integer, and let Γ be a fragmentable class of triangle-free graphs of maximum degree at most d. Then for $G \in \Gamma$, it can be determined in polynomial time whether or not G is a detachment of a complete graph.*

Proof. If G does not have $\binom{r}{2}$ edges for some r, then it cannot be a detachment of a complete graph. Otherwise, by Theorem 5.2, if r is larger than some constant we just have to check that G satisfies the degree partition extension condition. By Lemma 6.2 this can be done in polynomial time. If r is not large enough we use exhaustive search. □

7. Harmonious chromatic number

We now use the results of previous sections to prove some results about the harmonious chromatic number. As mentioned in Section 1, it is obvious that a graph G has a harmonious colouring with r colours if and only if the graph G' obtained by adding $\binom{r}{2} - |E(G)|$ isolated edges to G has an exact colouring with r colours, or equivalently is a detachment of K_r.

We will need the following.

Definition. Let m be a positive integer. Then we define $Q(m)$ to be the least positive integer k such that $\binom{k}{2} \geqslant m$. We also define $R(m)$ to be $\binom{Q(m)}{2} - m$.

Remark. It is easily calculated that

$$Q(m) = \left\lceil \frac{1 + \sqrt{8m+1}}{2} \right\rceil.$$

Also $0 \leqslant R(m) < Q(m) - 1$.

By comparing the number of edges of G with the number of colour pairs, it follows immediately that for any graph G with m edges, $h(G) \geqslant Q(m)$.

We first give a theorem which gives a useful equivalent of the degree partition extension condition. This theorem is due to the anonymous referee.

Theorem 7.1. *Let G be a graph with maximum degree d, $m = \binom{r}{2}$ edges, and n_i vertices of degree i and let $S = \{i : n_i \geqslant (d-1)(r-1)\}$. Then, for sufficiently large r, G satisfies the degree partition extension condition if and only if there exists a partial harmonious colouring c_0 of the rare vertices and an arbitrary (not necessarily proper) extension c of c_0 so that for each colour class i, the degree sum $\mathrm{ds}_c(i)$ is equivalent to $r - 1$ mod $\gcd(S)$.*

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Note. The bound $(d-1)(r-1)$ is best possible; this can be shown by the example of a graph with $(d-1)(r-1)-1$ degree 1 vertices, r a multiple of d, and all other vertices of degree d.

Proof. Necessity follows by considering the partition of G given by the degree partition extension condition. For sufficiency, take the colouring c, uncolour any vertex with (common) degree $j \in S$, and assign colour 1 to every uncoloured degree j vertex with $j \notin S$. Let $B_i = r - 1 - \mathrm{ds}_c(i)$ (which may be negative) be the remaining degree sum we need to achieve from the vertices with degree in S. Note that $g = \gcd(S)$ divides B_i for $i > 1$ since g divides all $j \in S$. But g divides $\sum B_i = \sum_{i \in S} in_i$, so must also divide B_1. We shall show that we can assume $\max_i B_i - \min_i B_i \leqslant dR_d + d^2(d - |S|)$. If there are more than g vertices with degree j assigned to colour i and j is a common degree, uncolour them g at a time, until there are at most $g-1$ such degree j vertices. Then $\max_i B_i - \min_i B_i \leqslant dR_d + d^2(d - |S|)$ since the degree sum of the rare vertices is at most dR_d and we have at most $(g-1) < d$ vertices of each of $(d - |S|)$ possible degrees coloured with i. Now recolour these groups of g vertices in turn with the colour i corresponding to the largest B_i. Then $\max_i B_i - \min_i B_i \leqslant \max(dR_d + d^2(d - |S|), gd) = dR_d + d^2(d - |S|)$. The following conditions now hold:

(a) $\gcd(S)$ divides all the B_i,
(b) $\max_i B_i - \min_i B_i \leqslant dR_d + d^2(d - |S|)$,
(c) $\sum_i B_i = \sum_{i \in S} in_i$,
(d) S contains the degree j with largest n_j.

First assume $|S| = 1$, so $S = \{j\}$ with n_j the largest value of n_i. Hence $\sum B_i = jn_j \geqslant jn/(d+1) \geqslant r(r-1)/d(d+1)$. But $\max B_i - \min B_i \leqslant dR_d + d^3$, so $r \min B_i \geqslant r(r-1)/d(d+1) - r(dR_d + d^3)$ which is positive for sufficiently large r. By (a) all the B_i are divisible by j so we can assign B_i/j vertices of degree j to each colour class i and we are done.

Now suppose $|S| \geqslant 2$. Pick the least common (smallest n_j) $j \in S$. Let $g = \gcd(S)$ and $g' = \gcd(S \setminus \{j\})$. Then by assigning at most $g' - 1$ degree j vertices to each colour class i we can ensure that g' divides B_i. (We are solving $xj \equiv B_i \pmod{g'}$ for x. We know there is a solution since $g = \gcd(j, g')$ divides B_i, hence there is a solution with $0 \leqslant x < g'$.) Since $n_j \geqslant (d-1)(r-1) \geqslant (g'-1)(r-1)$ we can repeat this process for all but at most one of the r colours without running out of degree j vertices. By (c) the last B_i minus the sum of the degrees of the remaining degree j vertices must be divisible by g', so we can colour degree j vertices with i so that g' divides this B_i as well. The sum of the degrees of the remaining degree j vertices is divisible by g', so the number of remaining degree j vertices is divisible by g'/g and we can partition them into groups of g'/g. Assigning these groups in turn to the largest B_i gives us new B_i' with g' dividing B_i', $\max B_i' - \min B_i' \leqslant dR_d + d^2(d - |S|) + \max((g'/g)j, (g'-1)j) \leqslant dR_d + d^2(d - |S| + 1)$, $\sum B_i' = \sum_{i \in S \setminus \{j\}} in_i$, and $S \setminus \{j\}$ still contains the most common degree. Thus the result follows by induction on $|S|$. □

Corollary 7.2. *Let d be a fixed positive integer, and let Γ be a fragmentable class of triangle-free graphs of maximum degree d. If $G \in \Gamma$ has m edges, n_i vertices of degree i, and if the*

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set $S = \{i : n_i \geqslant (d-1)(Q(m)-1)\}$ *has no common divisors* > 1, *then provided* m *is large enough,* $h(G) = Q(m)$.

Proof. Add $R(m)$ independent edges to G. This does not decrease the set S, so we still have $\gcd(S) = 1$. Let $c = c_0$ assign distinct colours to each rare degree vertex. Now apply the above Theorem 7.1 and Theorem 5.1. □

Similarly we can obtain the following upper bound on the harmonious chromatic number for large graphs.

Theorem 7.3. *Let* d *be a fixed positive integer, and let* Γ *be a fragmentable class of triangle-free graphs of maximum degree at most* d. *Let* $G \in \Gamma$, *with* m *edges, and for each* i, *let* n_i *be the number of vertices of degree* i *in* G. *Let,* $T = \{i : n_i \geqslant (d-1)(Q(m) + \lceil \frac{d}{2} \rceil - 1)\}$, *and set* $g = \gcd(T)$. *Then the harmonious chromatic number* $h(G)$ *is at most* $Q(m) + \lceil \frac{g}{2} \rceil$ *provided* m *is large enough. Furthermore, if* $g = 3$, *then* $h(G) \leqslant Q(m) + 1$ *if* m *is large enough.*

Proof. Let $r = Q(m) + k$ and let G' be the graph obtained by adding $\binom{r}{2} - |E(G)|$ edges to G. Then G' is an exact graph with $\binom{r}{2}$ edges. It suffices to show that if r is large enough, then G' satisfies the degree partition extension condition. Let L be the number of vertices of degree one in G'. Then

$$\begin{aligned} L &\geqslant 2\left[\binom{r}{2} - |E(G)|\right] \\ &= 2\left[\binom{Q(m)+k}{2} - |E(G)|\right] \\ &= 2\left[\binom{Q(m)}{2} + kQ(m) + \binom{k}{2} - |E(G)|\right] \\ &= 2\left[R(m) + kQ(m) + \binom{k}{2}\right] \\ &\geqslant 2kQ(m) + k(k-1). \end{aligned}$$

Now if $k = \lceil \frac{g}{2} \rceil$, we have $L \geqslant gQ(m) + k(k-1) \geqslant (g-1)(Q(m)+k)$ provided m is large enough. Hence provided r is large enough, $L \geqslant (g-1)r$. Now for each i, let n_i' be the number of vertices of degree i in G', and let $S = \{i : n_i' \geqslant (d-1)(r-1)\}$. It is clear that provided m is large enough, T is non-empty, so $g \leqslant d$. Hence $Q(m) + \lceil \frac{d}{2} \rceil - 1 \geqslant Q(m) + \lceil \frac{g}{2} \rceil - 1 = r - 1$, and since G' contains every vertex of G, it is clear that $T \subseteq S$. Hence $\gcd(S) | g$, in particular $\gcd(S) \leqslant g$. Now let c_0 be a colouring of the rare degree vertices of G' with distinct colours, and let c be an extension of this that, for each colour i, has at most $g-1$ vertices of degree one coloured i so that $\mathrm{ds}_c(i) \equiv r - 1 (\mathrm{mod}\ \gcd(S))$. Then by Theorem 7.1 and Theorem 5.1, we have that $h(G') = r$ if r is large enough. Hence $h(G) \leqslant r$ if m is large enough, as required.

If $g = 3$ and $k = 1$, we have $L \geqslant 2Q(m) = (g-1)(r-1)$. We proceed as above, except that this time we extend c_0 to c by first colouring each vertex of G whose degree is not in S; to do this we simply colour them in some order and for each vertex use a colour

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whose degree sum is currently minimal. This ensures that all degree sums will be bounded by a constant depending on d. Finally for each colour i, colour at most $g-1$ degree one vertices with colour i to ensure that $\mathrm{ds}_c(i) \equiv r-1 (\mathrm{mod}\ \gcd(S))$. It is possible that we may run out of degree one vertices when trying to do this for the last colour r, but in this case all remaining uncoloured vertices have degrees in S, from which it follows that if $\mathrm{ds}_c(i) \equiv r-1 (\mathrm{mod}\ \gcd(S))$ for $i = 1, \ldots, r-1$, then this holds for $i = r$ also. So as above we have $h(G) \leqslant h(G') = Q(m) + 1$. □

Remark. It is not hard to construct fragmentable classes of triangle-free graphs which are regular of degree d, with m edges and m arbitrarily large, such that $h(G) = Q(m) + \lceil \frac{d}{2} \rceil$, or, in the case $d = 3$, $h(G) = Q(m) + 1$.

Since every sufficiently large graph of some fixed genus has many vertices of degree at most 6, we get the following.

Theorem 7.4. *Let d, γ be fixed nonnegative integers, and let $\Gamma_{d,\gamma}$ be the class of graphs of maximum degree d and genus γ. Then if $G \in \Gamma_{d,\gamma}$ is triangle-free and has m edges, and m is large enough, $h(G) \leqslant Q(m) + 3$.*

Proof. We apply Theorem 7.3, noting that $g \leqslant 6$. The result follows. □

From Theorem 7.3 we immediately obtain the following.

Theorem 7.5. *Let d be a fixed positive integer, and let Γ be a fragmentable class of triangle-free graphs of maximum degree at most d. Then for $G \in \Gamma$, the harmonious chromatic number $h(G)$ can be determined in polynomial time.*

Proof. For each $r = Q(m), \ldots, Q(m) + \lceil \frac{d}{2} \rceil$, we test whether the graph G_r consisting of G plus $\binom{r}{2} - |E(G)|$ isolated edges is a detachment of K_r. By Theorem 7.3 there is some r for which G_r is a detachment of K_r; the least such r is $h(G)$. Since by Theorem 6.3 each test can be done in polynomial time, the whole algorithm runs in polynomial time also. □

Theorem 7.6. *Let $D \geqslant 2$ be a fixed, and let $L_D(N)$ be the*

$$\underbrace{N \times \ldots \times N}_{D}$$

lattice graph with vertex set $\{1, \ldots, N\}^D$ and two vertices $(x_1, \ldots, x_D)$, $(y_1, \ldots, y_D)$ adjacent if and only if $\sum_i |x_i - y_i| = 1$. Then if N is large enough $h(L_D(N)) = Q(m)$, where m is the number of edges of $L_D(N)$.

Proof. Note that the lattices are fragmentable and triangle-free, so we need only check that they satisfy the degree partition extension condition.

For $D \geqslant 3$ we use Corollary 7.2. Let n be the number of vertices of $L_D(N)$, then $n = N^D$. It is easily seen that $L_D(N)$ has $(N-2)^D$ vertices of degree $2D$, and $2D(N-2)^{D-1}$ vertices

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of degree $2D-1$. It is then immediate from Corollary 7.2 that $h(L_D(N)) = Q(m)$, provided N is large enough.

For the case $D = 2$, $L_2(N)$ has $m = 2N(N-1)$ edges, so that $Q(m) = 2N$. It was shown by Miller and Pritikin [18] that $h(L_2(N)) = Q(m)$ for all even N, so we deal only with the case N odd. Let L' be the graph obtained by adding N isolated edges to $L_2(N)$; then L' has $\binom{2N}{2}$ edges. Let $N = 2R+1$. Then L' has $2N = 4R+2$ vertices of degree 1, 4 of degree 2, $4N-8 = 8R-4$ of degree 3 and $N^2-4N+4 = 4R^2-4R+1$ of degree 4. Thus if n_i is the number of vertices of degree i in L' and $S = \{i : n_i \geqslant (d-1)(r-1)\}$, where $r = 2N$, then $4 \in S$. The only rare degree is 2, so colour these vertices with colour 1. Since the 4 vertices of degree 2 are the corners of the square lattice this gives a partial harmonious colouring. c_0. Then extend this colouring to a colouring c by giving $r = 2N$ vertices of degree 1 distinct colours. Then since N is odd, $\mathrm{ds}_c(i) \equiv 1 \equiv r-1 (\mathrm{mod}\, 4)$, so by Theorem 7.1 L' satisfies the degree partition extension condition, hence $h(L') = h(L_2(N)) = Q(m)$. □

7.1. Graphs of maximum degree at most 2

Theorem 7.7. *There exists an integer M such that if G is a graph with $m \geqslant M$ edges and maximum degree at most 2, and G consists of P paths, of total length L, and some cycles, then $h(G) = Q(m)$ if either* (i) *$Q(m)$ is odd, and $L + R(m) \neq 1, 2$, or* (ii) *$Q(m)$ is even, and $P + R(m) \geqslant Q(m)/2$. Otherwise $h(G) = Q(m) + 1$.*

Proof. First note that by Theorem 7.3, $h(G) \leqslant Q(m)+1$ if m is large enough. Let $r = Q(m)$. Next let G' be the graph formed from G by adding $R(m)$ isolated edges. Then G' has $P + R(m)$ paths of total length $L + R(m)$, and $\binom{r}{2}$ edges.

We now note that if neither (i) nor (ii) holds, then G' does not satisfy the degree partition extension condition. For then if $Q(m)$ is odd, we have that $r-1$ is even, so we need to partition the vertices into sets each of even total degree. Thus the vertices of degree one must occur in pairs in these sets. But G' has either one path of length one or two, with two vertices of degree one, which must therefore be in the same set, or two of length one, with 4 vertices of degree one which must be in at most two sets. In neither case do we obtain a partial harmonious colouring of the degree one vertices. If $Q(m)$ is even then $r-1$ is odd, hence we need at least $Q(m)$ vertices of odd degree to make a vertex partition with total degrees $r-1$ possible. But the number of vertices of odd degree is $2(P + R(m))$, and the result follows. Hence if neither (i) nor (ii) holds, then $h(G) = Q(m)+1$.

It remains to show that if (i) or (ii) holds, then $h(G) = Q(m)$. Form an amalgamation G'' of G' as follows. If $Q(m)$ is odd, then if there are any paths, join them all to form a single extra cycle. If $r = Q(m)$ is even, then join the endpoints of some paths until there are exactly $r/2$ paths.

Let $\varepsilon = \varepsilon_3$ be the constant given by Theorem 4.7. Let W be the set of vertices of G'' not contained in a triangle. Suppose first that $|W| \leqslant \varepsilon m/4$. Then we obtain a partial harmonious colouring of G'' with r colours as follows. If r is even, then colour the ends of the $r/2$ paths with each colour used once each, and let c_0 be this partial colouring. If r is odd, c_0 is vacuous. Let $\theta = \varepsilon r/2 + 2$. Let $C_1 = \emptyset$, $C_2 = \{1, \ldots, r\}$, X_2 the coloured vertices, and Y the uncoloured vertices of W. Then by Lemma 4.2 we can extend the

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partial colouring c_0 to a partial colouring c of the whole of W, with the degree sum of each colour odd and at most θ, provided that

$$r > 4 + 2\theta + |W|2/(\theta - 2).$$

But

$$\begin{aligned} 4 + 2\theta + |W|2/(\theta - 2) &\leqslant 4 + \varepsilon r + 4 + (\varepsilon m/2)/(\varepsilon r/2) \\ &\leqslant 4 + \varepsilon r + 4 + (r-1)/2, \end{aligned}$$

which is less than r. Now consider the graph U whose vertices are the colours $1, \ldots, r$ and edges the unused colour pairs. Then every vertex degree of U is even, and at least $r - 1 - \varepsilon r/2 - 2$, which is at least $(1 - \varepsilon)r$ if r is large enough. Hence by Theorem 4.7, U can be decomposed into triangles. Since the number of edges in U equals the number of edges in the triangles of G'', this allows us to complete an exact r-colouring of G''.

On the other hand, suppose that $|W| \geqslant \varepsilon m/4 \geqslant \varepsilon n/8$. The class of graphs of maximum degree 2 is fragmentable, hence for any $\varepsilon_0 > 0$, there is a constant C such that every graph of maximum degree 2 is (C, ε_0)-fragmentable. Now note that every vertex of W has an independent neighbourhood set, hence we can take $\varepsilon_1 = \varepsilon/8$ in Theorem 5.1. Then for a suitably chosen ε_0 and C, the graph G'' satisfies all the conditions of Theorem 5.1, and hence has an exact r-colouring.

Hence, in either case, if (i) or (ii) holds, $h(G) = Q(m)$ provided m is big enough. □

Remark. Theorem 7.7 also follows from recent work of Balister [2]. It appears very likely that the result is true for all m. This was proved for the case with no cycles by Georges [12], and follows from Balister's theorem for the cases where $Q(m)$ is odd, or there are no paths.

8. Conclusions

We have seen that the degree partition extension condition is both necessary and sufficient for sufficiently large bounded degree graphs to be detachments of a complete graph, in the case of, for example, triangle-free planar graphs. It is natural to consider whether the same might be true for all sufficiently large bounded degree graphs. Although this would be a very strong result, we know of no class of counter-examples. It is also interesting to note that even in the smallest nontrivial regular case, that of 14-vertex cubic graphs, only 107 out of the 540 such graphs are not detachments of K_7.

Another direction in which the results of this paper could be improved would be the reduction of the constants involved to more reasonable values. Unfortunately there seems to be little hope of this while, for example, the constants involved in Wilson's theorem have not been improved significantly. Nevertheless, it seems likely that the results proved in this paper for huge graphs remain true for graphs of much more modest size.

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