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RESEARCH ARTICLE

Imaginaries in equicharacteristic zero henselian fields

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Abstract

We classify the imaginaries in a large class of equicharacteristic zero henselian valued fields that contains all those with bounded inertia group, and more. To do so, we consider a mix of sorts introduced in earlier works of the two authors and prove elimination of imaginaries down to the field, the k-linear imaginaries, and the imaginaries of the value group.

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1. Introduction

In the model theory of valued fields, one of the most striking results is a theorem by Ax, Kochen, and, independently, Ershov, which states that the first-order theory of an equicharacteristic zero henselian valued field is completely determined by the first-order theories of its residue field k and of its value group Γ. A natural philosophy follows from this theorem: the model theory of a henselian valued field is controlled by its residue field and its value group.

In the early 2000s, Hrushovski asked if this philosophy also applies to the elimination of imaginaries: the classification of interpretable sets (quotients of definable sets by definable equivalence relations), or equivalently, the description of (rough) moduli spaces for families of definable sets. He proposed a classification of imaginaries reminiscent of the Ax-Kochen-Ershov principle. The goal of the present paper is to establish this classification for a broad class of henselian valued fields of equicharacteristic zero, including all those with bounded Galois group, that is, having finitely many extensions of any given degree. In fact, it suffices that the maximal unramified algebraic extension has bounded Galois group, in other words, when the inertia group is bounded.

The study of imaginaries in various henselian valued fields has been ongoing for the past 20 years, starting with the case of algebraically closed valued fields (ACVF) in the foundational work by Haskell, Hrushovski, and Macpherson [1]. They proved that in ACVF, every quotient can be described as a subset of products of certain specific quotients, known as the *geometric sorts* : the main field K, and, for all $n \in \mathbb{Z}_{>0}$, the space $\mathrm{Gr}_n := \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$ of free rank n $\mathcal{O}$-submodules of K^n (also known as the affine grassmanian of GL_n) and the space $\mathrm{Lin}_n = \coprod_{R \in \mathrm{Gr}_n} R/\mathfrak{m}R$, where $\mathcal{O}$ denotes the valuation ring and $\mathfrak{m} \subseteq \mathcal{O}$ is the unique maximal ideal. We say that ACVF eliminates imaginaries down to the geometric sorts. These results were later extended to other specific henselian fields, potentially with additional structure [2, 3, 4, 5].

As can be expected, the residue field and the value group create natural obstructions to the elimination of imaginaries in valued fields. In earlier work, the authors studied the residual obstructions [6] and the value group obstructions [7] independently. The main focus of the present work, and where its novelty and complexity lie, is to understand how they interact. This leads us to a classification under no assumptions on the residue field and very mild assumptions on the value group.

The classification of imaginaries in ACVF laid the groundwork for a rich "geometric model theory" of valued fields. However, the dependence on algebraically closed fields hides most of the more arithmetic phenomena. It is the authors' belief that the present classification of imaginaries and its methods will lead to a geometric model theory of valued fields that accounts for their arithmetic. This can be seen, for example, in their work [8] with Cubides Kovacsics on residue domination.

1.1. Obstructions arising from the value group

Let K be an equicharacteristic zero henselian valued field. When describing imaginaries in K, the imaginaries of the value group itself have to be accounted for. Moreover, the complexity of the value group also directly impacts the complexity of definable $\mathcal{O}$-modules and this also needs to be taken into account.

This can be done by introducing the *stabilizer sorts* which provide codes for all the definable $\mathcal{O}$-submodules of K^n, for any n. More precisely, let $\mathcal{C} = (\mathcal{C}_c)_{c \in \mathrm{Cut}^\star}$ be the (ind-)interpretable family of definable proper cuts in Γ – here a cut is said to be proper if it is neither of the cuts at infinity. For every $c \in \mathrm{Cut}^\star$, let I_c denote the $\mathcal{O}$-submodule $\{x \in K : v(x) \in \mathcal{C}_c\}$. For every tuple c in $\mathrm{Cut}^\star$, let Λ_c be the module $\sum_i I_{c_i} e_i$, where $(e_i)_{i<|c|}$ is the canonical basis of $K^{|c|}$.

The group B_n of upper triangular matrices acts on the set of all definable $\mathcal{O}$-submodules of K^n. We define

$$\mathrm{Mod}_c = \mathrm{B}_n/\mathrm{Stab}(\Lambda_c)$$

and we identify an element of Mod_c with the corresponding $\mathcal{O}$-submodule (see Notation 3.4.2).

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We also consider the (ind-)interpretable set

$$\mathrm{Mod} = \coprod_c \mathrm{Mod}_c$$

where c varies over the (ind-)interpretable set of finite tuples in $\mathrm{Cut}^\star$. Any definable $\mathcal{O}$-submodules of K^n is coded in $\mathrm{Mod} \cup K$ (see Corollary 3.7).

When the value group has bounded regular rank (i.e., there are at most countably many definable convex subgroups in any elementary extension[1]) and the residue field is algebraically closed, the second author [7, Theorem 5.12] showed that, together with the imaginaries of the value group, these are essentially the only new imaginaries.

1.2. Obstructions arising from the residue field

When the residue field is not algebraically closed, new obstructions arise. The residue field itself can have nontrivial imaginaries, but it might also induce linearly twisted imaginaries on interpretable k-vector spaces.

Let $R \subseteq K^n$ be a definable $\mathcal{O}$-submodule. Then the quotient $R/\mathfrak{m}R$ is a k-vector space of dimension $d \leq n$, on which k induces a nontrivial structure. Once we name a basis, $R/\mathfrak{m}R$ is definably isomorphic to some k^d, and hence imaginaries of $R/\mathfrak{m}R$ can be identified with imaginaries of k; but this identification is not canonical and depends on a choice of basis.

The structure $(\mathrm{k}, R/\mathfrak{m}R)$ can be seen as a structure in the language $\mathfrak{L}_{\mathrm{vect}}$ with two sorts:

- a sort for k with the structure induced by K;
- a vector space sort V with the (additive) group language;
- A function $\lambda : \mathrm{k} \times \mathrm{V} \to \mathrm{V}$ interpreted as scalar multiplication.

Given a set X interpretable without parameters in the $\mathfrak{L}_{\mathrm{vect}}$-theory of dimension d vector spaces, the interpretable sets $X^{(\mathrm{k}, R/\mathfrak{m}R)}$ have to be accounted for.[2]

Let $\mathrm{Cut}^{\star\star} = \mathrm{Cut}^\star \setminus \{\Gamma_{>\gamma} : \gamma \in \Gamma\}$ – unless the value group is discrete, in which case we set $\mathrm{Cut}^{\star\star} = \mathrm{Cut}^\star$. A module $R \subseteq K^n$ is said to be $\mathfrak{m}$-avoiding if it is in Mod_c, for some tuple c in $\mathrm{Cut}^{\star\star}$. The dimension of $R/\mathfrak{m}R$ only depends on c – it is equal to $d = |\{i : \mathcal{C}_{c_i} = \Gamma_{\leq\gamma}, \text{for some } \gamma \in \Gamma\}|$. For every quotient X as above, we define

$$\mathrm{Lin}_{c,X} = \coprod_{R \in \mathrm{Mod}_c} X^{(\mathrm{k}, R/\mathfrak{m}R)}$$

and the *(generalized)* k*-linear imaginaries*:

$$\mathrm{k}^{\mathrm{leq}} = \coprod_{c,X} \mathrm{Lin}_{c,X},$$

where c ranges over all finite tuples in $\mathrm{Cut}^{\star\star}$. Among those, we denote $\mathrm{Gr} = \coprod_c \mathrm{Mod}_c$ and $\mathrm{Lin} = \coprod_c \mathrm{Lin}_{c,\mathrm{V}}$. Along with K, these form the *(generalized) geometric sorts*, and they encode all definable $\mathcal{O}$-submodules of K^n, for any n (see Corollary 3.21).

Assuming that the value group is elementarily equivalent to $\mathbb{Q}$ or $\mathbb{Z}$, Hils and the first author [6, Theorem 6.1.1] show that, under the technical assumption that the residue field eliminates $\exists^\infty$, all new imaginaries essentially arise in this manner.

[1] We refer the reader to [9] for details on bounded regular rank groups.

[2] In [6], it is claimed that it suffices to consider interpretable sets of the form V/E, this claim appears to be incorrect.

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1.3. An imaginary Ax-Kochen-Ershov principle

In this paper, we provide a common generalization of [6, 7] by obtaining a general Ax-Kochen-Ershov principle for the classification of imaginaries, under a mild technical assumption on the value group (we refer the reader to Section 2.1 for notation and definitions related to imaginaries). However, note that dealing with both an arbitrary residue field and a very general value group introduces new issues that were not present in either earlier works. We give details on some of the new tools required to deal with these issues after stating the main theorems.

Definition 1.1. Given a structure M, tuples of variables x and y and a set of formulas $\Delta(x, y)$, a Δ-type $p(x)$ over M is a maximal finitely consistent set of formulas of the form $\phi(x, a)$ and $\neg\phi(x, a)$, where $\phi \in \Delta$ and a is a y-tuple in M. If $A \subseteq M$, we say that such a type is A-definable if for every $\phi \in \Delta$, the set $\{a \in M^y : \phi(x, a) \in p\}$ is A-definable.

Definition 1.2. We say that an ordered group G (potentially with additional structure) satisfies Property **D** if for every $A = \mathrm{acl}(A) \subseteq G^{\mathrm{eq}}$ and every finite set of A-formulas $\Delta(x, y)$ containing the formula $x < y_0$, any A-definable Δ-type $p(x)$ over G is contained in an A-definable complete type $q(x)$ over G.

This is a stronger property than the density of definable types. It holds in ordered abelian groups of bounded regular rank with no addition structure (see the second half of the proof of [7, Theorem 5.3]).

Our main results are the following. Let K be an equicharacteristic zero henselian valued field such that the value group is *either*:

- dense with property **D**;
- or, a discrete ordered abelian group of bounded regular rank – in which case, we add a constant for a uniformizer.

Theorem (Theorem 6.6). *Let K_{ac} be an expansion of K by angular components. Then K_{ac} weakly eliminates imaginaries down to $K \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$ – in other words, any interpretable set admits an interpretable finite cover by a subset of some cartesian power of $K \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$ (see Section 2.1 for precise definitions).*

Without an angular component the short exact sequence

$$1 \to \mathrm{k}^\star \to \mathrm{RV} = K^\star/(1+\mathfrak{m}) \to \Gamma \to 0$$

might not eliminate imaginaries down to $\mathrm{k}^{\mathrm{leq}}$ and Γ^{eq}, creating further obstructions (see Remark 6.5). They can be avoided by further assumption on Γ:

Theorem (Theorem 6.4). *Assume that for every $n \in \mathbb{Z}_{\geq 1}$ we have $[\Gamma : n\Gamma] < \infty$, and we add constants in* RV *so that $\Gamma/n\Gamma = \Gamma/nv(\mathrm{RV}(\mathrm{acl}(\emptyset)))$. Then K weakly eliminates imaginaries down to $K \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$.*

Remark 1.3. These theorems remain true when K comes with additional structure on k and, independently Γ – the main exception is that when the valuation is discrete, we require the valuation group to have no additional structure. Also, in the second theorem, the assumption on the finiteness of $\Gamma/n\Gamma$ can be replaced by asking that k is an algebraically closed field with no additional structure (see Corollary 4.11, this is essentially [7, Theorem 5.12]).

These results generalize all previously known weak elimination results in equicharacteristic zero (in particular, those of [6, 7]) and provides a definitive answer for, among others, all equicharacteristic zero henselian valued fields with bounded inertia group – meaning that the maximal unramified extension has bounded Galois group.

As a corollary (and an illustration) of these two results, when Γ is dense, we give a complete classification of (almost) k-internal sets – see Corollary 6.7. In the case of ACVF this classification is a cornerstone of the study of stable domination and the subsequent work of Hrushovski and Loeser [10] on Berkovich spaces. These new results are used in [11] to study residual domination in equicharacteristic zero henselian fields.

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1.4. Sketch of the proof

The proof of the main theorems proceeds in two main steps. The first one is a density result for definable types in the structure induced by the maximal unramified algebraic extension (Theorem 4.1). Density of definable (complete) types is at the core of many recent results on elimination of imaginaries. However, it cannot hold in an arbitrary equicharacteristic zero henselian field K because it might not hold in the residue field or the value group.

In [6, Theorem 3.1.3] Hils and the first author showed that definable types of the algebraic closure K^{a} are dense among definable sets in K, under the assumption that the residue field eliminates $\exists^{\infty}$. The second author ([7, Theorem 5.9]) proved density of definable types in the maximal unramified algebraic extension K^{ur}, assuming the value group has bounded regular rank. Theorem 4.1 generalizes both results and unifies them by proving that the definable types of K^{ur} are dense among the sets definable in K (assuming Property **D**). Note also that no hypothesis on the residue field is required anymore. A significant new challenge in this construction is to relate the germs of functions definable in K to those of functions definable in K^{ur} – see Section 4.3.

The second step of proof consists in studying the completions of the partial definable types constructed in the first step. We show that they are invariant over RV and k-vector spaces of the form $R/\mathfrak{m}R$, where R is a definable $\mathcal{O}$-submodules of K^n (Theorem 5.3). The bulk of the work (Proposition 5.16) revolves around showing that (generalized) geometric points can be lifted to the valued field by a sufficiently invariant type. This, in turn, relies heavily on the technical computation of germs of function taking values in sets of the form $R/\mathfrak{m}R$.

We describe such germs in three steps. First we first consider the case of valued fields with algebraically closed residue field (*cf.* Proposition 5.7). We then consider valued fields with dense value group and arbitrary residue field (Section 5.1.1). This relies on the characterization of the k-internal sets (Corollary 3.23). Lastly, we consider valued fields with discrete value group and arbitrary residue field (Section 5.1.2). In that case, we circumvent the characterization of the k-internal sets by considering a ramified extension with dense value group.

Together these two results imply a first elimination result relative to the imaginaries of RV and the k-vector spaces of the form $R/\mathfrak{m}R$ (Proposition 6.1). The main theorems follow by describing the imaginaries in RV with and without angular component.

1.5. Overview of the paper

Section 2 provides some preliminary reminders on imaginaries and the model theory of equicharacteristic zero henselian fields.

In Section 3, we introduce the stabilizer sorts and the generalized geometric sorts. Both provide codes for the definable $\mathcal{O}$-modules. We prove a unary decomposition for the stabilizer sorts (Proposition 3.10). The generalized geometric sorts – and the related notion of $\mathfrak{m}$-avoiding module – play a crucial role in classifying k-internal sets among the stabilizer sorts (Corollary 3.23), provided that the value group is dense.

In Section 4, we prove the density of definable types of the maximal unramified algebraic extension K^{ur} among definable sets of K (Theorem 4.1). In Section 5, we show that those definable types have invariant completions (Theorem 5.3). Finally, in Section 6, we wrap everything together and show the two main theorems.

2. Preliminaries

2.1. Model theoretic preliminaries

Throughout this text, if M is an $\mathfrak{L}$-structure, a set X is said to be "definable" if it is definable with parameters. If we specify that it is "$\mathfrak{L}$-definable" we mean that it is definable without parameters. Also, we extend definable sets canonically to elementary extensions of M and we distinguish the definable set X from the set $X(M)$ of its realizations in M.

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If $A \subseteq M$, we write $X(A)$ for the points of X whose coordinates are in A, rather than $X \cap \mathrm{dcl}(A)$. We change language too often to not be explicit with the definable closures at play.

We refer the reader to [12, Section 8.4] for a detailed exposition of the elimination of imaginaries. Let T be an $\mathfrak{L}$-theory. Consider the language $\mathfrak{L}^{\mathrm{eq}}$ obtained by adding to $\mathfrak{L}$ a new sort S_X for every $\mathfrak{L}$-definable set $X \subseteq Y \times Z$, where Y and Z are product of sorts, and a new symbol $f_X : Z \to S_X$. The $\mathfrak{L}^{\mathrm{eq}}$-theory T^{eq} is then obtained as the union of T, the fact that the f_X are surjective and that their fibers are the classes of the equivalence relation defined by $X_{z_1} = \{y \in Y : (y, z_1) \in X\} = X_{z_2}$.

Any $M \models T$ has a unique expansion to a model of T^{eq} denoted M^{eq} – whose points are called the *imaginaries*. A set is interpretable if it is definable in M^{eq}. Throughout this paper, when considering types, definable closures or algebraic closures, we will work in the $\mathfrak{L}^{\mathrm{eq}}$-structure, unless otherwise specified.

Given $M \models T$ and an $\mathfrak{L}(M)$-definable set X, we denote by $\ulcorner X \urcorner \subseteq M^{\mathrm{eq}}$ the intersection of all $A = \mathrm{dcl}(A) \subseteq M^{\mathrm{eq}}$ such that X is $\mathfrak{L}^{\mathrm{eq}}(A)$-definable. It is the smallest dcl-closed set of definition for X. Any dcl-generating subset of $\ulcorner X \urcorner$ is called *a code* of X. More generally, if $A \subseteq M^{\mathrm{eq}}$ is a set of parameters, any tuple e such that $\mathrm{dcl}(Ae) = \mathrm{dcl}(A \ulcorner X \urcorner)$ is called a code of X over A.

If $\mathcal{D}$ is a collection of sorts of $\mathfrak{L}^{\mathrm{eq}}$ (equivalently, a collection of $\mathfrak{L}$-interpretable sets) and $A \subseteq M^{\mathrm{eq}}$ is a set of parameters, we say that X is *coded* in $\mathcal{D}$ over A if it is $\mathfrak{L}^{\mathrm{eq}}(A \cup \mathcal{D}(\ulcorner X \urcorner))$-definable – that is, it admits a code in $\mathcal{D}$ over A.

The theory T is said to *eliminate imaginaries* down to $\mathcal{D}$ if, for every $M \models T$, every $\mathfrak{L}(M)$-definable set X is coded in $\mathcal{D}$ – equivalently, for every $e \in M^{\mathrm{eq}}$, there is some $d \in \mathcal{D}(\mathrm{dcl}(e))$ such that $e \in \mathrm{dcl}(d)$. Finally, we say that the theory T *weakly eliminates imaginaries* down to $\mathcal{D}$ if for every $e \in M^{\mathrm{eq}}$, there is some $d \in \mathcal{D}(\mathrm{acl}(e))$ such that $e \in \mathrm{dcl}(d)$.

If $p(x)$ is a (definable) partial type over some structure M and f and g are definable functions in M which are defined at realizations of p, we say that they have the same p-germ, and we write $[f]_p = [g]_p$ if $p(x) \vdash f(x) = g(x)$. When p is definable we write $[f]_p$ for class of p-germs as f varies in an $\mathfrak{L}$-definable family.

Lastly, we refer the reader to [13, Appendix A (Definition $A.2$ and $A.5$)] for a detailed presentation of expansions (called enrichments, there) and relative quantifier elimination. Note that in the present text, expansions do not allow adding new sorts.

2.2. *Equicharacteristic zero henselian fields*

Let $\mathrm{Hen}_{0,0}$ be the theory of equicharacteristic zero henselian valued fields (K, v) with no additional structure, in some language $\mathfrak{L}$. The exact language we use does not matter much since we will be working $\mathfrak{L}^{\mathrm{eq}}$. In this section, we recall some useful results about these structures. We denote by $\mathrm{RV}^\star$ the group $\mathrm{K}^\star/(1+\mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of the valuation ring $\mathcal{O} \subseteq \mathrm{K}$ and $\mathrm{rv} : \mathrm{K} \to \mathrm{RV} = \mathrm{RV}^\star \cup \{0\}$ the canonical projection (extended by $\mathrm{rv}(0) = 0$).

Theorem 2.1 [14, Theorem B]. *Let $M \models \mathrm{Hen}_{0,0}$ and $A \leq \mathrm{K}(M)$ be a subring. Every A-definable subset of $\mathrm{K}^x \times \mathrm{RV}^y$ is of the form $\{(x, y) : (\mathrm{rv}(P(x)), y) \in X\}$, for some tuple P in $A[x]$ and some $X \subseteq \mathrm{RV}^n$ which is $\mathrm{rv}(A)$-definable in the short exact sequence*

$$1 \to \mathrm{k}^\star \to \mathrm{RV}^\star \to \Gamma \to 0$$

where $\mathrm{k} = \mathcal{O}/\mathfrak{m}$ is the residue field and $\Gamma = \mathrm{v}(\mathrm{K})$ is the value group.

This remains true in RV-expansions – that is, when RV comes with additional structure.

From the result above, either by adding a section or proving a quantifier elimination result for short exact sequences, we can deduce the following:

Proposition 2.2. *Let $M \models \mathrm{Hen}_{0,0}$ and $A \leq \mathrm{K}(M)$ be a subring. The sets k and Γ are stably embedded (with respectively the structure of a field and an ordered group) and they are orthogonal. In other words, any M-definable subset of $\mathrm{k}^x \times \Gamma^y$ is a finite union of products $X \times Y$ where X is definable in the field k and Y is definable in the ordered group Γ.*

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Moreover, any A-definable $X \subseteq \Gamma^n$ is $\mathrm{v}(A)$-definable – we say that Γ is strongly stably embedded.

The first part is [15, corollary 2.25]. The second part follows [15, corollary 2.24], noting that adding a angular component does not grow the generated structure in Γ.

These results remain true in k- Γ-expansions – that is, when k comes with additional structure, and, independently so does Γ. They also hold if an angular component is added to the language.

Lemma 2.3. *Let M be an expansion of a model of $\mathrm{Hen}_{0,0}$ with strongly stably embedded value group Γ. For any subring $A \leq \mathrm{K}(M)$, we have*

$$\Gamma^{\mathrm{eq}}(\mathrm{acl}(A)) \subseteq \mathrm{acl}(\mathrm{v}(A)).$$

Proof. Let $X \subseteq \Gamma^n$ be $\mathrm{acl}(A)$-definable. Let $X_1 = X, \ldots, X_n$ be the $\mathrm{Aut}(M/A)$-conjugates of X over A. For every $x, y \in \Gamma^n$, we define xEy to hold if, for every i, $x \in X_i$ if and only if $y \in X_i$. This is an A-definable equivalence relation. By strong stable embeddedness, it is $\mathrm{v}(A)$-definable. Note also that E has finitely many equivalence classes and X is a union of classes. It follows that X is $\mathrm{acl}(\mathrm{v}(A))$-definable. □

Definition 2.4. Let $M \models \mathrm{Hen}_{0,0}$. Let $a \in \mathrm{K}(M)$ and let C be a cut in $\Gamma(M)$ – that is, an upwards closed subset. We define the generalized ball $b_C(a)$ of cut C around a to be $\{x \in \mathrm{K} : \mathrm{v}(x-a) \in C\}$. A generalized ball is said to be open if its cut is not of the form $\Gamma_{\geq\gamma}$, for each $\gamma \in \Gamma(M)$.

Let $\mathrm{B_g}$ denote the (ind-)interpretable set of (codes for) definable generalized balls.

Note that, for every $\gamma \in \Gamma(M)$, $b_{\Gamma_{>\gamma}}(a)$ is the open ball of radius γ around a, $b_{\Gamma_{\geq\gamma}}(a)$ is the closed ball of radius γ around a – and $b_\Gamma(a) = \mathrm{K}$ is also considered an open ball.

If $\Gamma(M)$ is discrete, $b_{\Gamma_{>\gamma}}(a) = b_{\Gamma_{\geq\gamma}}(a)$ is not considered to be an open generalized ball. Let M be an RV-expansion of a model of $\mathrm{Hen}_{0,0}$ and let $A \leq \mathrm{K}(M)$ be a subring. Let M^{a} denote the algebraic closure as a pure valued field.

Theorem 2.1 can also be refined for unary sets:

Proposition 2.5 [16, Proposition 3.6]. *Let $M \models \mathrm{Hen}_{0,0}$, let $A \leq \mathrm{K}(M)$ be a subring and let $X \subseteq \mathrm{K}\times\mathrm{RV}^n$ be A-definable. There exists a finite set $C \subseteq A^{\mathrm{a}} \cap \mathrm{K}(M)$ such that for every $\xi \in \mathrm{RV}^n$, $X_\xi = \{x \in \mathrm{K} : (x,\xi) \in X\} = \mathrm{rv}_C^{-1}(\mathrm{rv}_C(X_\xi))$ where $\mathrm{rv}_C(x) = (\mathrm{rv}(x-c))_{c\in C}$.*

It follows that for any generalized ball b that does not intersect C, either $b \cap X_\xi = b$ or $b \cap X_\xi = \emptyset$. We say that C prepares X.

This proposition remains true in RV-expansions. Also, it follows that $\mathrm{K}(\mathrm{acl}(A)) \subseteq A^{\mathrm{a}}$.

Lemma 2.6. *Let M be (an* RV*-expansion of) a model of $\mathrm{Hen}_{0,0}$, let $A \leq \mathrm{K}(M)$ and let b be an $\mathrm{acl}(A)$-definable generalized ball which is not an open ball. Then there exists $c \in A^{\mathrm{a}} \cap b(M)$ whose other $\mathrm{Aut}(M/A)$-conjugates are all outside of b.*

Moreover, if b is a ball, we may assume that no other $\mathrm{Aut}(M^{\mathrm{a}}/A)$-conjugate is in $b(M^{\mathrm{a}})$.

Proof. Note that any other $\mathrm{Aut}(M/A)$-conjugate of b is disjoint from b. Let B be the union of $\mathrm{Aut}(M/A)$-conjugates of b. It is an A-definable set. By Proposition 2.5, there exists $C \subseteq A^{\mathrm{a}} \cap \mathrm{K}(M)$ such that for any ball d disjoint from C, either $d \subseteq B$ or $d \cap B = \emptyset$. If $b \cap C = \emptyset$, let d be the largest ball containing b which is disjoint from C – that is, the open ball around b with radius $\min_{c\in C} \mathrm{v}(x-c)$, for any $x \in b$. Since d contains b and is disjoint from C, it is contained in B. So d is covered by finitely many disjoint subballs. As the residue field is infinite, this is impossible unless $d = b$, in which case b would be open, contradicting our assumption.

So $b \cap C \neq \emptyset$. Let $C_b = (\mathrm{Aut}(M/A) \cdot C) \cap b$. Since we are in equicharacteristic zero, the average c of C_b is in $b(M)$. By construction, no other $\mathrm{Aut}(M/A)$-conjugate of c is in b.

If b is a ball, then it is definable in M^{a} and its $\mathrm{Aut}(M^{\mathrm{a}}/A)$-orbit is also a finite set of disjoint balls. Let $C'_b = \mathrm{Aut}(M^{\mathrm{a}}/A) \cdot c$. Since M is henselian, the average c' of C'_b is in M. By construction again, the only $\mathrm{Aut}(M^{\mathrm{a}}/A)$-conjugate of c' in b is c'. □

Finally, when the residue field is algebraically closed, Theorem 2.1 can be further simplified:

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Theorem 2.7 [7, Corollary 2.33]. *Assume the residue field* $\mathrm{k}(M)$ *is algebraically closed. Every A-definable subset of* K^x *is of the form* $\mathrm{v}(P(x)) \in X$ *where* P *is a tuple in* $A[x]$ *and* $X \subseteq \Gamma^n$ *is* $\mathrm{v}(A)$*-definable in the ordered group structure. Moreover, this remains true in* Γ*-expansions – that is, when* Γ *comes with additional structure.*

3. Codes of $\mathcal{O}$-modules

3.1. The stabilizer sorts

Let M be an (expansion of a) valued field in some language $\mathfrak{L}$.

Notation 3.1. We fix an (ind-)$\mathfrak{L}$-interpretable family $\mathcal{C} = (\mathcal{C}_c)_{c \in \mathrm{Cut}}$ of cuts in Γ such that any M-definable cut is of the of form $\mathcal{C}_c$ for some unique $c \in \mathrm{Cut}(M)$. We will further assume that c is a canonical parameter for $\mathcal{C}_c$.

For every $c \in \mathrm{Cut}$, let I_c denote the $\mathcal{O}$-submodule $\{x \in \mathrm{K} : \mathrm{v}(x) \in \mathcal{C}_c\}$. Note that, by definition of $\mathcal{C}$, any $\mathfrak{L}(M)$-definable $\mathcal{O}$-submodule of K is of the form I_c for some unique $c \in \mathrm{Cut}(M)$. We also denote $\Delta_c = \{\gamma \in \Gamma : \gamma + \mathcal{C}_c = \mathcal{C}_c\}$ – it is a convex subgroup of Γ.

The following results are well-established and go back to Bauer's work on separated extensions.

Definition 3.2. A definable valuation v on an interpretable K-vector space V is a map to some interpretable set X with an order preserving action of Γ such that

- for every $\lambda \in \mathrm{K}$ and $x \in V$, $v(\lambda x) = \mathrm{v}(\lambda) + v(x)$;
- for every $x, y \in V$, $v(x + y) \geq \min\{v(x), v(y)\}$.

Note that we do not assume that $v(x) = v(0)$ implies $x = 0$, nor do we assume that the action of Γ on X is free.

Proposition 3.3. *Assume that* M *is definably spherically complete – that is, the intersection of any* M*-definable chain of balls is non empty.*

1. *For every* M*-definable valuation* v *on* K^n*, there exists a triangular basis* $(a_i)_{i<n}$ *of* K^n *such that, for all* i, $v(a_i) \in \mathrm{dcl}(\ulcorner v \urcorner)$ *and for every* $\lambda_i \in \mathrm{K}$*,*

$$v\Big(\sum_i \lambda_i a_i\Big) = \min_i \mathrm{v}(\lambda_i) \cdot v(a_i).$$

2. *Any* M*-definable* $\mathcal{O}$*-submodule* R *of* K^n *is of the form* $\sum_{i<n} \mathrm{I}_{c_i} a_i$*, where* a_i *is a triangular basis of* $\mathrm{K}^n(M)$ *and* $c_i \in \mathrm{Cut}(\ulcorner R \urcorner)$*.*

A basis as in the fist assertion is said to be separated. A module as in the second assertion is said to be of type $c = (c_i)_{i<n}$.

Proof. If M is (elementarily equivalent to a) maximally complete field, the first assertion is [7, Lemma 5.7]. If M is only definably spherically complete, the same proof works using [6, Claim 3.3.9] instead of [7, Fact 2.55].

Let us now prove the second assertion. For every $x \in \mathrm{K}^n$, we define $v_R(x) = \{\mathrm{v}(\lambda) : \lambda x \in R\}$ a (nonempty) cut of Γ. We order them by inclusion (so $\Gamma = v_R(0)$ is the maximal element and $\{\infty\}$ is the minimal element). Note that, for every $x \in \mathrm{K}$, $v_R(\lambda x) = -\mathrm{v}(\lambda) + v_R(x)$ and for this action of Γ on the set of cuts, v_R is an M-definable valuation.

By the first assertion, we can find a separated triangular basis $(a_i)_i$ of $\mathrm{K}^n(M)$, such that $v_R(a_i) \in \mathrm{dcl}(\ulcorner R \urcorner)$. Then $\sum_i \lambda_i a_i \in R$ if and only if $0 = \mathrm{v}(1) \in v_R(\sum_i \lambda_i a_i) = \min_i -\mathrm{v}(x_i) + v_R(a_i)$, that is, $\mathrm{v}(\lambda_i) \in v_R(a_i)$ for all i. Let $c_i \in \mathrm{Cut}(\mathrm{dcl}(\ulcorner R \urcorner))$ be such that $v_R(a_i) = \mathcal{C}_{c_i}$. We then have $R = \sum_i \mathrm{I}_{c_i} a_i$, as required. □

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Notation 3.4.

1. We write B_n to denote the set of $n \times n$ upper triangular and invertible matrices. We write $\mathrm{D}_n \leq \mathrm{B}_n$ for the subgroup of diagonal matrices and $\mathrm{U}_n \leq \mathrm{B}_n$ for the subgroup of unipotent matrices, that is upper triangular matrices with ones on the diagonal.
2. For every n-tuple c in Cut, we define Mod_c to be the interpretable set of modules of type c. Let $\Lambda_c = \sum \mathrm{I}_{c_i} e_i$ be the canonical module of type c, where e_i is the canonical basis of K^n. Then $\sum_{i<n} \mathrm{I}_{c_i} a_i = A \cdot \Lambda_c$ where $A \in \mathrm{B}_n$ is the upper triangular matrix of the a_i. In other words, B_n acts transitively on Mod_c and

$$\mathrm{Mod}_c \simeq \mathrm{B}_n / \mathrm{Stab}(\Lambda_c).$$

We will now identify Mod_c with this quotient of B_n and for every $s \in \mathrm{Mod}_c$, we write R_s for the $\mathcal{O}$-module of type c coded by s. Let $\mu_c : \mathrm{B}_n \to \mathrm{Mod}_c$ denote the natural quotient map.

If $\Delta \leq \Gamma$ is a (definable) convex subgroup, we write $\mathcal{O}_\Delta = \{x \in \mathrm{K} : \exists \delta \in \Delta \mathrm{v}(x) \geq \delta\}$ for the associated (definable) valuation ring. If $I, J \leq \mathrm{K}$ are two (definable) $\mathcal{O}$-submodules, let $(I : J)$ denote the (definable) $\mathcal{O}$-submodule $\{x \in \mathrm{K} : xJ \subseteq I\}$.

Proposition 3.5. *Let c be a tuple in* Cut. *For every $a \in \mathrm{B}_n$, we have*

$$a \in \mathrm{Stab}(\Lambda_c) \text{ if and only if } \begin{cases} a_{i,i} \in \mathcal{O}^\times_{\Delta_{c_i}} & \text{for all } i < n, \text{ and} \\ a_{i,j} \in (\mathrm{I}_{c_i} : \mathrm{I}_{c_j}) & \text{for all } i < j < n. \end{cases}$$

Proof. We proceed by induction on n. Write a as $\begin{pmatrix} a_{0,0} & b \\ 0 & e \end{pmatrix}$, with $e \in \mathrm{B}_{n-1}$, and c as (c_0, d), with $d \in \mathrm{Cut}^{n-1}$. If $a\Lambda_c \subseteq \Lambda_c$, then, considering the action on I_{c_0} and Λ_d, we see that $a_{0,0}\mathrm{I}_{c_0} \subseteq \mathrm{I}_{c_0}$, $b\Lambda_d \subseteq \mathrm{I}_{c_0}$ – so, considering the action on each I_{c_j}, for every $j > 0$, $a_{0,j}\mathrm{I}_{c_j} \subseteq \mathrm{I}_{c_0}$ – and $e\Lambda_d \subseteq \Lambda_d$; and the converse also holds.

Since $a\Lambda_c = \Lambda_c$ if, moreover, $a^{-1}\Lambda_c = \begin{pmatrix} a_{0,0}^{-1} & -a_{0,0}^{-1} b e^{-1} \\ 0 & e^{-1} \end{pmatrix} \Lambda_c \subseteq \Lambda_c$, it follows that we must further have $a_{0,0}\mathrm{I}_{c_0} = \mathrm{I}_{c_0}$, that is, $\mathrm{v}(a_{0,0}) \in \Delta_{c_0}$ and $e\Lambda_d = \Lambda_d$. These conditions are sufficient since, in that case, $a_{0,0}^{-1} b e^{-1} \Lambda_d = a_{0,0}^{-1} b \Lambda_d \subseteq a_{0,0}^{-1}\mathrm{I}_{c_0} = \mathrm{I}_{c_0}$. The claim now follows by induction. □

Definition 3.6. Let $\mathrm{Cut}^\star = \mathrm{Cut} \setminus \{\emptyset, \Gamma\}$ and Mod be the disjoint union of all the Mod_c where c is a tuple in $\mathrm{Cut}^\star$.

Note that any $s \in \mathrm{Mod}$ determines the unique c such that $s \in \mathrm{Mod}_c$.

Corollary 3.7. *Any M-definable $\mathcal{O}$-submodule R of K^n is coded in* $\mathrm{K} \cup \mathrm{Mod}$.

Proof. Let $V \subseteq \mathrm{K}^n$ be the K-span of R and $W = \{x \in \mathrm{K}^n : \mathrm{K}x \subseteq R\}$. Then, by [17, Lemma 4.3], V/W is $\mathrm{K}(\ulcorner R \urcorner)$-definably isomorphic to some K^r and R is entirely determined by its image in V/W. So we may assume $V = \mathrm{K}^r$ and $W = 0$ and hence that R is of type c with $c \in \mathrm{Cut}^\star$. By definition, it is coded in Mod_c. □

Remark 3.8. There is a lot of redundancy in Mod. If c and c' are tuples in Cut of the same length such that for every $i < n$, c'_i is a translate of c_i, then there is a natural bijection between Mod_c and $\mathrm{Mod}_{c'}$ given by the action of a diagonal matrix.

If there exists an (ind-)interpretable subset $\mathrm{Cut}' \subseteq \mathrm{Cut}$ such that any definable cut is of the form $a + \mathcal{C}_c$ for a unique $c \in \mathrm{Cut}'$, it follows that every M-definable $\mathcal{O}$-submodule of K^n is coded in $\mathrm{K} \cup \bigcup_{c \in \mathrm{Cut}' \setminus \{\emptyset, \Gamma\}} \mathrm{Mod}_c$. Similarly, we can replace Cut by Cut' in the definition of the geometric sorts (Definition 3.13).

This is the case, for example, in ordered abelian groups of bounded regular rank (*cf.* [7, Corollary 2.24]).

Remark 3.9. Any M-definable generalized ball is inter-definable with the sub- $\mathcal{O}$-module R of K^2 generated by $b \times \{1\}$; indeed $b = \{x \in \mathrm{K} : (x, 1) \in R\}$. So generalized balls are coded in Mod.

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Let us now describe the structure of Mod. The solvability of the upper triangular invertible matrices will play a central role in this description.

We go through the elements of an upper triangular matrix diagonal by diagonal starting with the middle diagonal, and in each diagonal, we proceed from top to bottom. In other words, we order pairs (i, j) such that $i \leq j < n$ first by $j - i$ and then by i. We will identify the set of such pairs with the set of non-negative integers $< n(n+1)/2$, according to that order.

For every pair (i, j), let $p_{i,j} : \mathrm{B}_n \to \mathrm{K}$ be the projection on coordinate (i, j). Let also $\varepsilon_{i,j} = 1$ if $i = j$ and 0 otherwise. For every pair ℓ, let $G_\ell = \{a \in \mathrm{B}_n : p_k(a) = \varepsilon_k,\ \forall k < \ell\}$. Then $G_0 = \mathrm{B}_n$ and $G_{n(n+1)/2} = \{\mathrm{id}\}$. By choice of the order, for every ℓ, $G_{\ell+1} \triangleleft G_\ell$ and p_ℓ induces an isomorphism from $G_\ell/G_{\ell+1}$ to $\mathbb{G}_{\mathrm{m}}$, if $\ell < n$, and to $\mathbb{G}_{\mathrm{a}}$ otherwise. Note also that $H_\ell = \{a \in G_\ell : p_k(a) = \varepsilon_k,\ \forall k > \ell\}$, is a section of p_ℓ restricted to G_ℓ and hence $G_\ell = G_{\ell+1} \rtimes H_\ell$.

Furthermore, we have $G_n = \mathrm{U}_n$, $\mathrm{B}_n = \mathrm{U}_n \rtimes \mathrm{D}_n$ and, for every $\ell \geq n$, G_ℓ is central in G_n module $G_{\ell+1}$ – actually modulo the next upper triangular group $G_{0,j}$ – if ℓ is a pair $(i, i + j - 1)$. In particular $G_\ell \trianglelefteq \mathrm{U}_n$.

We can now prove the following unary decomposition.

Proposition 3.10. *Let $s \in \mathrm{Mod}_c$. There exists a finite tuple $b = (b_\ell)_{\ell < n(n+1)/2}$ in M^{eq} – we identify each b_ℓ with a subset of some K^{r_ℓ} – and $cb_{<\ell}$-interpretable sets X_ℓ such that:*

- *for every ℓ, $b_\ell \in X_\ell$;*
- *$\mathrm{dcl}(s) = \mathrm{dcl}(cb)$;*
- *if $\ell < n$, then $X_\ell = \Gamma/\Delta_{c_i}$, where $\ell = (i, i)$;*
- *if $\ell \geq n$, then X_ℓ has a $cb_{<\ell}$-definable $\mathrm{K}/\mathrm{I}_\ell$-torsor structure where I_ℓ is a $cb_{<n}$-definable multiple of $(\mathrm{I}_{c_i} : \mathrm{I}_{c_j})$ and $\ell = (i, j)$.*

Moreover, for any $n \leq \ell < n(n+1)/2$ and any choice of $a_k \in b_k$, for $k < \ell$, there is a (uniformly) $ca_{<\ell}$-definable isomorphism of torsors $f_\ell : X_\ell \to \mathrm{K}/\mathrm{I}_\ell$ and a $ca_{<\ell}$-definable function $g_\ell : f_\ell(b_\ell) \to b_\ell$.

Proof. Let $F = \mathrm{Stab}(\Lambda_c)$, we identify s with a coset gF for some $g \in \mathrm{B}_n$. Let $d \in \mathrm{D}_n$ and $u \in \mathrm{U}_n$ be such that $g = ud$. Note that the map $B_n \to B_n/U_n \simeq D_n$ is the projection on the diagonal coordinates. Therefore, by Proposition 3.5, the image of F in B_n/U_n is identified with $F_{\mathrm{D}} = F \cap \mathrm{D}_n$. Writing $F_{\mathrm{U}} = F \cap \mathrm{U}_n$, we thus have $F = F_{\mathrm{U}} \rtimes F_{\mathrm{D}}$. By (the proof of) [18, Lemma 11.10], $\mathrm{dcl}(s) = \mathrm{dcl}(c, \ulcorner dF_{\mathrm{D}} \urcorner, \ulcorner uF_{\mathrm{U}}^d \urcorner)$, where $F_{\mathrm{U}}^d = dF_U d^{-1} = dFd^{-1} \cap \mathrm{U}_n$ does not depend on the choice of d in dF_D. Then, by Proposition 3.5, $\mathrm{D}_n/F_{\mathrm{D}} \simeq \prod_{\ell<n} \mathrm{K}^\star/\mathcal{O}^\star_{\Delta_\ell} \simeq \prod_{\ell<n} \Gamma/\Delta_i$ and, for every $\ell < n$, we chose $b_\ell = \mathrm{v}_{\Delta_\ell}(d_\ell)$ to be the ℓ-th coordinate in this product. Note that $\mathrm{dcl}(c^\ulcorner dF_D \urcorner) = \mathrm{dcl}(cb_{<n})$.

Now, for every $\ell \geq n$, note that $F_{\mathrm{U}}^d G_\ell = G_\ell F_{\mathrm{U}}^d$ is a subgroup, since G_ℓ is normal in U_n and moreover, $F_{\mathrm{U}}^d G_{\ell+1} \trianglelefteq F_{\mathrm{U}}^d G_\ell$ since for every $g \in G_\ell$, $(F_{\mathrm{U}}^d)^g \subseteq F_{\mathrm{U}}^d G_{\ell+1}$ by centrality of the sequence. For every $\ell \geq n$, let $X_\ell = uF_{\mathrm{U}}^d G_\ell / F_{\mathrm{U}}^d G_{\ell+1}$ for the right regular action and $b_\ell = uF_{\mathrm{U}}^d G_{\ell+1} \in X_\ell$. Then b_ℓ is s-definable, $X_n = \mathrm{U}_n/F_{\mathrm{U}}^d G_{n+1}$ is $cb_{<n}$-definable and, if $\ell > n$, the set $X_\ell = b_\ell/F_{\mathrm{U}}^d G_{n+1}$ is $cb_{<n}b_{\ell-1}$-definable. Also, X_ℓ is $cb_{<\ell}$-definably a (right) torsor for the group

$$\begin{aligned} F_{\mathrm{U}}^d G_\ell / F_{\mathrm{U}}^d G_{\ell+1} &\simeq G_\ell / (F_{\mathrm{U}}^d G_{\ell+1}) \cap G_\ell \\ &\simeq G_\ell / (F_{\mathrm{U}}^d \cap G_\ell) G_{\ell+1} \\ &\simeq \mathrm{K}/p_\ell(F_{\mathrm{U}}^d \cap G_\ell) \\ &\simeq \mathrm{K}/d_i d_j^{-1}(\mathrm{I}_{c_i} : \mathrm{I}_{c_j}), \end{aligned}$$

where $\ell = (i, j)$ and the last isomorphism follows from Proposition 3.5 – these isomorphisms are $cb_{<n}$-definable. Let $\mathrm{I}_\ell = d_i d_j^{-1}(\mathrm{I}_{c_i} : \mathrm{I}_{c_j})$, then I_ℓ only depends on $b_i = v_{\Delta_i}(d_i)$ and $b_j = \mathrm{v}_{\Delta_j}(d_j)$ and thus is $b_{<n}$-definable. Since $b_{n(n+1)/2-1} = uF_U^d$, we have $s \in \mathrm{dcl}(c, b_{<n}, b_{n(n+1)/2-1})$. This concludes the first part of the proposition.

Let us now fix $a_\ell \in b_\ell$, for all $\ell < n(n+1)/2$. Note that if $\ell < n$, $v(a_\ell) = \mathrm{v}_{\Delta_\ell}(d_\ell)$ and we may assume that $d = a_{<n}$. If $\ell = n$, as we saw above $X_n = \mathrm{U}_n/F_{\mathrm{U}}^d G_{n+1}$ is $ca_{<n}$-definably isomorphic to K/I_n. If $\ell > n$, since $a_{\ell-1} \in b_{\ell-1} = uF_{\mathrm{U}}^d G_\ell$, we have $a_{\ell-1} F_{\mathrm{U}}^d G_{\ell+1} \in X_\ell$. This gives rise to a $ca_{<n}a_{\ell-1}$-definable

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isomorphism

$$f_\ell : X_\ell \simeq F_{\mathrm{U}}^d G_\ell / F_{\mathrm{U}}^d G_{\ell+1} \simeq G_\ell / (F_{\mathrm{U}}^d G_{\ell+1}) \cap G_\ell \simeq \mathrm{K}/\mathrm{I}_\ell,$$

where the first isomorphism is induced by left multiplication by $a_{\ell-1}^{-1}$ and the third by the coordinate projection p_ℓ. Let $h_\ell : \mathrm{K} \to H_\ell$ be the section of p_ℓ. Then for every $x \in f_\ell(b_\ell)$, since $p_\ell(h_\ell(x))I_\ell = f_\ell(b_\ell)$, we have $h_\ell(x) \in a_{\ell-1}^{-1} u F_{\mathrm{U}}^d G_{\ell+1}$ and hence $g(x) = a_{\ell-1}s(x) \in b_\ell$. □

Recall that $\mu_c : B_n \to \mathrm{Mod}_c$ is the canonical projection.

Remark 3.11. Looking at the proof, all the operation applied to any upper triangular matrix representation of s are actually field operations. It follows that Proposition 3.10 can be refined as follows. If $A \leq \mathrm{K}(M)$ is a subfield, then:

- if $s \in \mu_c(\mathrm{B}_n(A))$, then for every ℓ, $b_\ell(A) \neq \emptyset$;
- if, for all $k < \ell$, the set $b_k(A)$ is nonempty, then I_ℓ is a translate of a c-definable cut by an element of $\mathrm{v}(A)$, we have $f_\ell(b_\ell)(A) \neq \emptyset$ and g_ℓ sends $f_\ell(b_\ell)(A)$ to $b_\ell(A)$;
- if, for all ℓ, the set $f_\ell(b_\ell)(A)$ is nonempty, then, by induction $b_\ell(A) \neq \emptyset$ and we can choose all $a_\ell \in b_\ell(A)$ – in particular, we can choose $d = a_{<n} \in D_n(A)$ and $u \in b_{n(n+1)/2-1}(A)$, so $s \in \mu_c(\mathrm{B}_n(A))$.

We conclude this section with one of our main uses for Proposition 3.10: characterizing parameter sets over which every definable module has a (triangular) basis. Recall that $\mathrm{B_g}$ is the (ind-)interpretable set of definable generalized balls.

Corollary 3.12. *Let $A \subseteq M^{\mathrm{eq}}$ and $C \subseteq \mathrm{K}(M)$ be a subfield. Assume that:*

1. *For every A-definable convex subgroup $\Delta \leq \Gamma$, we have $(\Gamma/\Delta)(\mathrm{dcl}(A)) \subseteq \mathrm{v}_\Delta(C)$.*
2. *For every $b \in \mathrm{B_g}(\mathrm{dcl}(AC))$ whose cut is a $\mathrm{v}(C)$-translate of an A-definable cut, we have $b(C) \neq \emptyset$.*

Then, for every tuple c in Cut, $\mathrm{Mod}_c(\mathrm{dcl}(A)) \subseteq \mu_c(\mathrm{B}_{|c|}(C))$.

Proof. Let $s \in \mathrm{Mod}_c(\mathrm{dcl}(A))$ – in particular $c \in \mathrm{Cut}(\mathrm{dcl}(A))$ – and $b = (b_\ell)_\ell$ be as in Proposition 3.10. By Remark 3.11, it suffices to show that, $f_\ell(b_\ell)(C) \neq \emptyset$. For $\ell < n$, since $b_\ell \in \Gamma/\Delta_\ell(\mathrm{dcl}(A))$, this follows from the first assumption. If $\ell \geq n$, by induction, $f_\ell(b_\ell) \in \mathrm{K}/\mathrm{I}_\ell$ is an AC-definable generalized ball whose cut is a $\mathrm{v}(C)$-translate of an A-definable cut; and we conclude by the second assumption. □

3.2. The geometric sorts

The goal of this section is to further simplify the codes of modules to something more akin to the geometric sorts of [1]. This will be crucial to classify k-internal sets, when the value group is non discrete, in Corollary 6.7.

Let M be an (expansion of a) valued field in some language $\mathfrak{L}$ with stably embedded residue field.

Definition 3.13.

1. Let $\mathrm{Cut}^{\star\star}$ denote $\mathrm{Cut}^\star \setminus \{\Gamma_{>\gamma} : \gamma \in \Gamma\}$ – unless Γ is discrete, in which case, $\mathrm{Cut}^{\star\star} = \mathrm{Cut}^\star$. Any module of type a tuple c in $\mathrm{Cut}^{\star\star}$ is said to be $\mathfrak{m}$-avoiding. Let Gr be the collection of codes for all $\mathfrak{m}$-avoiding modules; that is $\mathrm{Gr} = \coprod_{c \in \mathrm{Cut}^{\star\star}} \mathrm{Mod}_c$.
2. For every $\mathcal{O}$-module R, let $\mathrm{red}(R)$ denote the k-vector space $R/\mathfrak{m}R$ and let $\mathrm{red}_R : R \to \mathrm{red}(R)$ denote the canonical projection. We also define $\mathrm{Lin} = \coprod_{R \in \mathrm{Gr}} \mathrm{red}(R)$.
3. Let $\mathcal{G} = \mathrm{K} \cup \mathrm{Gr} \cup \mathrm{Lin}$ be the (generalized) geometric sorts.

Remark 3.14. For every $c \in \mathrm{Cut}^\star$, we have $\mathfrak{m}\mathrm{I}_c = \mathrm{I}_c$ if and only if $\mathcal{C}_c \neq \Gamma_{\geq\gamma}$ for each $\gamma \in \Gamma$. Indeed, if $\mathcal{C}_c$ does not have a minimal element, then, for every $x \in \mathrm{I}_c$, there exists $a \in \mathrm{I}_c$ such that $\mathrm{v}(a) < \mathrm{v}(x)$. Then $xa^{-1} \in \mathfrak{m}$ and hence $x = xa^{-1}a \in \mathfrak{m}\mathrm{I}_c$.

It follows that if c is some tuple in $\mathrm{Cut}^\star$ and R is an $\mathcal{O}$-module of type c. Then $\mathrm{red}(R)$ has dimension $|\{i : \exists\gamma \in \Gamma,\ \mathcal{C}_{c_i} = \Gamma_{\geq\gamma}\}|$ over k.

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Lemma 3.15. *Let $R \subseteq \mathrm{K}^n$ be an $\mathcal{O}$-module of type c for some tuple c in $\mathrm{Cut}^\star$. Then there exists an $\ulcorner R \urcorner$-definable $\mathfrak{m}$-avoiding module $\overline{R}$ containing R and such that $\mathfrak{m}R = \mathfrak{m}\overline{R}$. In particular, $\mathrm{red}(R) \subseteq \mathrm{red}(\overline{R})$ is a subspace and $\mathrm{dcl}(\ulcorner R \urcorner) = \mathrm{dcl}(\ulcorner \overline{R} \urcorner, \ulcorner \mathrm{red}(R) \urcorner)$.*

Proof. Let v_R be the valuation defined in the proof of Proposition 3.3. Recall that $R = \{x \in \mathrm{K}^n : \Gamma_{\geq 0} \subseteq v_R(x)\}$ and let $\overline{R} = \{x \in \mathrm{K}^n : \Gamma_{>0} \subseteq v_R(x)\}$. If e_i is a (triangular) basis such that $R = \sum_i \mathrm{I}_{c_i} e_i$, then $\overline{R} = \sum_i \overline{I}_{c_i} e_i$.

For every $\gamma \in \Gamma$, let $\gamma\mathcal{O} = \{x \in \mathrm{K} : \mathrm{v}(x) \geq \gamma\}$ and $\gamma\mathfrak{m} = \{x \in \mathrm{K} : \mathrm{v}(x) > \gamma\}$. We have $\overline{\gamma\mathcal{O}} = \gamma\mathcal{O}$ and hence $\mathfrak{m}\overline{\gamma\mathcal{O}} = \gamma\mathfrak{m}$. If I_c is neither $\gamma\mathcal{O}$ nor $\gamma\mathfrak{m}$ for any $\gamma \in \Gamma$, then $\mathfrak{m}\overline{I}_{c_i} = \mathfrak{m}\mathrm{I}_{c_i} = \mathrm{I}_{c_i}$. Finally, if Γ is dense then $\overline{\gamma\mathfrak{m}} = \gamma\mathcal{O}$ and $\mathfrak{m}\overline{\gamma\mathfrak{m}} = \gamma\mathfrak{m}$. It follows that $\mathfrak{m}\overline{R} = \mathfrak{m}R$.

The last assertion follows from the fact that $\mathrm{red}_R^{-1}(\mathrm{red}(R)) = R$. □

Let us now recall, following [18, Lemma 2.6.4], how to code definable subspaces of $\mathrm{red}(\overline{R})$. The following abstract conditions were isolated in [19].

Proposition 3.16. *Let M be some $\mathfrak{L}$-structure, k be some stably embedded $\mathfrak{L}$-definable field and $\bigcup_s V_s$ be a collection of finite-dimensional $\mathfrak{L}$-definable k-vector spaces which*

- *is closed under tensors: for every s, r, there is an $\mathfrak{L}$-definable injection from the interpretable set $V_s \otimes V_r$ into some V_t;*
- *is closed under duals: for every s, there is an $\mathfrak{L}$-definable injection from the interpretable set $V_s^\vee$ into some V_t;*
- *has flags: For every s, there exists r, t and a $\mathfrak{L}$-definable exact sequence $0 \to V_r \to V_s \to V_t \to 0$, with $\dim(V_r) = 1$.*

Then any definable subspace $W \subseteq V_s$, is coded in $\bigcup_s V_s$.

Proof. By (the proof of) [19, Proposition 5.2], W is coded in some projective space $\mathbb{P}(V_r)$. By [19, Lemma 5.6], given that the family has flags, $\mathbb{P}(V_r)$ is coded in $\bigcup_s V_s$. □

We consider once again an (expansion of a) valued field M in some language $\mathfrak{L}$.

Definition 3.17. For every $A \subseteq M^{\mathrm{eq}}$, let $\mathrm{Lin}_A = \coprod_{s \in \mathrm{Gr}(\mathrm{acl}(A))} \mathrm{red}(R_s)$ with its A-induced structure.

Before we prove that Proposition 3.16 can be applied to Lin_A, let us prove the following useful computation:

Lemma 3.18. *Let $I, J \leq \mathrm{K}$ be (M-definable) $\mathcal{O}$-submodules. If $(I : J) = \mathfrak{m}$, then $I = a\mathfrak{m}$ and $J = a\mathcal{O}$, for some $a \in \mathrm{K}$.*

Proof. We first argue that $\mathrm{v}(I) \subsetneq \mathrm{v}(J)$. If $\mathrm{v}(J) \subseteq \mathrm{v}(I)$, then $J \subseteq I$ so $\mathrm{Stab}(J) \subseteq (I : J)$, where $\mathrm{Stab}(J) = \{x \in K : xJ = J\}$. Since $\mathrm{Stab}(J) = \mathcal{O}_\Delta^\times$ for some definable convex subgroup Δ of the value group, and $\mathfrak{m}$ does not contain $\mathcal{O}_\Delta^\times$ then $\mathrm{v}(I) \subsetneq \mathrm{v}(J)$.

We aim to show that $\mathrm{v}(J)$ has a minimal element. Otherwise, given $\gamma \in \mathrm{v}(J) \backslash \mathrm{v}(I)$ there is some $\beta \in \mathrm{v}(J)$ such that $\beta < \gamma$, thus $\gamma = \beta + \delta$ where $\delta = \gamma - \beta > 0$. Take $x \in \mathfrak{m}$ and $y \in J$ such that $v(x) = \delta$ and $v(y) = \beta$. Consequently, $xy \notin I$ since $v(xy) = \gamma$, so $(I : J) \neq \mathfrak{m}$. Let γ_0 be the minimal element of $\mathrm{v}(J)$ and $a \in K$ such that $v(a) = \gamma_0$ thus $J = a\mathcal{O}$. To show that $I = a\mathfrak{m}$ it is sufficient to argue that $\mathrm{v}(J) \backslash \mathrm{v}(I) = \{\gamma_0\}$. If there is some $\beta \in \mathrm{v}(J) \backslash \mathrm{v}(I)$ such that $\beta \neq \gamma_0$, then $\beta - \gamma_0 > 0$. Take $x \in \mathfrak{m}$ such that $v(x) = \beta - \gamma_0$, then $x \in \mathfrak{m}$ but $ax \notin I$, hence $(J : I) \neq \mathfrak{m}$. Thus, $I = a\mathfrak{m}$, as required. □

Proposition 3.19. *For every $A \subseteq M^{\mathrm{eq}}$, Lin_A is a collection of finite dimensional k-vector spaces which is closed under tensors, duals and has flags.*

Proof. Let $R_1 \subseteq \mathrm{K}^n$ and $R_2 \subseteq \mathrm{K}^m$ be two $\mathrm{acl}(A)$-definable $\mathfrak{m}$-avoiding $\mathcal{O}$-modules. Let $f : \mathrm{K}^n \otimes \mathrm{K}^n \to \mathrm{K}^{nm}$ be the $\mathfrak{L}$-definable isomorphism induced by the canonical basis. By Lemma 3.15, we find an A-definable $\mathfrak{m}$-avoiding $\mathcal{O}$-module $R = \overline{f(R_1 \otimes R_2)}$ inducing an inclusion $\mathrm{red}(f(R_1 \otimes R_2)) \subseteq \mathrm{red}(R)$.

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Since $\mathrm{red}(R_1) \otimes \mathrm{red}(R_2)$ is A-definably isomorphic to $\mathrm{red}(R_1 \otimes R_2)$, we conclude that Lin_A is closed under tensors.

As for duals, for every $\mathcal{O}$-submodule $I \subseteq \mathrm{K}$, let $I^\vee = (\mathcal{O} : I) = \{x \in \mathrm{K} : xI \subseteq \mathcal{O}\}$. By Lemma 3.18, $I^\vee \neq \mathfrak{m}$. Let $g : (\mathrm{K}^n)^\vee \to \mathrm{K}^n$ be the $\mathfrak{L}$-definable isomorphism induced by the canonical basis composed with the map $\mathrm{K}^n \to \mathrm{K}^n$ reversing the order of the coordinates and let $(a_i^\vee)_i$ denote the dual basis. Then, as we reversed the order, $g(a_i^\vee)$ is an upper triangular basis. If $R = \sum_i \mathrm{I}_{c_i} a_i$ for some triangular basis $(a_i)_i$, then $g(\mathrm{Hom}_{\mathcal{O}}(R_1, \mathcal{O})) = \sum_i \mathrm{I}^\vee_{c_i} g(a_i^\vee)$, which is $\mathfrak{m}$-avoiding. This induces an A-definable isomorphism $\mathrm{red}(\mathrm{Hom}_{\mathcal{O}}(R_1, \mathcal{O})) \simeq \mathrm{red}(g(\mathrm{Hom}_{\mathcal{O}}(R_1, \mathcal{O})))$ showing that Lin_A is closed under duals.

Finally, regarding flags, let $R = \sum_i \mathrm{I}_{c_i} a_i \subset \mathrm{K}^n$ be an $\mathrm{acl}(A)$-definable $\mathfrak{m}$-avoiding $\mathcal{O}$-module, with a triangular. We find a flag for $\mathrm{red}(R)$ by induction on n. Let π be the projection on the last $n-1$ variables. Then, we have a split A-definable short exact sequence

$$0 \to \mathrm{I}_{c_0} a_0 \to R \to \pi(R) \to 0$$

which induces the following A-definable short exact sequence

$$0 \to \mathrm{I}_{c_0} a_0 / \mathfrak{m}\mathrm{I}_{c_0} a_0 \to R/\mathfrak{m}R \to \pi(R)/\mathfrak{m}\pi(R) \to 0.$$

If $\mathrm{I}_{c_0} = \mathfrak{m}\mathrm{I}_{c_0}$, then $\mathrm{red}(R) \simeq \mathrm{red}(\pi(R))$ and we conclude by induction on n. If not, $\mathrm{I}_{c_0} a_0 / \mathfrak{m}\mathrm{I}_{c_0} a_0$ is a dimension one k-vector space, and the above short exact sequence is a flag for $\mathrm{red}(R)$. □

Remark 3.20. Fix some $A \subseteq M^{\mathrm{eq}}$.

1. Recall that we assumed that k is stably embedded in M. It follows that Lin_A is stably embedded and its A-induced structure is definable with parameters in the structure with the $\mathfrak{L}$-induced structure on k and the vector space structure on each sort $R_s/\mathfrak{m}R_s$. Indeed, once we name a basis of every vector space in Lin_A, every definable subset in Lin_A can be identified with a definable subset in k. Similarly, if RV is stably embedded, so is $\mathrm{Lin}_A \cup \mathrm{RV}$.
2. Whenever k is a pure algebraically closed field, combining [19, Lemma 5.6] with Proposition 3.19, Lin_A, with its A-induced structure, eliminates imaginaries.

We can now improve Corollary 3.7:

Corollary 3.21. *Any M-definable $\mathcal{O}$-submodule R of type $c \in \mathrm{Cut}^\star$ is coded in $\mathcal{G}$.*

Conversely, any element $a + \mathfrak{m}s \in \mathrm{Lin}$ is coded by the $\mathcal{O}$-submodule generated by $(a + \mathfrak{m}s) \times \{1\}$. So any element of $\mathcal{G}$ is coded in $\mathrm{K} \cup \mathrm{Mod}$.

Proof. By Lemma 3.15, for some $\mathfrak{m}$-avoiding module $\overline{R}$ containing R such that $\mathfrak{m}R = \mathfrak{m}\overline{R}$, the lattice R is coded by $\ulcorner \overline{R} \urcorner \in \mathrm{Gr}$ and $\ulcorner \mathrm{red}(R) \urcorner$ where $\mathrm{red}(R) \leq \mathrm{red}(\overline{R})$ is a subspace. By Propositions 3.16 and 3.19, $\mathrm{red}(R)$ is coded in $\mathrm{Lin}_{\ulcorner R \urcorner} \subseteq \mathrm{Lin}$. □

One of the main reason for isolating the $\mathfrak{m}$-avoiding modules is the following result.

Proposition 3.22. *Let M be an* RV*-expansion of a model of* $\mathrm{Hen}_{0,0}$ *with stably embedded dense value group. If $X \subseteq \mathrm{Gr}$ is M-definable and orthogonal to Γ – meaning that any function $f : X \to \Gamma^{\mathrm{eq}}$ definable with parameters has finite image – then X is finite.*

Proof. Let us first consider the case of some M-definable $X \subseteq \mathrm{K}/\mathrm{I}_c$ for some $c \in \mathrm{Cut}^{\star\star}$. Let $Y \subseteq \mathrm{K}$ be the preimage of X. By Proposition 2.5, there exists a (nonempty) finite set $C \subseteq \mathrm{K}(M)$ such that, any ball b disjoint from C, is either contained in Y or is disjoint from it. If X is infinite, then there exists some $a \in Y$ such that $a + \mathrm{I}_c$ is disjoint from C. Let b be the maximal ball around a that is disjoint from C – that is, the ball $a + (a-c)\mathfrak{m}$ where $v(a-c) = \min_{c \in C} \mathrm{v}(a-c)$. Then $b \subseteq Y$ and b strictly contains b as $\mathrm{I}_c \neq d\mathfrak{m}$, for any d. It follows that the function $f : x \mapsto \mathrm{v}(x-a)$ induces a well-defined function on $(b \setminus \{a + \mathrm{I}_c\})/\mathrm{I}_c \subseteq X$ with infinite image, contradicting our hypothesis.

Let us now fix some $e \in X \subseteq \mathrm{Gr}$, in some elementary extension of M. Let b_ℓ be as in Proposition 3.10 and let us prove, by induction on ℓ, that $b_\ell \in M^{\mathrm{eq}}$. For all $\ell < n$, we have $b_\ell \in \Gamma/\Delta_\ell$, for some convex

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subgroup Δ_ℓ. Since $e \in X$ and X is orthogonal to Γ and $b_\ell \in \mathrm{dcl}(e)$, we must have $b_\ell \in M^{\mathrm{eq}}$. Let now $\ell \geq n$ and let us assume that $b_{<\ell} \in M$. We have $b_\ell \in X_\ell$ which is an M-definable torsor for some $\mathrm{K}/\mathrm{I}_\ell$ where, by Lemma 3.18, I_ℓ is not a multiple of $\mathfrak{m}$. Since $b_i \in \mathrm{dcl}(e)$, it is contained is an M-definable subset of $\mathrm{K}/\mathrm{I}_\ell$ which is orthogonal to Γ, and hence, by the first paragraph, finite. So $b_\ell \in M^{\mathrm{eq}}$.

Since $e \in \mathrm{dcl}(Mb)$, it follows that $e \in M^{\mathrm{eq}}$. As this holds for any $e \in X$ in some elementary extension, X is finite. □

Recall that a definable set X is (resp. almost) internal to another definable set Y if, over a model, X admits a one-to-one (resp. finite-to-one) map to Y^{eq}.

Corollary 3.23. *Let M be an* RV*-expansion of a model of* $\mathrm{Hen}_{0,0}$ *with dense value group. Assume that* k *and* Γ *are stably embedded and orthogonal. Let $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$ and let X be an almost* k*-internal A-definable subset of (some cartesian product of) $\mathcal{G}$, then $X \subseteq \mathrm{K}(A) \cup \mathrm{Gr}(A) \cup \mathrm{Lin}_A$.*

Proof. Any almost k-internal set is orthogonal to Γ. By Proposition 2.5, any infinite definable subset of K contains a ball and hence is not orthogonal to Γ, so if $X \subseteq \mathrm{K}$, then $X \subseteq \mathrm{K}(A)$. By Proposition 3.22, if $X \subseteq \mathrm{Gr}$, then $X \subseteq \mathrm{Gr}(A)$. Finally, if $X \subseteq \mathrm{Lin}$, then the projection of X to Gr is finite and hence $X \subset \coprod_{s \in \mathrm{Gr}(A)} \mathrm{red}(R_s) = \mathrm{Lin}_A$. □

Note that it is necessary to assume that $\Gamma(M)$ is dense. The pure valued field $\mathbb{C}((t))$ eliminates imaginaries down to $\mathcal{G}$. However, the interpretable set $\mathcal{O}/t^2\mathcal{O}$ is orthogonal to Γ (in fact, it is k-analyzable), but it is not k-internal; in particular, it cannot be a subset of $\mathrm{K}(A) \cup \mathrm{Gr}(A) \cup \mathrm{Lin}_A$.

4. Density of quantifier free definable types

Let M be a sufficiently saturated and homogeneous RV-expansion of a model of $\mathrm{Hen}_{0,0}$ with strongly stably embedded Γ (*cf.* Proposition 2.2). We realize types over $\mathbb{M}$ in some a sufficiently saturated and homogeneous $\mathbb{M} \succ M$. Let $M_1 = M^{\mathrm{ur}}$ be its maximal algebraic unramified extension (with the $\mathfrak{L}$-induced structure on Γ) and $M_0 = M^{\mathrm{a}}$ be its algebraic closure (as a pure valued field). Note that, by quantifier elimination, we have $M_0 \prec \mathbb{M}^{\mathrm{a}}$ and $M_1 \prec \mathbb{M}^{\mathrm{ur}}$.

In what follows, whenever we want to refer to the structure in $M_0 = M^{\mathrm{a}}$ (resp. $M_1 = M^{\mathrm{ur}}$, resp. M), we will indicate this by a 0 (resp. 1, resp. nothing): for example, acl_0, acl_1, or acl for the algebraic closure and $\mathrm{S}^0(M)$ for the space of types in M_0 over M.

We also assume that the language $\mathfrak{L}_i$ of M_i is morleyized, and we restrict ourselves to quantifier free $\mathfrak{L}_i$-formulas when interpreting them in a substructure. Note that it follows from quantifier elimination in M_1 (Theorem 2.7) that the trace on M of any $\mathfrak{L}_1$-definable set in M_1 is $\mathfrak{L}$-definable. Similarly, for $\mathfrak{L}_0$-definable sets. We also fix a sufficiently saturated extension of the triple (M_0, M_1, M) in which we realize types.

The goal of this section is to prove the following density result:

Theorem 4.1. *Assume that $\Gamma(M)$ satisfies Property* **D**. *Let $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$ and $X \subseteq \mathrm{K}^n$ be nonempty and $\mathfrak{L}(A)$-definable in M. Then there exists an $\mathfrak{L}(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable type $p \in \mathrm{S}^1(M)$ consistent with X.*

This statement was proved in [7, Theorem 5.9] when $M = M^{\mathrm{ur}}$ and Γ is an abelian ordered group of bounded regular rank. It also strengthens [6, Theorem 3.1.3] in two ways. The first is that it provides a definable type in a stronger reduct (i.e., a type in M^{ur} and not just in M^{a}). The second is that there is no hypothesis on the residue field: it is not required that the residue field eliminates $\exists^\infty$.

4.1. Codes of definable types

We start by proving the following useful fact allowing to compare types in M, $M_1 = M^{\mathrm{ur}}$ and $M_0 = M^{\mathrm{a}}$.

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Remark 4.2. Let $c = (c_1, \dots, c_n)$ be a finite tuple in Cut^{**}. Consider the natural map:

$$\mathrm{B}_n(M)/\mathrm{Stab}(\Lambda_c)(M) \to \mathrm{B}_n(M_1)/\mathrm{Stab}(\Lambda_c)(M_1).$$

So the map above identifies the code of the $\mathcal{O}(M)$-module $R = \sum_{i\leq n} a_i I_{c_i}(M)$ in M with the code of the $\mathcal{O}(M_1)$-module $R_1 = \sum_{i\leq n} a_i I_{c_i}(M_1) = \sum_{i\leq n} a_i \mathcal{O}(M_1)\cdot I_{c_i}(M)$ generated by R. Moreover, we have $R_1(M) = R$.

Recall the Definition 1.1 of a definable Δ-type.

Proposition 4.3. *Let $A = \mathrm{dcl}(A) \subseteq M^{\mathrm{eq}}$ and $p(x) \in \mathrm{S}^{\varepsilon}_{\mathrm{K}^{|x|}}(M)$ be finitely satisfiable in M and $\mathfrak{Q}(A)$-definable, for $\varepsilon = 0$ or 1. Then p has a unique extension $q_\varepsilon \in \mathrm{S}^{\varepsilon}(M_\varepsilon)$. Moreover, q_1 is $\mathfrak{Q}_1(\mathcal{G}(A)\cup\Gamma^{\mathrm{eq}}(A))$-definable and $q_0|_{M_1}$ is $\mathfrak{Q}_1(\mathcal{G}(A))$-definable.*

We follow the proof of [7, Theorem 5.9], *mutatis mutandis*.

Proof. The uniqueness of q_ε follows from [6, Lemma 3.3.7]; note that q_ε is finitely satisfiable in M.

For every integer $d \geq 0$, let $V_d \simeq \mathrm{K}^\ell$ be the space of polynomials in $\mathrm{K}(M)[x]_{\leq d}$ of degree less or equal than d (in each x_i). It comes with an $\mathfrak{Q}(A)$-definable valuation defined by $v(P) \leq v(Q)$ if $p(x) \vdash \mathrm{v}(P(x)) \leq \mathrm{v}(Q(x))$.

By Proposition 3.3 there exists a separated basis $(P_i)_{i\leq \ell} \in V_d$ such that, for every i, $\gamma_i = v(P_i) \in \mathrm{dcl}(\ulcorner v \urcorner) \subseteq A$. By [6, Claim 3.3.5], $(P_i)_i$ is also a separated basis of $V_d^1 = \mathrm{K}(M_1)[x]_{\leq d}$ with the valuation v_1 where $v_1(P) \leq v_1(Q)$ if $q_1(x) \vdash \mathrm{v}(P(x)) \leq \mathrm{v}(Q(x))$. It follows that $v_1(V_d^1) = v(V_d) = \bigcup_i \gamma_i + \Gamma(M)$ which we identify in Γ^{eq} over $\mathcal{G}(A)$ with disjoint copies of Γ.

By Corollary 3.7 the definable $\mathcal{O}$-modules $R_i = \{P \in V_d : v(P) \geq \gamma_i\}$ are coded by some tuple e_i in $\mathrm{Mod}(A)$. We identify e_i with the code $e_i^1 \in \mathrm{Mod}^1(M)$ of $R_i^1 = \{P \in V_d^1 : v_1(P) \geq \gamma_i\}$ via the map in Remark 4.2. Furthermore, the R_i^1 entirely determine v_1 which is therefore coded in $\mathcal{G}(A)$ (*cf.* Corollary 3.21). Since $q_0|_{M_1}$ is entirely determined by the valuations v_1 on V_d^1, the proposition is proved in that case.

To conclude, let us prove the definability of q_1. By Theorem 2.7, any $\mathfrak{Q}_1$-formula $\phi(x,y)$ (with variables in K) is equivalent to one of the form $\psi(\mathrm{v}(P(x,y)))$, where $P \in \mathbb{Z}[x,y]$ is a tuple. Let $X_\phi = \{v_1(P(x,a)) \in v_1(V_d^1) : q_1(x) \vdash \psi(\mathrm{v}(P(x,a)))\}$. Note that if $v_1(P(x,b)) = v_1(P(x,a))$, then $q_1(x) \vdash \mathrm{v}(P(x,b)) = \mathrm{v}(P(x,a))$ and hence $q_1(x) \vdash \psi(\mathrm{v}(P(x,b)))$ if and only if $v_1(P(x,b)) \in X_\phi$. So it suffices to show that the X_ϕ are $\mathfrak{Q}_1(\mathcal{G}(A)\cup\Gamma^{\mathrm{eq}}(A))$-definable, but this follows immediately from the identification $v_1(V_d^1) = \bigcup_i \gamma_i + \Gamma(M)$. □

From now on, we will identify $\mathfrak{Q}(A)$-definable types in $\mathrm{S}^1(M)$ that are finitely satisfiable in M with their unique extension to $M_1 = M^{\mathrm{ur}}$.

We now note that coding definable $\mathfrak{Q}_0$-types already allows us to code some imaginaries, namely certain germs of functions into the space of balls – see Section 2.1 for the definition of germs.

Definition 4.4. Let B be a chain of balls in M_0 (including points and K itself). We define the generic η_B of the generalized ball $\bigcap_{b\in B} b$, to be the $\mathfrak{Q}_0(M_0)$-type generated by:

$$\{x \in b : b \in B\} \cup \{x \notin b' : b' \text{ is a ball in } M_0 \text{ and } \forall b \in B,\ b' \text{ is a strict subball of } b\}.$$

If b is a generalized ball, we write η_b for the type η_B where B is the set of balls containing b. If b is a definable generalized ball in M, $\eta_b|_M(x)$ is an $\mathfrak{Q}(\ulcorner b \urcorner)$-definable type (finitely) consistent with $x \in b$.

Lemma 4.5. *Assume that $\Gamma(M)$ is dense. Let $a \in \mathrm{K}(N)$, for some $N \succ M$, be such that $p_0(x) = \mathrm{tp}_0(a/M)$ is $\mathfrak{Q}(M)$-definable. Let $b(a)$ be an $\mathfrak{Q}_0(Ma)$-definable open ball whose radius is in $\Gamma(N)$ – where b implicitly denotes some $\mathfrak{Q}_0(M)$-definable map. Then the germ $[b]_{p_0}$ is coded in $\mathcal{G}(M)$ over $\mathcal{G}(\ulcorner p_0 \urcorner)$.*

Proof. We may assume that N^{a} is $|M|^+$-saturated. Let $q_0(x,y)$ be the $\mathfrak{Q}(M)$-definable $\mathfrak{Q}_0(M)$-type of ac where c is generic in $b(a)$ over Ma – that is $c \models \eta_b|_{Ma}$. Note that, since we are in equicharacteristic

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zero, $b(a)$ has a point in N and, in fact, since $b(a)$ is open and $\Gamma(N)$ is dense, the generic of $b(a)$ in N^a is satisfiable in N. It follows that q_0 is finitely satisfiable in M.

By Proposition 4.3, p_0 and q_0 are coded in $\mathcal{G}(M)$. Moreover, for every $\sigma \in \mathrm{Aut}(M/\ulcorner p \urcorner)$ and $ac \models q_0$, we have $\sigma(q_0) = q_0$ if and only if $b(a) = b^\sigma(a)$. So $[b]_p$ is coded by $\mathcal{G}(\ulcorner q_0 \urcorner)$ over $\mathcal{G}(\ulcorner p_0 \urcorner)$. □

4.2. Unary subsets of K

We first consider the case of Theorem 4.1 when $X \subseteq \mathrm{K}$. The proof proceeds as in [7, Theorem 5.3] where, in the unary case, the hypothesis that $M = M^{\mathrm{ur}}$ is not used.

Lemma 4.6. *Let $A = \mathrm{acl}(A) \subseteq M$ and $X \subseteq \mathrm{K}$ be $\mathfrak{L}(A)$-definable in M. There exists an $\mathfrak{L}_1(\mathrm{B_g}(A))$-definable chain of balls E in M_1 such that the $\mathfrak{L}_1(\mathrm{B_g}(A))$-definable type $\eta_E|_{M_1} \in \mathrm{S}^0(M_1)$ is consistent with X.*

This follows from [6, Section 3] which states a relative version of that statement. However, since the machinery set up for the relative version of the statement is rather heavy, let us sketch a proof. A version of this proof can also be found in [7, Theorem 5.5].

Proof. Let $\mathrm{B}(M_1)$ be the set of all open and closed balls (including points and K itself) in M_1. Given $b_1, b_2 \in \mathrm{B}$ we write $b_1 \trianglelefteq b_2$ if $b_1 \cap X \subseteq b_2 \cap X$. This is a preorder with associated equivalence $\equiv$ and the associated order is a tree $\mathcal{T}$ if we remove the class $E_\emptyset$ of balls that don't intersect X.

Note that any $\equiv$-class $E \neq E_\emptyset$, the generalized ball $b_E = \bigcap_{b \in E} b$ is defined by knowing a point in b_E and the set $\{\mathrm{v}(x - y) : x, y \in b_E\}$ which is definable in $\Gamma(M) = \Gamma(M_1)$. So b_E and E are $\mathfrak{L}_1(M)$-definable. It follows that E is coded in $\mathrm{B_g}(M)$ and that the generic type $\eta_E|_{M_1}(x) \in \mathrm{S}^0(M_1)$ is $\mathfrak{L}_1(\ulcorner E \urcorner)$-definable. If the type η_E is not consistent with X, by compactness, $X \cap b_E$ is covered by finitely many disjoint balls of M_0. But then the smallest ball b_0 in M_0 containing $X \cap b_E$ is closed with radius in $\Gamma(M) = \Gamma(M_1)$ and $X \cap b_E$ is covered by finitely many maximal open subballs of b_0 which are indeed balls of M_1. It follows that E has finitely many direct predecessors for $\trianglelefteq$, each of them in $\mathrm{acl}(A, \ulcorner E \urcorner)$.

Starting with the class of K and proceeding by induction, either the lemma holds or the tree $\mathcal{T}$ has an initial infinite discrete finitely branching tree. All the elements of this initial tree are $\mathfrak{L}_1(\mathrm{B_g}(A))$-definable $\equiv$-classes.

By Proposition 2.5, there exists a finite set $C \subseteq \mathrm{K}(M)$ preparing X and can find an $\equiv$-class E in the infinite initial discrete tree such that $b_E \cap C = \emptyset$. Then $b_E \subseteq X$ and hence X is consistent with the $\mathfrak{L}_1(\mathrm{B_g}(A))$-definable type $\eta_E|_{M_1}$. □

Let us now show that the type $\eta_E|_{M_1}$ can then be completed to an $\mathfrak{L}_1(\mathrm{B_g}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable type $p_1 \in \mathrm{S}^1(M_1)$ consistent with X:

Lemma 4.7. *Assume that $\Gamma(M)$ satisfies Property* **D**. *Let $A = \mathrm{dcl}(A) \subseteq M^{\mathrm{eq}}$, let X be $\mathfrak{L}(A)$-definable and let E be an $\mathfrak{L}_1(\mathrm{B_g}(A))$-definable chain of balls in M_1 such that $\eta_E|_{M_1}$ is consistent with X. Then, there exists an $\mathfrak{L}_1(\mathrm{B_g}(A) \cup \Gamma^{\mathrm{eq}}(\mathrm{acl}(A)))$-definable type $p \in \mathrm{S}^1(M_1)$ containing $\eta_E|_{M_1}$ and consistent with X.*

Proof. Let $b_E = \bigcap_{b \in E} b$ and $c \in X$ realize $\eta_E|_{M_1}$. Then $\gamma = \mathrm{v}(c - a)$ does not depend on $a \in b_E(M_1)$. If $\gamma \in \Gamma(M)$, then b_E is a closed ball and $\eta_E|_{M_1}$ generates a complete type over M_1.

If not, let C be the $\mathfrak{L}_1(\Gamma^{\mathrm{eq}}(A))$-definable cut of γ over $\Gamma(M_1)$. For every $a \in b_E(M_1)$, let $Y_a = \mathrm{v}(X - a) = \{v(x - a) : x \in x\}$ – it contains γ. Let $\eta_C(x) = \{x < \gamma : \gamma \in C\} \cup \{x > \gamma : \gamma \notin C\}$ be the $\mathfrak{L}_1(\Gamma^{\mathrm{eq}}(A))$-definable generic type of C over $\Gamma(M_1)$ and let $r(x) = \eta_C(x) \cup \{x \in Y_a : a \in b_E(M_1)\}$. Then r is $\mathfrak{L}_1(\Gamma^{\mathrm{eq}}(A))$-definable and $\gamma \models r$. By property **D**, r is contained an $\mathfrak{L}_1(\Gamma^{\mathrm{eq}}(A))$-definable complete type $q(x)$ over $\Gamma(M_1)$.

Let $\delta \models q$ and fix some $a \in b_E(M_1)$. Since $\delta \in Y_a$, there exists $c \in X$ such that $\mathrm{v}(c - a) = \delta$, and since $\delta \models \eta_C$, we have $c \models \eta_E|_{M_1}$. So the type $p(x) = \eta_E|_{M_1}(x) \cup q(\mathrm{v}(x - a))$ is consistent with X. It is complete by Theorem 2.7, and it is $\mathfrak{L}_1(\mathrm{B_g}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable. □

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Corollary 4.8. *Assume that* $\Gamma(M)$ *satisfies Property* **D**. *Let* $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$. *Then any* $\mathfrak{L}(A)$*-definable subset of* K *is consistent with an* $\mathfrak{L}(\mathrm{B_g}(A) \cup \Gamma^{\mathrm{eq}}(A))$*-definable type* $p \in \mathrm{S}^1(M)$.

4.3. Germs of functions

To prove density of definable types in general, we now wish to proceed by transitivity. However, since we are working with definable $\mathfrak{L}_1$-types, we first need to address the potential difference between acl and acl_1. For every tuple a in M^{eq}, let a_{K} enumerate $a \cap \mathrm{K}$.

Proposition 4.9. *Assume that* Γ *has property* **D**. *Let* $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$ *and let* X *be pro-* $\mathfrak{L}(A)$*-definable in* M *in the sorts* $\mathcal{G} \cup \Gamma^{\mathrm{eq}}$. *Let* $p(x)$ *be an infinitary* $\mathfrak{L}(A)$*-definable* $\mathfrak{L}_1(M)$*-type which is consistent with* X. *Assume that for any* $a \models p$, *we have* $a \subseteq \mathrm{acl}_1(Ma_{\mathrm{K}})$. *Then, for every* $\mathfrak{L}(A)$*-definable one-to-finite correspondence* F *into* $\mathcal{G} \cup \Gamma^{\mathrm{eq}}$ *defined at all realizations of* p *in* X, *there exists an* $\mathfrak{L}(A)$*-definable* $q(xy) \in \mathrm{S}^1(M)$ *containing* $p(x)$ *and there exists* $ac \models q$ *with* $a \in X$ *such that* $c \in \mathrm{acl}_1(Ma_{\mathrm{K}}c_{\mathrm{K}})$ *and* $F(a) \cap \mathrm{dcl}_1(ac) \neq \emptyset$.

Note that if $a \in \mathcal{G}(M) \cup \Gamma^{\mathrm{eq}}(M)$, then as the induced $\mathfrak{L}_1$-structure on M is $\mathfrak{L}$-definable, then $\mathrm{acl}_1(a) \cap (\mathcal{G}(M) \cup \Gamma^{\mathrm{eq}}(M)) \subseteq \mathrm{acl}(a)$. Furthermore, if $a \subseteq \mathrm{K}(M)$, $\mathrm{K}(\mathrm{acl}(a)) = \mathbb{Q}(a)^{\mathrm{a}} \cap \mathrm{K}(M) \subseteq \mathrm{K}(\mathrm{acl}_1(a))$.

Proof. If the proposition holds for F, then for some $\mathfrak{L}_1$-definable map, we have $f(ac) \in F(a)$. So replacing p by q and X by $x \in X \wedge f(xy) \in F(x)$, the hypothesis of the proposition stills holds and we may assume that if $a \in X$ realizes p then $F(a) \cap \mathrm{dcl}_1(a) \neq \emptyset$. The proof now proceeds by proving the proposition for various specific F and changing p and X as above, until we exhaust all possible F. To do so, it suffices, by Corollary 3.12, to add a point to a_K in every Aa-algebraic generalized ball and a lift of every element in $\Gamma/\Delta(\mathrm{dcl}(Aa))$, for any convex $\Delta \leq \Gamma$.

We first consider the case where F is almost definable in M_1.

Claim 4.9.1. If there exists an $\mathfrak{L}_1(M)$-definable one-to-finite correspondence G such that $p(x) \wedge x \in X \vdash F(x) \subseteq G(x)$, then the proposition holds.

Proof. Since $p \in \mathrm{S}^1(M)$, the cardinal of $G(x)$ is constant when x varies over realizations of p; and we may assume that it is minimal. Then for every other such G', we have $p(x) \wedge x \in X \vdash F(x) \subseteq G(x) \cap G'(x)$ and hence $p(x) \wedge x \in X \vdash G(x) = G'(x)$. In other words, for some (and hence for every) $a \models p$, $G(a) = G'(a)$. This holds in particular of any $G' = \sigma(G)$, where $\sigma \in \mathrm{Aut}(M/A)$ and thus $[G]_p \in \mathrm{dcl}(A) = A$ – recall that we assumed M to be sufficiently saturated and homogeneous. Let $G_0 \subseteq G$ be minimal $\mathfrak{L}_1(M)$-definable such that $p(x) \vdash \emptyset \neq G_0(x) \subseteq G(x)$. Then $[G_0]_p \in \mathrm{acl}(A) = A$. The type $q(xy) = p(x) \wedge y \in G_0(x)$ is complete as cardinality of G_0 is minimal and it is $\mathfrak{L}(M)$-definable and $\mathrm{Aut}(M/A)$-invariant; so it is $\mathfrak{L}(A)$-definable. By construction, for any $ac \models q$, we have $c \in G_0(a) \subseteq \mathrm{acl}_1(Ma) \subseteq \mathrm{acl}_1(Ma_K)$. Moreover, if $q(xy)$ is not consistent with $x \in X \wedge y \in F(x)$, then $p(x) \wedge x \in X \vdash F(x) \subseteq G(x) \setminus G_0(x)$, contradicting the minimality of G. So the type q is as required. □

Let us now assume that the co-domain of F is Γ^{eq}. By strong stable embeddedness of Γ and Lemma 2.3, for every $a \models p$,

$$\Gamma^{\mathrm{eq}}(\mathrm{acl}(Aa)) \subseteq \Gamma^{\mathrm{eq}}(\mathrm{acl}(Ma)) = \Gamma^{\mathrm{eq}}(\mathrm{acl}(Ma_{\mathrm{K}})) \subseteq \mathrm{acl}_1(\mathrm{v}(M(a_{\mathrm{K}})))$$

and hence, by compactness, there exists an $\mathfrak{L}_1(M)$-definable one-to-finite correspondence G such that $p(x) \vdash F(x) \subseteq G(x)$. We now conclude with Claim 4.9.1. As indicated at the start of the proof, we may therefore assume that a contains all of $\Gamma^{\mathrm{eq}}(\mathrm{acl}(Aa))$.

Claim 4.9.2. Let $b : p \to \mathrm{B_g}$ be $\mathfrak{L}_1$-definable. Then there exists an $\mathfrak{L}(A)$-definable type $q(xy) \in \mathrm{S}^1(M)$ finitely satisfiable in M containing $p(x) \cup \{y \models \eta_{b(x)}|_{Mx}\}$.

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Proof. Let $a \models p$. Since $\Gamma^{\mathrm{eq}}(\mathrm{acl}_1(b(a))) \subseteq \Gamma^{\mathrm{eq}}(\mathrm{acl}_1(a)) \subseteq \Gamma^{\mathrm{eq}}(\mathrm{acl}(a)) \subseteq a$, by Lemma 4.7 applied with M elementary extension of M_1 containing a, there exists an $\mathfrak{L}_1(a)$-definable type $r_a(y)$ containing $\eta_{b(a)}(y)$ and $y \in b(a)$. Let $c \models r_a|_{Ma}$. Then $\mathrm{tp}_1(ac/M)$ is as required. □

For every $\mathfrak{L}(Aa)$-definable convex subgroup $\Delta \leq \Gamma$ and $\gamma \in (\Gamma/\Delta)(\mathrm{acl}(Aa)) \subseteq a$, the set $\mathrm{v}_\Delta^{-1}([\gamma, \infty])$ is an $\mathfrak{L}_1(a)$-definable generalized ball. Let q be as in Claim 4.9.2 applied to any $\mathfrak{L}$-definable function b such that $b(a) = \mathrm{v}_\Delta^{-1}([\gamma, \infty])$. If $ac \models q$, we have $\mathrm{v}_\Delta(c) = \gamma$. Replacing p by q, we may assume that $\mathrm{v}_\Delta(a_{\mathrm{K}})$ contains all of $(\Gamma/\Delta)(\mathrm{acl}(Aa))$.

If the co-domain of F is k, let $\eta_k(y) \in \mathrm{S}^1(M)$ be the $\mathfrak{L}$-definable generic of k – that is, the only nonalgebraic type concentrating on k. If $p(x) \otimes \eta_{\mathrm{k}}(y)$ is not consistent with $x \in X \wedge y \in F(y)$, then there exists an $\mathfrak{L}_1(M)$-definable one-to-finite correspondence G such that $p(x) \wedge x \in X \vdash F(x) \subseteq G(x)$, by Claim 4.9.1 and Claim 4.9.2, we find $q(xy)$ consistent with $x \in X \wedge \mathrm{res}(y) \in F(x)$. On the other hand, if $p(x) \otimes \eta_{\mathrm{k}}(y)$ is consistent with $x \in X \wedge y \in F(x)$, let $q(xz) = p(x) \otimes \eta_{\mathcal{O}}(z)|_M$, then, by hypothesis, $q(xz)$ is consistent with $x \in X \wedge \mathrm{res}(z) \in F(x)$. Changing X and p we may therefore assume that $\mathrm{k}(\mathrm{acl}(Aa)) \subseteq \mathrm{res}(a_{\mathrm{K}})$.

We may also assume that a_{K} is a field. Now, if $\xi \in \mathrm{RV}(\mathrm{acl}(Aa))$, then $\mathrm{v}(\xi) \in \Gamma(\mathrm{acl}(Aa)) \subseteq \mathrm{v}(a_{\mathrm{K}})$. Let $c \in a$ be such that $\mathrm{v}(c) = \mathrm{v}(\xi)$, then $\xi \mathrm{rv}(c)^{-1} \in \mathrm{k}(\mathrm{acl}(Aa)) \subseteq \mathrm{res}(a_{\mathrm{K}})$. It follows that $\mathrm{RV}(\mathrm{acl}(Aa)) \subseteq \mathrm{rv}(a_{\mathrm{K}})$.

Let us now assume that the co-domain of F is the set of generalized balls that are not open balls. For any $a \models p$ and $b \in F(a) \subseteq \mathrm{acl}(Aa) \subseteq \mathrm{acl}(Ma_{\mathrm{K}})$, by Lemma 2.6, we have $b(\mathrm{acl}_1(Ma_{\mathrm{K}})) \supseteq b(\mathrm{acl}(Ma_{\mathrm{K}})) \neq \emptyset$. Moreover, the radius of b is in $\Gamma^{\mathrm{eq}}(\mathrm{acl}(Aa)) \subseteq \mathrm{acl}_1(Ma_{\mathrm{K}})$, by strong stable embeddedness. Hence, $b \in \mathrm{acl}_1(Ma)$ and we conclude with Claim 4.9.1. So we may assume that $b \in a$. Applying Claim 4.9.2, we may further assume that b has a point in a.

Now, if $b \in \mathrm{acl}(Aa)$ is an open ball, the smallest closed ball around b has a point $c \in a$ and $b - c \in \mathrm{RV}(\mathrm{acl}(Aa)) \subseteq \mathrm{rv}(a_{\mathrm{K}})$ also has a point in a, hence so does b. Recall that we already assumed that, for every $\mathrm{acl}(Aa)$-definable convex subgroup $\Delta \leq \Gamma$, $\Gamma/\Delta(\mathrm{acl}(Aa)) \subseteq \mathrm{v}_\Delta(a_{\mathrm{K}})$. By Corollary 3.12, we have $\mathcal{G}(\mathrm{acl}(Aa)) \subseteq \mathrm{dcl}_1(a)$.

Recall that to get to that point, we have replaced $p(x)$ by a type $q(xy)$ consistent with X and such that if $ac \models q$, then $c \in \mathrm{acl}(Ma_{\mathrm{K}}c_{\mathrm{K}})$. So the proposition is proved. □

Applying Proposition 4.9 to all possible F – as we have actually done in the proof of the proposition – we get:

Corollary 4.10. *Let $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$, let X be $\mathfrak{L}(A)$-definable and let $p \in \mathrm{S}^1(M)$ be $\mathfrak{L}(A)$-definable and consistent with X. Then there exists $a \models p$ in X such that*

$$\mathrm{tp}_1(\mathcal{G}(\mathrm{acl}(Aa)) \cup \Gamma^{\mathrm{eq}}(\mathrm{acl}(Aa))/M)$$

is $\mathfrak{L}(A)$-definable.

The proof of Theorem 4.1 is now a standard induction.

Proof of Theorem 4.1. Recall that, by Proposition 4.3, we can identify $\mathfrak{L}(A)$-definable types in $\mathrm{S}^1(M)$ which are finitely satisfiable in M with their unique $\mathfrak{L}_1(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable extension to M_1. Let $N \prec M$ be small and contain A. Note that by Theorem 2.7, $N_1 = N^{\mathrm{ur}} \prec M_1$. So it suffices to prove the theorem over N.

We proceed by induction on the integer n such that $X \subseteq \mathrm{K}^n$. Let $\pi : \mathrm{K}^n \to \mathrm{K}$ be the projection on the first coordinate. By Corollary 4.8, there exists an $\mathfrak{L}(A)$-definable type $p \in \mathrm{S}^1(M)$ consistent with $\pi(X)$. Let $a \in X$ realize $p|_N$ and let c enumerate $\mathcal{G}(\mathrm{acl}(Aa)) \cup \Gamma^{\mathrm{eq}}(\mathrm{acl}(Aa))$. By Corollary 4.10, we may assume that $\mathrm{tp}_1(c/N)$ is $\mathfrak{L}(A)$-definable.

By induction, there exists an $\mathfrak{L}_1(c)$-definable $r_c(z) \in \mathrm{S}^1(M_1)$ consistent with X_a. In particular $r_c|_M$ is $\mathfrak{L}(c)$-definable. Let $d \in X_a$ realize $r_c|_M$. Then $q = \mathrm{tp}_1(acd/N)$ is $\mathfrak{L}(A)$-definable and $ad \in X$. But q is $\mathfrak{L}(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable, by Proposition 4.3, concluding the proof. □

In the case $M = M^{\mathrm{ur}}$, we can deduce a slight generalization of [7, Theorem 5.12]:

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Corollary 4.11. *Assume that:*

- $\Gamma(M)$ *satisfies Property* **D***;*
- $\mathrm{k}(M)$ *is a pure algebraically closed field.*

Then M weakly eliminates imaginaries down to $\mathcal{G} \cup \Gamma^{\mathrm{eq}}$.

Proof. Fix some $e \in M^{\mathrm{eq}}$. Let $A = \mathrm{acl}(e)$ and let f be an $\mathfrak{L}$-definable map with domain K^n and such that $X = f^{-1}(e) \neq \emptyset$. By Theorem 4.1, there exists an $\mathfrak{L}(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable type $p(x) \in \mathrm{S}^1(M)$ consistent with X. But since $M = M_1$, this type is complete and we have $p(x) \vdash f(x) = e$. It follows that $e \in \mathrm{dcl}(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$ proving weak elimination. □

5. Invariant extensions

In this section, we will consider the invariance of types over stably embedded definable sets, for example, RV. This gives rise to several notions of invariance isolated in [6, Section 4.2].

Whenever $D = \bigcup_i D_i$ is ind-$\mathfrak{L}$-definable and stably embedded, we denote by D^{eq} the ind-$\mathfrak{L}$-interpretable union of all $\mathfrak{L}$-interpretable sets X that admit an $\mathfrak{L}$-definable surjection $\prod_j D_{i_j} \to X$.

Let M be sufficiently saturated and homogeneous. Let $C \subseteq M$ be potentially large and let D be ind-$\mathfrak{L}$-definable and stably embedded. Let p be a partial type over M (closed under implication).

Definition 5.1. We say that the type p:

1. is $\mathrm{Aut}(M/C)$-invariant if for every $\sigma \in \mathrm{Aut}(M/C)$, we have $p = \sigma(p)$;
2. has $\mathrm{Aut}(M/C)$-invariant D-germs if it is $\mathrm{Aut}(M/C)$-invariant and so is the p-germ of every (relatively) M-definable map $f : p \to D^{\mathrm{eq}}$;
3. is $\mathrm{Aut}(M/D)$-invariant if it has $\mathrm{Aut}(M/D(M))$-invariant D-germs.

A nice property of the stronger notion is that it is transitive – *cf.* [6, Lemma 4.2.4]:

Lemma 5.2. *Let $N \succ M$ be saturated*[3] *strictly larger than M. Let $p \in S(M)$ have $\mathrm{Aut}(M/C)$-invariant D-germs, let $a \models p$ in N and let $q \in S(N)$ be $\mathrm{Aut}(N/CD(N)a)$-invariant. Then $q|_M$ is $\mathrm{Aut}(M/C)$-invariant.*

Moreover, if q has $\mathrm{Aut}(N/CD(N)a)$-invariant E-germs, for some (ind-)$\mathfrak{L}$-definable set E, then $q|_M$ has $\mathrm{Aut}(M/C)$-invariant E-germs.

The main goal of this section is to prove the following statement. Recall (Definition 3.17) that $\mathrm{Lin}_A = \coprod_{s \in \mathrm{Gr}(\mathrm{acl}(A))} \mathrm{red}(R_s)$ and that a bounded regular rank group is an ordered abelian group with countably many definable convex subgroups.

Theorem 5.3. *Let M be sufficiently saturated and homogeneous* RV*-expansion of a model of* $\mathrm{Hen}_{0,0}$ *such that the value group Γ and residue field are stably embedded and orthogonal and Γ is either:*

- *dense with property* **D***;*
- *or, a pure discrete ordered abelian group of bounded regular rank – in that case we also add a constant for a uniformizer π.*

Let $M_0 = M^{\mathrm{a}}$, let $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$ be small and let a be a tuple in $\mathrm{K}(N)$, for some $N \succ M$. Assume that $\mathrm{tp}_0(a/M)$ is $\mathrm{Aut}(M/\mathcal{G}(A))$-invariant, then $\mathrm{tp}(a/M)$ is $\mathrm{Aut}(M/\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$-invariant.

We follow the general strategy of [6, Section 4]. The main new challenge is to prove the equivalent (Proposition 5.16) of [6, Corollary 4.4.6] in the present setting since the geometric sorts are now larger.

[3] We assume such models exist.

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5.1. Germs of functions into the linear sorts

One important ingredient of the proof of Proposition 5.16 is a description of the germ of certain functions into the linear sorts (*cf.* Lemmas 5.9 and 5.13). We proceed in three steps. First, we consider the case of a valued field with algebraically closed residue field. Then we consider valued fields with dense value groups (and arbitrary residue fields). Finally, we consider valued fields with discrete value groups for which a serious obstruction arises: the classification of k-internal sets given in Corollary 3.23 does not hold for discrete value groups. This can be circumvented by considering a ramified extension with dense value group.

Let M be sufficiently saturated and homogeneous (RV-expansion of a) model of $\mathrm{Hen}_{0,0}$ in a language $\mathfrak{L}$, whose value group Γ is stably embedded, has property **D** and is orthogonal to k, which is itself stably embedded.

We first prove that germs of functions into the linear sorts are internal to the residue field (see Proposition 5.7). We follow the general strategy of [6, Lemma 3.4.1] and first prove a result on the growth of dcl in $\mathrm{Lin}_A^{\mathrm{eq}}$. Recall the definition of an open generalized ball (Definition 2.4).

Lemma 5.4. *Let $A \subset M^{\mathrm{eq}}$ be small and let U be an open ∞-A-definable generalized ball. Let a be a generic element of U over A in some $N \succ M$. Then* $\mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa)) \subseteq \mathrm{acl}(A)$.

Proof. Let $f : U \to \mathrm{Lin}_A^{\mathrm{eq}}$ be some (relatively) A-definable function and let $\mathrm{cut}(U)$ be the cut of U. Fix some $e \in U(M)$. For now, we work over M, so we can assume that $e = 0$ and we identify $\mathrm{Lin}_A^{\mathrm{eq}}$ with k^{eq}. For every $\gamma \in \mathrm{cut}(U)$ and $d \in \mathrm{k}^{\mathrm{eq}}$, let $X_{d,\gamma} = \{x \in U : \mathrm{v}(x) = \gamma \text{ and } f(x) = d\}$. Then by Proposition 2.5, there exists a finite set $C \subseteq \mathrm{K}(M)$ which does not depend on γ or d, such that for every ball b, if $b \cap C = \emptyset$, then $b \cap X_{d,\gamma} \neq \emptyset$ implies that $b \subseteq X_{d,\gamma}$.

If $a \in U$ is not in the smallest ball b containing $C \cap U(M)$ and 0, then the open ball of radius $\mathrm{v}(a)$ around a – that is $\mathrm{rv}(a)$ – is entirely contained in $X_{f(a),\mathrm{v}(a)}$. In other words, f induces a well-defined function $\overline{f} : \mathrm{rv}(U \setminus b) \to \mathrm{k}^{\mathrm{eq}}$.

Claim 5.4.1. Let $f : \mathrm{RV} \to k^{\mathrm{eq}}$ be M-definable. Then there are finitely many $\gamma_i \in \Gamma(M)$ such that $f(\{x \in \mathrm{RV} : \mathrm{v}(x) \neq \gamma_i\})$ is finite.

Proof. Let $N \succ M$ be sufficiently saturated and homogeneous. For any choice of $\alpha \in \mathrm{k}$ and $\gamma \in \Gamma(N) \setminus \Gamma(M)$, we find an automorphism $\sigma \in \mathrm{Aut}(\mathrm{RV}(N)/\mathrm{RV}(M), \mathrm{k}(N))$ such that, if $\mathrm{v}(x) = \gamma$, then $\sigma(x) = \alpha \cdot x$. First, as $\mathrm{k}^\times$ is divisible and thus injective, we find a group morphism $h : \Gamma \to \mathrm{k}^\times$ sending γ to α and $\Gamma(M)$ to 0. This h induces an automorphism $\sigma \in \mathrm{Aut}(\mathrm{RV}(N)/\mathrm{RV}(M), \mathrm{k}(N))$ defined by $\sigma(x) = h(\mathrm{v}(x)) \cdot x$.

Let $x, y \in \mathrm{RV}$ be such that $\mathrm{v}(x) = \mathrm{v}(y) \notin \Gamma(M)$. Then, by the above paragraph, there is an automorphism σ fixing k and $\mathrm{RV}(M)$, and hence f, and such that $\sigma(x) = y$. It follows that $f(x) = \sigma(f(x)) = f(\sigma(x)) = f(y)$. By compactness, there are finitely many $\gamma_i \in \Gamma(M)$ such that f induces a function $\Gamma \setminus \bigcup_i \gamma_i \to \mathrm{k}^{\mathrm{eq}}$. This function has finite image by orthogonality of k and Γ. □

Thus, we have found an M-definable closed ball $b' \subset U$ such that $|f(U \setminus b')| = n < \infty$. By compactness, there exists an A-definable $Z \supseteq U$ such that $|f(Z \setminus b')| = n$. If $A = M$, the proposition is proved. In general, we show that b' can be replaced by a generalized A-definable ball. If there are two such M-definable closed balls with empty intersection, then $f(Z)$ is finite. If not, they form a chain which is A-definable. Hence, their intersection is an A-definable generalized sub-ball B of U such that $f(U \setminus B)$ is finite.

In both cases, if a is generic in U over A, $f(a)$ is in a finite A-definable set. In other words, $\mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa)) \subseteq \mathrm{acl}(A)$. □

Proposition 5.5. *Let $A \subseteq M^{\mathrm{eq}}$ and let a be a tuple in* $\mathrm{K}(M)$. *There is a countable tuple $c \in \mathrm{Lin}_A(\mathrm{dcl}(Aa))$ such that*

$$\mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa)) \subseteq \mathrm{acl}(Ac).$$

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Proof. Let us first assume that $|a| = 1$. Let $W = \{b : a \in b \text{ and } b \text{ is a } A\text{-definable generalized ball }\}$. Then a is generic over A in the ∞-A-definable generalized ball

$$U = \bigcap_{b \in W} b.$$

If U is open, by Lemma 5.4, we have $\mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa)) \subseteq \mathrm{acl}(A)$. If U is closed, let $c_0 = \mathrm{res}_U(a) \in \mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa))$. Let U_0 be the intersection of all Ac_0-definable generalized balls containing a. Either U_0 is open, or we set $c_1 = \mathrm{res}_{U_0}(a) \in \mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa))$. We continue this process unless U_i is open and we set $U_\omega = \bigcap_j U_j$. Then a is generic in the $Ac_{\geq 0}$-definable open generalized ball U_ω. By Lemma 5.4, we have $\mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa)) \subseteq \mathrm{acl}(Ac_{\geq 0})$, concluding the proof.

Let us now assume that $n > 1$ and proceed by induction. Let $d \in \mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa_{<n}))$ be such that $\mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa_{<n})) \subseteq \mathrm{acl}(Ad)$. By the case $n = 1$, let also $c \in \mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa))$ be such that $\mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa)) \subseteq \mathrm{acl}(Aa_{<n}c)$. For every $e \in \mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa))$, there exists an $Aa_{<n}$-definable one-to-finite correspondence f such that $e \in f(c)$. We have $\ulcorner f \urcorner \in \mathrm{Lin}_A^{\mathrm{eq}}(\mathrm{dcl}(Aa_{<n})) \subseteq \mathrm{acl}(Ad)$ and hence $e \in \mathrm{acl}(Adc)$. □

Lemma 5.6. *Let M be an $\mathfrak{L}$-structure, let X and D be $\mathfrak{L}$-definable sets and let $a \in X(N)$, for some $N \succ M$. Assume there exists a countable tuple c in M such that $D(\mathrm{dcl}(Ma)) \subseteq \mathrm{acl}(D(M)ac)$. Then, for every $\mathfrak{L}$-definable family $(f_\lambda)_{\lambda \in \Lambda} : X \to D$, there exists an $\mathfrak{L}(M)$-definable one-to-finite correspondence $(g_\delta)_{\delta \in D^m} : X \to D$ such that, for every $\lambda \in \Lambda(M)$, there exists a $\delta \in D^m(M)$ with $f_\lambda(a) \in g_\delta(a)$.*

In particular, if $p \in \mathrm{S}(M)$ is definable, the interpretable set $\{[f_\lambda]_p : \lambda \in \Lambda\}$ is almost D-internal.

Proof. The existence of g follows by compactness, in a sufficiently saturated model of the pair (N, M). Now, if $p \in \mathrm{S}(M)$ is definable, for every $\lambda \in \Lambda(M)$, let $Y_\lambda = \{\delta \in D^m : p(x) \vdash f_\lambda(x) \in g_\delta(x)\}$. Then $h([f_\lambda]_p) = \ulcorner Y_\lambda \urcorner$ lies in an interpretable D-internal set. Moreover, since $p(x) \vdash f_\lambda(x) \in \bigcap_{\delta \in Y_\lambda} g_\delta(x)$ which is finite, the map h is finite-to-one. □

Proposition 5.7. *Assume that* k *is stable. Let $A \subseteq M^{\mathrm{eq}}$ and let $p(x) \in \mathrm{S}(M)$ be A-definable concentrating on K^n for some n. Let f be an M-definable function. Assume that for every $a \models p$, we have $f(a) \in \mathrm{Lin}_{Aa}$, then $[f]_p$ lies in an almost* k*-internal A-definable set.*

Proof. We may assume that $A = \mathrm{acl}(A)$. As k is stable, there exists a $\mathrm{acl}(Aa)$-definable type q of bases for Lin_{Aa}. Let $b \models q|_{Aa}$. Since $\mathrm{tp}(\mathrm{acl}(Aa)/M)$ is A-definable, $\mathrm{tp}(ab/M)$ is also A-definable. Moreover, if $g(ab) \in \mathrm{k}$ enumerates the coordinates of $f(a)$ in the basis b, we have $f(a) = h(b, g(ab))$ where h is A-definable.

By Proposition 5.5, there exists a countable tuple c in M such that $\mathrm{k}(\mathrm{dcl}(Mab)) \subseteq \mathrm{acl}(M\mathrm{k}(\mathrm{dcl}(abc)))$. As k is stably embedded, it follows that $\mathrm{k}(\mathrm{dcl}(Mab)) \subseteq \mathrm{acl}(\mathrm{k}(M)abc)$. Therefore, by Lemma 5.6, the germ $[g]_q$ lies in an A-interpretable almost k-internal set. Moreover, if $\sigma \in \mathrm{Aut}(M/A[g]_q)$, then $\sigma(g)(ab) = g(ab)$ and hence

$$\sigma(f)(a) = h(b, \sigma(g(ab))) = h(c, g(ab)) = f(a).$$

So $[f]_p \in \mathrm{dcl}(A[g]_q)$ also lies in an A-interpretable almost k-internal set. □

5.1.1. Dense value groups

As previously, let $M_0 = M^{\mathrm{a}}$ and $M_1 = M^{\mathrm{ur}}$.

Lemma 5.8. *We have $\mathcal{G}(\mathrm{dcl}_1(M)) = \mathcal{G}(M)$.*

Proof. Let $A = \mathrm{K}(M)$. Note that $\Gamma(M_1) = \mathrm{v}(A)$. Also, any $\mathfrak{L}_1(A)$-definable ball b contains a point in $a \in \mathrm{K}(M_1) \subseteq A^{\mathrm{a}}$. Since A is henselian, the Galois-conjugates of a in M_1 over A are all in the generalized ball b and their mean d is fixed by $\mathrm{Gal}(M_1/A)$. Since the extension $A \leq \mathrm{K}(M_1)$ is normal, $d \in A$. So we can apply Corollary 3.12 (in M_1) to see that $\mathcal{G}(\mathrm{dcl}_1(M)) \subseteq \bigcup_{c \in \mathrm{Cut}} \mu_c(\mathrm{B}_n(A)) = \mathcal{G}(M) \subseteq \mathcal{G}(\mathrm{dcl}_1(M))$. □

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If we further assume that the value group is dense, what we have done so far is enough to show that germ of $\mathfrak{L}_0$-definable open balls are coded in the linear part.

Lemma 5.9. *Assume $\Gamma(M)$ is dense. Let $A \subseteq \mathcal{G}(M)$ and let a be a tuple of* K*-points in $N \succ M$ be such that $p = \mathrm{tp}_0(a/M)$ is $\mathfrak{L}(A)$-definable. Let $b(a)$ be an open $\mathfrak{L}_0(Ma)$-definable ball whose radius is in $\Gamma(\mathrm{dcl}_1(Aa))$. Then, in the structure M_1, the germ $[b]_p$ is coded in $\mathcal{G}(\mathrm{acl}(A)) \cup \mathrm{Lin}_A(M)$ over A.*

Proof. By Proposition 4.3 and property **D**, we may assume that $tp_1(a/M_1)$ is $\mathfrak{L}_1(\mathrm{acl}_1(A))$-definable. We have $b(a) \in \mathrm{Lin}_{Aa}$, so, by Proposition 5.7 applied in M_1, the germ $[b]_p$ lies in an $\mathfrak{L}_1(A)$-definable k-internal set. On the other hand, by Lemma 4.5, it is coded by some $e \in \mathcal{G}(M_1)$ over A. It now follows from Corollary 3.23 that $e \in \mathcal{G}(\mathrm{acl}_1(A)) \cup \mathrm{Lin}_{\mathrm{acl}_1(A)}(M_1)$. Since $e \in \mathrm{dcl}_1(M)$, by Lemma 5.8, we have $e \in \mathcal{G}(M)$ and hence $e \in \mathcal{G}(\mathrm{acl}(A)) \cup \mathrm{Lin}_A(M)$. □

5.1.2. Discrete value groups

We now assume that $\Gamma(M)$ is a pure, discrete ordered abelian group of bounded regular rank. We add a constant π for a uniformizer in M. We introduce $M_1' = M_1[\pi^{1/\infty}]$ the extension of M_1 obtained by adding n-th roots of π for all $n > 0$. We assume the language $\mathfrak{L}_1'$ of M_1' is morleyized and we restrict ourselves to quantifier free $\mathfrak{L}_1'$-formulas when interpreting them in a substructure. We write acl'_1 and dcl'_1 to indicate the algebraic and definable closure in M_1'.

Lemma 5.10. *The definable convex subgroups of $\Gamma(M_1')$ are exactly the convex hulls of definable convex subgroups of $\Gamma(M_1)$ and $\Gamma(M_1')$ has bounded regular rank. Furthermore, the definable cuts in $\Gamma(M_1')$ are exactly the upward closures of definable cuts in $\Gamma(M_1)$ and the cuts above or below a point of $\Gamma(M_1')$.*

Proof. Fix some $n \in \omega$. Since $\Gamma(M)$ has bounded regular rank, for each n there is a finite sequence of convex subgroups $0 < \Delta_1 < \cdots < \Delta_k = \Gamma(M)$ such that Δ_{i+1}/Δ_i is n-regular. Note that $\Gamma(M_1') = \mathbb{Q}v(\pi) + \Gamma(M)$ and $\overline{\Delta}_i = \mathbb{Q}v(\pi) + \Delta_i$ is also a convex subgroup of $\Gamma(M_1')$. Then $\overline{\Delta}_1$ is n-divisible and $\overline{\Delta}_{i+1}/\overline{\Delta}_i$ is n-regular as it is isomorphic to Δ_{i+1}/Δ_i. Consequently, for each $n < \omega$, $\Gamma(M_1')$ has the same n-regular rank than $\Gamma(M)$, thus by [20, Proposition 2.3] $\Gamma(M_1')$ is of bounded regular rank and each $\overline{\Delta}_i$ is definable in $\Gamma(M_1')$. Furthermore, the map Δ to $\overline{\Delta}$ is a one to one correspondence between the convex subgroups of $\Gamma(M)$ and $\Gamma(M_1')$.

Let $S \subseteq \Gamma(M)$ be a definable cut and $\Delta_S = \{\gamma \in \Gamma(M) : \gamma + S = S\}$. By [9, Fact 3.2] Δ_S is a convex definable subgroup of $\Gamma(M)$, and it is the maximal convex subgroup such that S is a union of Δ_S-cosets. If $\Delta_S = \{0\}$, then there exists a $\gamma \in S$ such that $\gamma - \mathrm{v}(\pi) \notin S$. It follows that S is the cut below γ and so is its upwards closure in $\Gamma(M_1')$. If $\Delta_S \neq \{1\}$, then S can be identified with a subset of $\Gamma(M)/\Delta_S$ which is isomorphic to $\Gamma(M_1')/\overline{\Delta}_S$ and hence the upwards closure of S in $\Gamma(M_1')$ is definable.

Conversely, let $S' \subseteq \Gamma(M_1')$ be a definable cut and $\overline{\Delta}_{S'} = \{\gamma \in \Gamma(M_1') : \gamma + S' = S'\}$. If $\overline{\Delta}_{S'} \neq \{0\}$, then, as above, S' is the upward closure of $S' \cap \Gamma(M)$ which is definable. If $\overline{\Delta}_{S'} = \{0\}$, then, by [9, Proposition 3.3], S' is of the form $nx \square \beta$ for some $\beta \in \Gamma(M_1')$ and $\square \in \{>, \geq\}$. Growing n, we may assume that $\beta \in \Gamma(M)$. Moreover, since $\overline{\Delta}_{S'} = \{0\}$, for some $\gamma \in S'$, $\gamma - \mathrm{v}(\pi) \notin S'$. As $(\gamma - \mathrm{v}(\pi), \gamma] \cap \Gamma(M) \neq \emptyset$, we may assume that $\gamma \in \Gamma(M)$. Then $\beta = n\gamma - i\mathrm{v}(\pi)$, for some i, and S' is the cut above or below $\gamma - n^{-1}i\mathrm{v}(\pi)$. □

It follows, by quantifier elimination in bounded regular rank group ([9, Theorem 2.17]), that $M_1' \prec \mathbb{M}^{\mathrm{ur}}[\pi^{1/\infty}]$. Also, we can naturally identify the set $\mathcal{G}(M_1)$ with a subset of $\mathcal{G}(M_1')$. We do, however, have to code the imaginaries of M that M_1' believes to be geometric:

Lemma 5.11. *Let $R \in \mathrm{Gr}(\mathrm{dcl}'_1(M_1))$.*

1. *There is a $Q \in \mathrm{Gr}(M_1)$ such that $R \in \mathrm{dcl}'_1(Q)$ and Q is definable from R in the pair (M_1', M_1).*
2. *For every $e \in \mathrm{red}(R)(\mathrm{dcl}'_1(M_1))$, there exists $\varepsilon \in \mathrm{red}(Q)(M_1)$ such that $e \in \mathrm{dcl}'_1(\varepsilon)$ and ε is definable from e in the pair (M_1', M_1).*

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Proof. For some $n \geq 1$, we have $R \in \mu_c(\mathrm{B}_m(\mathrm{K}(M_1)[\varpi]))$, where c is a tuple in $\mathrm{Cut}^{\star\star}$ and $\varpi^n = \pi$. Let $f_\varpi : \mathrm{K}(M_1)^n \to \mathrm{K}(M_1)[\varpi]$ send a to $\sum_{i<n} a_i \varpi^i$. Then for every $a \in \mathrm{K}(M_1)^n$, $\mathrm{v}(f_\varpi(a)) = \min_i \mathrm{v}(a_i) + n^{-1} i \mathrm{v}(\pi)$. It follows that the preimage of $R(\mathrm{K}(M_1)[\varpi])$ by f_ϖ is an $\mathfrak{L}(M_1)$-definable $\mathcal{O}$-submodule $Q(M_1)$ with $Q \in \mathrm{Gr}(M_1)$. If ϖ' is another n-th root of π, then $\varpi' = \sigma(\varpi)$ for some $\sigma \in \mathrm{Aut}(M_1'/M_1)$. Then, since $\sigma(R) = R$, we have

$$f_{\varpi'}^{-1}(R(\mathrm{K}(M_1)[\varpi'])) = \sigma(f_\varpi^{-1}(R(\mathrm{K}(M_1)[\varpi]))) = \sigma(Q(M_1)) = Q(M_1).$$

So Q does not depend on the choice of ϖ' and it is definable from R in the pair (M_1', M_1). Also, since f_ϖ is linear, it induces a surjective map $Q(M_1') \to R(M_1')$, whose image does not depend on ϖ. It follows that $R \in \mathrm{dcl}_1'(Q)$.

Let us now consider some $e \in \mathrm{red}(R)(\mathrm{dcl}_1'(M_1))$. Growing n, we may assume that $e \in \mathrm{red}(R)(\mathrm{K}(M_1)[\varpi])$. Let ε be the preimage of e under the bijection $\mathrm{red}(Q)(M_1) \to \mathrm{red}(R)(\mathrm{K}(M_1)\varpi)$ induced by f. As above, ε does not depend on the choice of ϖ and it has the required properties. □

We can now prove a variant of Proposition 4.3:

Lemma 5.12. *Let $A = \mathrm{dcl}(A) \subseteq M^{\mathrm{eq}}$ and let $a \in N \succ M$ be such that $\mathrm{tp}_0(a/M)$ is $\mathfrak{L}(A)$-definable. Then $\mathrm{tp}_0(a/M)$ has a unique extension to $\mathrm{S}^0(M_1')$ and this extension is $\mathfrak{L}_1'(\mathcal{G}(A))$-definable.*

Proof. The uniqueness follows from Proposition 4.3 – in fact, there is a unique extension to M_0. Let $d \geq 0$, $V_d = \mathrm{K}[x]_{\leq d}$ and v be the valuation on V_d defined by $v(P) \leq v(Q)$ if $\mathrm{v}(P(a)) \leq \mathrm{v}(Q(a))$. By Proposition 3.3, the space $V_d(M)$ admits a separated basis $(P_i)_{i \leq \ell} \in V_d(M)$. By [6, Claim 3.3.5], it is also a separated basis of $V_d(M_1')$.

For every i, j, let $C_{i,j} = \{\gamma \in \Gamma : v(P_i) + \gamma v(P_j)\}$. If the stabilizer Δ of $C_{i,j}$ is not 0, then, since $\Gamma/\Delta(M) = \Gamma/\Delta(M_1')$ (by Lemma 5.10), $C_{i,j}(M)$ is co-initial in $C_{i,j}(M_1')$ which is indeed definable. If this stabilizer is 0, since $\Gamma(M_1)$ is discrete, $C_{i,j}$ has a minimal element $\gamma_{i,j} \in \Gamma(M)$ and $v(P_j) = v(P_i) + \gamma_{i,j}$. So v is indeed definable in M_1'. Moreover, the $\mathcal{O}_1'$-module $R_i(M_1') = \{P \in V_d(M_1') : v(P) \geq v(P_i)\}$ is the $\mathcal{O}_1'$-module generated by $R_i(M) = \{P \in V_d(M) : v(P) \geq v(P_i)\}$ whose codes we identify as in Proposition 4.3 via the natural inclusion map. □

We can now recover the equivalent of Lemma 5.9 in the case of a discrete value group:

Lemma 5.13. *Let $A \subseteq \mathcal{G}(M)$ and let a be a tuple of* K*-points in $N \succ M$ such that $p = \mathrm{tp}_0(a/M)$ is $\mathfrak{L}(A)$-definable. Let $b(a)$ be an open $\mathfrak{L}_0(Ma)$-definable ball whose radius is in $\Gamma(\mathrm{dcl}_1'(Aa))$. Then, in the structure M_1, the germ $[b]_p$ is coded in $\mathcal{G}(\mathrm{acl}(A)) \cup \mathrm{Lin}_A(M)$ over A.*

Proof. Given Proposition 4.3 and Lemma 5.8, we may assume that $M = M_1$. Growing M_1, we may also assume that M_1 is sufficiently saturated and homogeneous. By Lemma 5.12, the type $\mathrm{tp}_0(a/M_1')$ is $\mathfrak{L}_1'(A)$-definable. Now, applying Lemma 5.9 in M_1', the germ $[b]_p$ is coded in $\mathcal{G}(\mathrm{acl}_1'(A)) \cup \mathrm{Lin}_{\mathrm{acl}_1'(A)}(M_1')$ over A. In other words, there are some tuple t in $\mathrm{K}(\mathrm{acl}_1'(A)) \cap \mathrm{dcl}_1'(M_1) = \mathrm{K}(\mathrm{acl}(A))$, some $R \in \mathrm{Gr}(\mathrm{acl}_1'(A)) \cap \mathrm{dcl}_1'(M_1)$ and some $e \in \mathrm{red}(R)(\mathrm{dcl}_1'(M_1))$ which code $[f]_p$ over A. Let Q and ε be as in Lemma 5.11. Now, any automorphism of $\sigma \in \mathrm{Aut}(M_1/A)$ (extended in any way to M_1') fixes Q if and only if it fixes R – so $Q \in \mathrm{Gr}(\mathrm{acl}_1(A))$ – and σ fixes $[f]_p$ if and only if it fixes t, R and e, if and only if it fixes t,Q and ε. □

5.2. Invariant resolutions

Let M be as in Theorem 5.3. As before, let $M_0 = M^{\mathrm{a}}$ and $M_1 = M^{\mathrm{ur}}$. Given a subset A of $\mathcal{G}$, our goal is now to find a subset C of K whose type over M is invariant over A and some stably embedded set. This follows from the following lemma:

Lemma 5.14. *Let $N \succ M$, let $D \subseteq M$ be potentially large, let a be a tuple in $\mathrm{K}(N)$ and let ρ be a pro-$\mathfrak{L}_1(M)$-definable map. Assume that $\mathrm{rv}(M(a)) \subseteq \mathrm{dcl}_1(D\rho(a))$ and that $p_1 = \mathrm{tp}_1(a/M)$ and $[\rho]_{p_1}$ are $\mathrm{Aut}(M/D)$-invariant. Then $p = \mathrm{tp}(a/M)$ has $\mathrm{Aut}(M/D)$-invariant* RV*-germs.*

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This is essentially [6, Lemma 4.2.5] in a slightly different context and the proof is identical. The main ingredient is elimination of quantifier down to RV – see Theorem 2.1.

Proof. Let N_1 be a large saturated elementary extension of M_1 containing N. Fix $\sigma \in \mathrm{Aut}(M/D)$. Since $p_1 = \mathrm{tp}_1(a/M)$ is $\mathrm{Aut}(M/D)$ invariant, there is an $\mathfrak{L}_1$-elementary embedding $\tau : M(a) \to M(a)$ extending σ. Because $[\rho]_{p_1}$ is $\mathrm{Aut}(M/D)$-invariant, we have $\rho(a) = \sigma(\rho)(a)$. Consequently, since $\mathrm{rv}(M(a)) \subseteq \mathrm{dcl}_1(D\rho(a))$, $\tau|_{\mathrm{rv}(M(a))}$ is the identity map. By Theorem 2.1 in N_1, extending τ by the identity on $\mathrm{RV}(N_1)$ yields an $\mathfrak{L}_1$-elementary embedding. Since RV is stably embedded, this embedding further extends to an element τ of $\mathrm{Aut}(N_1/D, \mathrm{RV}(N_1), a)$ – *cf.* [12, Lemma 10.1.5].

By Theorem 2.1 (in M now), $\tau|_{M(a)\cup \mathrm{RV}(N)}$ is $\mathfrak{L}$-elementary. Consequently, $\mathrm{tp}(a, M) = \mathrm{tp}(a, \sigma(M))$ and we conclude that $\sigma(p) = p$, as required. Lastly, we argue that $\mathrm{tp}(a/M)$ has $\mathrm{Aut}(M/C)$-invariant RV-germs. Let $X \subseteq \mathrm{RV}^n$ be $\mathfrak{L}(Ma)$-definable. Then, by Theorem 2.1, it is $\mathfrak{L}(\mathrm{rv}(M(a)))$-definable and hence $X(N) = \tau(X(N)) = \tau(X)(N)$. Equivalently, σ fixes the p-germ of any $\mathfrak{L}(M)$-definable function $f : p \to \mathrm{RV}^{\mathrm{eq}}$. □

Let us now describe how RV grows when adding one field element:

Lemma 5.15. *Let $A \subseteq \mathcal{G}(M) \cup \Gamma^{\mathrm{eq}}(M)$ contain $\mathcal{G}(\mathrm{acl}(A))$ and let a be a tuple of* K*-points in $N \succ M$ such that $p = \mathrm{tp}_1(a/M_1)$ is $\mathfrak{L}(A)$-definable. Let $b(a)$ be an $\mathfrak{L}_1(Aa)$-definable generalized ball. If $\Gamma(M)$ is discrete, we assume that the cut of $b(a)$ is $\mathfrak{L}'_1(\mathcal{G}(A)a)$-definable. Let $c \in N$ realize the generic $\eta_{b(a)}|_{M_1 a}$ – that is, c is in $b(a)$ but not in any proper generalized* $\mathrm{acl}_1(M_1 a)$*-definable sub-ball. Let $q = \mathrm{tp}_1(ac/M_1)$. Then there is a pro-$\mathfrak{L}_1(M)$-definable map ρ into some power of* RV *such that $[\rho]_q \in \mathrm{dcl}(A, \mathrm{Lin}_A(M))$ and $\mathrm{rv}(M(ac)) \subseteq \mathrm{dcl}_1(\mathrm{rv}(M(a)), \rho(ac))$.*

Proof. We proceed by cases.

Claim 5.15.1 [6, Lemma 4.3.10]. If $b(a)$ is not closed, there is a pro- $\mathfrak{L}_0(M)$-definable map ρ into some power of RV such that $[\rho]_q \in \mathrm{dcl}(A)$ and $\mathrm{rv}(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}(M(a)), \rho(ac))$.

Note that in equicharacteristic zero, condition (2) of [6, Lemma 4.3.10] is verified as soon as $b(a)$ is not closed. So we may assume that $b(a)$ is closed. By Lemma 2.6, there is an $\mathfrak{L}_0(Ma)$-definable finite set $G(a) \subseteq \mathrm{K}$ such that $G(a) \cap b(a) = \{g\}$ is a singleton.

Claim 5.15.2 [6, Lemma 4.3.13]. If $b(a)$ is closed and $G(a) \subseteq \mathrm{K}$ is an $\mathfrak{L}_0(Ma)$-definable finite set such that $G(a) \cap b(a) = \{g(a)\}$ is a singleton, then

$$\mathrm{rv}(M(ac)) \subseteq \mathrm{dcl}_1(\mathrm{rv}(M(a)), \mathrm{rv}(c - g(a))).$$

Note that such a $G(a)$ always exists by Lemma 2.6. Let $\rho(ac) = \mathrm{rv}(c - g) \in \mathrm{dcl}_0(Mac)$. We have $\gamma = \mathrm{v}(c - g) \in \Gamma(\mathrm{dcl}_1(Aa))$ as it is the radius of b. If $\Gamma(M)$ is discrete, it is in $\mathrm{dcl}'_1(\mathcal{G}(A)a)$ by assumption. So, by Lemmas 5.9 and 5.13, we have $[h]_q \in \mathrm{dcl}_1(A \cup \mathrm{Lin}_A(M))$. Then, as required, we have $[\rho]_q \in \mathrm{dcl}_1(A, \mathrm{Lin}_A(M)) \cap \mathcal{G}(M)$ and $\mathrm{rv}(M(ac)) \subseteq \mathrm{dcl}_1(\mathrm{rv}(M(a)), \rho(ac))$. □

We can now prove the existence of sufficiently invariant resolutions of geometric points:

Proposition 5.16. *Let $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$. There exists $C \subseteq \mathrm{K}(N)$, for some $N \succ M$, with:*

1. *$\mathcal{G}(A) \subseteq \mathrm{dcl}_1(C, \Gamma(M))$;*
2. *$\mathrm{tp}_1(C/M)$ is $\mathfrak{L}(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable;*
3. *$\mathrm{tp}(C/M)$ has $\mathrm{Aut}(M/\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$-invariant* RV*-germs.*

Proof. By transfinite induction, we construct a tuple c in $N \succ M$ and a (pro-) $\mathfrak{L}_1(M)$-definable function ρ such that:

- $p_1 = \mathrm{tp}_1(c/M)$ is finitely satisfiable in M and $\mathfrak{L}(A)$-definable;
- $\mathrm{rv}(M(c)) \subseteq \mathrm{dcl}_1(\mathrm{RV}(M), \rho(c))$;
- $[\rho]_{p_1} \in \mathrm{dcl}_1(\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$;

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- any $\mathfrak{L}_1(\mathcal{G}(A)c)$-definable generalized ball b has a point in C;
- for all $\mathfrak{L}_1(\mathcal{G}(A)c)$-definable convex subgroup $\Delta \leq \Gamma$, $\Gamma/\Delta(\mathrm{dcl}_1(Ac)) \subseteq \mathrm{v}_\Delta(\mathrm{K}(C))$.

Note that $\mathrm{tp}_1(\mathrm{acl}_1(Ac) \cap N/M)$ is definable over $\mathcal{G}(\mathrm{acl}(A)) \cup \Gamma(\mathrm{acl}(A))^{\mathrm{eq}} \subseteq A$ (*cf.* Proposition 4.3). Let $b(c)$ be an $\mathfrak{L}_1(\mathcal{G}(A)c)$-definable generalized ball whose cut is a $\mathrm{v}(c)$-translate of an $\mathfrak{L}(A)$-definable cut. By property **D**, the generic $\eta_{b(c)}$ can be extended to a complete $\mathfrak{L}(\mathrm{acl}_1(Ac) \cap N)$-definable $\mathfrak{L}_1$-type – and this type is finitely satisfiable in N. Using Lemma 5.15, we can thus add a generic of $b(c)$ to c. We then iterate this construction.

Given such a tuple c, by Corollary 3.12 applied in M_1, we have $\mathcal{G}(A) \subseteq \mathrm{dcl}_1(c, \Gamma(A)^{\mathrm{eq}}) \subseteq \mathrm{dcl}_1(c, \Gamma(M))$. Moreover, the type $\mathrm{tp}(c/M)$ has $\mathrm{Aut}(M/A, \mathrm{RV}(M), \mathrm{Lin}_A(M))$-invariant RV-germs by Lemma 5.14. □

We deduce Theorem 5.3 from Proposition 5.16 and the machinery of [6, Section 4].

Proposition 5.17 ([6, Corollary 4.4.1] and Lemma 5.14). *Let $A \subseteq \mathrm{K}(M)$ and $R \subseteq \mathrm{RV}(M)$. There exists $C \subseteq \mathrm{K}(N)$, for some $N \succ M$, such that and $\mathrm{tp}(C/M)$ is $\mathrm{Aut}(M/A, \mathrm{RV})$-invariant and $R \subseteq \mathrm{dcl}_0(AC)$.*

Proposition 5.18 [6, Corollary 4.4.3]. *Let $A \subseteq \mathrm{K}(M)$. There exists $C \prec N$, for some $N \succ M$, such that and $\mathrm{tp}(C/M)$ is $\mathrm{Aut}(M/A, \mathrm{RV})$-invariant and C contains a realization of every $\mathfrak{L}(A)$-type.*

Proposition 5.19 [6, Corollary 4.3.17]. *Let $A \subseteq M^{\mathrm{eq}}$ be small, let $C \prec M$ contain a realization of every $\mathfrak{L}(A)$-type and let $a \in \mathrm{K}(N)$, for some $N \succ M$ be such that $\mathrm{tp}_0(a/M)$ is $\mathrm{Aut}(M/A)$-invariant. Then $\mathrm{tp}(a/M)$ is $\mathrm{Aut}(M/C, \mathrm{RV})$-invariant.*

Proof of Theorem 5.3. Fix $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$ and a in some elementary extension of M such that $p_0 = \mathrm{tp}_0(a/M)$ is $\mathrm{Aut}(M/\mathcal{G}(A))$-invariant.

- By Proposition 5.16, we find $C \subseteq \mathrm{K}(N)$, for some $N \succ M$ (sufficiently saturated and homogeneous), such that $\mathrm{tp}(C/M)$ has $\mathrm{Aut}(M/\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$-invariant RV-germs and $\mathcal{G}(A) \subseteq \mathrm{dcl}_1(C\gamma)$ for some (infinite) tuple γ in $\Gamma(M)$.
- By Proposition 5.17 and transitivity (Lemma 5.2), growing C, we may assume that $\gamma \in \mathrm{v}(C)$. By Proposition 5.18 and transitivity, we can further assume that $C \prec N$ contains a realization of every type over $\mathcal{G}(A)$.
- We may assume that $a \models p_0|_N$ – *cf.* [6, Claim 4.4.7]. Then $\mathrm{tp}_0(a/N)$ is $\mathrm{Aut}(N/C)$-invariant and $\mathrm{tp}(a/N)$ is $\mathrm{Aut}(N/C, \mathrm{RV})$-invariant by Proposition 5.19.

By transitivity $\mathrm{tp}(a/M)$ is $\mathrm{Aut}(M/\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$-invariant. □

6. Eliminating imaginaries

Following the general strategy of [6, Theorem 6.1.1], we can now deduce elimination of imaginaries. Let M be a sufficiently saturated and homogeneous, and as in Theorem 5.3. Let $M_0 = M^{\mathrm{a}}$ and $M_1 = M^{\mathrm{ur}}$.

Proposition 6.1. *Let $e \in M^{\mathrm{eq}}$ and $A = \mathrm{acl}(e)$. Then*

$$e \in \mathrm{dcl}(\mathcal{G}(A), (\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}}(A)).$$

Proof. We may assume M is sufficiently saturated and homogeneous. There is an $\mathfrak{L}$-definable map f and a tuple m in $K(M)$ such that $f(m) = e$. Let $X = f^{-1}(e)$. By Theorem 4.1 we can find a type $p \in \mathrm{S}^1(M)$ such that:

- $p \cup X$ is consistent;
- p is $\mathfrak{L}_1(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable.

Take $a \in X$ satisfying p. Then $\mathrm{tp}_0(a/M)$ is $\mathfrak{L}(\mathcal{G}(A))$-definable. By Theorem 5.3, the type $q = \mathrm{tp}(a/M)$ is $\mathrm{Aut}(M/\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$-invariant. So, for every automorphism $\sigma \in \mathrm{Aut}(M/\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$, we have $e = \sigma(e)$ since $q = \sigma(q) \vdash \sigma(e) = f(x) = e$.

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As $\mathrm{RV} \cup \mathrm{Lin}_A$ is stably embedded (*cf.* Remark 3.20), it follows (e.g., [6, Lemma 4.2.3]) that

$$e \in \mathrm{dcl}(\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M)).$$

So there is a $\mathfrak{L}(\mathcal{G}(A))$-definable function g and a tuple c in $\mathrm{RV}^m(M) \times \mathrm{Lin}_A^n(M)$ such that $g(c) = e$. Let $Y = g^{-1}(e)$. This is an A-definable subset of $\mathrm{RV}^m \times \mathrm{Lin}_A^n$. Consequently, $\ulcorner Y \urcorner \in (\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}}(A)$ and

$$e \in \mathrm{dcl}(\mathcal{G}(A), \ulcorner Y \urcorner) \subseteq \mathrm{dcl}(\mathcal{G}(A) \cup (RV \cup \mathrm{Lin}_A)^{\mathrm{eq}}(A)),$$

as required. □

We now want to describe the imaginaries in $\mathrm{RV} \cup \mathrm{Lin}_A$.

Proposition 6.2 [6, Proposition 5.3.1]. *Further assume that M is a* k-Γ*-expansion of* $\mathrm{Hen}_{0,0}$ *and that for every $n \in \mathbb{Z}_{\geq 2}$, one has $[\Gamma : n\Gamma] < \infty$ and the preimage in* RV *of any coset of $n\Gamma$ contains a point which is algebraic over $\emptyset$. Then, for $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$, we have*

$$(\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}}(A) \subseteq \mathrm{dcl}(\Gamma^{\mathrm{eq}}(A) \cup \mathrm{Lin}_A^{\mathrm{eq}}(A)).$$

Finally, let us relate $\mathrm{Lin}_A^{\mathrm{eq}}$ to the linear imaginaries $\mathrm{k}^{\mathrm{leq}}$:

Lemma 6.3. *Let $A = \mathrm{dcl}(A) \subseteq M$. Then $\mathrm{Lin}_A^{\mathrm{eq}}(A) \subseteq \mathrm{dcl}(\mathrm{k}^{\mathrm{leq}}(A))$.*

Proof. Recall that Lin_A is a stably embedded collection of k-vector spaces – see Remark 3.20. Take $e \in \mathrm{Lin}_A^{\mathrm{eq}}(A)$. Then e is the code of a definable set $X \subseteq \prod_i \mathrm{red}(R_i)$ where $a \in \prod_j \mathrm{red}(R'_j)$, the family $(X_a)_a$ is $\mathfrak{L}(A)$-definable and R_i and R'_j are A-definable $\mathfrak{m}$-avoiding module. Then $R = \prod R_i \times \prod_j R'_j$ is an A-definable $\mathfrak{m}$-avoiding module. Adding zero coordinates, we may assume that we have $X_a \subseteq \mathrm{red}(R)$ and $a \in \mathrm{red}(R)$.

For every basis b of $\mathrm{red}(R)$, the set X_a is $\mathfrak{L}_{\mathrm{vect}}(\mathrm{k}^{\mathrm{eq}}(A)ab)$-definable in $(\mathrm{k}, \mathrm{red}(R))$. Replacing a with $ab \in \mathrm{red}(R)^{d+1}$, we may assume that X_a is $\mathfrak{L}_{\mathrm{vect}}(a)$-definable. Let aEa' be the equivalence relation defined by $X_a = X'_a$ and let c be the type of R. Then $e \in \mathrm{dcl}(\mathrm{Lin}_{c,V^{d+1}}(A))$. □

We can now prove our main results. Let M be as in Theorem 5.3.

Theorem 6.4. *Assume that M is a* k- Γ*-expansion of* $\mathrm{Hen}_{0,0}$ *and that, for every $n \in \mathbb{Z}_{\geq 2}$, one has $[\Gamma : n\Gamma] < \infty$ and the preimage in* RV *of any coset of $n\Gamma$ contains a point which is algebraic over $\emptyset$. Then M weakly eliminates imaginaries down to $K \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$.*

Proof. Let $e \in M^{\mathrm{eq}}$ and $A = \mathrm{acl}(A)$. By Proposition 6.1, we have

$$e \in \mathrm{dcl}(\mathcal{G}(A) \cup (\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}}(A)).$$

By Proposition 6.2, we have

$$(\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}}(A) \subseteq \mathrm{dcl}(\Gamma^{\mathrm{eq}}(A), \mathrm{Lin}_A^{\mathrm{eq}}(A)) \subseteq \mathrm{dcl}(\Gamma^{\mathrm{eq}}(A), \mathrm{k}^{\mathrm{leq}}(A)),$$

where the last inclusion follows from Lemma 6.3. □

Remark 6.5. In general, imaginaries in the short exact sequence $\mathrm{k}^\times \to \mathrm{RV}^\times \to \Gamma$ might be more complicated. We can embed the short exact sequence $\mathrm{k}^\times(M) \to \mathrm{RV}^\times(M) \to \Gamma(M)$ into a sequence $\mathrm{k}^\times(M) \to H \to \mathbb{Q} \otimes \Gamma(M)$ – this is clear after adding a section in some $\aleph_1$-saturated elementary extension. A refinement of [6, Theorem 5.1.4] shows that any set X definable in $\mathrm{RV}(M)$ is weakly coded in $\Gamma^{\mathrm{eq}}(M) \cup (H_\Delta^{\mathrm{eq}} \cap \mathrm{dcl}(M))$, where Δ is the divisible hull of $\Gamma(\ulcorner X \urcorner)$.

The elements of $H_\Delta^{\mathrm{eq}} \cap \mathrm{dcl}(\mathrm{RV})$ correspond to M-definable subsets of H_δ for some tuple δ in $\Delta \subseteq \mathbb{Q} \otimes \Gamma(M)$. These are the imaginaries that we need to add, or classify, to obtain a version of Theorem 6.4 without any finiteness assumption on $\Gamma/n\Gamma$.

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If $[\Gamma : n\Gamma] < \infty$ for every $n \geq 2$ and we add constants as in Theorem 6.4, then any H_δ, for $\delta \in \Delta$, is canonically isomorphic to some H_γ with $\gamma \in \Gamma(\ulcorner X \urcorner)$; and we recover Theorem 6.4. In contrast, if k is a pure algebraically closed field, the k-linear structure H_Δ eliminates imaginaries and $H_\Delta \cap \mathrm{dcl}(M) = H_{\Gamma(\ulcorner X \urcorner)}$, yielding back Corollary 4.11.

Theorem 6.6. *Assume that M admits $\mathfrak{L}$-definable angular components. Then M weakly eliminates imaginaries down to $K \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$.*

Proof. Let $e \in M^{\mathrm{eq}}$ and $A = \mathrm{acl}(A)$. By Proposition 6.1, $e \in \mathrm{dcl}(\mathcal{G}(A), (\mathrm{RV} \cup \mathrm{Lin}_A^{\mathrm{eq}})(A))$. Since RV is $\mathfrak{L}$-definably isomorphic to $\mathrm{k}^\times \times \Gamma$, then $(\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}} \subseteq (\Gamma \cup \mathrm{Lin}_A)^{\mathrm{eq}}$. The statement now follows from orthogonality of Γ and Lin_A and Lemma 6.3. □

As an illustration, we conclude this paper with the complete classification of (almost) k-internal sets, when the value group is dense.

Corollary 6.7. *Let M be as in Theorem 6.4 or Theorem 6.6 and assume that $\Gamma(M)$ is dense. Let $A \subseteq M^{\mathrm{eq}}$ and X be A-definable. The following statements are equivalent:*

1. *X is* k*-internal;*
2. *X is almost* k*-internal;*
3. *X is orthogonal to Γ;*
4. $X \subseteq \mathrm{dcl}(\mathrm{acl}(A), \mathrm{Lin}_A)$.

Proof. The fourth statement is a particular case of the first statement. The second statement is a particular case of the first, and it implies the third since k and Γ are orthogonal. There remains to prove that if X is orthogonal to Γ then it is a subset of $\mathrm{acl}(A) \cup \mathrm{Lin}_A^{\mathrm{eq}}$. By Theorems 6.4 and 6.6, any element $a \in X$ is weakly coded by some tuple η in $\mathrm{K} \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$. Then η also lies on a A-definable set orthogonal to Γ.

There remains to show that $\eta \in \mathrm{dcl}(\mathrm{acl}(A) \cup \mathrm{Lin}_A^{\mathrm{eq}})$. We may assume that η is a single point. Since X is orthogonal to Γ, if $\eta \in \mathrm{K} \cup \Gamma^{\mathrm{eq}}$, then $\eta \in \mathrm{acl}(A)$. If $\eta \in \mathrm{Gr}$, then, by Proposition 3.22, $\eta \in \mathrm{acl}(A)$. Finally, if $\eta \in \mathrm{red}(R_s)^{\mathrm{eq}}$, for some $s \in \mathrm{Gr}$, then $s \in \mathrm{acl}(A)$ and hence $\eta \in \mathrm{Lin}_A^{\mathrm{eq}}$. □

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