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On uniformly continuous surjections between function spaces

Ali Eysen and Vesko Valov

Abstract. We consider uniformly continuous surjections between $C_p(X)$ and $C_p(Y)$ (resp., $C_p^*(X)$ and $C_p^*(Y)$) and show that if X has some dimensional-like properties, then so does Y. In particular, we prove that if $T : C_p(X) \to C_p(Y)$ is a continuous linear surjection and $\dim X = 0$, then $\dim Y = 0$. This provides a positive answer to a question raised by Kawamura–Leiderman [11, Problem 3.1].

1 Introduction

All spaces in this article, if not said otherwise, are Tychonoff spaces and all maps are continuous. By $C(X)$ (resp., $C^*(X)$), we denote the set of all continuous (resp., continuous and bounded) real-valued functions on a space X. We write $C_p(X)$ (resp., $C_p^*(X)$) for the spaces $C(X)$ (resp., $C^*(X)$) endowed with the pointwise topology. More information about the spaces $C_p(X)$ can be found in [20]. By dimension, we mean the *covering dimension* dim defined by finite functionally open covers (see [2]). According to that definition, we have $\dim X = \dim \beta X$, where βX is the Čech–Stone compactification of X.

After the striking result of Pestov [18] that $\dim X = \dim Y$ provided that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, and Gul'ko's [7] generalization of Pestov's theorem that the same is true for uniformly continuous homeomorphisms, a question arose whether $\dim Y \leq \dim X$ if there is continuous linear surjection from $C_p(X)$ onto $C_p(Y)$ (see [1]). This was answered negatively by Leiderman–Levin–Pestov [14] and Leiderman–Morris–Pestov [15]. On the other hand, it was shown in [14] that if there is a linear continuous surjection $C_p(X) \to C_p(Y)$ such that X and Y are compact metrizable spaces and $\dim X = 0$, then $\dim Y = 0$. The last result was extended for arbitrary compact spaces by Kawamura–Leiderman [10] who also raised the question whether this is true for any Tychonoff spaces X and Y. In this article, we provide a positive answer to that question and discuss the situation when the surjection $T : C_p(X) \to C_p(Y)$ is uniformly continuous. Let us note that the preservation of dimension under linear homeomorphisms doesn't hold for function

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spaces with the uniform norm topology. Indeed, according to the classical result of Milutin [17] if X and Y are any uncountable metrizable compacta, then there is a linear homeomorphism between the Banach spaces $C(X)$ and $C(Y)$.

Suppose $E_p(X) \subset C_p(X)$ and $E_p(Y) \subset C_p(Y)$ are subspaces containing the zero functions on X and Y, respectively. Recall that a map $\varphi : E_p(X) \to E_p(Y)$ is *uniformly continuous,* if for every neighborhood U of the zero function in $E_p(Y)$, there is a neighborhood V of the zero function in $E_p(X)$ such that $f, g \in E_p(X)$ and $f - g \in V$ implies $\varphi(f) - \varphi(g) \in U$. Evidently, if $E_p(X)$ and $E_p(Y)$ are linear spaces, then every linear continuous map between $E_p(X)$ and $E_p(Y)$ is uniformly continuous. If $f \in C_p(X)$ is a bounded function, then $\|f\|$ stands for the supremum norm of f. The notion of c-good maps was introduced in [6] (see also [5]), where c is a positive number. A map $\varphi : E_p(X) \to E_p(Y)$ is *c-good* if for every bounded function $g \in E_p(Y)$ there exists a bounded function $f \in E_p(X)$ such that $\varphi(f) = g$ and $\|f\| \leq c\|g\|$.

Everywhere below, by $D(X)$, we denote either $C^*(X)$ or $C(X)$. Here is one of our main results.

Theorem 1.1 *Let $T : D_p(X) \to D_p(Y)$ be a c-good uniformly continuous surjection for some $c > 0$. Then ,Y is zero-dimensional provided so is X.*

Corollary 1.2 *Suppose there is a linear continuous surjection from $C_p^*(X)$ onto $C_p^*(Y)$. Then, Y is zero-dimensional provided that so is X.*

Corollary 1.2 follows from Theorem 1.1 because every linear continuous surjection between $C_p^*(X)$ and $C_p^*(Y)$ is c-good for some $c > 0$ (see Proposition 3.3). Moreover, as one of the referees observed, it is not always possible to have a continuous linear surjection $T : C_p^*(X) \to C_p(Y)$ with $C_p(Y) \neq C_p^*(Y)$. Indeed, we consider the composition $T \circ i$, where $i : C_p(\beta X) \to C_p^*(X)$ is the restriction map. Then, according to [21, Proposition 2], Y is pseudocompact and $C_p(Y) = C_p^*(Y)$.

We consider properties $\mathcal{P}$ of σ-compact metrizable spaces such that:

(a) if $X \in \mathcal{P}$ and $F \subset X$ is closed, then $F \in \mathcal{P}$;
(b) if X is a countable union of closed subsets each having the property $\mathcal{P}$, then $X \in \mathcal{P}$;
(c) if $f : X \to Y$ is a perfect map with countable fibers and $Y \in \mathcal{P}$, then $X \in \mathcal{P}$.

For example, *zero-dimensionality*, *strongly countable-dimensionality,* and *C-space property* are finitely multiplicative properties satisfying conditions (a)–(c). Another two properties of this type, but not finitely multiplicative, are *weakly infinite-dimensionality*, or *$(m - C)$-spaces* in the sense of Fedorchuk [4]. The definition of all dimension-like properties mentioned above, and explanations that they satisfy conditions (a)–(c) can be found at the end of Section 2.

For σ-compact metrizable spaces, we have the following version of Theorem 1.1 (actually Theorem 1.3 below is true for any topological property $\mathcal{P}$ satisfying conditions (a)–(c)).

Theorem 1.3 *Suppose X and Y are σ-compact metrizable spaces and let $T : D_p(X) \to D_p(Y)$ be a c-good uniformly continuous surjection for some $c > 0$. If all finite powers of X are weakly infinite-dimensional or $(m - C)$-spaces, then all finite powers of Y are also weakly infinite-dimensional or $(m - C)$-spaces.*

Krupski [11] proved similar result for σ-compact metrizable spaces: if $T : C_p(X) \to C_p(Y)$ is a continuous open surjection and all powers of X are weakly

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infinite-dimensional or $(m-C)$-spaces, then Y is also weakly infinite-dimensional of has the property $m-C$. Theorem 1.3 was established in [6] in the special case when X, Y are compact metrizable spaces and the property is either zero-dimensionality or strong countable-dimensionality.

The notion of c-good maps is crucial in the proof of the above results. One of the referees observed that if $X = Y$ is the space of natural numbers, then the continuous linear surjection $T : C_p(X) \to C_p(Y)$, $T(f)(n) = \frac{1}{n+1} f(n)$, is not c-good for any c. Therefore, as the referee suggested, the more interesting question is whether the existence of a linear continuous surjection $T : C_p(X) \to C_p(Y)$ implies the existence of another linear continuous surjection $S : C_p(X) \to C_p(Y)$ which is c-good for some $c > 0$. In that connection, let us note that more general notion was considered in our recent paper [3]: $T : D_p(X) \to D_p(Y)$ is *inversely bounded* if for every norm bounded sequence $\{g_n\} \subset C^*(Y)$ there exists a norm bounded sequence $\{f_n\} \subset C^*(X)$ with $T(f_n) = g_n$ for all n. Evidently, every c-good map is inversely bounded. It was established in [3] that uniformly continuous inversely bounded surjections $T : D_p(X) \to D_p(Y)$, where X and Y are metrizable, preserve any one of the properties zero-dimensionality, countable-dimensionality, or strong countable-dimensionality. Most probably Theorem 1.1 remains true if the surjection T is uniformly continuous and inversely bounded.

Finally, here is the theorem which provides a positive answer to the question of Kawamura–Leiderman [10, Problem 3.1] mentioned above.

Theorem 1.4 *Let $T : C_p(X) \to C_p(Y)$ be a linear continuous surjection. If* $\dim X = 0$, *then* $\dim Y = 0$.

2 Preliminary results

In this section, we prove Proposition 2.1 which is used in the proofs of Theorems 1.1 and 1.3. Our proof is based on the idea of support introduced by Gul'ko [7] and the extension of this notion introduced by Krupski [12].

Let $\mathbb{Q}$ be the set of rational numbers. A subspace $E(X) \subset C(X)$ is called a QS-algebra [7] if it satisfies the following conditions: (i) if $f, g \in E(X)$ and $\lambda \in \mathbb{Q}$, then all functions $f+g$, $f \cdot g$, and λf belong to $E(X)$ and (ii) for every $x \in X$ and its neighborhood U in X there is $f \in E(X)$ such that $f(x) = 1$ and $f(X \setminus U) = 0$.

We are using the following facts from [7].

(2.1) If X has a countable base and $\Phi \subset C(X)$ is a countable set, then there is a countable QS-algebra $E(X) \subset C(X)$ containing Φ. Moreover, it follows from the proof of [7, Proposition 1.2] that $E(X) \subset C^*(X)$ provided that $\Phi \subset C^*(X)$.

(2.2) If U is an open set in X, $x_1, x_2, \dots, x_k \in U$ and $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{Q}$, then there exists $f \in E(X)$ such that $f(x_i) = \lambda_i$ for each i and $f(X \setminus U) = 0$.

(2.3) We consider the following condition for a QS-algebra $E(X)$ on X: For every compact set $K \subset X$ and an open set W containing K there exists $f \in E(X)$ with $f|K = 1$, $f|(X \setminus W) = 0$ and $f(x) \in [0,1]$ for all $x \in X$. Note that if X has a countable base $\mathcal{B}$, then there is a countable QS-algebra $E(X)$ on X satisfying that condition. Indeed, we can assume that $\mathcal{B}$ is closed under finite unions and find $U, V \in \mathcal{B}$ such that $K \subset V \subset \overline{V} \subset U \subset \overline{U} \subset W$. Then, consider the set

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Φ of all functions $f_{U,V} : X \to [0,1]$, where $\overline{V} \subset U$ with $U, V \in \mathcal{B}$, such that $f_{V,U}|\overline{V} = 1$ and $f_{V,U}|(X \setminus U) = 0$. According to (2.1), Φ can be extended to a countable QS-algebra $E(X)$ on X.

Everywhere below, we denote by $\overline{\mathbb{R}}$ the extended real line $[-\infty, \infty]$.

Proposition 2.1 *Let $\overline{X}$ and $\overline{Y}$ be metrizable compactifications of X and Y, and $H \subset \overline{X}$ be a σ-compact space containing X. Suppose $E(H)$ is a QS-algebra on H satisfying condition (2.3), $E(X) = \{\overline{f}|X : \overline{f} \in E(H)\}$ and $E(Y) \subset C(Y)$ is a family such that every $g \in E(Y)$ is extendable to a map $\overline{g} : \overline{Y} \to \overline{\mathbb{R}}$ and $E(\overline{Y}) = \{\overline{g} : g \in E(Y)\}$ contains a QS-algebra on $\overline{Y}$. Let also $\varphi : E_p(X) \to E_p(Y)$ be a uniformly continuous surjection which is c-good for some $c > 0$.*

If all finite powers of H have a property $\mathcal{P}$ satisfying conditions (a)–(c), then there exists a σ-compact set $Y_\infty \subset \overline{Y}$ containing Y such that all finite powers of Y_∞ have the same property $\mathcal{P}$.

Proof We fix a countable base $\mathcal{B}$ of H which is closed under finite unions, and denote by f the restriction $\overline{f}|X$ of any $\overline{f} \in E(H)$. For every $y \in \overline{Y}$, there is a map $\alpha_y : E(H) \to \overline{\mathbb{R}}$, $\alpha_y(\overline{f}) = \overline{\varphi(f)}(y)$. Since φ is uniformly continuous, so is the map $\beta_y : E_p(X) \to \mathbb{R}$, $\beta_y(f) = \varphi(f)(y)$. Let $H = \bigcup_k H_k$ be the union of an increasing sequence $\{H_k\}$ of compact sets. Following Krupski [12], for every $y \in \overline{Y}$ and every $p, k \in \mathbb{N}$, we define the families

$$\mathcal{A}^k(y) = \{K \subset H_k : K \text{ is closed and } a(y,K) < \infty\}$$

and

$$\mathcal{A}^k_p(y) = \{K \subset H_k : K \text{ is closed and } a(y,K) \le p\},$$

where

$$a(y,K) = \sup\{|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| : \overline{f}, \overline{g} \in E(H), |\overline{f}(x) - \overline{g}(x)| < 1 \ \forall x \in K\}.$$

Possibly, some or both of the values $\alpha_y(\overline{f}), \alpha_y(\overline{g})$ from the definition of $a(y,K)$ could be $\pm\infty$. That's why we use the following agreements:

(2.4) $\infty + \infty = \infty, \infty - \infty = -\infty + \infty = 0, -\infty - \infty = -\infty$.

Note that $a(y, \varnothing) = \infty$ since φ is surjective.

Using that $E(X)$ and $E(H)$ are QS-algebras on X and H, and following the arguments from Krupski's paper [12] (see also the proofs of [7, Proposition 1.4] and [16, Proposition 3.1]), one can establish the following claims (for the sake of completeness, we provide the proofs).

Claim 1 *For every $y \in Y$, there is $p, k \in \mathbb{N}$ such that $\mathcal{A}^k_p(y)$ contains a finite nonempty subset of X.* ■

This claim follows from the proof of [12, Proposition 2.1]. Indeed, since φ is uniformly continuous there is $p \in \mathbb{N}$ and a finite set $K \subset X$ such that if $f, g \in E(X)$ and $|f(x) - g(x)| < 1/p$ for every $x \in K$, then $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| = |\varphi(f)(y) - \varphi(g)(y)| < 1$. Take arbitrary $\overline{f}, \overline{g} \in E(H)$ with $|\overline{f}(x) - \overline{g}(x)| < 1$ for every $x \in K$ and consider the functions $\overline{f}_m = \overline{f} + \frac{m}{p}(\overline{g} - \overline{f}) \in E(H)$ for each $m = 0, 1, \dots, p$. Obviously

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$|f_m(x) - f_{m+1}(x)| < 1/p$ for all $x \in K$, so $|\alpha_y(\overline{f}_m) - \alpha_y(\overline{f}_{m+1})| < 1$. Consequently, $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| \le \sum_{m=0}^{p-1} |\alpha_y(\overline{f}_m) - \alpha_y(\overline{f}_{m+1})| < p$. Because K is finite, there is $k \in \mathbb{N}$ with $K \subset H_k$. Hence, $K \in \mathcal{A}_p^k(y)$.

Consider the sets $Y_p^k = \{y \in \overline{Y} : \mathcal{A}_p^k(y) \neq \varnothing\}$, $p, k \in \mathbb{N}$.

Claim 2 *Each Y_p^k is a closed subset of $\overline{Y}$.*

We use the proof of [12, Lemma 2.2]. Suppose $y \notin Y_p^k$. Since $y \in Y_p^k$ iff $H_k \in \mathcal{A}_p^k(y)$, $H_k \notin \mathcal{A}_p^k(y)$. So, there exist $\overline{f}, \overline{g} \in E(H)$ with $|\overline{f}(x) - \overline{g}(x)| < 1$ for all $x \in H_k$ and $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| > p$. Then, $V = \{z \in \overline{Y} : |\alpha_z(\overline{f}) - \alpha_z(\overline{g})| > p\}$ is a neighborhood of y with $V \cap Y_p^k = \varnothing$.

Claim 3 *Every set $Y_{p,q}^k = \{y \in Y_p^k : \exists K \in \mathcal{A}_p^k(y) \text{ with } |K| \le q\}$, $p, q, k \in \mathbb{N}$, is closed in Y_p^k.*

Following the proof of [12, Lemma 2.3], we first show that the set $Z = \{(y, K) \in Y_p^k \times [H_k]^{\le q} : K \in \mathcal{A}_p^k(y)\}$ is closed in $Y_p^k \times [H_k]^{\le q}$, where $[H_k]^{\le q}$ denotes the space of all subsets $K \subset H_k$ of cardinality $\le q$ endowed with the Vietoris topology. Indeed, if $(y, K) \in Y_p^k \times [H_k]^{\le q} \setminus Z$, then $K \notin \mathcal{A}_p^k(y)$. Hence, $a(y, K) > p$ and there are $\overline{f}, \overline{g} \in E(H)$ such that $|\overline{f}(x) - \overline{g}(x)| < 1$ for all $x \in K$ and $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| > p$. Let $U = \{z \in Y_p^k : |\alpha_z(\overline{f}) - \alpha_z(\overline{g})| > p\}$ and $V = \{x \in H_k : |\overline{f}(x) - \overline{g}(x)| < 1\}$. The set $U \times < V >$ is a neighborhood of (y, K) in $Y_p^k \times [H_k]^{\le q}$ disjoint from Z (here $< V >= \{F \in [H_k]^{\le q} : F \subset V\}$). Since $Y_p^k \times [H_k]^{\le q}$ is compact and $Y_{p,q}^k$ is the image of Z under the projection $Y_p^k \times [H_k]^{\le q} \to Y_p^k$, $Y_{p,q}^k$ is closed in Y_p^k.

For every k, let $Y_k = \bigcup_{p,q} Y_{p,q}^k$. Obviously, $Y_k \subset \{y \in \overline{Y} : \mathcal{A}^k(y) \neq \varnothing\}$. Since $H_k \subset H_{k+1}$ for all k, the sequence $\{Y_k\}$ is increasing. It may happen that $Y_k = \varnothing$ for some k, but Claim 1 implies that $Y \subset \bigcup_k Y_k$.

Claim 4 *For every $y \in Y_k$, the family $\mathcal{A}^k(y)$ is closed under finite intersections and $a(y, K_1 \cap K_2) \le a(y, K_1) + a(y, K_2)$ for all $K_1, K_2 \in \mathcal{A}^k(y)$.*

We follow the proof of [12, Lemma 2.5] to show that $K_1 \cap K_2 \in \mathcal{A}^k(y)$ for any $K_1, K_2 \in \mathcal{A}^k(y)$. Let $\overline{f}, \overline{g} \in E(H)$ with $|\overline{f}(x) - \overline{g}(x)| < 1$ for all $x \in K_1 \cap K_2$ and $U = \{x \in H : |\overline{f}(x) - \overline{g}(x)| < 1\}$. Take an open set W in H containing K_1 with $W \cap K_2 \subset U$ and choose $\overline{u} \in E(H)$ such that $\overline{u}|K_1 = 1$, $\overline{u}|(H \setminus W) = 0$ and $\overline{u}(x) \in [0, 1]$ for all $x \in H$ (see condition (2.3)). Then, $\overline{h} = \overline{u} \cdot (\overline{f} - \overline{g}) + \overline{g} \in E(H)$, $\overline{h}|K_1 = \overline{f}|K_1$, $\overline{h}|(K_2 \setminus W) = \overline{g}|(K_2 \setminus W)$, and $|\overline{h}(x) - \overline{g}(x)| < 1$ for $x \in K_2$. Since $K_1 \in \mathcal{A}^k(y)$ and $\overline{h}|K_1 = \overline{f}|K_1$, we have $|\alpha_y(\overline{f}) - \alpha_y(\overline{h})| \le a(y, K_1) < \infty$. Similarly, $K_2 \in \mathcal{A}^k(y)$ and $|\overline{h}(x) - \overline{g}(x)| < 1$ for $x \in K_2$ imply $|\alpha_y(\overline{h}) - \alpha_y(\overline{g})| \le a(y, K_2) < \infty$. Therefore,

$$|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| \le |\alpha_y(\overline{f}) - \alpha_y(\overline{h})| + |\alpha_y(\overline{h}) - \alpha_y(\overline{g})| \le a(y, K_1) + a(y, K_2).$$

Note that the last inequality is true if some of $\alpha_y(\overline{f}), \alpha_y(\overline{h}), \alpha_y(\overline{g})$ are $\pm\infty$. Indeed, if $\alpha_y(\overline{f}) = \pm\infty$, then $|\alpha_y(\overline{f}) - \alpha_y(\overline{h})| < \infty$ implies $\alpha_y(\overline{h}) = \pm\infty$. Consequently, $\alpha_y(\overline{g}) = \pm\infty$ because $|\alpha_y(\overline{h}) - \alpha_y(\overline{g})| < \infty$. Similarly, if $\alpha_y(\overline{h}) = \pm\infty$ or $\alpha_y(\overline{g}) = \pm\infty$, then the other two are also $\pm\infty$. Hence, $a(y, K_1 \cap K_2) \le a(y, K_1) +$

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$a(y, K_2)$, which means that $K_1 \cap K_2 \neq \varnothing$ (otherwise, $a(y, K_1 \cap K_2) = \infty$) and $K_1 \cap K_2 \in \mathcal{A}^k(y)$.

Since each family $\mathcal{A}^k(y)$, $y \in Y_k$, consists of compact subsets of H_k, $K(y,k) = \bigcap \mathcal{A}^k(y)$ is nonempty and compact.

Claim 5 *For every $y \in Y_k$, the set $K(y,k)$ is a nonempty finite subset of H_k with $K(y,k) \in \mathcal{A}^k(y)$. Moreover, if $y \in Y$, then there exists k such that $y \in Y_k$ and $K(y,k) \subset X$.*

Let $y \in Y_k$. We already observed that $K(y,k)$ is compact and nonempty. Since $y \in Y^k_{p,q}$ for some p, q, $\mathcal{A}^k(y)$ contains finite sets. Hence, $K(y,k)$ is also finite and $K(y,k) \in \mathcal{A}^k(y)$ because it is an intersection of finitely many elements of $\mathcal{A}^k(y)$. If $y \in Y$, then by Claim 1, there is k such that $\mathcal{A}^k(y)$ contains a finite subset of X. Since $K(y,k)$ is the minimal element of $\mathcal{A}^k(y)$, it is also a subset of X.

Following [7], for every k, we define $M^k(p,1) = Y^k_{p,1}$ and $M^k(p,q) = Y^k_{p,q} \setminus Y^k_{2p,q-1}$ if $q \geq 2$.

Claim 6 *$Y_k = \bigcup\{M^k(p,q) : p, q = 1, 2, \dots\}$ and for every $y \in M^k(p,q)$, there exists a unique set $K_{kp}(y) \in \mathcal{A}^k(y)$ of cardinality q such that $a(y, K_{kp}(y)) \leq p$.*

Since $M^k(p,q) \subset Y^k_{p,q} \subset Y_k$, $\bigcup\{M^k(p,q) : p, q = 1, 2, \dots\} \subset Y_k$. If $y \in Y_k$, then $K(y,k) \in \mathcal{A}^k(y)$ is a finite subset of H_k. Assume $|K(y,k)| = q$ and $a(y, K(y,k)) \leq p$ for some p, q. So, $y \in Y^k_{p,q}$. Moreover, $y \notin Y^k_{2p,q-1}$, otherwise, there would be $K \in \mathcal{A}^k(y)$ with $a(y,K) \leq 2p$ and $|K| \leq q-1$. The last inequality contradicts the minimality of $K(y,k)$. Hence, $y \in M^k(p,q)$ which shows that $Y_k = \bigcup\{M^k(p,q) : p, q = 1, 2, \dots\}$.

Suppose $y \in M^k(p,q)$. Then, there exists a set $K \in \mathcal{A}^k(y)$ with $a(y,K) \leq p$ and $|K| \leq q$. Since, $y \notin Y^k_{2p,q-1}$, $|K| = q$. If there exists another $K' \in \mathcal{A}^k(y)$ with $a(y,K') \leq p$ and $|K'| = q$, then $K \cap K' \neq \varnothing$, $|K \cap K'| \leq q-1$ and, by Claim 4, $a(y, K \cap K') \leq a(y,K) + a(y,K') \leq 2p$. This means that $y \in Y^k_{2p,q-1}$, a contradiction. Hence, there exists a unique $K_{kp}(y) \in \mathcal{A}^k(y)$ such that $a(y, K_{kp}(y)) \leq p$ and $|K_{kp}(y)| = q$.

For every q, let $[H_k]^q$ denote the set of all q-points subsets of H_k endowed with the Vietoris topology.

Claim 7 *The map $\Phi_{kpq} : M^k(p,q) \to [H_k]^q$, $\Phi_{kpq}(y) = K_{kp}(y)$, is continuous.*

Because $K_{kp}(y) \subset H_k$ consists of q points for all $y \in M^k(p,q)$, it suffices to show that if $K_{kp}(y) \cap U \neq \varnothing$ for some open $U \subset H$, then there is a neighborhood V of y in $\overline{Y}$ with $K_{kp}(z) \cap U \neq \varnothing$ for all $z \in V \cap M^k(p,q)$. We can assume that $K_{kp}(y) \cap U$ contains exactly one point x_0.

Let $q \geq 2$, so $K_{kp}(y) = \{x_0, x_1, \dots, x_{q-1}\}$. Since $y \notin Y^k_{2p,q-1}$ we have $a(y,K) > 2p$, where $K = \{x_1, \dots, x_{q-1}\}$. Hence, there are $\overline{f}, \overline{g} \in E(H)$ such that $|\overline{f}(x) - \overline{g}(x)| < 1$ for all $x \in K$ and $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| > 2p$. The last inequality implies $\overline{f}(x_0) \neq \overline{g}(x_0)$, otherwise, $a(y, K_{kp}(y))$ would be greater than $2p$ (recall that $y \in M^k(p,q)$ implies $a(y, K_{kp}(y)) \leq p$). So, at least one of the numbers $\overline{f}(x_0), \overline{g}(x_0)$ is not zero. Without loss of generality, we can assume that $\overline{f}(x_0) > 0$, and let r be a rational number with $\frac{-1+\delta}{\overline{f}(x_0)} < r < \frac{1+\delta}{\overline{f}(x_0)}$, where $\delta = \overline{f}(x_0) - \overline{g}(x_0)$. Then, $-1 < (1-r)\overline{f}(x_0) - \overline{g}(x_0) < 1$, and choose $\overline{h}_1 \in E(H)$ such that $\overline{h}_1(x_0) = r$ and $\overline{h}_1(x) = 0$ for all $x \notin U$. Consider the

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function $\overline{h} = (1 - \overline{h}_1)\overline{f}$. Clearly, $\overline{h}(x_0) = (1-r)\overline{f}(x_0)$ and $\overline{h}(x) = \overline{f}(x)$ if $x \notin U$. Hence, $\overline{h} \in E(H)$ and $|\overline{h}(x) - \overline{g}(x)| < 1$ for all $x \in K_{kp}(y)$. This implies $|\alpha_y(\overline{h}) - \alpha_y(\overline{g})| \le p$. Then,

$$|\alpha_y(\overline{f}) - \alpha_y(\overline{h})| \ge |\alpha_y(\overline{f}) - \alpha_y(\overline{g})| - |\alpha_y(\overline{h}) - \alpha_y(\overline{g})| > 2p - p = p.$$

Observe that it is not possible $\alpha_y(\overline{f}) = \alpha_y(\overline{h}) = \pm\infty$ because $|\alpha_y(\overline{h}) - \alpha_y(\overline{g})| \le p$ would imply $\alpha_y(\overline{g}) = \pm\infty$. Then, $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| = 0$, a contradiction.

The set $V = \{z \in \overline{Y} : |\alpha_z(\overline{f}) - \alpha_z(\overline{h})| > p\}$ is a neighborhood of y. Since $\overline{h}(x) = \overline{f}(x)$ for all $x \notin U$, $K_{kp}(z) \cap U = \varnothing$ for some $z \in V \cap M^k(p,q)$ would imply $|\alpha_z(\overline{h}) - \alpha_z(\overline{f})| \le p$, a contradiction. Therefore, $K_{kp}(z) \cap U \neq \varnothing$ for $z \in V \cap M^k(p,q)$.

If $q = 1$, then $K_{kp}(y) = \{x_0\}$ and $K(y,k) = K_{kp}(y)$. So, $H_k \backslash U \notin \mathcal{A}^k(y)$ (otherwise, $K(y,k) \subset H_k \backslash U$). Hence, there exist $\overline{f}, \overline{g} \in E(H)$ such that $|\overline{f}(x) - \overline{g}(x)| < 1$ for all $x \in H_k \backslash U$ and $|\alpha_y(\overline{f}) - \alpha_y(\overline{g})| > 2p$. Define $\overline{h} \in E(H)$ as in the previous case and use the same arguments to complete the proof.

Since $Y^k_{p,q}$ are compact subsets of $\overline{Y}$, each $M^k(p,q)$ is a countable union of compact subsets $\{F^k_n(p,q) : n = 1,2,\ldots\}$ of $\overline{Y}$. So, by Claim 6, $Y_k = \bigcup\{F^k_n(p,q) : n,p,q = 1,2,\ldots\}$. According to Claim 7, all maps $\Phi^n_{kpq} = \Phi_{kpq}|F^k_n(p,q) : F^k_n(p,q) \to [H_k]^q$ are continuous. Moreover, since $Y \subset \bigcup_k Y_k$, $Y \subset \bigcup\{F^k_n(p,q) : n,p,q,k = 1,2,\ldots\}$.

Claim 8 *The fibers of $\Phi^n_{kpq} : F^k_n(p,q) \to [H_k]^q$ are finite.*

We follow the arguments from the proof of [6, Theorem 4.2]. Fix $z \in F^k_n(p,q)$ for some n,p,q,k and let $A = \{y \in F^k_n(p,q) : K_{kp}(y) = K_{kp}(z)\}$. Since Φ^n_{kpq} is a perfect map, A is compact. Suppose A is infinite, so it contains a convergent sequence $S = \{y_m\}$ of distinct points. Because $E(\overline{Y})$ contains a QS-algebra Γ on $\overline{Y}$, for every y_m there exist its neighborhood U_m in $\overline{Y}$ and a function $\overline{g}_m \in \Gamma$, $\overline{g}_m : \overline{Y} \to [0,2p]$ such that: $U_m \cap S = \{y_m\}$, $\overline{g}_m(y_m) = 2p$, and $\overline{g}_m(y) = 0$ for all $y \notin U_m$. Since φ is c-good, for each m, there is $f_m \in E(X)$ with $\varphi(f_m) = \overline{g}_m|Y = g_m$ and $\|f_m\| \le c\|g_m\|$. So, $\|\overline{f}_m\| \le 2pc$, $m = 1,2,\ldots$ and the sequence $\{\overline{f}_m\}$ is contained in the compact set $[-2pc, 2pc]^H$. Hence, $\{\overline{f}_m\}$ has an accumulation point in $[-2pc,2pc]^H$. This implies the existence of $i \neq j$ such that $|\overline{f}_i(x) - \overline{f}_j(x)| < 1$ for all $x \in K_{kp}(z)$. Consequently, since $K_{kp}(y_j) = K_{kp}(z)$, $|\alpha_{y_j}(\overline{f}_j) - \alpha_{y_j}(\overline{f}_i)| \le p$. On the other hand, $\alpha_{y_j}(\overline{f}_j) = \overline{\varphi(f_j)}(y_j) = \overline{g}_j(y_j) = 2p$ and $\alpha_{y_j}(\overline{f}_i) = \overline{\varphi(f_i)}(y_j) = \overline{g}_i(y_j) = 0$, so $|\alpha_{y_j}(\overline{f}_j) - \alpha_{y_j}(\overline{f}_i)| = 2p$, a contradiction.

Now, we can complete the proof of Proposition 2.1. Suppose H has a property $\mathcal{P}$ satisfying conditions (a)–(c). Then so does H^q_k for each k,q because H^q_k is closed in H^q. We claim that the space $[H_k]^q$ also has the property $\mathcal{P}$. Indeed, let $\mathcal{C}$ be a countable base of H_k. For every q-tuple $(U_1,\ldots,U_q)$ of elements of $\mathcal{C}$ with pairwise disjoint closures, the closed set

$$W(U_1,\ldots,U_q) = \{\{x_1,\ldots,x_q\} : x_i \in \overline{U}_i, i = 1,\ldots,q\} \subset [H_k]^q$$

is homeomorphic to the closed subset $\overline{U}_1 \times \cdots \times \overline{U}_q$ of H^q_k. Hence, $W(U_1,\ldots,U_q) \in \mathcal{P}$. Clearly, the space $[H_k]^q$ can be covered by countably many sets of the form

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$W(U_1, \dots, U_q)$, therefore, it belongs to $\mathcal{P}$. Finally, since the maps $\Phi^n_{kpq} : F^k_n(p,q) \to [H_k]^q$ are perfect and have finite fibers, each $F^k_n(p,q)$ has the property $\mathcal{P}$. Therefore, by condition (b), $Y_\infty = \bigcup\{F^k_n(p,q) : n,p,q,k = 1,2,\dots\}$ has the property $\mathcal{P}$.

It remains to show that all powers of Y_∞ also have the property $\mathcal{P}$. Simplifying the notations, we observed that $Y_\infty = \bigcup_{m=1}^\infty F_m$, where every F_m is a compact set admitting a map Φ_m with finite fibers onto a compact subset of H^m. Then, for every k, we have

$$Y^k_\infty = \bigcup_{(m_1,m_2,\dots,m_k)} F_{m_1} \times F_{m_2} \times \cdots \times F_{m_k}.$$

Consequently, $\prod_{i=1}^k \Phi_{m_i} : \prod_{i=1}^k F_{m_i} \to H^{m_1+m_2+\cdots+m_k}$ is a map which fibers are products of k-many finite sets. Hence, the fibers of $\prod_{i=1}^k \Phi_{m_i}$ are finite. Because $H^{m_1+m_2+\cdots+m_k} \in \mathcal{P}$, so is $\prod_{i=1}^k F_{m_i}$. Finally, by property (b), $Y^k_\infty \in \mathcal{P}$.

All definitions below, except that one of $(m-C)$-spaces, can be found in [2]. A normal space X is called strongly countable-dimensional if X can be represented as a countable union of closed finite-dimensional subspaces. Recall that a normal space X is weakly infinite-dimensional if for every sequence $\{(A_i, B_i)\}$ of pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \dots$ such that L_i is a partition between A_i and B_i and $\bigcap_i L_i = \varnothing$. A normal space X is a C-space if for every sequence $\{\mathcal{G}_i\}$ of open covers of X there exists a sequence $\{\mathcal{H}_i\}$ of families of pairwise disjoint open subsets of X such that for $i = 1, 2, \dots$ each member of $\mathcal{H}_i$ is contained in a member of $\mathcal{G}_i$ and the union $\bigcup_i \mathcal{H}_i$ is a cover of X. The $(m-C)$-spaces, where $m \geq 2$ is a natural number, were introduced by Fedorchuk [4]: A normal space X is an $(m-C)$-space if for any sequence $\{\mathcal{G}_i\}$ of open covers of X such that each $\mathcal{G}_i$ consists of at most m elements, there is a sequence of disjoint open families $\{\mathcal{H}_i\}$ such that each $\mathcal{H}_i$ refines $\mathcal{G}_i$ and $\bigcup_i \mathcal{H}_i$ is a cover of X. The $(2-C)$-spaces are exactly the weakly infinite-dimensional spaces and for every m we have the inclusion $(m+1)-C \subset m-C$. Moreover, every C-space is $m-C$ for all m.

It is well known that the class of metrizable strongly countable-dimensional spaces contains all finite-dimensional metrizable spaces and is contained in the class of metrizable C-spaces. The last inclusion follows from the following two facts: (i) every finite-dimensional paracompact space is a C-space [2, Theorem 6.3.7] and (ii) every paracompact space which is a countable union of its closed C-spaces is also a C-space [8, Theorem 4.1]. Moreover, every C-space is weakly infinite-dimensional [2, Theorem 6.3.10].

In the class of σ-compact metrizable spaces, the zero-dimensionality satisfies all conditions (a)–(c), see Theorems 1.5.16, 1.2.2, and 1.12.4 from [2]. The strong countable-dimensionality also satisfies all these conditions, condition (c) follows easily from [2, Theorem 1.12.4]. For C-space, this follows from mentioned above fact that a countable union of closed compact C-spaces is a C-space [8, Theorem 4.1] and the following results of Hattori–Yamada [9]: the class of compact C-spaces is closed under finite products any perfect preimage of a C-space with C-space fibers is a C-space. Finally, if a σ-compact space X is weakly infinite-dimensional, then obviously every closed subset of X, as well as any countable union of closed subsets of X have the same property (see [2, Theorem 6.1.6]). Condition (c) follows from the following result of Pol [19, Theorem 4.1]: If $f : X \to Y$ is a continuous map between compact metrizable

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spaces such that Y is weakly infinite-dimensional and each fiber $f^{-1}(y)$, $y \in Y$, is at most countable, then X is weakly infinite-dimensional. The validity of conditions (a)–(c) for $(m-C)$-spaces in the class of σ-compact metrizable spaces follows from the following results [4]: The $(m-C)$-space property is hereditary with respect to closed subsets, a countable union of closed $(m-C)$-spaces is also $m-C$. Moreover, for $(m-C)$-spaces, Krupski [11, Lemma 4.5] established an analog of the cited above Pol's result. Therefore, condition (c) holds for the property $m-C$ in the class of σ-compact metrizable spaces.

3 Uniformly continuous surjections

In this section, we prove Theorems 1.1 and 1.3.

For every space X, let $\mathcal{F}_X$ be the class of all maps from X onto second countable spaces. For any two maps $h_1, h_2 \in \mathcal{F}_X$, we write $h_1 > h_2$ if there exists a continuous map $\theta : h_1(X) \to h_2(X)$ with $h_2 = \theta \circ h_1$. If $\Phi \subset C(X)$, we denote by $\triangle\Phi$ the diagonal product of all $f \in \Phi$. Clearly, $(\triangle\Phi)(X)$ is a subspace of the product $\prod\{\mathbb{R}_f : f \in \Phi\}$, and let $\pi_f : (\triangle\Phi)(X) \to \mathbb{R}_f$ be the projection. Following [7], we call a set $\Phi \subset C(X)$ *admissible* if the family $\pi(\Phi) = \{\pi_f : f \in \Phi\}$ is a QS-algebra on $(\triangle\Phi)(X)$. We are using the following facts:

(3.1) $\dim X \le n$ if and only if for every $h \in \mathcal{F}_X$, there exists a $h_0 \in \mathcal{F}_X$ such that $\dim h_0(X) \le n$ and $h_0 > h$ [18].

(3.2) If $\dim X \le n$ and $\Phi \subset C(X)$ is countable, then there exists a countable set $\Theta \subset C(X)$ containing Φ with $\dim(\triangle\Theta)(X) \le n$. Moreover, it follows from the proof of [7, Lemma 2.2] that we can choose $\Theta \subset C^*(X)$ provided that $\Phi \subset C^*(X)$.

(3.3) For every countable $\Phi' \subset C(X)$, there is a countable admissible set Φ containing Φ' such that $(\triangle\Phi)(X)$ is homeomorphic to $(\triangle\Phi')(X)$. According to the proof of [7, Lemma 2.4], Φ could be taken to be a subset of $C^*(X)$ if $\Phi' \subset C^*(X)$. Moreover, we can assume that Φ' satisfies condition (2.3), so $\pi(\Phi)$ also satisfies that condition.

(3.4) If $\{\Psi_n\}$ is an increasing sequence of admissible subsets of $C(X)$, then $\Psi = \bigcup_n \Psi_n$ is also admissible (see [7, Lemma 2.5]).

We also need the following lemmas.

Lemma 3.1 *Let X be a zero-dimensional separable metrizable space and $E(X)$ be a countable subfamily of $C^*(X)$. Then, there exists a metrizable zero-dimensional compactification $\overline{X}$ of X such that each $f \in E(X)$ can be extended over $\overline{X}$.*

Proof Let X_0 be a metrizable compactification of X and for every $f \in E(X)$ denote by Z_f the closure of $f(X)$ in $\mathbb{R}$. Consider the diagonal product h of the maps j and $\triangle\{f : f \in E(X)\}$, where $j : X \hookrightarrow X_0$ is the embedding. Then, the closure X_1 of $h(X)$ in the product $X_0 \times \prod_{f\in E(X)} Z_f$ is a compactification of X such that every $f \in E(X)$ can be extended over X_1. Let $\theta : \beta X \to X_1$ be the map witnessing that βX is a compactification of X larger than X_1. Since $\dim \beta X = 0$, by the Mardešić factorization theorem [2, Theorem 3.4.1] there is a metrizable compactum $\overline{X}$ and maps $\nu : \beta X \to \overline{X}$ and $\eta : \overline{X} \to X_1$ such that $\dim \overline{X} = 0$ and $\theta = \eta \circ \nu$. Evidently, $\nu | i(X)$ is a homeomorphism, where $i : X \hookrightarrow \beta X$ is the embedding, so $\overline{X}$ is a compactification of X.

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Because every $f \in E(X)$ is extendable to a function $\overline{f} : X_1 \to \mathbb{R}$, the composition $\overline{f} \circ \eta$ is an extension of f over $\overline{X}$. ∎

Lemma 3.2 *Let X be a separable metrizable space and $E(X)$ be a countable subfamily of $C(X)$. Then there exists a metrizable compactification $\overline{X}$ of X such that each $f \in E(X)$ can be extended to a map $\overline{f} : \overline{X} \to \overline{\mathbb{R}}$. Moreover, if $\dim X = 0$, then we can assume that $\dim \overline{X} = 0$.*

Proof The proof is similar to the proof of Lemma 3.1. The only difference is that for every $f \in E(X)$, we consider Z_f to be the closure of $f(X)$ in $\overline{\mathbb{R}}$. ∎

Proof of Theorem 1.1 Let $T : D_p(X) \to D_p(Y)$ be a uniformly continuous c-good surjection. Everywhere below, for $f \in C^*(X)$, let $\overline{f} \in C(\beta X)$ be its extension; similarly, if $g \in C(Y)$ then $\overline{g} \in C(\beta Y, \overline{\mathbb{R}})$ is the extension of g. According to (3.1), it suffices to prove that for every $h \in \mathcal{F}_Y$, there is $h_0 \in \mathcal{F}_Y$ such that $\dim h_0(Y) = 0$ and $h_0 \succ h$. So, we fix $h \in \mathcal{F}_Y$ and let $\overline{h} : \beta Y \to \overline{h(Y)}$ be an extension of h, where $\overline{h(Y)}$ is a metrizable compactification of $h(Y)$. We will construct by induction two sequences $\{\Psi_n\}_{n\geq 1} \subset C(\beta X)$ and $\{\Phi_n\}_{n\geq 1} \subset C(\beta Y, \overline{\mathbb{R}})$ of countable sets, countable QS-algebras $\mathcal{A}_n$ on $(\triangle\Psi_n)(\beta X)$ and countable QS-algebras Λ_n on $Y'_n = (\triangle\Phi'_n)(\beta Y)$, where $\Phi'_n = \{\overline{T(f)} : \overline{f} \in \Psi_n\}$, satisfying the following conditions for every $n \geq 1$:

(3.5) $\Phi_1 \subset C(\beta Y)$ is admissible and $\triangle\Phi_1 \succ \overline{h}$;
(3.6) $\Phi_n \subset \Phi_{n+1} = \Phi'_n \cup \{\lambda \circ (\triangle\Phi'_n) : \lambda \in \Lambda_n\}$;
(3.7) each Ψ_n is admissible, $\dim(\triangle\Psi_n)(\beta X) = 0$ and $\Psi_n \subset \Psi_{n+1}$;
(3.8) $\mathcal{A}_n$ is a QS-algebra on $(\triangle\Psi_n)(\beta X)$ satisfying condition (2.3);
(3.9) Λ_{n+1} contains $\{\lambda \circ \delta_n : \lambda \in \Lambda_n\}$, where $\delta_n : Y'_{n+1} \to Y'_n$ is the surjective map generated by the inclusion $\Phi'_n \subset \Phi'_{n+1}$;
(3.10) for every $\overline{g} \in \Phi_n \cap C(\beta Y)$, there is $\overline{f}_g \in \Psi_n$ with $\|f_g\| \leq c\|g\|$ and $T(f_g) = g$.

Since $\overline{h}(\beta Y)$ is a separable metrizable space, by (3.3), there is a countable admissible set $\Phi_1 \subset C(\beta Y)$ with $\triangle\Phi_1 \succ \overline{h}$. Choose a countable set $\Psi'_1 \subset C(\beta X)$ such that for every $\overline{g} \in \Phi_1$, there is $\overline{f}_g \in \Psi'_1$ with $\|f_g\| \leq c\|g\|$ and $T(f_g) = g$. Next, use (3.2) to find countable $\Theta_1 \subset C(\beta X)$ containing Ψ'_1 such that $\dim(\triangle\Theta_1)(\beta X) = 0$. Finally, by (3.3), we can extend Θ_1 to a countable admissible set $\Psi_1 \subset C(\beta X)$ such that $(\triangle\Psi_1)(\beta X)$ is homeomorphic to $(\triangle\Theta_1)(\beta X)$ and the QS-algebra $\mathcal{A}_1 = \pi(\Psi_1)$ on $(\triangle\Psi_1)(\beta X)$ satisfies condition (2.3). Evidently, $\dim(\triangle\Psi_1)(\beta X) = 0$.

Suppose Φ_k and Ψ_k are already constructed for all $k \leq n$. Then, $\Phi'_n = \{\overline{T(f)} : \overline{f} \in \Psi_n\}$ is a countable set in $C(\beta Y, \overline{\mathbb{R}})$. Because $\Psi_{n-1} \subset \Psi_n$, $\Phi'_{n-1} \subset \Phi'_n$. So, there is a surjective map $\delta_{n-1} : Y'_n \to Y'_{n-1}$ (see (3.9)). Since $\triangle\Phi'_n \succ \triangle\Phi'_{n-1}$, we have $\delta_{n-1}(y) = \triangle\Phi'_{n-1}((\triangle\Phi'_n)^{-1}(y))$ for all $y \in Y'_n$. Choose a countable QS-algebra Λ_n on Y'_n containing the family $\{\lambda \circ \delta_{n-1} : \lambda \in \Lambda_{n-1}\}$ and let $\Phi_{n+1} = \Phi'_n \cup \{\lambda \circ (\triangle\Phi'_n) : \lambda \in \Lambda_n\}$. Next, take a countable set $\Psi'_{n+1} \subset C(\beta X)$ containing Ψ_n such that for every $\overline{g} \in \Phi_{n+1} \cap C(\beta Y)$, there is $\overline{f}_g \in \Psi'_{n+1}$ with $\|f_g\| \leq c\|g\|$ and $T(f_g) = g$. Then, by (3.2), there is countable $\Theta_{n+1} \subset C(\beta X)$ containing Ψ'_{n+1} with $\dim(\triangle\Theta_{n+1})(\beta X) = 0$. Finally, according to (3.3), we extend Θ_{n+1} to a countable admissible set $\Psi_{n+1} \subset C(\beta X)$ such that $(\triangle\Psi_{n+1})(\beta X)$ is homeomorphic to $(\triangle\Theta_{n+1})(\beta X)$ and the QS-algebra $\mathcal{A}_{n+1} = \pi(\Psi_{n+1})$ on $(\triangle\Psi_{n+1})(\beta X)$ satisfies condition (2.3). This completes the induction.

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Let $\Psi = \bigcup_n \Psi_n$, $X_0 = (\triangle\Psi)(X)$, $\overline{X}_n = (\triangle\Psi_n)(\beta X)$, and $\overline{X}_0 = (\triangle\Psi)(\beta X)$. Similarly, let $\Phi = \bigcup_n \Phi_n$, $Y_0 = h_0(Y)$, and $\overline{Y}_0 = (\triangle\Phi)(\beta Y)$, where $h_0 = (\triangle\Phi)|Y$. Both Ψ and Φ are countable and Ψ is an admissible subset of $C(\beta X)$ (see (3.4)). Hence, the family $E(\overline{X}_0) = \{\pi_{\overline{f}} : \overline{f} \in \Psi\}$ is a countable QS-algebra on $\overline{X}_0$. Moreover, the family $E(Y_0) = \{\pi_{\overline{g}}|Y_0 : \overline{g} \in \Phi\}$ is extendable over $\overline{Y}_0$. Since $\Psi_n \subset \Psi_{n+1}$ for every n, there are maps $\theta_n^{n+1} : \overline{X}_{n+1} \to \overline{X}_n$. Because $\Psi = \bigcup_n \Psi_n$, the space $\overline{X}_0$ is the limit of the inverse sequence $S_X = \{\overline{X}_n, \theta_n^{n+1}\}$ with $\dim \overline{X}_n = 0$ for all n. Hence, $\overline{X}_0$ is also zero-dimensional (see [2, Theorem 3.4.11]. Observe also that $E(\overline{X}_0) = \bigcup_{n=1}^{\infty}\{h \circ \theta_n : h \in \mathcal{A}_n\}$.

Let us show that $E(\overline{X}_0)$ ia a QS-algebra satisfying condition (2.3). Because $E(\overline{X}_0)$ is the union of the increasing sequence of the families $\{h \circ \theta_n : h \in \mathcal{A}_n\}$ and each $\mathcal{A}_n$ is a QS-algebra, $E(\overline{X}_0)$ is closed under sums, multiplications, and multiplications by rational numbers. It remains to show that for every $x \in \overline{X}_0$ and every its neighborhood $U \subset \overline{X}_0$, there is $\overline{f} \in E(\overline{X}_0)$ such that $\overline{f}(x) = 1$ and $\overline{f}(\overline{X}_0 \setminus U) = 0$. But that follows from the proof of the more general condition (2.3). To show that $E(\overline{X}_0)$ satisfies (2.3), take a compact set $K \subset \overline{X}_0$ and an open set $W \subset \overline{X}_0$ containing K. Because $\overline{X}_0$ is the limit of the inverse sequence $S_X = \{\overline{X}_n, \theta_n^{n+1}\}$, there are n, a compact set $K_n \subset \overline{X}_n$ and an open set $W_n \subset \overline{X}_n$ containing K_n such that $K \subset \theta_n^{-1}(K_n) \subset \theta_n^{-1}(W_n) \subset W$, where $\theta_n : \overline{X}_0 \to \overline{X}_n$ denotes the n-th projection in S_X. Then, according to condition (3.8), there is $f_n \in \mathcal{A}_n$ with $f_n|K_n = 1$, $f_n|(\overline{X}_n \setminus W_n) = 0$ and $f_n : \overline{X}_n \to [0,1]$. Finally, observe that the function $\overline{f} = f_n \circ \theta_n : \overline{X}_0 \to [0,1]$ belongs to $E(\overline{X}_0)$ and $\overline{f}$ satisfies the conditions $\overline{f}|K = 1$ and $\overline{f}|(\overline{X}_0 \setminus W) = 0$.

It follows from the construction that $\Phi = \{\overline{T(f)} : \overline{f} \in \Psi\}$ and for every $\overline{g} \in \Phi \cap C(\beta Y)$, there is $\overline{f}_g \in \Psi$ with $T(f_g) = g$ and $\|f_g\| \le c\|g\|$. Observe that for all $\overline{f} \in \Psi$ and $\overline{g} \in \Phi$, we have $f = \pi_{\overline{f}} \circ (\triangle\Psi)|X$ and $g = \pi_{\overline{g}} \circ (\triangle\Phi)|Y$. Therefore, there is a surjective map $\varphi : E_p(X_0) \to E_p(Y_0)$ defined by $\varphi(\pi_f) = \pi_{T(f)}$, where π_f and π_g denote, respectively, the functions $\pi_{\overline{f}}|X_0$ and $\pi_{\overline{g}}|Y_0$. Moreover, $\|f\| = \|\pi_f\|$ and $\|g\| = \|\pi_g\|$ for all $\overline{f} \in \Psi$ and $\overline{g} \in \Phi \cap C(\beta Y)$. This implies that φ is a c-good surjection.

Let's show that φ is uniformly continuous. Suppose

$$V = \{\pi_g : |\pi_g(y_i)| < \varepsilon \ \forall i \le k\}$$

is a neighborhood of the zero function in $E_p(Y_0)$. Take points $\overline{y}_i \in Y$ with $h_0(\overline{y}_i) = y_i$, $i = 1, 2, \dots, k$, and let $\widetilde{V} = \{g \in D_p(Y) : |g(\overline{y}_i)| < \varepsilon \ \forall i \le k\}$. Since T is uniformly continuous, there is a neighborhood

$$\widetilde{U} = \{f \in D_p(X) : |f(\overline{x}_j)| < \delta \ \forall j \le m\}$$

of the zero function in $D_p(X)$ such that $f - f' \in \widetilde{U}$ implies $T(f) - T(f') \in \widetilde{V}$ for all $f, f' \in D_p(X)$. Let $x_j = (\triangle\Psi)(\overline{x}_j)$ and

$$U = \{\pi_f : |\pi_f(x_j)| < \delta \ \forall j \le m\}.$$

Obviously, $\pi_f - \pi_{f'} \in U$ implies $f - f' \in \widetilde{U}$. Hence, $T(f) - T(f') \in \widetilde{V}$, which yields $\varphi(\pi_f) - \varphi(\pi_{f'}) \in V$.

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Finally, we can show that $E(\overline{Y}_0)$ contains a QS-algebra on $\overline{Y}_0$. Since $\Lambda_n \subset C(Y'_n)$ is a QS-algebra on Y'_n, it separates the points and the closed sets in Y'_n. So, Y'_n is homeomorphic to $(\triangle\Lambda_n)(Y'_n)$. This implies that $Y_{n+1} = (\triangle\Phi_{n+1})(\beta Y)$ is homeomorphic to Y'_n. Therefore, Λ_n can be considered as a QS-algebra on Y_{n+1}. On the other hand, $\overline{Y}_0$ is the limit of the inverse sequence $S_Y = \{Y_{n+1}, \gamma^{n+2}_{n+1}, n \geq 1\}$, where $\gamma^{n+2}_{n+1} : Y_{n+2} \to Y_{n+1}$ is the surjective map generated by the inclusion $\Phi_{n+1} \subset \Phi_{n+2}$. According to condition (3.9), we can also assume that Λ_{n+1} contains the family $\{\lambda \circ \gamma^{n+2}_{n+1} : \lambda \in \Lambda_n\}$. Denote by $\gamma_{n+1} : \overline{Y}_0 \to Y_{n+1}$ the projections in S_Y, and let $\Gamma_n = \{\lambda \circ \gamma_{n+1} : \lambda \in \Lambda_n\}$. Then, $\{\Gamma_n\}_{n\geq 1}$ is an increasing sequence of countable families and $\Gamma_n \subset E(\overline{Y}_0) = \{\pi_{\overline{g}} : \overline{g} \in \Phi\}$ for every n. We claim that $\Gamma = \bigcup_n \Gamma_n$ is a QS-algebra on $\overline{Y}_0$. Indeed, Γ is closed under addition, multiplication, and multiplication by rational numbers. Because for every $y \in \overline{Y}_0$ and its neighborhood $V \subset \overline{Y}_0$, there is n and an open set $V_n \subset Y_{n+1}$ such that $y_n = \gamma_{n+1}(y) \in V_n$ and $\gamma^{-1}_{n+1}(V_n) \subset V$, there exists $\lambda \in \Lambda_n$ with $\lambda(y_n) = 1$ and $\lambda(Y_{n+1}\setminus V_n) = 0$. Then, $g = \lambda \circ \gamma_{n+1} \in \Gamma$, $g(y) = 1$, and $g(\overline{Y}_0\setminus V) = 0$.

To prove Theorem 1.1, we apply Proposition 2.1 with $H = \overline{X}_0$ to find a σ-compact set $Y_\infty \subset \overline{Y}_0$ containing Y_0 with $\dim Y_\infty = 0$. Therefore, by [2, Proposition 1.2.2], $\dim Y_0 = 0$. ∎

Proposition 3.3 [13] *For every linear continuous surjective map $T : C^*_p(X) \to C^*_p(Y)$, there is $c > 0$ such that T is c-good.*

Proof By the Closed Graph Theorem, T considered as a map between the Banach spaces $C^*_u(X)$ and $C^*_u(Y)$, both equipped with the sup-norm, is continuous. Then, T induced a linear isomorphism T_0 between $C^*_u(X)/K$ and $C^*_u(Y)$, where K is the kernel of T. So, for every $g \in C^*_u(Y)$, we have $\|T_0^{-1}(g)\| \leq \|T_0^{-1}\| \cdot \|g\|$. Because

$$\|T_0^{-1}(g)\| = \inf\{\|f - h\| : h \in K\},$$

where $f \in C^*_u(X)$ with $T(f) = g$, there exists $h_g \in K$ such that $\|f - h_g\| \leq 2\|T_0^{-1}(g)\|$. Hence, $\|f - h_g\| \leq 2\|T_0^{-1}\| \cdot \|g\|$. Since $T(f - h_g) = T(f) = g$, we obtain that T is c-good with $c = 2\|T_0^{-1}\|$. ∎

Proof of Theorem 1.3 Following the proof of Theorem 1.1, we construct two sequences $\{\Psi_n\}_{n\geq 1} \subset C(\beta X)$ and $\{\Phi_n\}_{n\geq 1} \subset C(\beta Y, \overline{\mathbb{R}})$ of countable sets and countable QS-algebras $\mathcal{A}_n$ on $\triangle\Psi_n(\beta X)$ and Λ_n on $Y'_n = (\triangle\Phi'_n)(\beta Y)$ satisfying the conditions (3.5)–(3.10) except (3.7). Because X and Y are separable metrizable spaces, we can choose countable sets Ψ_1 and Φ_1 such that $(\triangle\Phi_1)|Y$ and $(\triangle\Psi_1)|X$ are homeomorphisms.

Then, following the notations from the proof of Theorem 1.1, we have that X_0 and Y_0 are homeomorphic to X and Y, respectively. Moreover, there exists a uniformly continuous c-good surjection $\varphi : E_p(X_0) \to E_p(Y_0)$ such that $E(\overline{X}_0)$ is a QS-algebra on $\overline{X}_0$ satisfying condition (2.3), $E(\overline{Y}_0) \subset C(\overline{Y}_0, \overline{\mathbb{R}})$ and $E(\overline{Y}_0)$ contains a countable QS-algebra Γ on $\overline{Y}_0$. According to the proof of Proposition 2.1 with $H = X_0$, there is a σ-compact set $Y_\infty \subset \overline{Y}_0$ containing Y_0 which a countable union of closed sets $F \subset Y_\infty$ such that F admits a map with finite fibers into a finite power of X_0. Let $Y_0 = \bigcup_m Y_m$ with each Y_m being compact. Then, Y_0 can also be represented as a countable union of compact sets each admitting a map with finite powers into a finite power of X_0. Now, we apply the following fact which was actually used in the proof of Proposition 2.1:

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Assume all powers of a σ-compact metrizable space P have a property $\mathcal{P}$ satisfying conditions (a)–(c). If a metrizable space Z is the union of countably many compact sets Z_n such that each Z_n admits a map with finite fibers into a finite power P^{k_n}, then all finite powers of Z also have the property $\mathcal{P}$. Since all finite powers of X_0 have the property $\mathcal{P}$, where $\mathcal{P}$ is either weakly infinite-dimensionality or the $(m-C)$-space property, by mentioned above fact, all finite powers of Y_0 are either weakly infinite-dimensional or have the $(m-C)$-space property. ∎

4 Proof of Theorem 1.4

We consider topological properties $\mathcal{P}$ of separable metrizable spaces satisfying condition (b) from the introduction section plus the following three:

(a′) if $X \in \mathcal{P}$, then $F \in \mathcal{P}$ for every subset $F \subset X$;
(c′) if $f : X \to Y$ is a perfect map between metrizable spaces with zero-dimensional fibers and $Y \in \mathcal{P}$, then $X \in \mathcal{P}$;
(d′) $\mathcal{P}$ is closed under finite products.

The zero-dimensionality, the countable-dimensionality, and the strong countable dimensionality satisfy these conditions (see [2]).

If $E(X) \subset C(X)$ is a QS-algebra on X, then the family $LE(X)$ of all finite linear combinations $\sum_{i=1}^{k} \lambda_i \cdot f_i$ with $f_i \in E(X)$ and $\lambda_i \in \mathbb{R}$ is called the *linear hull* of $E(X)$.

Proposition 4.1 *Let X and Y be separable metrizable spaces and $E(X) \subset C(X)$ be a countable QS-algebra on X and $E(Y) \subset C(Y)$ be a countable family. Suppose there are metrizable compactifications $\overline{X}$ and $\overline{Y}$ of X and Y and a countable base $\mathcal{B}$ of $\overline{X}$ such that:*

- *every $f \in E(X)$ can be extended to a map $\overline{f} : \overline{X} \to \overline{\mathbb{R}}$ and for every finite open cover $\gamma = \{U_i : i = 1, 2, \dots, k\}$ of $\overline{X}$ with elements from $\mathcal{B}$, there exists a partition of unity $\{\overline{f}_i : i = 1, 2, \dots, k\}$ subordinated to γ with $f_i \in E(X)$;*
- *every $g \in E(Y)$ can be extended to a map $\overline{g} : \overline{Y} \to \overline{\mathbb{R}}$ and the set of all real-valued elements of $E_p(\overline{Y}) = \{\overline{g} : g \in E(Y)\}$ is dense in $C_p(\overline{Y})$;*
- *for every compact set $K \subset \overline{X}$ and every open set $W \subset \overline{X}$ containing K, there is $f \in E(X)$ such that $\overline{f}(K) = 1$, $\overline{f}(\overline{X} \setminus W) = 0$, and $f(x) \in [0,1]$ for all $x \in X$.*

If $\overline{X}$ has a property $\mathcal{P}$ satisfying conditions (a'), (b), (c'), and (d'), and $\varphi : E_p(X) \to E_p(Y)$ is a linear continuous surjection such that φ can be continuously extended over $LE_p(X)$, then $Y \in \mathcal{P}$.

Proof Let $E(\overline{X}) = \{\overline{f} : f \in E(X)\}$. Every $y \in \overline{Y}$ generates a map $l_y : E(\overline{X}) \to \overline{\mathbb{R}}$, $l_y(\overline{f}) = \overline{\varphi(f)}(y)$. Assuming the equalities from (2.4), for any $\overline{f}_1, \overline{f}_2 \in E(\overline{X})$ and $x \in \overline{X}$, we can write $\overline{f}_1(x) + \overline{f}_2(x)$ but not always $\overline{f}_1 + \overline{f}_2 = \overline{f_1 + f_2}$. Also, if $\overline{f}_1 + \overline{f}_2 \in E(\overline{X})$, it is possible that $l_y(\overline{f}_1 + \overline{f}_2) \neq l_y(\overline{f}_1) + l_y(\overline{f}_2)$. If $\lambda \in \mathbb{R}\setminus\{0\}$ and $\overline{\lambda \cdot f} \in E(\overline{X})$, then $l_y(\overline{\lambda \cdot f}) = \lambda \cdot l_y(\overline{f})$ (here, $\lambda \cdot (\pm\infty) = \pm\infty$ if $\lambda > 0$ and $\lambda \cdot (\pm\infty) = \mp\infty$ if $\lambda < 0$). In case $\lambda = 0$, we have $l_y(\overline{0 \cdot f}) = 0$. More general, if $\overline{h} \in C(\overline{X})$ such that $h \cdot f \in E(X)$ for some $f \in E(X)$, then $l_y(\overline{h \cdot f})$ is well defined.

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The *support of* l_y, $y \in \overline{Y}$, is defined to be the set $supp(l_y)$ of all $x \in \overline{X}$ satisfying the following condition [22]: for every neighborhood $U \subset \overline{X}$ of x, there is $f \in E(X)$ such that $\overline{f}(\overline{X}\setminus U) = 0$ and $l_y(\overline{f}) \neq 0$. Obviously, $supp(l_y)$ is closed in $\overline{X}$.

Claim 9 *$supp(l_y)$ is non-empty for all $y \in \overline{Y}$.* ■

Indeed, because the set of real-valued functions from $E(\overline{Y})$ is dense in $C_p(\overline{Y})$, for every $y \in \overline{Y}$, there is $g \in E(\overline{Y})$ with $g(y) \neq 0$. Then, take $f_y \in E(X)$ such that $\varphi(f_y) = g|Y$, so $l_y(\overline{f}_y) = g(y)$. If $supp(l_y) = \varnothing$, every $x \in \overline{X}$ has a neighborhood V_x such that $l_y(\overline{f}) = 0$ for any $f \in E(X)$ with $\overline{f}(\overline{X}\setminus V_x) = 0$. Passing to smaller neighborhoods, we can assume that $V_x \in \mathcal{B}$ for all x. Hence, there is a finite open cover $\gamma = \{V_{x_1}, \dots, V_{x_k}\}$ of $\overline{X}$ and a partition of unity $\mu = \{\overline{h}_1, \dots, \overline{h}_k\} \subset E(\overline{X})$ subordinated to γ. Then, $f_y \cdot h_i \in E(X)$ and $\overline{f_y \cdot h_i}|(\overline{X}\setminus V_{x_i}) = 0$, which implies $l_y(\overline{f_y \cdot h_i}) = 0$ for all i. Because μ is a partition of unity, $\overline{f}_y = \sum_{i=1}^k \overline{f_y \cdot h_i}$. So, $l_y(\overline{f}_y) = \sum_{i=1}^k l_y(\overline{f_y \cdot h_i}) = 0$, a contradiction.

Claim 10 *Let $U \subset \overline{X}$ be a neighborhood of $supp(l_y)$ and $f, g \in E(X)$ with $\overline{f}|U = \overline{g}|U$. Then, $l_y(\overline{f}) = l_y(\overline{g})$. In particular, $\overline{f}(U) = 0$ implies $l_y(\overline{f}) = 0$.*

First, let $\overline{f}(U) = 0$ for some $\overline{f} \in E(\overline{X})$. We can assume that U is a finite union of open sets V_i from $\mathcal{B}$, $i = 1, \dots, k$. Every $x \in \overline{X}\setminus U$ has a neighborhood V_x such that $l_y(\overline{g}) = 0$ for any $g \in E(X)$ with $\overline{g}(\overline{X}\setminus V_x) = 0$. As in the proof of Claim 9, we can assume that $V_x \in \mathcal{B}$, and choose another neighborhood $U_x \in \mathcal{B}$ with $\overline{U}_x \subset V_x$. Take a finite open cover $\{U_{x_1}, \dots, U_{x_m}\}$ of $\overline{X}\setminus U$. Then, $\gamma = \{V_1, \dots, V_k, U_{x_1}, \dots, U_{x_m}\}$ is an open cover of $\overline{X}$ consisting of elements from $\mathcal{B}$. Hence, there exits a partition of unity $\{\overline{h}_1, \dots, \overline{h}_k, \overline{\theta}_1, \dots, \overline{\theta}_m\} \subset E(\overline{X})$ subordinated to γ. Then, $h_i \cdot f, \theta_j \cdot f \in E(X)$ for all i, j and $f = \sum_{i=1}^k h_i \cdot f + \sum_{j=1}^m \theta_j \cdot f$. Take a sequence $\{y_n\} \subset Y$ with $\lim_n y_n = y$. So, $\varphi(f)(y_n) = \sum_{i=1}^k \varphi(h_i \cdot f)(y_n) + \sum_{j=1}^m \varphi(\theta_j \cdot f)(y_n)$. Observe that $(h_i \cdot f)(x) = 0$ for all $x \in X$, so $\varphi(h_i \cdot f)$ is the zero function on Y and $\varphi(h_i \cdot f)(y_n) = 0$, $i = 1, \dots, k$. Hence, $\varphi(f)(y_n) = \sum_{j=1}^m \varphi(\theta_j \cdot f)(y_n)$. On the other hand, $(\theta_j \cdot f)(x) = 0$ for all $x \in X\setminus U_{x_j}$. So, $(\overline{\theta_j \cdot f})(x) = 0$ for all $x \in \overline{X}\setminus V_{x_j}$ because $\overline{X}\setminus V_{x_j} \subset \overline{X\setminus U_{x_j}}$. This implies that $l_y(\overline{\theta_j \cdot f}) = 0$ for all $j = 1, \dots, m$. Since, $l_y(\overline{\theta_j \cdot f}) = \lim_n \varphi(\theta_j \cdot f)(y_n) = 0$, we have

$$l_y(\overline{f}) = \lim_n \varphi(f)(y_n) = \lim_n \sum_{j=1}^m \varphi(\theta_j \cdot f)(y_n) = 0.$$

Suppose now that $\overline{f}|U = \overline{g}|U$ for some $f, g \in E(X)$. Then, $f(x) - g(x) = 0$ for all $x \in U \cap X$. Consequently, $(\overline{f - g})(x) = 0$ for all $x \in U$ and, according to the previous paragraph, $l_y(\overline{f - g}) = 0$. Hence, for every sequence $\{y_n\} \subset Y$ converging to y, we have

$$l_y(\overline{f - g}) = \lim_n \varphi(f - g)(y_n) = \lim_n \varphi(f)(y_n) - \lim_n \varphi(g)(y_n) = l_y(\overline{f}) - l_y(\overline{g}).$$

Claim 11 *If $supp(l_{y_0}) \cap U \neq \varnothing$ for some open $U \subset \overline{X}$ and $y_0 \in \overline{Y}$, then y_0 has a neighborhood $V \subset \overline{Y}$ such that $supp(l_y) \cap U \neq \varnothing$ for every $y \in V$.*

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Let $x_0 \in supp(l_{y_0}) \cap U$ and $\overline{f} \in E(\overline{X})$ be such that $\overline{f}(\overline{X}\setminus W) = 0$ and $l_{y_0}(\overline{f}) \neq 0$, where W is a neighborhood of x_0 with $\overline{W} \subset U$. Assuming the claim is not true, we can find a sequence $\{y_n\} \subset \overline{Y}$ converging to y_0 such that $supp(l_{y_n}) \cap U = \varnothing$ for every n. Since $\overline{X}\setminus\overline{W}$ is a neighborhood of each $supp(l_{y_n})$, by Claim 10, $l_{y_n}(\overline{f}) = 0$. Because $\lim_n l_{y_n}(\overline{f}) = l_{y_0}(\overline{f})$, we have $l_{y_0}(\overline{f}) = 0$, a contradiction.

Following the notations from the proof of Proposition 2.1, for every $y \in \overline{Y}$, we define

$$a(y) = \sup\{|l_y(\overline{f})| : \overline{f} \in E(\overline{X}) \text{ and } |\overline{f}(x)| < 1 \ \forall x \in supp(l_y)\}.$$

Claim 12 *If $y \in Y$ then $supp(l_y) = \{x_1(y), x_2(y), \dots, x_q(y)\}$ is a finite non-empty subset of X. Moreover, there exist real numbers $\lambda_i(y)$, $i = 1, \dots, q$, such that $\sum_{i=1}^{q} |\lambda_i(y)| = a(y)$ and $l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(y))$ for all $\overline{f} \in E(\overline{X})$.*

Let $\widetilde{\varphi} : LE_p(X) \to LE_p(Y)$ be the continuous extension of φ. Then, every l_y, $y \in Y$, can be continuously extended to a linear functional $\widetilde{l}_y : LE_p(X) \to \mathbb{R}$, $\widetilde{l}_y(\sum_{i=1}^{k} \lambda_i \cdot f_i) = \sum_{i=1}^{k} \lambda_i \cdot l_y(f_i)$. Since $\widetilde{\varphi}$ is uniformly continuous, according to the proof of Claim 1 from Proposition 2.1, for every $y \in Y$, there exists a finite set $K = \{x_1(y), x_2(y), \dots, x_q(y)\} \subset X$ such that

$$\sup\{|\widetilde{l}_y(f)| : f \in LE(X) \text{ and } |f(x)| < 1 \ \forall x \in K\} < \infty.$$

Because $\widetilde{l}_y$ is linear, we can show that $\widetilde{l}_y(g) = 0$ for any $g \in LE(X)$ with $g(x_i(y)) = 0$, $i = 1, \dots, q$. Since $E(X)$ is a QS-algebra, for every i, there is a function $g_i \in E(X)$ such that $g_i(x_i(y)) = 1$ and $g_i(x_j(y)) = 0$ with $j \neq i$. Now, for every $f \in E(X)$, the function $g = f - \sum_{i=1}^{q} g_i \cdot f(x_i(y))$ belongs to $LE(X)$ and $g(x_i(y)) = 0$ for all i. So, $\widetilde{l}_y(g) = 0$ and $l_y(f) = \sum_{i=1}^{q} \lambda_i(y) f(x_i(y))$, where $\lambda_i(y) = l_y(g_i)$. Because $y \in Y$, we can also write

$$l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(y))$$

with $\lambda_i(y) = l_y(\overline{g}_i)$. Note that each $\lambda_i(y)$ is a real number because $l_y(g_i) = \varphi(g_i)(y)$. The last equality is valid for all $\overline{f} \in E(\overline{X})$ and shows that $supp(l_y) = \{x_i(y) : \lambda_i(y) \neq 0\} \subset K$. Note that, by Claim 9, $supp(l_y) \neq \varnothing$.

To complete the proof of Claim 12, assume that $supp(l_y) = K$, and for every natural k take a function $f_k \in E(X)$ with $f_k(x_i(y)) = \varepsilon_i(1 - 1/k)$, where $\varepsilon_i = 1$ if $\lambda_i(y) > 0$ and $\varepsilon_i = -1$ if $\lambda_i(y) < 0$. Clearly, $|f_k(x_i(y))| < 1$ for all i, k and $\lim_k l_y(\overline{f}_k) = \sum_{i=1}^{q} |\lambda_i(y)|$. Hence, $\sum_{i=1}^{q} |\lambda_i(y)| \leq a(y)$. The reverse inequality $a(y) \leq \sum_{i=1}^{q} |\lambda_i(y)|$ follows from $l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(y))$, $\overline{f} \in E(\overline{X})$.

For every $p, q \in \mathbb{N}$, let $Y_{p,q} = \{y \in \overline{Y} : |supp(l_y)| \leq q \text{ and } a(y) \leq p\}$.

Claim 13 *Every set $Y_{p,q}$ is closed in $\overline{Y}$.*

Let $\{y_n\}$ be a sequence in $Y_{p,q}$ converging to $y \in \overline{Y}$. Suppose $y \notin Y_{p,q}$. Then either $supp(l_y)$ contains at least $q + 1$ points or $a(y) > p$. If $supp(l_y)$ contains at least $q + 1$ points $x_1, x_2, \dots, x_{q+1}$, we choose disjoint neighborhoods O_i of x_i, $i = 1, \dots, q + 1$. By Claim 11, there is a neighborhood V of y such that $supp(l_z) \cap O_i \neq \varnothing$ for all i and

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$z \in V$. This implies that $supp(l_{y_n})$ has at least $q+1$ points for infinitely many n's, a contradiction.

If $y \notin Y_{p,q}$ and $a(y) > p$, then there exists $\overline{f} \in E(\overline{X})$ such that $|\overline{f}(x)| < 1$ for all $x \in supp(l_y)$ and $|l_y(\overline{f})| > p$. Take a neighborhood U of $supp(l_y)$ with $U \subset \{x \in \overline{X} : |\overline{f}(x)| < 1\}$. Choose another neighborhood W of $supp(l_y)$ with $\overline{W} \subset U$. Next, there is $h \in E(X)$ such that $\overline{h}(\overline{W}) = 1$, $\overline{h}(\overline{X}\setminus U) = 0$, and $h(x) \in [0,1]$ for all $x \in X$. Then, $\overline{g} = \overline{h \cdot f} \in E(\overline{X})$ and $|\overline{g}(x)| < 1$ for all $x \in \overline{X}$. Moreover, $\overline{g}|W = \overline{f}|W$. So, by Claim 10, $|l_y(\overline{g})| = |l_y(\overline{f})| > p$. Therefore, $V = \{z \in \overline{Y} : |l_z(\overline{g})| > p\}$ is a neighborhood of y in $\overline{Y}$ with $V \cap Y_{p,q} = \varnothing$, a contradiction.

According to Claim 12, for every $y \in Y$, then there exist $p, q \in \mathbb{N}$ and real numbers $\lambda_i(y)$ such that for all $\overline{f} \in E(\overline{X})$, we have $l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(y))$ with $a(y) = \sum_{i=1}^{q} |\lambda_i(y)| \le p$, where $\{x_1(y), \dots, x_q(y)\}$ is the support $supp(l_y)$. Hence, $Y \subset \bigcup\{Y_{p,q} : p, q \in \mathbb{N}\}$. For every $p \ge 1$ and $q \ge 2$, we define

$$M(p,1) = Y_{p,1} \text{ and } M(p,q) = Y_{p,q}\setminus Y_{2p,q-1}.$$

Some of the sets $M(p,q)$ could be empty, but $Y \subset \bigcup\{M(p,q) : p, q = 1, 2, \dots\}$. Indeed, by Claim 12, there exist p, q with $|supp(l_y)| = q$ and $a(y) \le p$. Then, $y \in M(p,q)$. Since each $Y_{p,q}$ is closed, $M(p,q) = \bigcup_{n=1}^{\infty} F'_n(p,q)$ such that each $F'_n(p,q)$ is a compact subset of $\overline{Y}$. We define $F_n(p,q) = \overline{Y \cap F'_n(p,q)}$. Then, $Y \subset \bigcup\{F_n(p,q) : n, p, q = 1, 2, \dots\}$. Obviously, $supp(l_y)$ consists of q different points for any $y \in M(p,q)$. So, we have a map $S_{p,q} : M(p,q) \to [\overline{X}]^q$, $S_{p,q}(y) = supp(l_y)$. According to Claim 11, $S_{p,q}$ is continuous when $[\overline{X}]^q$ is equipped with the Vietoris topology. For every $y \in M(p,q)$, let $S_{p,q}(y) = \{x_i(y)\}_{i=1}^{q}$. Everywhere below, we consider the restriction $S_{p,q}|F_n(p,q)$ and for every $z \in F_n(p,q)$ denote by $A(z) = \{y \in F_n(p,q) : S_{p,q}(y) = S_{p,q}(z)\}$ the fiber $S_{p,q}^{-1}(S_{p,q}(z))$ generated by z. Since $F_n(p,q)$ is compact and $S_{p,q}$ is continuous, each $A(z)$ is a compact subset of $F_n(p,q)$.

Claim 14 *Let $z \in Y \cap F_n(p,q)$. Then, for every $y \in A(z)$, there are real numbers $\{\lambda_i(y)\}_{i=1}^{q}$ such that $l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(z))$, where $S_{p,q}(z) = \{x_i(z)\}_{i=1}^{q}$, and $\sum_{i=1}^{q} |\lambda_i(y)| \le p$ for all $\overline{f} \in E(\overline{X})$. Moreover, each λ_i is a continuous real-valued function on $A(z)$.*

Choose neighborhoods O_i of $x_i(z)$ in $\overline{X}$ with disjoint closures and functions $\overline{g}_i \in E(\overline{X})$ with $\overline{g}_i|O_i = 1$ and $\overline{g}_i|O_j = 0$ if $j \ne i$ (this can be done by choosing $g_i \in E(X)$ with $g_i(O_i \cap X) = 1$ and $g_i(O_j \cap X) = 0$ when $j \ne i$). According to the proof of Claim 12, for every $y \in A(z) \cap Y$ and the real numbers $\lambda_i(y) = l_y(\overline{g}_i)$, we have $l_y(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(z))$ for all $\overline{f} \in E(\overline{X})$. Let's show this is true for all $y \in A(z)$. So, fix $y \in A(z)\setminus Y$ and take a sequence $\{y_m\} \subset Y \cap F_n(p,q)$ converging to y. Then, by Claim 12, $S_{p,q}(y_m) = \{x_i(y_m)\}_{i=1}^{q} \subset X$ and there are real numbers $\{\lambda_i(y_m)\}_{i=1}^{q}$ such that $l_{y_m}(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y_m)\overline{f}(x_i(y_m))$ for all $\overline{f} \in E(\overline{X})$. On the other hand, since the map $S_{p,q}$ is continuous, each sequence $\{x_i(y_m)\}_{m=1}^{\infty}$ converges to $x_i(y) = x_i(z)$, $i = 1, \dots, q$. So, we can assume that $\{x_i(y_m)\}_{m=1}^{\infty} \subset O_i$. Consequently, $l_{y_m}(\overline{g}_i) = \lambda_i(y_m)$ and $\lim_m \lambda_i(y_m) = l_y(\overline{g}_i)$. Since $\{y_m\} \subset Y \cap Y_{p,q}$, $\sum_{i=1}^{q} |\lambda_i(y_m)| \le p$ for each m (see Claim 12). Hence, $\sum_{i=1}^{q} |l_y(\overline{g}_i)| \le p$. Denoting $\lambda_i(y) = l_y(\overline{g}_i)$, we obtain $\sum_{i=1}^{q} |\lambda_i(y)| \le p$. The last inequality means that all $\lambda_i(y)$ are real numbers. Since

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$\lim_m \overline{f}(x_i(y_m)) = \overline{f}(x_i(y))$ for all $\overline{f} \in E(\overline{X})$ and each $\overline{f}(x_i(y))$ is a real number (recall that $z \in Y$, so by Claim 12, $x_i(z) = x_i(y) \in X$), we have $l_y(\overline{f}) = \lim l_{y_m}(\overline{f}) = \sum_{i=1}^{q} \lambda_i(y)\overline{f}(x_i(y))$.

Finally, the equality $\lambda_i(y) = \overline{\varphi(g_i)}(y)$ implies that λ_i are continuous on $A(z)$.

Claim 15 *Let* $A(z) = \{y \in F_n(p,q) : S_{p,q}(y) = S_{p,q}(z)\}$ *with* $z \in Y \cap F_n(p,q)$. *Then, there is a linear continuous map* $\varphi_z : C_p(S_{p,q}(z)) \to C_p(A(z))$ *such that* $\varphi_z(C(S_{p,q}(z)))$ *is dense in* $C_p(A(z))$.

Following the previous notations, for every $h \in C(S_{p,q}(z))$ and $y \in A(z)$, we define $\varphi_z(h)(y) = \sum_{i=1}^{q} \lambda_i(y)h(x_i(z))$. Because λ_i are continuous real-valued functions on $A(z)$, so is each $\varphi_z(h)$. Continuity of φ_z with respect to the pointwise convergence topology is obvious. Let's show that $\varphi_z(C(S_{p,q}(z)))$ is dense in $C_p(A(z))$. Indeed, take $\theta \in C_p(A(z))$ and its neighborhood $V \subset C_p(A(z))$. Then extend θ to a function $\overline{\theta} \in C(\overline{Y})$. Because the set of real-valued elements of $E_p(\overline{Y})$ is dense in $C_p(\overline{Y})$, there is $\overline{g} \in E_p(\overline{Y})$ with $\overline{g}|A(z) \in V$, so $\overline{g}(y) \in \mathbb{R}$ for all $y \in \overline{Y}$. Next, choose $f \in E(X)$ such that $\varphi(f) = g$. Since $z \in Y$, each $x_i(z) \in X$. So, all $\overline{f}(x_i(z))$ are real numbers. Then, $h = \overline{f}|S_{p,q}(z) \in C(S_{p,q}(z))$ and, according to Claim 14, we have $\varphi_z(h) = \overline{g}|A(z)$.

Claim 16 *The fibers* $A(z)$ *of the map* $S_{p,q} : F_n(p,q) \to [\overline{X}]^q$ *are zero-dimensional for all* $z \in Y \cap F_n(p,q)$.

Since $|S_{p,q}(z)| = q$, $C_p(S_{p,q}(z))$ is isomorphic to $\mathbb{R}^q$. Hence, Claim 15 together with basic facts from linear algebra imply that $|A(z)| \leq q$, in particular, $A(z)$ is zero-dimensional.

Now, we can complete the proof of Proposition 4.1. As in the proof of Proposition 2.1, using that $\overline{X}^q \in \mathcal{P}$, we can show that $[\overline{X}]^q \in \mathcal{P}$. Therefore, condition (c') implies that each $F_n(p,q)$ has also the property $\mathcal{P}$, and so does $Y_n(p,q) = F_n(p,q) \cap Y$. Finally, since Y is the union of its closed subsets $Y_n(p,q)$, we conclude that $Y \in \mathcal{P}$.

Lemma 4.2 *For every countable set* $\Phi' \subset C(\beta Y, \overline{\mathbb{R}})$ *there is a countable set* $\Phi \subset C(\beta Y, \overline{\mathbb{R}})$ *containing* Φ' *such that* $(\triangle\Phi)(\beta Y)$ *is homeomorphic to* $(\triangle\Phi')(\beta Y)$ *and the set of real-valued elements of* $E_\Phi = \{\pi_{\overline{g}} : \overline{g} \in \Phi\}$ *is dense in* $C_p((\triangle\Phi)(\beta Y))$.

Proof Let $\phi' = \triangle\Phi'$. Since Φ' is countable, $\phi'(\beta Y)$ is a metrizable compactum. Hence, by [7, Proposition 1.2], there is a countable QS-algebra $E \subset C(\phi'(\beta Y))$. Let $\Phi = \Phi' \cup \{g \circ \phi' : g \in E\}$. Since the functions of E separate the points and closed subsets of $\phi'(\beta Y)$, $(\triangle\Phi)(\beta Y)$ is homeomorphic to $\phi'(\beta Y)$. Since E is a QS-algebra on $\phi'(\beta Y)$, E is a dense subset of $C_p(\phi'(\beta Y))$. Clearly, E is a subset of E_Φ and consists of real-valued functions. ∎

Lemma 4.3 *Let* X *be a zero-dimensional space and* $\Psi' \subset C(X)$ *be a countable set. Then, there is a countable admissible set* $\Psi \subset C(X)$ *containing* Ψ' *and a zero-dimensional metrizable compactification* $\overline{X}_\Psi$ *of* $X_\Psi = (\triangle\Psi)(X)$ *having a countable base* $\mathcal{B}$ *such that:*

- $\overline{X}_\Psi = (\triangle\overline{\Psi})(\beta X)$ *with* $\overline{\Psi} = \{\overline{f} : f \in \Psi\} \subset C(\beta X, \overline{\mathbb{R}})$;
- *each* π_f, $f \in \Psi$, *is extendable to a map* $\overline{\pi}_f : \overline{X}_\Psi \to \overline{\mathbb{R}}$;

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- *$E(X_\Psi) = \{\pi_f : f \in \Psi\}$ is a countable QS-algebra on X_Ψ and $E(\overline{X}_\Psi) = \{\overline{\pi}_f : f \in \Psi\}$ contains a countable QS-algebra on $\overline{X}_\Psi$ satisfying condition (2.3);*
- *for every finite open cover γ of $\overline{X}_\Psi$ with elements from $\mathcal{B}$, the family $E(\overline{X}_\Psi)$ contains a partition of unity subordinated to γ.*

Proof We first choose a countable admissible set $\Psi_0 \subset C(X)$ such that $\dim(\triangle\Psi_0)(X) = 0$ and $\Psi' \subset \Psi_0$ (see (3.2) and (3.3)). Then, by Lemma 3.2, there is a metrizable compactification Z_0 of $X_0 = (\triangle\Psi_0)(X)$ with $\dim Z_0 = 0$ such that each π_f, $f \in \Psi_0$, is extendable to a map $\overline{\pi}_f : Z_0 \to \overline{\mathbb{R}}$. Choose a countable QS-algebra $C_0 \subset C(Z_0)$ satisfying condition (2.3) (that can be done because Z_0 has a countable base, see the explanations in (2.3)). Let $\mathcal{B}_0$ be a countable base for Z_0 and for every finite open cover γ of Z_0 consisting of open sets from $\mathcal{B}_0$ fix a partition of unity α_γ subordinated to γ. Because the family Ω_0 of all finite open covers of Z_0 consisting of elements of $\mathcal{B}_0$ is countable, so is the family $E(Z_0) = \{\overline{f} : f \in \Psi_0\} \cup \{\alpha_\gamma : \gamma \in \Omega_0\} \cup C_0$. The set $E_0 = \{h|X_0 : h \in E(Z_0)\}$ may not be a QS-algebra on X_0 but, according to [7, Proposition 1.2], there exists a countable QS-algebra Θ_1 on X_0 containing E_0 as a proper subset. For every $h \in \Theta_1$, the function $h \circ (\triangle\Psi_0) : X \to \mathbb{R}$ can be extended to a map $\widetilde{h} : \beta X \to \overline{\mathbb{R}}$ and let $P_h = \widetilde{h}(\beta X)$. Because $\{h|X_0 : h \in C_0\} \subset \Theta_1$ and it separates the points and the closed sets of X_0, by Lemma 3.2, there is a metrizable compactification Z_1 of X_0 such that $\dim Z_1 = 0$, each $h \in \Theta_1$ is extendable to a map $\overline{h} : Z_1 \to \overline{\mathbb{R}}$. The compactification Z_1 is a closed subset of the product $Z_0 \times \prod_{h\in\Theta_1} P_h$. Then, the projection $Z_0 \times \prod_{h\in\Theta_1} P_h \to Z_0$ provides a map $\theta_0^1 : Z_1 \to Z_0$ such that $\theta_0^1 \circ j_1 = j_0$ with $j_1 : X_0 \to Z_1$ and $j_0 : X_0 \to Z_0$ being the corresponding embeddings.

Next, fix a base $\mathcal{B}_1$ on Z_1 containing all sets $(\theta_0^1)^{-1}(U)$, $U \in \mathcal{B}_0$, which is closed under finite intersections, and consider the family Ω_1 of all finite open covers of Z_1 consisting of sets from $\mathcal{B}_1$. For each $\gamma \in \Omega_1$ fix a partition of unity α_γ subordinated to γ and let $E(Z_1) = \{\overline{h} : h \in \Theta_1\} \cup \{\alpha_\gamma : \gamma \in \Omega_1\} \cup C_1$ and $E_1 = \{f|X_0 : f \in E(Z_1)\}$, where $C_1 \subset C(Z_1)$ is a countable QS-algebra of Z_1 satisfying condition (2.3) such that $\{h \circ \theta_0^1 : h \in C_0\} \subset C_1$. We construct by induction an increasing sequence $\{\Theta_n\}$ of countable QS-algebras on X_0, zero-dimensional metrizable compactifications Z_n of X_0 with a countable base $\mathcal{B}_n$, countable QS-algebras $C_n \subset C(Z_n)$ on Z_n, countable families $E(Z_n) \subset C(Z_n, \overline{\mathbb{R}})$, and surjective maps $\theta_n^{n+1} : Z_{n+1} \to Z_n$ such that:

(1) $\mathcal{B}_{n+1}$ contains all inverse images $(\theta_n^{n+1})^{-1}(U)$, $U \in \mathcal{B}_n$, and is closed under finite intersections;
(2) C_{n+1} satisfies condition (2.3) and $\{h \circ \theta_n^{n+1} : h \in C_n\} \subset C_{n+1}$;
(3) every Z_{n+1} is a zero-dimensional metrizable compactification of X_0 such that every $h \in \Theta_{n+1}$ is extendable to a map $\overline{h} \in C(Z_{n+1}, \overline{\mathbb{R}})$;
(4) $E(Z_{n+1}) = \{\overline{h} : h \in \Theta_{n+1}\} \cup \{\alpha_\gamma : \gamma \in \Omega_{n+1}\} \cup C_{n+1}$, where Ω_{n+1} is the family of all finite open covers γ of Z_{n+1} with elements from $\mathcal{B}_{n+1}$ and α_γ is a partition of unity subordinated to γ;
(5) $E_n = \{f|X_0 : f \in E(Z_n)\} \subset \Theta_{n+1}$.

If the construction is performed for all $k \le n$, let $E_n = \{h|X_0 : \overline{h} \in E(Z_n)\}$ and Θ_{n+1} be a countable QS-algebra on X_0 with $E_n \subset \Theta_{n+1}$. For every $h \in \Theta_{n+1}$, the function $h \circ (\triangle\Psi_0) : X \to \mathbb{R}$ can be extended to a map $\widetilde{h} : \beta X \to \overline{\mathbb{R}}$ and let $P_h = \widetilde{h}(\beta X)$.

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By Lemma 3.2, there exists a metrizable compactification Z_{n+1} of X_0 such that $\dim Z_{n+1} = 0$ and each $h \in \Theta_{n+1}$ is extendable to a map $\overline{h} : Z_{n+1} \to \overline{\mathbb{R}}$. The space Z_{n+1} is a closed subset of $Z_n \times \prod_{h\in\Theta_{n+1}} P_h$. So, the projection $Z_n \times \prod_{h\in\Theta_{n+1}} P_h \to Z_n$ determines a map $\theta_n^{n+1} : Z_{n+1} \to Z_n$ such that $\theta_n^{n+1} \circ j_{n+1} = j_n$, where $j_{n+1} : X_0 \to Z_{n+1}$ and $j_n : X_0 \to Z_n$ are the corresponding embeddings. Next, choose a base $\mathcal{B}_{n+1}$ of Z_{n+1}, a countable QS-algebra $C_{n+1} \subset C(Z_{n+1})$, and a countable family $E(Z_{n+1})$ satisfying conditions (1)–(5).

Since $\{\Theta_n\}$ is an increasing sequence of QS-algebras on X_0, $\Theta = \bigcup_n \Theta_n$ is also a countable QS-algebra on X_0, and the limit space Z of the inverse sequence $S = \{Z_n, \theta_{n-1}^n\}$ is zero-dimensional. Because $\theta_n^{n+1} \circ j_{n+1} = j_n$ with $j_{n+1} : X_0 \to Z_{n+1}$ and $j_n : X_0 \to Z_n$ being embeddings, there is an embedding $j : X_0 \to Z$ such that $\theta_n \circ j = j_n$, where $\theta_n : Z \to Z_n$ is the projection in S. So, Z is a compactification of X_0 and every $h \in \Theta$ is extendable to a map $\overline{\overline{h}} : Z \to \overline{\mathbb{R}}$. Indeed, if $h \in \Theta_n$, then h can be extended to a map $\overline{h} : Z_n \to \overline{\mathbb{R}}$ and $\overline{\overline{h}} = \overline{h} \circ \theta_n$ is an extension of h over Z. Hence, $\Theta = \{g|Z : g \in E(Z)\}$, where $E(Z) = \bigcup_n \{h \circ \theta_n : h \in E(Z_n)\}$.

Denote $C_n' = \{h \circ \theta_n : h \in C_n\}$, $n \geq 0$. Because $\{h \circ \theta_{n-1}^n : h \in C_{n-1}\} \subset C_n$, the sequence $\{C_n'\}$ is increasing and $C = \bigcup_n C_n'$ is closed under multiplications, additions, and additions by rational numbers. To show it a QS-algebra on Z, we need to prove that for every point $z \in Z$ and its neighborhood $U \subset Z$, there is $h \in C$ such that $h(z) = 1$ and $h(Z\backslash U) = 0$. But that follows from condition (2.3), so let show that C satisfies condition (2.3). To this end, take a compact set $K \subset Z$ and an open set $W \subset Z$ containing K. Then, there exist n, a compact set $K_n \subset Z_n$ and open set $W_n \subset Z_n$ containing K_n such that $K \subset \theta^{-1}(K_n) \subset \theta^{-1}(W_n) \subset W$. Since C_n satisfies condition (2.3), there is $h \in C_n$ with $h|K_n = 1$, $h|(Z_n\backslash W_n) = 0$ and $h(z) \in [0,1]$ for all $z \in Z_n$. Then, $h' = \theta_n \circ h \in C$, $h' : Z \to [0,1]$, $h'|K = 1$ and $h'|(X\backslash W) = 0$.

Recall that all finite intersections $U = \bigcap_{i=1}^k \theta_i^{-1}(U_i)$ with $U_i \in \mathcal{B}_i$ form a base $\mathcal{B}$ for Z. Because of the choice of all $\mathcal{B}_i$, see condition (1), $\mathcal{B}$ consists of all sets of the form $U = \theta_n^{-1}(U_n)$ with $U_n \in \mathcal{B}_n$, $n \in \mathbb{N}$. Let's show that for any finite open cover γ of Z consisting of sets from $\mathcal{B}$, the set $E(Z)$ contains a partition of unity subordinated to γ. Indeed, for any such a cover $\gamma = \{U_1, \ldots, U_k\}$, there is n and a cover $\gamma_n = \{U_1^n, \ldots, U_k^n\} \in \Omega_n$ such that $U_i = \theta_n^{-1}(U_i^n)$. So, there is a partition of unity $\alpha_{\gamma_n} = \{h_i^n : i = 1, \ldots, k\}$ subordinated to γ_n with $\alpha_{\gamma_n} \subset E(Z_n)$. Then, $\{h_i^n \circ \theta_n : i = 1, \ldots, k\}$ is a partition of unity subordinated to γ and it is contained in $E(Z)$. Finally, let $\Psi = \{h \circ \triangle\Psi_0 : h \in \Theta\}$, $\overline{\Psi} = \{\overline{f} : f \in \Psi\} \subset C(\beta X, \overline{\mathbb{R}})$, and $\overline{X}_\Psi = (\triangle\overline{\Psi})(\beta X)$. Since C is a QS-algebra on Z, it separates the points and the closed sets in Z. Moreover, $C \subset E(Z)$, which means that $\overline{X}_\Psi$ is homeomorphic to Z. ∎

Proof of Theorem 1.4 Let $T : C_p(X) \to C_p(Y)$ be a continuous linear surjection and $\dim X = 0$. To show that $\dim Y = 0$, it suffices to prove that for every $h \in \mathcal{F}_Y$, there exists $h_0 \in \mathcal{F}_Y$ with $h_0 > h$ and $\dim h_0(Y) = 0$. To this end, fix $h \in \mathcal{F}_Y$. Following the proof of Theorem 1.1 and using Lemmas 4.2 and 4.3, we are constructing two increasing sequences of countable sets $\{\overline{\Psi}_n\} \subset C(\beta X, \overline{\mathbb{R}})$, $\{\overline{\Phi}_n\} \subset C(\beta Y, \overline{\mathbb{R}})$, metrizable compactifications $\overline{X}_n = (\triangle\overline{\Psi}_n)(\beta X)$ and $\overline{Y}_n = (\triangle\overline{\Phi}_n)(\beta Y)$ of the spaces $X_n = (\triangle\Psi_n)(X)$ and $Y_n = (\triangle\Phi_n)(Y)$, where $\Psi_n = \overline{\Psi}_n|X$ and $\Phi_n = \overline{\Phi}_n|Y$, countable bases $\mathcal{B}_n'$ and $\mathcal{B}_n''$ for $\overline{X}_n$ and $\overline{Y}_n$ and continuous surjections $\theta_n^{n+1} : \overline{X}_{n+1} \to \overline{X}_n$, $\delta_n^{n+1} : \overline{Y}_{n+1} \to \overline{Y}_n$ satisfying the following conditions (everywhere below, if $f \in C(X)$ then $\overline{f} : \beta X \to \overline{\mathbb{R}}$ denotes its extension):

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(4.1) $\triangle\Phi_1 \succ h$, $\Phi_n \subset \{T(f) : f \in \Psi_n\} \subset \Phi_{n+1}$ and $\Psi_n \subset \Psi_{n+1}$;
(4.2) each π_f, $f \in \Psi_n$, is extendable to a map $\overline{\pi}_f : \overline{X}_n \to \overline{\mathbb{R}}$;
(4.3) Ψ_n is admissible, $\dim \overline{X}_n = 0$ and $\mathcal{B}'_n$ satisfies condition (1) from Lemma 4.3;
(4.4) $E(\overline{X}_n) = \{\overline{\pi}_f : f \in \Psi_n\}$ contains a countable QS-algebra $C_n \subset C(\overline{X}_n)$ on $\overline{X}_n$ such that C_n satisfies condition (2) from the proof of Lemma 4.3;
(4.5) for every finite open cover γ of $\overline{X}_n$, consisting of sets from $\mathcal{B}'_n$, there exists a partition of unity α_γ subordinated to γ with $\alpha_\gamma \subset E(\overline{X}_n)$;
(4.6) every π_g, $g \in \Phi_n$, is extendable to a map $\overline{\pi}_g : \overline{Y}_n \to \overline{\mathbb{R}}$;
(4.7) $\mathcal{B}''_n$ contains all inverse images $(\delta^n_{n-1})^{-1}(U)$, $U \in \mathcal{B}''_{n-1}$, and is closed under finite intersections;
(4.8) the set of real-valued functions from $E(\overline{Y}_n) = \{\overline{\pi}_g : g \in \Phi_n\}$ is dense in $C_p(\overline{Y}_n)$.

Since $h(Y)$ is a separable metrizable space, there is a countable set $\Phi'_1 \subset C(Y)$ with $h = \triangle\Phi'_1$. Let $\overline{\Phi}'_1 = \{\overline{g} : g \in \Phi'_1\} \subset C(\beta Y, \overline{\mathbb{R}})$. By Lemma 4.2, there is a countable set $\overline{\Phi}_1 \subset C(\beta Y, \overline{\mathbb{R}})$ containing $\overline{\Phi}'_1$ such that $(\triangle\overline{\Phi}_1)(\beta Y)$ is homeomorphic to $(\triangle\overline{\Phi}'_1)(\beta Y)$ and $\{\pi_{\overline{g}} : \overline{g} \in \overline{\Phi}_1\}$ contains a dense subset of $C_p(\overline{Y}_1)$, where $\overline{Y}_1 = (\triangle\overline{\Phi}_1)(\beta Y)$. Let $\Phi_1 = \{g : \overline{g} \in \overline{\Phi}_1\}$, $Y_1 = (\triangle\Phi_1)(Y)$, and $E(\overline{Y}_1) = \{\overline{\pi}_g : g \in \Phi_1\}$. So, Φ_1 satisfies conditions (4.6) and (4.7). Next, choose a countable set $\Psi'_1 \subset C(X)$ with $T(\Psi'_1) = \Phi_1$ and apply Lemma 4.3 to find a countable admissible set Ψ_1 containing Ψ'_1 and a metrizable compactification $\overline{X}_1$ of $X_1 = \triangle\Psi_1(X)$ satisfying conditions (4.2)–(4.5).

Suppose the construction is done for all $k \le n$. Let $\Phi'_{n+1} \subset C(Y)$ be a countable set containing $T(\Psi_n)$ and denote $\overline{\Phi'}_{n+1} = \{\overline{g} : g \in \Phi'_{n+1}\} \subset C(\beta Y, \overline{\mathbb{R}})$. By Lemma 4.2, there is a countable set $\overline{\Phi}_{n+1} \subset C(\beta Y, \overline{\mathbb{R}})$ containing $\overline{\Phi'}_{n+1}$ such that $(\triangle\overline{\Phi}_{n+1})(\beta Y)$ is homeomorphic to $(\triangle\overline{\Phi'}_{n+1})(\beta Y)$ and $\{\pi_{\overline{g}} : \overline{g} \in \overline{\Phi}_{n+1}\}$ contains a dense subset of $C_p(\overline{Y}_{n+1})$, where $\overline{Y}_{n+1} = (\triangle\overline{\Phi}_{n+1})(\beta Y)$. Let $\Phi_{n+1} = \{g : \overline{g} \in \overline{\Phi}_{n+1}\}$ and $Y_{n+1} = (\triangle\Phi_{n+1})(Y)$. Note that $\Phi_n \subset \Phi_{n+1}$ because $\Phi_n \subset T(\Psi_n)$. Next, choose a countable set $\Psi'_{n+1} \subset C(X)$ with $T(\Psi'_{n+1}) = \Phi_{n+1}$ and apply Lemma 4.3 to find a countable admissible set Ψ_{n+1} containing $\Psi'_{n+1} \cup \Psi_n$ and a metrizable compactification $\overline{X}_{n+1}$ of $X_{n+1} = (\triangle\Psi_{n+1})(X)$ satisfying conditions (4.1)–(4.5). Because $\overline{\Psi}_n \subset \overline{\Psi}_{n+1}$, $\triangle\overline{\Psi}_{n+1} \succ \triangle\overline{\Psi}_n$. Hence, there exists a map $\theta^{n+1}_n : \overline{X}_{n+1} \to \overline{X}_n$ defined by $\theta^{n+1}_n = \triangle\overline{\Psi}_n((\triangle\overline{\Psi}_{n+1})^{-1}(x))$. This completes the induction.

As in the proof of Theorem 1.1, we denote $\overline{X}_0 = (\triangle\overline{\Psi})(\beta X)$, $\overline{Y}_0 = (\triangle\overline{\Phi})(\beta Y)$, and $h_0 = \triangle\Phi$, where $\Psi = \bigcup_n \Psi_n$ and $\Phi = \bigcup_n \Phi_n$. Clearly, $\Phi = \{T(f) : f \in \Psi\}$. Since $\overline{X}_0$ is the limit space of the inverse sequence $S_X = \{\overline{X}_n, \theta^{n+1}_n\}$ and $\dim \overline{X}_n = 0$ for all n, $\dim \overline{X}_0 = 0$. Moreover, $E(X_0) = \{\pi_f : f \in \Psi\}$ is a countable QS-algebra on X_0 such that every π_f is extendable to a continuous map $\overline{\pi}_f : \overline{X}_0 \to \overline{\mathbb{R}}$. Denote $E(\overline{X}_0) = \{\overline{\pi}_f : f \in \Psi\}$. Let also $C = \bigcup_n \{h \circ \theta_n : h \in C_n\} \subset E(\overline{X}_0)$. The same arguments as in the proof of Lemma 4.3 show that C is a QS-algebra on $\overline{X}_0$ satisfying condition (2.3). Let $\mathcal{B}'$ be the base of $\overline{X}_0$ generated by the bases $\mathcal{B}'_n$. The arguments from the proof of Lemma 4.3 also provide that for every finite open cover γ of $\overline{X}_0$, consisting of open sets from $\mathcal{B}'$ there exists a partition of unity α_γ subordinated to γ with $\alpha_\gamma \subset E(\overline{X}_0)$.

Because $\Phi_n \subset \Phi_{n+1}$, $\overline{\Phi}_n \subset \overline{\Phi}_{n+1}$. This implies $\triangle\Phi_{n+1} \succ \triangle\Phi_n$. So, for every n, there is map $\delta^{n+1}_n : \overline{Y}_{n+1} \to \overline{Y}_n$ defined by $\delta^{n+1}_n(y) = \triangle\Phi_n((\triangle\Phi_{n+1})^{-1}(y))$. Then,

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$\overline{Y}_0$ is the limit of the inverse sequence $S_Y = \{\overline{Y}_n, \delta_n^{n+1}\}$. Let $E(Y_0) = \{\pi_g : g \in \Phi\}$ and $E(\overline{Y}_0) = \{\overline{\pi}_g : g \in \Phi\}$. We claim that the set of real-valued elements of $E_p(\overline{Y}_0)$ is dense in $C_p(\overline{Y}_0)$. Indeed, every projection $\delta_n : \overline{Y}_0 \to \overline{Y}_n$ induces a continuous map $\delta_n^* : C_p(\overline{Y}_n) \to C_p(\overline{Y}_0)$ defined by $\delta_n^*(h) = h \circ \delta_n$. Because the set of real-valued functions from $E(\overline{Y}_n) = \{\overline{\pi}_g : g \in \Phi_n\}$ is dense in $C_p(\overline{Y}_n)$ and $E_p(\overline{Y}_0) = \bigcup_n \delta_n^*(E(\overline{Y}_n))$, it suffices to show that $\bigcup_n \delta_n^*(C_p(\overline{Y}_n))$ is dense in $C_p(\overline{Y}_0)$. To this end, let $O = \{f \in C_p(\overline{Y}_0) : |f(\overline{y}_i) - f_0(\overline{y}_i)| < \varepsilon_i, i = 1, \dots, k\}$ be a neighborhood of some $f_0 \in C_p(\overline{Y}_0)$, where all $\overline{y}_i$ are different points from $\overline{Y}_0$. Since the base $\mathcal{B}''$ of $\overline{Y}_0$ consists of all sets of the form $\delta_n^{-1}(U_n)$, $U_n \in \mathcal{B}''_n$, there is n and different points $y_i \in \overline{Y}_n$ such that $\delta_n(\overline{y}_i) = y_i$ and $\delta_n^{-1}(U_i) \subset f_0^{-1}(V_i)$, where $U_i \in \mathcal{B}''_n$ and V_i is the open interval $(f_0(\overline{y}_i) - \varepsilon_i, f_0(\overline{y}_i) + \varepsilon_i)$. Then, the set $W = \{h \in C_p(\overline{Y}_n) : h(y_i) \in V_i, i = 1, 2, \dots, k\}$ is open in $C_p(\overline{Y}_n)$. So, it contains a function $h_0 \in E(\overline{Y}_n)$. Hence, $h_0 \circ \delta_n$ is a function from $\bigcup_n \delta_n^*(C_p(\overline{Y}_n)) \cap O$. So, the set of all real-valued functions from $E(\overline{Y}_0)$ is dense in $C_p(\overline{Y}_0)$.

Because T is linear, so is the map $\varphi : E_p(X_0) \to E_p(Y_0)$ defined by $\varphi(\pi_f) = \pi_{T(f)}$. The arguments from the proof of Theorem 1.1 show that φ is continuous, but we are going to prove the more general fact that φ has a continuous extension over the linear hull $LE_p(X_0)$. For every $f \in LE(X_0)$, let $f^* = f \circ (\triangle\Psi) \in C(X)$. Evidently, if $f = \sum_{i=1}^k \lambda_i \cdot f_i \in LE(X_0)$, then $f^* = \sum_{i=1}^k \lambda_i \cdot f_i^*$. So, we have another description of the map φ: $\varphi(f) = \pi_{T(f^*)}$, $f \in E(X_0)$. ∎

Claim 17 *Let $f = \sum_{i=1}^k \lambda_i \cdot f_i \in LE(X_0)$ with $f_i \in E(X_0)$ for all i. If $y^* \in Y$ and $(\triangle\Phi)(y^*) = y$, then $T(f^*)(y^*) = \sum_{i=1}^k \lambda_i \cdot T(f_i^*)(y^*) = \sum_{i=1}^k \lambda_i \cdot \varphi(f_i)(y)$.*

It suffices to show that $T(f^*)(y^*) = \varphi(f)(y)$ for all $f \in E(X_0)$. And that is true because $f^* \in \Psi$, so $T(f^*) \in \Phi$ and $T(f^*)(y^*) = \varphi(f)(y)$.

We define $\widetilde{\varphi} : LE(X_0) \to LE(Y_0)$ by $\widetilde{\varphi}(\sum_{i=1}^k \lambda_i \cdot f_i) = \sum_{i=1}^k \lambda_i \cdot \varphi(f_i)$, where $\lambda_i \in \mathbb{R}$ and $f_i \in E(X_0)$. The continuity of $\widetilde{\varphi}$ with respect to the pointwise topology is equivalent of the continuity of all linear functionals $\widetilde{l}_y : LE_p(X_0) \to \mathbb{R}$ defined by $\widetilde{l}_y(f) = \widetilde{\varphi}(f)(y)$, $y \in Y_0$. So, fix $y_0 \in Y_0$ and $f_0 = \sum_{i=1}^k \lambda_i \cdot f_i \in LE(X_0)$ with $f_i \in E(X_0)$ such that $\widetilde{l}_{y_0}(f_0) \in V$ for some open interval $V \subset \mathbb{R}$. Then, $f_0^* = \sum_{i=1}^k \lambda_i \cdot f_i^*$ and, by Claim 17, we have

$$T(f_0^*)(y_0^*) = \sum_{i=1}^k \lambda_i \cdot T(f_i^*)(y_0^*) = \sum_{i=1}^k \lambda_i \cdot \varphi(f_i)(y_0) = \widetilde{l}_{y_0}(f_0),$$

where $y_0^* \in Y$ with $(\triangle\Phi)(y_0^*) = y_0$. Consequently, there is a neighborhood $W^* = \{g \in C_p(X) : |g(x_j^*) - f_0^*(x_j^*)| < \eta_j, j = 1, 2, \dots, m\}$ of f_0^* in $C_p(X)$ such that $T(g)(y_0^*) \in V$ for all $g \in W^*$. Observe that

$$f_0^*(x_j^*) = \sum_{i=1}^k \lambda_i \cdot f_i^*(x_j^*) = \sum_{i=1}^k \lambda_i \cdot f_i(x_j) = f_0(x_j),$$

where $x_j = (\triangle\Psi)(x_j^*)$. So, $W = \{f \in LE(X_0) : |f(x_j) - f_0(x_j)| < \eta_j, j = 1, 2, \dots, m\}$ is a neighborhood of f_0 in $LE_p(X_0)$. If $g \in W$, then $g = \sum_{s=1}^m \lambda_s \cdot g_s$ for some $g_s \in E(X_0)$. Hence, $g^* = \sum_{s=1}^m \lambda_s \cdot g_s^* \in W^*$, which means that $T(g^*)(y_0^*) \in V$. Finally,

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according to Claim 17, $T(g^*)(y_0^*) = \sum_{s=1}^{m} \lambda_s \cdot \varphi(g_s)(y_0) = \widetilde{l}_{y_0}(g) \in V$. Thus, $\widetilde{\varphi}: LE_p(X_0) \to LE_p(Y_0)$ is continuous.

Therefore, the spaces X_0, $\overline{X}_0$, Y_0, and $\overline{Y}_0$ satisfy the assumptions of Proposition 4.1. So, we apply Proposition 4.1 with $X = X_0$ and $Y = Y_0$ to conclude that $\dim Y_0 = 0$. That completes the proof of Theorem 1.4.

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