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Asset Substitution and Structured Financing

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Abstract

This article shows how structured financing can be used to solve the asset substitution problem in a dynamic setting. Structuring induces the firm's owner to optimally choose the first best operating strategy even though the owner's value function might be locally convex (concave), which would ordinarily lead to overinvestment (underinvestment) in risky projects. This result is demonstrated in two different continuous time settings—one that is based on the risk-shifting framework of Leland (1998) and one that generalizes the scaled return model of Green (1984). It is shown that the contractual nature of the structuring is a key determinant of the issuing firm's dynamic asset volatility. Furthermore, unlike nonstructured financing, the default (conversion) probability of a structured debt security may be increasing (decreasing) in the firm's total assets. Structured securities are therefore hedge assets, which potentially explains the popularity of structured securities among investors and third-party issuers.

I. Introduction

A key insight of Jensen and Meckling (1976) is that external financing can alter the firm's optimal operating strategy such that equityholders no longer maximize the firm's total value. In the case of debt financing, the equityholders retain a convex claim on the firm's assets and thus have an incentive to overinvest in risky assets after the debt is in place. The resulting reduction in total firm value is known as the agency cost of asset substitution. Green (1984) shows in a single-period setting how this cost can be eliminated via capital structure design.[1] Instead of using straight debt financing, the firm should use convertible debt financing and balance the convex and concave regions of the contract to create a security that is locally equity-like. However, as Hennessy and Tserlukevich (2008) demonstrate,

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[1]Related early works include Smith and Warner (1979), Barnea, Haugen, and Senbet (1980), and Haugen and Senbet (1981), who show how bond covenants, callable debt, and stock options, respectively, can mitigate the cost of asset substitution.

convertible debt cannot eliminate the cost of asset substitution in a dynamic setting, since equity is always risk-loving when the firm is near bankruptcy. Thus, it remains an open question whether or not the asset substitution problem can be solved via capital structure design in a dynamic setting. This article resolves the question by showing how structured financing can induce the first best strategy even if the equityholders' value function is locally convex or concave.

According to McCann and Cilia (1994), "structured securities are hybrids, having components of straight debt instruments and derivatives intertwined." Thus, structured finance employs the tools of financial engineering to facilitate the capital raising process. Structured securities are used in lieu of conventional securities, such as straight debt, and have exotic features that are included to satisfy the financing need of the issuer or the investment need of the buyer.[2] Typical examples of structured securities include exchangeable bonds (Swieringa and Morse (1985), Barber (1993)), liquid yield option notes (McConnell and Schwartz (1986)), equity-indexed notes (Chen and Sears (1990), Finnerty (1993), Gorton and Pennacchi (1993), and Stoimenov and Wilkens (2005)), gold-indexed securities (Chidambaran, Fernando, and Spindt (2001)), callable warrants (Schultz (1993)), reverse convertible debentures (Flannery (2005)), reverse exchangeable securities (Benet, Giannetti, and Pissaris (2006)), and death spiral convertibles (Hillion and Vermaelen (2004)).

This article focuses on a particular type of structured security in which the payoff-relevant features are correlated with the issuing firm's assets.[3] Table 1 lists several motivating examples. Each security in the table has a payoff that is tied to a contractually specified index. In turn, the index is correlated with the performance of the issuing firm's assets. For example, a silver mining firm (Sunshine Mining) specifies a silver index in its bond indenture, a financial brokerage (Oppenheimer) specifies an index that is tied to the NYSE trading volume, and an oil firm (Marathon Oil) specifies an oil index. In other cases, the connection is less direct but equally relevant: The chemical firm Shin Etsu, which uses crude oil as an input to production, specifies an index that is inversely related to oil prices. Smithson and Chew (1992) and Chidambaran et al. (2001) argue that the securities in Table 1 can be viewed as bundled hedges. Bundling a hedge with a debt offering allows the issuing firm to credibly commit to the hedging strategy that maximizes the firm's total value (Smith and Stulz (1985)).

This article identifies the particular bundled hedging strategy that should be used to solve the asset substitution problem. In other words, rather than study particular cases of bundling as in Chidambaran et al. (2001), this article derives an optimal bundling strategy. In the simpler dynamic settings considered here, the optimal bundling strategy completely eliminates the agency cost of dynamic asset substitution. In more complex settings, such as those that involve contracting

[2] Although this article focuses on the issuer's financing problem (i.e., the need to raise capital while simultaneously solving the asset substitution problem), it is shown below that structured securities are hedge assets that offer unique payoff patterns. Thus, structured financing helps to satisfy the buyer's investment need, which explains the interest from third-party issuers such as investment banks.

[3] This type of security is contained in the broader class of securities analyzed in Manso, Strulovici, and Tchistyi (2006). However, these authors do not examine the asset substitution problem.

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TABLE 1

Sample of Indexed Note Offerings (1980–2005)

Table 1 shows several examples of structured securities. Each security in the table has a payoff that is tied to a contractually specified index. The specified index is correlated with the performance of the issuing firm's assets. [a]Chidambaran, Fernando, and Spindt (1999); [b]Smithson and Chew (1992); [c]The New York Times (July 16, 1986); [d]American Metal Market (Nov. 30, 1988); [e]Moody's Special Report, Global Credit Research (Jan. 1996); [f]Bloomberg Financial Markets; [g]Zurich Financial Services, Press Release (Jan. 15, 2002); [h]J.P. Morgan Chase, Prospectus Term Sheet (Dec. 1, 2005).

Issuing Company	Year of Issue	Description
Sunshine Mining	1980	Silver-indexed notes[a]
Oppenheimer & Co.	1981	Indexed to NYSE trading volume[b]
Marathon Oil	1986	Oil-indexed notes[c]
Pegasus Gold	1986	Gold-indexed notes[b]
Salomon Brothers	1986	Indexed to the S&P 500[b]
Merrill Lynch	1987	Indexed to the NYSE Composite[b]
Eastman Kodak	1987	Gold-indexed notes[a]
Magma Copper	1988	Copper-indexed notes[d]
Presidio Oil & Gas	1989	Natural-gas-indexed notes[e]
Shin Etsu Chemical	1991	Inverse oil-indexed notes[b]
Freeport-McMoRan	1993	Gold-indexed notes[a]
Sallie Mae	1997	CPI-indexed notes[f]
Zurich Financial	2002	Indexed to the Swiss Market Index (SMI)[g]
J.P. Morgan Chase	2005	Indexed to the Russell 2000[h]

constraints, the strategy mitigates but does not completely eliminate the agency cost. In both cases, the strategy involves structured securities whose payoff-relevant features are correlated with the firm's total assets, as in Table 1. Furthermore, the correlation is implemented via a contractual structuring variable that serves the role of an index.

While straight debt with a fixed face value ordinarily leads to overinvestment, a structured debt contract in which the face value varies with the firm's total assets mitigates the equityholders' incentive to overinvest in risky assets. In the simpler dynamic settings considered here, the equityholders are induced to always choose the first best operating strategy, and thus the agency cost of asset substitution is 0. A similar result holds if the equityholders' value function is locally concave, which would ordinarily lead to underinvestment, or, if the value function has both convex and concave regions, which would ordinarily lead to dynamic risk shifting.

When a firm issues nonstructured straight debt, the equityholders' (debtholders') exposure involves limited downside (upside) but substantial upside (downside). The oppositeness of these exposures creates an incentive for equityholders to deviate from the first best. Structuring the debt can eliminate this incentive by dynamically altering the upside and downside exposures of both equityholders and debtholders. To see this, consider a firm that is financed with equity and structured debt. The structured debt's face value depends on the performance of the firm's assets. In particular, suppose the face value's elasticity with respect to the firm's assets is greater than 1. As the assets increase (decrease) by 1%, the face value of the structured debt increases (decreases) by more than 1%, which in turn increases (decreases) the likelihood of default. If default occurs, the equity is worthless and the debtholders take ownership of the firm. Unlike the case of nonstructured debt, the structured debtholders take ownership when the assets are at a relatively high level, which gives the debtholders upside exposure. If, instead, default does not occur, the equityholders retire the debt and retain ownership of the firm. However, unlike the case of nonstructured debt, the equityholders retain ownership when the assets are at a relatively low level, which gives the

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equityholders downside exposure. By issuing a structured debt security with the proper dynamic elasticity, the equityholders' upside and downside exposures can be adjusted dynamically so that the equityholders always find it optimal to choose the first best operating strategy.

To illustrate the robustness of structured financing for solving the asset substitution problem, two different dynamic settings are considered. The first is a modified version of Leland's (1998) risk-shifting model.[4] In this setting, the firm's owner can costlessly switch back and forth between two different drift-diffusion pairs that govern the evolution of the firm's total assets. Switching back and forth through time captures the dynamics of asset substitution, as described in Childs, Mauer, and Ott (2005) and Hennessy and Tserlukevich (2008). Furthermore, since switching affects the asset drift, asset substitution can be value-dissipating, as in Mauer and Sarkar (2005). However, in contrast to the existing literature, this article uses a finite time horizon and does not rely on numerical methods to solve the problem.

The second setting is a generalized version of Green's (1984) two-project model. Green's (1984) analysis exhibits the following properties: i) the first best strategy involves equal investment in the two projects; ii) the optimal strategy in the presence of debt financing involves overinvestment in the riskier project; and iii) a properly designed convertible debt contract can restore the first best strategy. This article constructs a continuous time model that replicates features i) and ii), but does not replicate iii) for the reasons discussed in Hennessy and Tserlukevich (2008). It is then shown how structured financing can restore iii). In other words, while a nonstructured convertible debt contract solves the asset substitution problem in the single-period model of Green (1984), a structured convertible debt contract (or some other structured contract, as shown later) is required to solve the asset substitution problem in a dynamic setting.

Childs et al. (2005) and Ju and Ou-Yang (2005) show that the agency cost of asset substitution can be mitigated if a firm has sufficient financial flexibility to dynamically rebalance its short-term debt level. Since the structured debt contracts in this article have contractual features (e.g., face values) that are time-varying, structuring can be viewed as a substitute for dynamic debt rebalancing. However, in contrast to the dynamic debt rebalancing literature, all the structured securities in this article are long-term in nature. The firm finances its operations only once and thus does not require the flexibility to revisit the capital markets in the future for additional funding. Instead, the original financing stays in place, but the payoff-relevant parameters change dynamically as the firm's total assets change.

While the asset substitution problem was initially studied in a single-period setting,[5] the profession has recently taken an increased interest in studying the effects of dynamic asset substitution. For instance, Mello and Parsons (1992),

[4]Related (but not identical) risk-shifting models can be found in Carlson and Lazrak (2006), Chesney and Gibson-Asner (2001), Childs, Mauer, and Ott (2005), Décamps and Djembissi (2007), Ericsson (2000), Hennessy and Tserlukevich (2008), Mauer and Triantis (1994), and Subramanian (2005).

[5]See Barnea et al. (1980), Haugen and Senbet (1981), Gavish and Kalay (1983), Green (1984), Green and Talmor (1986), John (1987), Spatt and Sterbenz (1993), and Nachman and Noe (1996).

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Leland (1998), Ericsson (2000), Décamps and Djembissi (2007), Mauer and Sarkar (2005), and Parrino and Weisbach (1999) focus on quantifying the agency cost of dynamic asset substitution. In contrast to these papers, the current paper shows how structured financing can be used to resolve the bondholder-stockholder agency conflict in a dynamic setting. Unlike the existing literature, the main focus here is on solving the asset substitution problem rather than measuring its cost. The agency cost of dynamic asset substitution is equal to 0 in the presence of a properly designed capital structure, at least for the simpler dynamic settings considered here.

The remainder of the article is organized as follows. Section II modifies the risk-shifting framework of Leland (1998) and shows how a properly designed capital structure can eliminate the agency cost of asset substitution. The main result is summarized in Proposition 1, and explicit analytical solutions are presented in Tables 2–4 for a variety of parameter restrictions. Section III then generalizes Green's (1984) model to a continuous time setting. Explicit analytical solutions for this setting are presented in Table 5. Section IV examines the characteristics of structured contracts and shows how the standard intuition concerning default probabilities, default costs, conversion probabilities, and expected yields may be incorrect for structured securities. Section V offers concluding remarks and directions for future research.

II. A Model with Two Risk-Return Choices

Consider a continuous time model in which uncertainty is represented by a one-dimensional Brownian motion, z_t, that is defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Time is indexed by $t \in [0, T]$, and $\{\mathcal{F}_t : t \in [0, T]\}$ is the augmented filtration that is generated by z_t. The owner of a firm has access to a single risky project that requires an investment of $V_0 > 0$ at time 0. The project produces a payoff of V_T at time T, where V_t follows the process

$$dV_t = \mu_t V_t dt + \sigma_t V_t dz_t, \tag{1}$$

for $t \in [0, T]$. The project does not produce intermediate cash flows, nor does it require intermediate investments. If undertaken, the project in equation (1) will comprise the total assets of the firm.

The owner's initial funds, denoted by W, satisfy $0 < W < V_0$. Thus, to undertake the project, the owner must raise $V_0 - W$ via external financing. The final payoff V_T is shared at time T by the owner and the external financier, where the sharing rule is determined by the capital structure put in place at time 0. As in Green (1984) and Green and Talmor (1986), it is assumed that there are no taxes, coupons, or debt amortization. Furthermore, unlike Black and Cox (1976) and Leland (1998), there is no possibility for the firm to reorganize prior to time T. While formally incorporating these features is possible, their omission simplifies the setting so that an explicit analytical solution is possible.

Two features of the model raise comment. First, why would the firm want to use debt financing as opposed to outside equity? In the absence of taxes, structured debt financing and outside equity are equally successful in solving the asset substitution problem. But once taxes are included, which are ignored only for

simplicity, structured debt is the more attractive solution due to the tax deductibility of interest payments.[6] In a world with taxes, structured debt maintains the tax advantages of nonstructured debt but also has desirable equity-like features that allow the firm's owner to solve the asset substitution problem. Second, why would the firm want to use structured debt as opposed to nonstructured debt, with a covenant requiring the owner to choose the first best? By using structured debt, the firm's owner can design his residual claim so that it is optimal to choose the first best operating strategy for all $t \in [0, T]$. Since the owner has no incentive to deviate from the first best, the debtholders do not need to monitor the owner's actions. On the other hand, using nonstructured debt with restrictive covenants does not eliminate the owner's incentive to expropriate value from the debtholders. Only with continuous monitoring, which may be costly, will the debtholders be assured that the owner follows the first best strategy for all $t \in [0, T]$. For these reasons, this article focuses on structured financing as the most attractive type of security for solving the asset substitution problem.

The owner's operating discretion is modeled by allowing the owner to choose between two drift-diffusion pairs. At each t, the owner chooses $\{\mu_t, \sigma_t\}$ to be either $\{\mu_1, \sigma_1\}$ or $\{\mu_2, \sigma_2\}$, where μ_1, σ_1, μ_2, and σ_2 are positive constants. The owner can switch back and forth between the pairs $\{\mu_1, \sigma_1\}$ and $\{\mu_2, \sigma_2\}$ without cost, where the switching rule is $\mathcal{F}_t$-measurable. Switching back and forth allows the owner to engage in dynamic asset substitution.[7]

To capture the notion that the owner operates his firm in an economy that is populated by risk-averse investors, the stochastic discount factor at time t is

$$m_t = e^{-rt-\frac{1}{2}\lambda^2 t-\lambda z_t}, \tag{2}$$

where $r > 0$ is the constant instantaneous interest rate and $\lambda > 0$ is the constant instantaneous risk premium. Using equation (2), the time t value of the firm is determined by evaluating the expectation $\mathrm{E}_t[(m_T/m_t)V_T]$, where $\mathrm{E}_t[\cdot] = \mathrm{E}[\cdot|\mathcal{F}_t]$ denotes expectation under the P probability measure. For example, if the pair $\{\mu_1, \sigma_1\}$ is chosen for all $s \in [t, T]$, the firm's value at time t is

$$\mathrm{E}_t\left[\frac{m_T}{m_t}V_T\right] = V_t e^{(\mu_1-\sigma_1\lambda-r)(T-t)}. \tag{3}$$

If, instead, the manager chooses the pair $\{\mu_2, \sigma_2\}$ for all $s \in [t, T]$, the time t firm value is $V_t e^{(\mu_2-\sigma_2\lambda-r)(T-t)}$. Thus, the firm value is maximized by choosing the drift-diffusion pair at each t that has the highest risk-adjusted instantaneous return. In other words, the owner should always choose the pair that satisfies $\max[\mu_1 - \sigma_1\lambda, \mu_2 - \sigma_2\lambda]$. This is the first best strategy.

Since the initial firm value under the first best strategy is

$$V_0 e^{(\max[\mu_1-\sigma_1\lambda,\mu_2-\sigma_2\lambda]-r)T},$$

[6]This remains true in the case of zero-coupon debt, since the amortization of the original issue discount is treated as interest for tax purposes.

[7]Other types of switching models have also been used. For example, Mello and Parsons (1992) and Mauer and Triantis (1994) use a switching model for the firm's decision to operate or shut down, while Childs et al. (2005) use a switching model for the firm's decision to exercise or reverse its investment in growth options.

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the net present value (NPV) at time 0 is

$$V_0 e^{(\max[\mu_1-\sigma_1\lambda,\mu_2-\sigma_2\lambda]-r)T} - V_0. \tag{4}$$

The NPV in expression (4) is positive if $\max[\mu_1 - \sigma_1\lambda, \mu_2 - \sigma_2\lambda] > r$. It is assumed that this inequality holds, since otherwise the owner's best course of action is to invest everything in the riskless asset.

Equation (2) can also be written as $m_t = e^{-rt}\,\mathrm{E}_t[dQ/dP]$, where $dQ/dP = e^{-(1/2)\lambda^2 T-\lambda z_T}$ defines a new probability measure Q such that $dz_t = dz_t^Q - \lambda dt$, and z_t^Q is a standard Brownian motion under Q. The process for V_t under the risk-adjusted measure Q is

$$dV_t = (\mu_t - \sigma_t\lambda)\,V_t dt + \sigma_t V_t dz_t^Q, \tag{5}$$

and thus equation (3) can be rewritten as $\mathrm{E}_t^Q[e^{-r(T-t)}V_T]$. This notation is used in the sequel.

A. Agency Cost of Asset Substitution

The literature defines the agency cost of asset substitution as the difference between the first best firm value and the firm value achieved in the presence of external financing. When external funds are used to finance the firm, the owner chooses a drift-diffusion pair at each t to maximize his residual claim on the firm's assets. If the owner's optimal strategy in the presence of external financing differs from the first best strategy, the total firm value will be less than the first best value. Since the external financing is fairly priced at time 0, the reduction in total firm value at time 0 is borne entirely by the owner. This produces a positive agency cost of asset substitution, and thus the owner fails to capture the full amount of the NPV in expression (4). On the other hand, if the owner's choice of capital structure creates a residual claim whose optimizing strategy coincides with the first best strategy, the total firm value in the presence of external financing will coincide with the first best firm value. In this case, the owner captures the full amount of the NPV in expression (4), and the agency cost of asset substitution is 0. The next section shows how a zero agency cost can be achieved by using structured financing.

B. Structured External Financing Contracts

In this paper, a structured security is an external financing contract whose payoff-relevant features (e.g., the face value or the conversion value) are allowed to covary with the firm's assets.[8] To formally describe a structured security, let $\{\theta_t : t \in [0,T]\}$ be $\mathcal{F}_t$-measurable, and define the structuring variable ξ_t as

$$\xi_t = \exp\left[-\frac{1}{2}\int_0^t \theta_s^2 ds + \int_0^t \theta_s dz_s^Q\right]. \tag{6}$$

[8] Under this definition, a structured security can also be a hybrid security, since it may contain both debt- and equity-like features. However, not all hybrids are structured (e.g., a nonstructured convertible bond). Structured securities are therefore a generalized class of security with hybrids and nonhybrids arising as special cases.

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It is assumed that the functional form of θ_t is specified by the owner at time 0 and thereafter cannot be altered. In other words, $\{\theta_t : t \in [0, T]\}$ is part of the contract's design, which is strictly enforceable. Note that ξ_t and V_t are correlated, since z_t^Q drives both processes. Thus, ξ_t serves the role of an index, as in Table 1.

The owner finances the firm by issuing a structured contract whose time T payoff is

$$F(V_T ; \xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n), \tag{7}$$

where $K_1, K_2, \ldots, K_n$ are constants. The contract F is nondecreasing in V_T, has no intermediate cash flows (zero coupon), and is homogeneous of degree one in all of its arguments. If the owner specifies $\theta_t = 0$ for all t, then $\xi_T = 1$, and expression (7) reduces to a nonstructured contract. At the other extreme, if the owner specifies a nonzero θ_t for all t, ξ_T varies with the state of nature, and $\xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n$ are correlated with the performance of the firm's assets at time T.

For example, let $n = 2$, and suppose that F is a structured convertible bond whose payoff is

$$\min [V_T, \xi_T K_1] + \max [V_T - \xi_T K_2, 0]. \tag{8}$$

The face value $\xi_T K_1$ and the conversion value $\xi_T K_2$ are correlated with V_T. In the special case of $\theta_t = 0$ for all t, the structured convertible bond reduces to a nonstructured convertible bond in which the bondholders take control of the firm upon conversion.[9]

By defining the structuring variable as in equation (6) and specifying an appropriate $\{\theta_t : t \in [0, T]\}$ at time 0, the owner can be induced to always choose the first best operating strategy. To see this, suppose the owner issues a structured security at time 0. Once the security is issued, the owner no longer has direct control over ξ_t, since it depends only on the path of z_t^Q, which is governed by nature, and on $\{\theta_t : t \in [0, T]\}$, which cannot be altered since it is part of the issued security's design. However, due to the homogeneity of F and the form of equation (6), ξ_T can be factored out of the debt's payoff and used as a Radon-Nikodym derivative to define a new probability measure (say, R) that is equivalent to the risk-adjusted measure Q. The structured security and the owner's residual claim are valued by the market using R. By cleverly specifying $\{\theta_t : t \in [0, T]\}$ at time 0, the owner can choose the particular R that induces the first best operating choice (e.g., $\{\mu_2, \sigma_2\}$) for all t. Essentially, the owner designs the firm's capital structure at $t = 0$ such that the incentive for dynamic asset substitution is eliminated for all $t \in [0, T]$.

More specifically, ξ_t can be written as $\xi_t = \mathrm{E}_t^Q[dR/dQ]$, where $dR/dQ = \xi_T$ defines a new probability measure R such that $dz_t^Q = dz_t^R + \theta_t dt$ and z_t^R is a standard Brownian motion under R. Thus, the value of the owner's residual claim is

[9]Nonstructured convertible bonds are analyzed in Ingersoll (1977) and Brennan and Schwartz (1980), (1982). While Ingersoll ((1977), sect. 7) analyzes the case of a time-varying conversion value, which is related to the term $\xi_T K_2$ in expression (8), he does not examine the asset substitution problem.

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$$(9)\qquad \mathrm{E}_t^Q\left[e^{-r(T-t)}\left(V_T - F(V_T\,;\xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n)\right)\right]$$
$$= \mathrm{E}_t^R\left[\frac{\xi_t}{\xi_T} e^{-r(T-t)}\left(V_T - F(V_T\,;\xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n)\right)\right]$$
$$= \xi_t\,\mathrm{E}_t^R\left[e^{-r(T-t)}\left(V_T\xi_T^{-1} - F(V_T\xi_T^{-1}\,; K_1, K_2, \ldots, K_n)\right)\right],$$

which follows since F is homogeneous. The owner's optimization problem is therefore

$$(10)\qquad \max_{\{\mu_s,\sigma_s\},\, s\in[t,T]} \xi_t\,\mathrm{E}_t^R\left[e^{-r(T-t)}\left(G_T - F(G_T\,; K_1, K_2, \ldots, K_n)\right)\right],$$

where $\{\mu_s, \sigma_s\}$ is either $\{\mu_1, \sigma_1\}$ or $\{\mu_2, \sigma_2\}$ at each s, and where $G_t = V_t\xi_t^{-1}$ follows the process

$$(11)\qquad dG_t = (\mu_t - \sigma_t\lambda)G_t dt + (\sigma_t - \theta_t)\,G_t dz_t^R$$

and $G_0 = V_0$. Letting $f(G_t, t) = \mathrm{E}_t^R\left[e^{-r(T-t)}\left(G_T - F(G_T\,; K_1, K_2, \ldots, K_n)\right)\right]$, Bellman's principle of stochastic control can be used to restate the problem in expression (10) as

$$(12)\quad 0 = \max_{\{\mu_t,\sigma_t\}}\left[\xi_t\left(\frac{\partial f}{\partial t} - rf + (\mu_t - \sigma_t\lambda)G_t\frac{\partial f}{\partial G} + \frac{1}{2}(\sigma_t - \theta_t)^2 G_t^2\frac{\partial^2 f}{\partial G^2}\right)\right],$$

subject to the terminal condition $f(G_T, T) = G_T - F(G_T\,; K_1, K_2, \ldots, K_n)$. Two observations can be made about equation (12). First, since ξ_t factors out of the Bellman equation, the owner's effective value function is simply $f(G_t, t)$. Second, note that the terms multiplying $\partial^2 f/\partial G^2$ include θ_t. This is a key property, since a judicious choice of θ_t can induce the owner to choose the first best strategy even if the sign of $\partial^2 f/\partial G^2$ would ordinarily lead the owner to deviate from the first best.

For example, suppose $\{\mu_2, \sigma_2\}$ is the first best pair, and let $\sigma_2 < \sigma_1$. If the owner finances the firm with nonstructured straight debt, $\theta_t = 0$ and the classical asset substitution problem arises. The owner's equity stake resembles a call option on the firm's assets, and thus $\partial^2 f/\partial G^2$ is positive. In this case, the owner has an incentive to abandon the first best pair $\{\mu_2, \sigma_2\}$ in favor of the higher volatility second best pair $\{\mu_1, \sigma_1\}$. From equation (12), this incentive will be strongest when $\partial^2 f/\partial G^2$ is large relative to $\partial f/\partial G$. Choosing the second best pair produces a lower risk-adjusted return, which multiplies $\partial f/\partial G$, but this may be more than offset by the higher volatility, which multiplies $\partial^2 f/\partial G^2$. To eliminate this incentive, the owner can credibly commit to the first best pair by issuing structured debt with $\theta_t = \sigma_1 + \sigma_2$ for all t. To see that this value for θ_t is appropriate, note that $\partial^2 f/\partial G^2$ is also positive for (nonconvertible) structured debt. Thus, the owner can achieve the higher volatility σ_1 by choosing the lower volatility, first best pair $\{\mu_2, \sigma_2\}$. This follows because σ_2 is the argument that maximizes

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$(\sigma_t - (\sigma_1 + \sigma_2))^2$, which in turn multiplies $\partial^2 f/\partial G^2$ in equation (12). This simple example illustrates the logic that underlies structuring.[10]

Given the above framework, three parametric cases are worth analyzing: i) $\mu_2 - \sigma_2\lambda > \mu_1 - \sigma_1\lambda$ and $\sigma_2 > \sigma_1$; ii) $\mu_2 - \sigma_2\lambda > \mu_1 - \sigma_1\lambda$ and $\sigma_2 < \sigma_1$; and iii) $\mu_2 - \sigma_2\lambda = \mu_1 - \sigma_1\lambda$ and $\sigma_2 > \sigma_1$. All three cases involve $\sigma_2 \neq \sigma_1$, since otherwise there is no risk-shifting problem to study. Furthermore, the three cases are exhaustive, since the omitted cases can be handled by reversing the subscripts 1 and 2. With these cases in mind, the article's first result is presented in Proposition 1.

Proposition 1. Let $\partial f/\partial G > 0$. Then the agency cost of asset substitution is 0 under each set of conditions listed below:

i) Let $\mu_2 - \sigma_2\lambda > \mu_1 - \sigma_1\lambda$ and $\sigma_2 > \sigma_1$. Furthermore, let $\theta_t = 0$ when $\partial^2 f/\partial G^2 \geq 0$, and let $\theta_t = \sigma_1 + \sigma_2$ when $\partial^2 f/\partial G^2 < 0$. Then the owner optimally chooses $\{\mu_2, \sigma_2\}$ for all t, which is the first best strategy.

ii) Let $\mu_2 - \sigma_2\lambda > \mu_1 - \sigma_1\lambda$ and $\sigma_2 < \sigma_1$. Furthermore, let $\theta_t = \sigma_1 + \sigma_2$ when $\partial^2 f/\partial G^2 \geq 0$, and let $\theta_t = 0$ when $\partial^2 f/\partial G^2 < 0$. Then the owner optimally chooses $\{\mu_2, \sigma_2\}$ for all t, which is the first best strategy.

iii) Let $\mu_2 - \sigma_2\lambda = \mu_1 - \sigma_1\lambda$ and $\sigma_2 > \sigma_1$. Since $\mu_2 - \sigma_2\lambda = \mu_1 - \sigma_1\lambda$, the owner always achieves the first best. If $\theta_t = 0$ when $\partial^2 f/\partial G^2 < 0$ or if $\theta_t = \sigma_1 + \sigma_2$ when $\partial^2 f/\partial G^2 \geq 0$, the owner implements the first best by optimally choosing $\{\mu_1, \sigma_1\}$ at time t. If $\theta_t = 0$ when $\partial^2 f/\partial G^2 \geq 0$ or if $\theta_t = \sigma_1 + \sigma_2$ when $\partial^2 f/\partial G^2 < 0$, the owner implements the first best by optimally choosing $\{\mu_2, \sigma_2\}$ at time t.

The proof of the proposition is verified by examining the Bellman equation in (12). The owner's choice of the drift-diffusion pair affects two terms in equation (12): the $\partial f/\partial G$ term and the $\partial^2 f/\partial G^2$ term. For case i) of the proposition, the owner chooses $\{\mu_2, \sigma_2\}$ at each t, which is the first best strategy. This produces the highest risk-adjusted return, which multiplies $\partial f/\partial G > 0$. Furthermore, it maximizes σ_t^2 when $\partial^2 f/\partial G^2 \geq 0$, and it minimizes $(\sigma_t - (\sigma_1 + \sigma_2))^2$ when $\partial^2 f/\partial G^2 < 0$.

For case ii), the owner's problem in equation (12) is again solved by choosing the first best pair $\{\mu_2, \sigma_2\}$ at each t. This produces the highest risk-adjusted return, it maximizes $(\sigma_t - (\sigma_1 + \sigma_2))^2$ when $\partial^2 f/\partial G^2 \geq 0$, and it minimizes σ_t^2 when $\partial^2 f/\partial G^2 < 0$.

Since $\mu_2 - \sigma_2\lambda = \mu_1 - \sigma_1\lambda$ in case iii), the agency cost of asset substitution is always 0.[11] However, the owner's optimal strategy is affected by the sign of $\partial^2 f/\partial G^2$ and the value of θ_t. If $\theta_t = 0$ for all t, the owner optimally chooses $\{\mu_2, \sigma_2\}$ if $\partial^2 f/\partial G^2 \geq 0$ and $\{\mu_1, \sigma_1\}$ if $\partial^2 f/\partial G^2 < 0$. This is the case of pure dynamic risk shifting. Alternatively, if $\theta_t = \sigma_1 + \sigma_2$ for all t, the owner optimally

[10]When $\partial^2 f/\partial G^2 > 0$, it is tempting to argue that the owner should specify a very large value for θ_t. Although a large value induces the first best operating choice in the above example, it does not increase the value of the owner's claim, since the debt is fairly priced at time 0. The debt's initial face value simply adjusts to reflect the large value for θ_t. This issue is revisited when structures that do not rely on the project parameters are discussed.

[11]Using the terminology of Nachman and Noe (1996), asset substitution is *value neutral* when $\mu_2 - \sigma_2\lambda = \mu_1 - \sigma_1\lambda$. For the case of $\mu_2 - \sigma_2\lambda \neq \mu_1 - \sigma_1\lambda$, *value-dissipating* asset substitution can be avoided by using a capital structure that satisfies condition i) or ii) of Proposition 1.

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chooses $\{\mu_1, \sigma_1\}$ if $\partial^2 f/\partial G^2 \geq 0$ and $\{\mu_2, \sigma_2\}$ if $\partial^2 f/\partial G^2 < 0$. This strategy also involves dynamic risk shifting, but now the drift-diffusion pairs are reversed relative to the sign of $\partial^2 f/\partial G^2$. Dynamic risk shifting will also occur when $\partial^2 f/\partial G^2$ does not change signs but θ_t changes back and forth between 0 and $\sigma_1 + \sigma_2$. On the other hand, structuring can be used to induce the owner to optimally commit to a particular drift-diffusion pair. If $\theta_t = 0$ when $\partial^2 f/\partial G^2 \geq 0$, and $\theta_t = \sigma_1 + \sigma_2$ when $\partial^2 f/\partial G^2 < 0$, the owner's optimal strategy is to choose $\{\mu_2, \sigma_2\}$ for all t. Alternatively, if $\theta_t = \sigma_1 + \sigma_2$ when $\partial^2 f/\partial G^2 \geq 0$, and $\theta_t = 0$ when $\partial^2 f/\partial G^2 < 0$, the owner's optimal strategy is to choose $\{\mu_1, \sigma_1\}$ for all t. Many of these sub-cases are illustrated later.

C. Notation and a Preliminary Result

Note that the owner's Bellman equation in (12) involves the instantaneous variance $(\sigma_t - \theta_t)^2$. If this variance is constant, both the owner's residual claim and the value of the issued security can be expressed in terms of the Black and Scholes (1973) formula.[12] Let $C_{BS}(G_t, K, T - t, \delta, \sigma)$ denote the Black and Scholes (1973) European call option formula, where G_t is the underlying asset, K is the strike price, $T - t$ is the time to expiration, δ is the effective discount rate, and σ is the volatility parameter. These five inputs to the Black and Scholes (1973) formula will vary on a case-by-case basis depending on θ_t and the owner's optimal strategy. For example, the effective discount rate may be $\mu_1 - \sigma_1\lambda$ or $\mu_2 - \sigma_2\lambda$, the volatility parameter may be σ_1 or σ_2, and the underlying asset may be V_t (if θ_t is always zero) or G_t (if θ_t is nonzero).

If $(\sigma_t - \theta_t)^2$ is not constant, the Black and Scholes (1973) framework is no longer valid. In this case, given the values of θ_t in Proposition 1, the instantaneous variance may periodically switch back and forth between σ_1^2 and σ_2^2. To handle this case, a preliminary result is presented that is used in the remainder of the article. Let $\sigma_{\max} = \max[\sigma_1, \sigma_2]$, $\sigma_{\min} = \min[\sigma_1, \sigma_2]$, and define $g(t)$ as

$$g(t) = Le^{-c(T-t) - \frac{1}{2}\sigma_1\sigma_2(T-t)}, \tag{13}$$

where c and L are positive constants. Furthermore, let $D(G_t, t)$ denote the smooth continuous solution to the partial differential equation

$$0 = \begin{cases} \dfrac{\partial D}{\partial t} - cD + cG_t\dfrac{\partial D}{\partial G} + \dfrac{1}{2}\sigma_{\max}^2 G_t^2 \dfrac{\partial^2 D}{\partial G^2}, & G_t \leq g(t) \\ \dfrac{\partial D}{\partial t} - cD + cG_t\dfrac{\partial D}{\partial G} + \dfrac{1}{2}\sigma_{\min}^2 G_t^2 \dfrac{\partial^2 D}{\partial G^2}, & G_t > g(t) \end{cases}, \tag{14}$$

subject to the terminal condition $D(G_T, T) = \max(0, G_T - K_1) - \max(0, G_T - K_2)$, where $K_2 > K_1$. A closed-form solution for $D(G_t, t)$ is given in the Appendix. Under the restriction that $K_1 = Le^{-b\sigma_{\max}}$ and $K_2 = Le^{b\sigma_{\min}}$ for some constant $b > 0$,[13] it is also shown in the Appendix that $\partial^2 D/\partial G^2 > 0$ if $G_t < g(t)$,

[12] For all of the capital structures considered, $\mu_t - \sigma_t\lambda$ is constant along the optimal path and coincides with the first best risk-adjusted return. Thus, the instantaneous variance is the sole determinant of the validity of the Black and Scholes (1973) framework.

[13] This restriction is weak, since for any given values of $\sigma_{\min}$, $\sigma_{\max}$, K_1, and $K_2 > K_1$, the constants L and b can always be chosen to solve the pair of equations $K_1 = Le^{-b\sigma_{\max}}$ and $K_2 = Le^{b\sigma_{\min}}$.

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$\partial^2 D/\partial G^2 < 0$ if $G_t > g(t)$, and $\partial^2 D/\partial G^2 = 0$ if $G_t = g(t)$. Thus, $g(t)$ is the boundary that separates the convex and concave regions of $D(G_t, t)$. In the sequel, the function $D(G_t, t)$ appears when the owner issues a structured convertible debt contract and $(\sigma_t - \theta_t)^2$ is not constant. To show an explicit dependence on the parameters, the function $D(G_t, t)$ is written as

$$D(G_t, t) = D(G_t, K_1, K_2, L, T - t, c, \sigma_{\max}, \sigma_{\min}). \tag{15}$$

D. Capital Structures with Analytical Solutions

Using the results of Section II.C, Proposition 1 is illustrated for several simple capital structures. Case i) is illustrated in Table 2: In Panel A, the owner finances the firm by issuing nonstructured debt, while in Panel B the owner instead issues structured convertible debt. In each panel, the structuring factor θ_t is chosen, so that the owner is induced to optimally choose the first best pair $\{\mu_2, \sigma_2\}$ for all t. For instance, since the owner's residual claim in Panel A of Table 2 is convex, the owner has an incentive to choose the high-volatility pair. Since this is also the first best pair, the structuring factor is set equal to 0. In Panel B, the owner's residual claim is expressed in terms of the function $D(G_t, t)$, which switches from convexity to concavity as G_t crosses $g(t)$ from below. Thus, letting $\theta_t = 0$ when $G_t \leq g(t)$ and $\theta_t = \sigma_1 + \sigma_2$ when $G_t > g(t)$, the owner is induced to optimally choose $\{\mu_2, \sigma_2\}$ for all t.

TABLE 2

Feasible Capital Structures for

$$\mu_2 - \sigma_2\lambda > \mu_1 - \sigma_1\lambda \quad \text{and} \quad \sigma_2 > \sigma_1$$

Two different capital structures are illustrated for the case in which the pair $\{\mu_2, \sigma_2\}$ exhibits both a higher risk-adjusted return and a higher volatility relative to $\{\mu_1, \sigma_1\}$. Both capital structures are consistent with the first best strategy, which is to choose $\{\mu_2, \sigma_2\}$ for all t. The restriction on L (Panel B of Table 2) ensures that the owner's claim is locally equity-like (a zero second derivative) at time 0. The financing restrictions on K (Panel A) and b (Panel B) ensure that the issued security is fairly priced at time 0. Thus, the owner captures the first best NPV, and the agency cost of external financing is 0. C_{BS} denotes the Black-Scholes (1973) formula, D is given in the text, and $G_t = V_t \xi_t^{-1}$.

Panel A. Nonstructured Debt	
Owner's payoff at T	$\max[0, V_T - K]$
Financier's payoff at T	$\min[V_T, K]$
Structuring factor θ_t	0
Optimal strategy at t	$\{\mu_2, \sigma_2\}$
Owner's value at t	$e^{(\mu_2 - \sigma_2\lambda - r)(T-t)} C_{\text{BS}}(V_t, K, T - t, \mu_2 - \sigma_2\lambda, \sigma_2)$
Financing restriction on K	$V_0 - W = e^{(\mu_2 - \sigma_2\lambda - r)T}[V_0 - C_{\text{BS}}(V_0, K, T, \mu_2 - \sigma_2\lambda, \sigma_2)]$
Panel B. Structured Convertible Debt	
Owner's payoff at T	$\max[0, V_T - \xi_T K_1] - \max[0, V_T - \xi_T K_2]$
Financier's payoff at T	$\min[V_T, \xi_T K_1] + \max[0, V_T - \xi_T K_2]$
Parameter conditions	$K_1 = Le^{-b\sigma_2}$ and $K_2 = Le^{b\sigma_1}$
Structuring factor θ_t	$\begin{matrix} 0 & \text{if} \quad G_t \leq g(t) \\ \sigma_1 + \sigma_2 & \text{if} \quad G_t > g(t) \end{matrix}$, $g(t) = Le^{-(\mu_2 - \sigma_2\lambda)(T-t) - \frac{1}{2}\sigma_1\sigma_2(T-t)}$
Optimal strategy at t	$\{\mu_2, \sigma_2\}$
Owner's value at t	$\xi_t e^{(\mu_2 - \sigma_2\lambda - r)(T-t)} D(G_t, Le^{-b\sigma_2}, Le^{b\sigma_1}, L, T - t, \mu_2 - \sigma_2\lambda, \sigma_2, \sigma_1)$
Local equity restriction on L	$L = V_0 e^{(\mu_2 - \sigma_2\lambda + \frac{1}{2}\sigma_1\sigma_2)T}$
Financing restriction on b	$V_0 - W = e^{(\mu_2 - \sigma_2\lambda - r)T}\left[V_0 - D(V_0, Le^{-b\sigma_2}, Le^{b\sigma_1}, L, T, \mu_2 - \sigma_2\lambda, \sigma_2, \sigma_1)\right]$

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The financing restrictions on K (Panel A of Table 2) and b (Panel B of Table 2) ensure that the issued security is fairly priced at time 0. Thus, the owner captures the first best NPV in expression (4), and the agency cost of asset substitution is 0. In Panel B, the concave and convex regions of the structured convertible debt contract are balanced against one another at time 0 by choosing L such that $g(0) = V_0$. This restriction on L ensures that the owner's value function is locally equity-like (i.e., a zero second derivative) at time 0 and is consistent with Green (1984).

Table 3 analyzes case ii) of Proposition 1. In this case, the asset substitution problem is solved if the owner issues either structured debt (Panel A of Table 3) or structured convertible debt (Panel B of Table 3). As in Table 2, each panel specifies θ_t such that the owner is induced to choose $\{\mu_2, \sigma_2\}$ for all t, which is the first best strategy. However, the structuring factors are different than their counterparts in Table 2, since now the first best pair $\{\mu_2, \sigma_2\}$ has the lower volatility.

TABLE 3

Feasible Capital Structures for

$$\mu_2 - \sigma_2\lambda \;>\; \mu_1 - \sigma_1\lambda \quad \text{and} \quad \sigma_2 \;<\; \sigma_1$$

Two different capital structures are illustrated for the case in which the pair $\{\mu_2, \sigma_2\}$ exhibits a higher risk-adjusted return but a lower volatility relative to $\{\mu_1, \sigma_1\}$. Both capital structures are consistent with the first best strategy, which is to choose $\{\mu_2, \sigma_2\}$ for all t. The restriction on L (Panel B of Table 3) ensures that the owner's claim is locally equity-like (a zero second derivative) at time 0. The financing restrictions on K (Panel A) and b (Panel B) ensure that the issued security is fairly priced at time 0. Thus, the owner captures the first best NPV, and the agency cost of external financing is 0. C_{BS} denotes the Black-Scholes (1973) formula, D is given in the text, and $G_t = V_t\xi_t^{-1}$.

Panel A. Structured Debt

Owner's payoff at T	$\max[0, V_T - \xi_T K]$
Financier's payoff at T	$\min[V_T, \xi_T K]$
Structuring factor θ_t	$\sigma_1 + \sigma_2$
Optimal strategy at t	$\{\mu_2, \sigma_2\}$
Owner's value at t	$\xi_t e^{(\mu_2 - \sigma_2\lambda - r)(T-t)} C_{BS}(G_t, K, T-t, \mu_2 - \sigma_2\lambda, \sigma_1)$
Financing restriction on K	$V_0 - W = e^{(\mu_2 - \sigma_2\lambda - r)T}[V_0 - C_{BS}(V_0, K, T, \mu_2 - \sigma_2\lambda, \sigma_1)]$

Panel B. Structured Convertible Debt

Owner's payoff at T	$\max[0, V_T - \xi_T K_1] - \max[0, V_T - \xi_T K_2]$
Financier's payoff at T	$\min[V_T, \xi_T K_1] + \max[0, V_T - \xi_T K_2]$
Parameter conditions	$K_1 = Le^{-b\sigma_1}$ and $K_2 = Le^{b\sigma_2}$
Structuring factor θ_t	$\begin{matrix}\sigma_1 + \sigma_2 & \text{if} & G_t \le g(t) \\ 0 & \text{if} & G_t > g(t)\end{matrix}$, $g(t) = Le^{-(\mu_2 - \sigma_2\lambda)(T-t) - \frac{1}{2}\sigma_1\sigma_2(T-t)}$
Optimal strategy at t	$\{\mu_2, \sigma_2\}$
Owner's value at t	$\xi_t e^{(\mu_2 - \sigma_2\lambda - r)(T-t)} D(G_t, Le^{-b\sigma_1}, Le^{b\sigma_2}, L, T-t, \mu_2 - \sigma_2\lambda, \sigma_1, \sigma_2)$
Local equity restriction on L	$L = V_0 e^{(\mu_2 - \sigma_2\lambda + \frac{1}{2}\sigma_1\sigma_2)T}$
Financing restriction on b	$V_0 - W = e^{(\mu_2 - \sigma_2\lambda - r)T}\left[V_0 - D(V_0, Le^{-b\sigma_1}, Le^{b\sigma_2}, L, T, \mu_2 - \sigma_2\lambda, \sigma_1, \sigma_2)\right]$

If the firm is financed with structured debt (Panel A of Table 3), the owner's residual claim is a call option on the firm's assets with a strike price that is continuously adjusted to reflect the debt's outstanding face value. To see this, recall that $G_t = V_t\xi_t^{-1}$, and note that C_{BS} is homogeneous of degree one. Thus, the owner's value at time t in Panel A of Table 3 can be rewritten as

$$e^{(\mu_2 - \sigma_2\lambda - r)(T-t)} C_{BS}(V_t, \xi_t K, T-t, \mu_2 - \sigma_2\lambda, \sigma_1). \tag{16}$$

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The underlying asset is now V_t, and the strike price is $\xi_t K$, where the latter quantity represents the time t face value of the structured debt. Note that structuring allows the owner to cherry-pick the best feature of each drift-diffusion pair. The claim in expression (16) is valued using the higher risk-adjusted return, $\mu_2 - \sigma_2\lambda$, and the higher volatility, σ_1. Structuring allows the owner to credibly commit to $\{\mu_2, \sigma_2\}$ for all t, yet still enjoy the favorable valuation associated with the higher volatility. Of course there is a trade-off, since $\xi_t K$ positively correlates with V_t, which alters the set of states for which expression (16) finishes in the money, relative to a call option with a constant strike price.

Table 4 illustrates case iii) of Proposition 1. Although the first best NPV is always attained, the capital structure and the structuring factor have a profound effect on the owner's optimal strategy. In turn, this affects the volatility pattern of the firm's total assets. Table 4 shows four alternative convertible debt capital structures. Panel A illustrates the case of $\theta_t = 0$ (i.e., nonstructured convertible debt). In this case, the owner optimally chooses the high-volatility pair if $V_t \leq g(t)$ and otherwise chooses the low-volatility pair. This is a case of pure dynamic risk shifting. The other panels show that structuring can induce the owner to optimally choose the opposite dynamic risk-shifting strategy (Panel B), to always choose the high-volatility pair (Panel C), or to always choose the low-volatility pair (Panel D). Thus, the type of external financing (e.g., convertible debt) is neither a good predictor of the owner's optimal operating strategy nor a good predictor of the volatility characteristics of the firm's assets. The salient feature is the structure of debt.

E. Operating Policy Agreement

It is worth noting that structured debt does not necessarily produce an operating policy agreement between the owner and the debtholders, because when the owner's value function is locally concave (convex), the debtholders' value function is locally convex (concave). This follows because the total firm value always has a zero second derivative. Although structured debt induces the owner to optimally choose the first best operating policy, this choice may not be the preferred choice of the debtholders. Thus, control rights are important: The solution to the asset substitution problem is contingent on the assumed control rights. If control rights are transferred from the owner to the debtholders without changing the capital structure, the asset substitution problem resurfaces. Only in special cases will structured debt simultaneously solve the asset substitution problem and align the operating policy choices of the owner and the debtholders. One instance of alignment occurs when $\theta_t = \frac{1}{2}(\sigma_1 + \sigma_2)$. In this knife edge case, the owner's choice of operating policy no longer depends on the sign of $\partial^2 f / \partial G^2$ (see equation (12)). A second special case is discussed in Section IV.F, where it is shown how to structure debt in order to avoid costly default.

F. Additional Structuring Issues

The prior analysis involved two assumptions that are worthy of discussion. First, ξ_t in equation (6) was assumed to be perfectly instantaneously correlated

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TABLE 4

Feasible Convertible Debt Structures for

$$\mu_2 - \sigma_2\lambda = \mu_1 - \sigma_1\lambda \quad \text{and} \quad \sigma_2 > \sigma_1$$

Four different convertible debt structures are illustrated for the case in which the pairs $\{\mu_1, \sigma_1\}$ and $\{\mu_2, \sigma_2\}$ exhibit the same risk-adjusted return but have different volatilities. For each structure, the owner's payoff at T is $\max[0, V_T - \xi_T K_1] - \max[0, V_T - \xi_T K_2]$ and the financier's payoff at T is $\min[V_T, \xi_T K_1] + \max[0, V_T - \xi_T K_2]$. The parameters K_1 and K_2 satisfy $K_1 = Le^{-b\sigma_2}$ and $K_2 = Le^{b\sigma_1}$, while the local equity restriction on L is

$$L = V_0 e^{(\mu_2 - \sigma_2\lambda + \frac{1}{2}\sigma_1\sigma_2)T}.$$

The financing restriction on b, which ensures that the issued security is fairly priced at time 0, is $V_0 - W = e^{(\mu_2 - \sigma_2\lambda - r)T}V_0 - f_0$, where $f_0 = f(V_0, 0)$ is the owner's value at time 0. The four structures induce different strategies, but all four deliver the first best NPV (since $\mu_2 - \sigma_2\lambda = \mu_1 - \sigma_1\lambda$). Thus, the agency cost of external financing is 0. D is given in the text, and $G_t = V_t\xi_t^{-1}$.

Panel A. Nonstructured Convertible Debt	
Structuring factor θ_t	0
Optimal strategy at t	$\begin{matrix}\{\mu_2, \sigma_2\} & \text{if} & V_t \le g(t) \\ \{\mu_1, \sigma_1\} & \text{if} & V_t > g(t)\end{matrix}$, $g(t) = Le^{-(\mu_2 - \sigma_2\lambda)(T-t) - \frac{1}{2}\sigma_1\sigma_2(T-t)}$
Owner's value at t	$e^{(\mu_2 - \sigma_2\lambda - r)(T-t)} \times D(V_t, Le^{-b\sigma_2}, Le^{b\sigma_1}, L, T-t, \mu_2 - \sigma_2\lambda, \sigma_2, \sigma_1)$
Panel B. Structured Convertible Debt	
Structuring factor θ_t	$\sigma_1 + \sigma_2$
Optimal strategy at t	$\begin{matrix}\{\mu_1, \sigma_1\} & \text{if} & G_t \le g(t) \\ \{\mu_2, \sigma_2\} & \text{if} & G_t > g(t)\end{matrix}$, $g(t) = Le^{-(\mu_2 - \sigma_2\lambda)(T-t) - \frac{1}{2}\sigma_1\sigma_2(T-t)}$
Owner's value at t	$\xi_t e^{(\mu_2 - \sigma_2\lambda - r)(T-t)} D(G_t, Le^{-b\sigma_2}, Le^{b\sigma_1}, L, T-t, \mu_2 - \sigma_2\lambda, \sigma_2, \sigma_1)$
Panel C. Structured Convertible Debt	
Structuring factor θ_t	$\begin{matrix}0 & \text{if} & G_t \le g(t) \\ \sigma_1 + \sigma_2 & \text{if} & G_t > g(t)\end{matrix}$, $g(t) = Le^{-(\mu_2 - \sigma_2\lambda)(T-t) - \frac{1}{2}\sigma_1\sigma_2(T-t)}$
Optimal strategy at t	$\{\mu_2, \sigma_2\}$
Owner's value at t	$\xi_t e^{(\mu_2 - \sigma_2\lambda - r)(T-t)} \times D(G_t, Le^{-b\sigma_2}, Le^{b\sigma_1}, L, T-t, \mu_2 - \sigma_2\lambda, \sigma_2, \sigma_1)$
Panel D. Structured Convertible Debt	
Structuring factor θ_t	$\begin{matrix}\sigma_1 + \sigma_2 & \text{if} & G_t \le g(t) \\ 0 & \text{if} & G_t > g(t)\end{matrix}$, $g(t) = Le^{-(\mu_1 - \sigma_1\lambda)(T-t) - \frac{1}{2}\sigma_1\sigma_2(T-t)}$
Optimal strategy at t	$\{\mu_1, \sigma_1\}$
Owner's value at t	$\xi_t e^{(\mu_1 - \sigma_1\lambda - r)(T-t)} D(G_t, Le^{-b\sigma_2}, Le^{b\sigma_1}, L, T-t, \mu_1 - \sigma_1\lambda, \sigma_2, \sigma_1)$

with V_t even though some of the securities in Table 1 appear to use indices that are imperfectly correlated with the firm's assets. Second, it was assumed that the firm could design its structured debt contract using the specific project parameters σ_1 and σ_2. Neither assumption is critical for the article's main result: Structured debt can eliminate the agency cost of asset substitution.

Concerning the first assumption, suppose equation (6) is generalized to take the form

$$\tilde{\xi}_t = \exp\left[-\frac{1}{2}\int_0^t \theta_s^2 ds + \int_0^t \theta_s dz_s^Q - \frac{1}{2}\int_0^t \beta_s^2 ds + \int_0^t \beta_s dw_s^Q\right],$$

where $\{\beta_t : t \in [0, T]\}$ is exogenous and the Brownian motion w_t^Q is independent of z_t^Q. Thus, $\tilde{\xi}_t$ is imperfectly correlated with the firm's assets in equation (5). If the firm structures its debt using $\tilde{\xi}_t$ instead of ξ_t, the owner's Bellman equation in (12) remains the same, with the exception of the $\partial^2 f/\partial G^2$ term. In particular, the quantity $(\sigma_t - \theta_t)^2$ must be replaced by $\beta_t^2 + (\sigma_t - \theta_t)^2$. This has no

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impact on the ability of structured debt to solve the asset substitution problem. The owner is still able to specify $\{\theta_t : t \in [0, T]\}$ as part of the issued security's design in order to induce the first best, as in Proposition 1. However, the variance is now higher by β_t^2, which impacts the market's valuation of the structured security. In turn, this affects the debt's initial face value via the financing constraint.

Concerning the second assumption, Proposition 1 can be generalized to allow for structures that do not depend explicitly on the project parameters. To see this, suppose $\{\mu_2, \sigma_2\}$ is the first best pair, and let $\sigma_2 > \sigma_1$. This corresponds to Table 2 and case i) of Proposition 1. If $\partial^2 f/\partial G^2 \geq 0$, then *any* value of θ_t that is less than $\frac{1}{2}(\sigma_1 + \sigma_2)$ will induce the owner to choose the first best. Since θ_t is less than $\frac{1}{2}(\sigma_1 + \sigma_2)$, the distance between θ_t and the owner's choice σ_t is maximized by choosing $\sigma_t = \sigma_2$. For simplicity, case i) of Proposition 1 uses $\theta_t = 0$ when $\partial^2 f/\partial G^2 \geq 0$. On the other hand, if $\partial^2 f/\partial G^2 < 0$, then *any* value of θ_t greater than $\frac{1}{2}(\sigma_1 + \sigma_2)$ will induce the owner to choose the first best. This follows because the distance between $\theta_t > \frac{1}{2}(\sigma_1 + \sigma_2)$ and the owner's choice σ_t is minimized by choosing $\sigma_t = \sigma_2$. Thus, the values of $\theta_t = 0$ and $\theta_t = \sigma_1 + \sigma_2$ used in case i) of Proposition 1 are sufficient, but not necessary, for the result to be true. The owner can use any pair of constants as long as the smaller (larger) constant is less (greater) than the average volatility. Given a particular pair of constants, the face value of the structured debt is adjusted such that the debt is fairly valued at time 0. Thus, the owner's initial equity value is not affected by the chosen pair of constants. A similar argument can be used to generalize cases ii) and iii) of the proposition, which correspond to Tables 3 and 4, respectively.

The previous generalization suggests that structured debt may also be useful for solving the asset substitution problem when there is uncertainty about the project parameters. For example, let $\{\mu_2, \sigma_2\}$ be the first best pair, but suppose the owner does not know the value of σ_2. For simplicity, assume σ_2 takes values in the interval $[\sigma_{2L}, \sigma_{2H}]$, where $\sigma_{2L} < \sigma_{2H}$. There are three cases of interest, two of which can be solved with structuring. First suppose that $\sigma_{2L} > \sigma_1$, where σ_1 is known. To induce the first best choice in this case, θ_t should be less than $\frac{1}{2}(\sigma_1 + \sigma_{2L})$ if $\partial^2 f/\partial G^2 \geq 0$ and greater than $\frac{1}{2}(\sigma_1 + \sigma_{2H})$ if $\partial^2 f/\partial G^2 < 0$. Thus, the debt is structured by using a pair of constants for θ_t such that the distance between the two constants is large enough to bracket all possible values of the average volatility. Next, suppose that $\sigma_1 > \sigma_{2H}$. This is similar to the prior case, except that now the two inequalities involving $\partial^2 f/\partial G^2$ must be reversed. Lastly, suppose that $\sigma_{2H} > \sigma_1 > \sigma_{2L}$. In this case, it does not appear that structured debt can solve the asset substitution problem. Regardless of the value of θ_t or the sign of $\partial^2 f/\partial G^2$ (unless the sign is 0), the owner may prefer to deviate from the first best.

III. Continuous Time Extension of Green (1984)

In Section II, the owner's operating choice is limited to the pairs $\{\mu_1, \sigma_1\}$ and $\{\mu_2, \sigma_2\}$. Limiting the operating choice in this manner does not drive the paper's results. This is shown by first generalizing Green's (1984) one-period model,

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which involves a continuum of possible operating choices, to a continuous time setting. It is then shown how structured financing induces the first best. Structured financing is therefore a robust tool for solving the asset substitution problem.

In Green ((1984), p. 117), the firm's assets are equal to

$$k(I_1)[1+R_1]+k(I_2)[1+R_2], \tag{17}$$

where $k(\cdot)$ is a scale function, R_j is the rate of return on project j, and I_j is the amount allocated to project j. Green (1984) assumes that $R_2 = R_1 + z$, where z is unpriced noise. The two projects have identical risk-adjusted expected returns, but project 2 is riskier than project 1 due to z.

To extend the model to a continuous time setting, consider two correlated risky projects, $V_{1,t}$ and $V_{2,t}$, whose evolutions are described by

$$dV_{1,t} = (\mu-\sigma\lambda)V_{1,t}dt+\sigma V_{1,t}dz^Q_{1,t} \quad \text{and} \tag{18}$$

$$dV_{2,t} = (\mu-\sigma\lambda)V_{2,t}dt+\sigma V_{2,t}dz^Q_{1,t}+\nu V_{2,t}dz^Q_{2,t}, \tag{19}$$

where $z^Q_{1,t}$ and $z^Q_{2,t}$ are independent Brownian motions under the risk-adjusted measure Q. In equations (18)–(19), μ is the drift parameter under the true probability measure, σ and ν are the volatility parameters, and λ is a risk premium.[14] It is assumed that $\mu-\sigma\lambda > r$, where r is the interest rate. In addition, note that only the risk associated with $z_{1,t}$ is priced. Thus, the two projects have identical risk-adjusted instantaneous returns, but project 2 is exposed to the unpriced idiosyncratic risk $z_{2,t}$. These assumptions are consistent with those in Green ((1984), pp. 118–119).

Following expression (17), the projects' returns are combined using a concave scale function $k(\cdot)$.[15] However, rather than work with the total return as in expression (17), it is convenient in the continuous time model to work instead with the instantaneous return. The firm's total assets, denoted by X_t, are assumed to follow the process

$$\frac{dX_t}{X_t} = k(I_{1,t})\frac{dV_{1,t}}{V_{1,t}}+k(I_{2,t})\frac{dV_{2,t}}{V_{2,t}}, \tag{20}$$

where $I_{j,t} \geq 0$ for $j = 1, 2$ is the capital stock allocated to project j at time t. The left-hand side of equation (20) is the instantaneous return on the firm's total assets, while the right-hand side of equation (20) is the continuous time counterpart to expression (17).

In Green's (1984) single-period model, the capital allocation decision occurs only once, at the beginning of the period. In contrast, it is necessary in the continuous time model to allow the owner to allocate the firm's capital dynamically

[14]The notation in this section only partially overlaps with that in Section II. For example, λ is still a constant risk premium, but $\mathcal{F}_t$ is now taken to be the augmented filtration generated by $z^Q_{1,t}$ and $z^Q_{2,t}$. Furthermore, the volatility parameters are now σ and ν, instead of σ_1 and σ_2.

[15]The function $k(\cdot)$ satisfies $k(0)=0$, $k' > 0$, $k'' < 0$, $\lim_{I\to\infty} k(I)=A < \infty$, and $\lim_{I\to 0} k'(I)=\infty$. These assumptions are consistent with those in Green ((1984), p. 117).

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at each t. To achieve this, it is assumed that $I_{1,t} + I_{2,t} = I$ for all t, where I is the constant total capital stock. The owner purchases capital (such as a factory) at time 0 by spending an amount equal to I. Assuming perfect divisibility, the owner must then choose how to allocate this capital for each $t \in [0, T]$. The capital allocated to project j at time t is scaled using $k(\cdot)$ and earns the instantaneous return $dV_{j,t}/V_{j,t}$, as in equation (20). For simplicity, the capital is assumed to be nondepreciable over $[0, T]$, and thus I is a constant. Substituting equations (18)–(19) into equation (20), the process for the total assets under Q is[16]

$$\frac{dX_t}{X_t} = (\mu - \sigma\lambda)\left[k(I_{1,t}) + k(I_{2,t})\right] dt + \sigma\left[k(I_{1,t}) + k(I_{2,t})\right] dz^Q_{1,t} + \nu k(I_{2,t}) dz^Q_{2,t}, \tag{21}$$

where it is assumed that $X_0 = I$. Since the projects in equations (18)–(19) do not produce intermediate cash flows, there are no intermediate cash flows associated with X_t. The owner simply invests $X_0 = I$ at time 0 and then operates the firm by allocating I between $I_{1,t}$ and $I_{2,t}$ for all $t \in [0, T]$. If external financing is used to finance a portion of the initial investment I, the final payoff X_T is shared at time T between the owner and the external financier. The sharing rule is determined by the external financing contract that is put in place at time 0.

A. First Best Strategy

The first best strategy is the one that maximizes the total firm value. This strategy corresponds to the case in which the owner's initial funds are equal to I and external financing is not needed. The owner's Bellman problem in this case is

$$0 = \max_{\{I_{1,t}, I_{2,t}\}} \left[\frac{\partial f}{\partial t} - rf + (\mu - \sigma\lambda)\left[k(I_{1,t}) + k(I_{2,t})\right] X_t \frac{\partial f}{\partial X} + \frac{1}{2}\left(\sigma^2\left[k(I_{1,t}) + k(I_{2,t})\right]^2 + \nu^2\left[k(I_{2,t})\right]^2\right) X_t^2 \frac{\partial^2 f}{\partial X^2}\right], \tag{22}$$

subject to the constraint $I_{1,t} + I_{2,t} = I$ and the terminal condition $f(X_T, T) = X_T$. Due to the terminal condition, the value function f should have the property $\partial^2 f/\partial X^2 = 0$. The first-order condition is therefore

$$0 = (\mu - \sigma\lambda)\left[k'(I^*_{1,t}) - k'(I^*_{2,t})\right] X_t \frac{\partial f}{\partial X},$$

which has the solution $I^*_{1,t} = I^*_{2,t} = 0.5I$. Since $k'' < 0$, the second-order condition is satisfied. Thus, the first best strategy involves equal allocation to the two projects for all t. This result coincides with Green ((1984), p. 120). Since $I^*_{1,t} = I^*_{2,t} = 0.5I$, the total firm value at time t is $X_t e^{2(\mu - \sigma\lambda)k(0.5I)(T-t) - r(T-t)}$, which solves equation (22).

[16] Note that equation (21) depends only on X_t and not on $V_{1,t}$ or $V_{2,t}$. This is due to the assumption in equation (20). If an alternative assumption is used, such as $X_t = k(I_{1,t})V_{1,t} + k(I_{2,t})V_{2,t}$, the process for X_t depends explicitly on $V_{2,t}$. In this case, it can be shown that structuring still solves the asset substitution problem, but the closed-form solutions to the problem, such as those in Table 5, are lost.

B. Nonstructured Debt Financing and Overinvestment

Now suppose the owner has initial funds equal to W, where $0 < W < I$. The owner, thus, raises $I - W$ via external debt financing. If the owner issues a nonstructured debt contract with final payoff $\min[X_T, K]$, the owner's Bellman problem is

$$0 = \max_{\{I_{1,t},I_{2,t}\}} \left[\frac{\partial f}{\partial t} - rf + (\mu - \sigma\lambda)\left[k(I_{1,t}) + k(I_{2,t})\right] X_t \frac{\partial f}{\partial X} + \frac{1}{2}\left(\sigma^2 \left[k(I_{1,t}) + k(I_{2,t})\right]^2 + \nu^2 \left[k(I_{2,t})\right]^2\right) X_t^2 \frac{\partial^2 f}{\partial X^2}\right], \tag{23}$$

subject to $I_{1,t} + I_{2,t} = I$ and the terminal condition $f(X_T, T) = \max[0, X_T - K]$. The first-order condition implies that

$$\frac{k'(\hat{I}_{2,t})}{k'(\hat{I}_{1,t})} = \frac{\left[(\mu - \sigma\lambda) X_t \frac{\partial f}{\partial X} + \sigma^2 \left[k(\hat{I}_{1,t}) + k(\hat{I}_{2,t})\right] X_t^2 \frac{\partial^2 f}{\partial X^2}\right]}{\left[(\mu - \sigma\lambda) X_t \frac{\partial f}{\partial X} + \sigma^2 \left[k(\hat{I}_{1,t}) + k(\hat{I}_{2,t})\right] X_t^2 \frac{\partial^2 f}{\partial X^2} + \nu^2 k(\hat{I}_{2,t}) X_t^2 \frac{\partial^2 f}{\partial X^2}\right]}. \tag{24}$$

Since the owner's claim f is increasing and convex, $\partial f/\partial X$ and $\partial^2 f/\partial X^2$ are both positive. Thus, the fraction on the right-hand side of equation (24) is less than 1, implying that $k'(\hat{I}_{2,t}) < k'(\hat{I}_{1,t})$. Due to the concavity of $k(\cdot)$, it is concluded that $\hat{I}_{2,t} > \hat{I}_{1,t}$ for all t. The owner overinvests in the riskier project relative to the first best strategy, which produces a strictly positive agency cost of asset substitution. This coincides with Proposition 1 in Green ((1984), p. 124).

Suppose instead that the owner finances the firm by issuing nonstructured convertible debt. In this case, the terminal condition for f changes, but expressions (23)–(24) are otherwise unaltered. Thus, as claimed by Hennessy and Tserlukevich (2008), nonstructured convertible debt does not restore the first best strategy. The owner can choose the parameters of the convertible bond so that $\partial^2 f/\partial X^2 = 0$ at time 0, which implies equal initial allocation to the two projects. However, for any $t > 0$, it may be the case that $\partial^2 f/\partial X^2 > 0$, which leads to overinvestment in the riskier project, or it may be the case that $\partial^2 f/\partial X^2 < 0$, which leads to underinvestment in the riskier project. Thus, there is a strict departure between the continuous time framework and the one-period model of Green (1984). This departure is resolved by using structured financing.

C. Structured External Financing Contracts

Define the structuring variable ξ_t as

$$\xi_t = \exp\left[-\frac{1}{2}\int_0^t \theta_{1,s}^2 ds - \frac{1}{2}\int_0^t \theta_{2,s}^2 ds + \int_0^t \theta_{1,s} dz_{1,s}^Q + \int_0^t \theta_{2,s} dz_{2,s}^Q\right], \tag{25}$$

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where $\theta_{1,t}$ and $\theta_{2,t}$ are $\mathcal{F}_t$-measurable structuring factors whose functional forms are chosen at time 0 as part of the issued security's design. The owner finances $I - W$ at time 0 by issuing a contract whose payoff at time T is $F(X_T; \xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n)$. Like expression (7), the contract F has no intermediate cash flows, is nondecreasing in X_T, and is homogeneous of degree one in all of its arguments.

Note that ξ_t can be written as $\xi_t = \mathrm{E}_t^Q[dR/dQ]$, where $dR/dQ = \xi_T$ defines a new probability measure R such that $dz_{1,t}^Q = dz_{1,t}^R + \theta_{1,t}dt$, $dz_{2,t}^Q = dz_{2,t}^R + \theta_{2,t}dt$, and $z_{1,t}^R$ and $z_{2,t}^R$ are independent Brownian motions under R. Thus, the value of the owner's residual claim is

$$
\begin{aligned}
(26)\quad & \mathrm{E}_t^Q\left[e^{-r(T-t)}\left(X_T - F(X_T; \xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n)\right)\right] \\
&= \mathrm{E}_t^R\left[\frac{\xi_t}{\xi_T} e^{-r(T-t)}\left(X_T - F(X_T; \xi_T K_1, \xi_T K_2, \ldots, \xi_T K_n)\right)\right] \\
&= \xi_t\, \mathrm{E}_t^R\left[e^{-r(T-t)}\left(X_T \xi_T^{-1} - F(X_T \xi_T^{-1}; K_1, K_2, \ldots, K_n)\right)\right],
\end{aligned}
$$

which follows since F is homogeneous. The owner's optimization problem is therefore

$$
(27)\qquad \max_{\{I_{1,s}, I_{2,s}\},\, s \in [t,T]} \xi_t\, \mathrm{E}_t^R\left[e^{-r(T-t)}\left(G_T - F(G_T; K_1, K_2, \ldots, K_n)\right)\right],
$$

subject to $I_{1,s} + I_{2,s} = I$ for all $s \in [t, T]$. The process for $G_t = X_t \xi_t^{-1}$ is

$$
\begin{aligned}
(28)\quad \frac{dG_t}{G_t} &= (\mu - \sigma\lambda)\left[k(I_{1,t}) + k(I_{2,t})\right] dt \\
&\quad + \left[\sigma\left(k(I_{1,t}) + k(I_{2,t})\right) - \theta_{1,t}\right] dz_{1,t}^R + \left[\nu k(I_{2,t}) - \theta_{2,t}\right] dz_{2,t}^R,
\end{aligned}
$$

where $G_0 = X_0$. Letting $f(G_t, t) = \mathrm{E}_t^R\left[e^{-r(T-t)}\left(G_T - F(G_T; K_1, K_2, \ldots, K_n)\right)\right]$, Bellman's principle of stochastic control is used to restate expression (27) as

$$
\begin{aligned}
(29)\quad 0 &= \max_{\{I_{1,t}, I_{2,t}\}}\left[\xi_t\left(\frac{\partial f}{\partial t} - rf + (\mu - \sigma\lambda)\left[k(I_{1,t}) + k(I_{2,t})\right] G_t \frac{\partial f}{\partial G}\right.\right. \\
&\quad \left.\left. + \frac{1}{2}\left(\left[\sigma\left(k(I_{1,t}) + k(I_{2,t})\right) - \theta_{1,t}\right]^2 + \left[\nu k(I_{2,t}) - \theta_{2,t}\right]^2\right) G_t^2 \frac{\partial^2 f}{\partial G^2}\right)\right],
\end{aligned}
$$

subject to $I_{1,t} + I_{2,t} = I$ and the terminal condition $f(G_T, T) = G_T - F(G_T; K_1, K_2, \ldots, K_n)$. Note that equation (29) differs from equation (23) due to the presence of ξ_t, $\theta_{1,t}$, and $\theta_{2,t}$. The factors $\theta_{1,t}$ and $\theta_{2,t}$ will play an important role in solving the asset substitution problem.

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1. Optimal Structuring with the Black-Scholes Pricing Framework

The first-order condition to the Bellman problem in equation (29) can be written as

$$(30)\quad \frac{k'(I^*_{2,t})}{k'(I^*_{1,t})} = \left[(\mu-\sigma\lambda)\,G_t f_G + \sigma G_t^2 f_{GG}\left(\sigma\left[k\left(I^*_{1,t}\right)+k\left(I^*_{2,t}\right)\right]-\theta_{1,t}\right)\right]/$$
$$\left[(\mu-\sigma\lambda)\,G_t f_G + \sigma G_t^2 f_{GG}\left(\sigma\left[k\left(I^*_{1,t}\right)+k\left(I^*_{2,t}\right)\right]-\theta_{1,t}\right) + \nu G_t^2 f_{GG}\left(\nu k\left(I^*_{2,t}\right)-\theta_{2,t}\right)\right],$$

where $f_G = \partial f/\partial G$ and $f_{GG} = \partial^2 f/\partial G^2$. If the owner specifies $\theta_{2,t} = \nu k(0.5I)$ for all t, the solution to equation (30) is $I^*_{1,t}=I^*_{2,t}=0.5I$, which is the first best strategy. Note that the volatility coefficient of the idiosyncratic risk in equation (28) vanishes along the optimal path when $\theta_{2,t} = \nu k(0.5I)$. While the owner deviates from the first best in the absence of structuring, as shown in Section III.B, the structured contract eliminates the idiosyncratic risk from the owner's problem and induces the owner to choose the first best. The second-order condition depends on the sign of

$$(31)\quad 2(\mu-\sigma\lambda)k''(0.5I)G_t f_G + \left[\nu^2\left(k'(0.5I)\right)^2 + 4\sigma^2 k(0.5I)k''(0.5I) - 2\sigma k''(0.5I)\theta_{1,t}\right]G_t^2 f_{GG},$$

where $\theta_{2,t} = \nu k(0.5I)$ has been used. By letting

$$(32)\quad \theta_{1,t} = \frac{\nu^2\left(k'(0.5I)\right)^2 + 4\sigma^2 k(0.5I)k''(0.5I)}{2\sigma k''(0.5I)} = 2\sigma k(0.5I) + \frac{\nu^2\left(k'(0.5I)\right)^2}{2\sigma k''(0.5I)},$$

the terms that multiply f_{GG} in expression (31) vanish. Thus the second-order condition for a maximum is satisfied regardless of the sign of f_{GG}. Furthermore, when $\theta_{2,t} = \nu k(0.5I)$ and $\theta_{1,t}$ is given by equation (32), equation (28) has constant volatility, which implies that the Black-Scholes (1973) pricing framework is valid. However, alternative forms of $\theta_{1,t}$ may destroy the Black-Scholes (1973) pricing, as shown next.

2. Optimal Structuring with Risk Shifting

Suppose the owner finances the firm by issuing a structured convertible bond whose payoff is $\min[X_T, \xi_T K_1] + \max[0, X_T - \xi_T K_2]$. From equation (30), the first-order condition is satisfied by letting $\theta_{2,t} = \nu k(0.5I)$ and $I^*_{1,t} = I^*_{2,t} = 0.5I$ for all t, which is the first best strategy. Although $\theta_{1,t}$ in equation (32) induces the Black-Scholes (1973) pricing, it is possible to specify a different $\theta_{1,t}$ that still satisfies expression (31) but induces pricing that is consistent with risk shifting. To see this, note that since $\mu - \sigma\lambda > 0$, $k'' < 0$, and $\partial f/\partial G > 0$, the first term in

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expression (31) is always negative. However, the second term in expression (31) depends on the quantity

$$\nu^2 \left(k'(0.5I)\right)^2 + 4\sigma^2 k(0.5I)k''(0.5I). \tag{33}$$

If expression (33) is negative, the owner can structure the convertible bond using

$$\theta_{1,t} = \begin{cases} 0, & G_t \leq g(t) \\ 2\sigma k(0.5I) + \dfrac{\nu^2 \left(k'(0.5I)\right)^2}{2\sigma k''(0.5I)}, & G_t > g(t) \end{cases}. \tag{34}$$

The reasoning is that since $G_t \leq g(t)$ corresponds to $\partial^2 f/\partial G^2 \geq 0$, the owner can let $\theta_{1,t}=0$ and still be assured that expression (31) is negative. However, $G_t > g(t)$ corresponds to $\partial^2 f/\partial G^2 < 0$. In this case, letting $\theta_{1,t}$ equal equation (32) assures that the $\partial^2 f/\partial G^2$ terms in expression (31) vanish. This leads to risk shifting in equation (28).

On the other hand, if expression (33) is positive, the owner can structure the convertible bond using

$$\theta_{1,t} = \begin{cases} 2\sigma k(0.5I) + \dfrac{\nu^2 \left(k'(0.5I)\right)^2}{2\sigma k''(0.5I)}, & G_t \leq g(t) \\ 0, & G_t > g(t) \end{cases}, \tag{35}$$

where the reasoning is similar to the prior case. In equations (34)–(35), the switching boundary $g(t)$ is

$$g(t) = Le^{-2(\mu-\sigma\lambda)k(0.5I)(T-t)-\frac{1}{2}\sigma_{\min}\sigma_{\max}(T-t)}, \tag{36}$$

where $L > 0$,

$$\sigma_{\min} = \min\left[2\sigma k(0.5I), -\frac{\nu^2 \left(k'(0.5I)\right)^2}{2\sigma k''(0.5I)}\right], \quad \text{and}$$

$$\sigma_{\max} = \max\left[2\sigma k(0.5I), -\frac{\nu^2 \left(k'(0.5I)\right)^2}{2\sigma k''(0.5I)}\right].$$

Note that both $\sigma_{\min}$ and $\sigma_{\max}$ are positive, since $k'' < 0$.

Since expression (33) depends on $k(\cdot)$, the owner's choice of how to structure the convertible bond depends on the firm's scale function. Two firms that face the same pair of projects in equations (18)–(19) and are otherwise identical except for their scale functions may have different capital structures. Expression (33) may be positive for one firm and negative for the other.

Panel A of Table 5 shows the case when expression (33) is negative and $\theta_{1,t}$ is given by equation (34). The negative value of expression (33) implies that

$$\sigma_{\min} = -\frac{\nu^2 \left(k'(0.5I)\right)^2}{2\sigma k''(0.5I)} \quad \text{and} \quad \sigma_{\max} = 2\sigma k(0.5I).$$

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Using these values of $\sigma_{\min}$ and $\sigma_{\max}$ to construct $K_1 = Le^{-b\sigma_{\max}}$ and $K_2 = Le^{b\sigma_{\min}}$, the value of the owner's residual claim is expressed in terms of function D in equation (15). At time 0, the concave and convex regions of the structured convertible bond are balanced against one another by choosing L in equation (36) such that $g(0) = G_0$. Thus, $\partial^2 f/\partial G^2 = 0$ at time 0. Given this value for L, the financing restriction on the parameter b guarantees that the issued security is fairly priced at time 0. The owner captures the first best NPV, and the agency cost of external financing is 0.

TABLE 5

Feasible Convertible Debt Structures for the Dynamic Version of Green (1984)

Two convertible debt structures are illustrated for the case in which a scale function $k(\cdot)$ is used to aggregate the project returns. The restriction on $k(\cdot)$ and its derivatives, which is given in the top row of each panel, ensures that the second-order condition is satisfied. The owner's payoff at T is $\max[0, X_T - \xi_T K_1] - \max[0, X_T - \xi_T K_2]$, and the financier's payoff at T is $\min[X_T, \xi_T K_1] + \max[0, X_T - \xi_T K_2]$. Each capital structure induces the owner to undertake the first best strategy of equal investment in the two projects for all t. Thus, the agency cost of external financing is 0. For each panel

$$K_1 = Le^{-b\sigma_{\max}}, \quad K_2 = Le^{b\sigma_{\min}}, \quad \text{and} \quad g(t) = Le^{-2(\mu-\sigma\lambda)k(0.5I)(T-t)-\frac{1}{2}\sigma_{\min}\sigma_{\max}(T-t)},$$

where $\sigma_{\min}$ and $\sigma_{\max}$ are given in the table. The financing restriction on b ensures that the issued security is fairly priced at time 0, while the restriction

$$L = G_0 e^{[2(\mu-\sigma\lambda)k(0.5I)+\frac{1}{2}\sigma_{\min}\sigma_{\max}]T}$$

ensures that the owner's claim is locally equity-like (a zero second derivative) at time 0. The function D is given in the text, and $G_t = X_t \xi_t^{-1}$.

Panel A. Structured Convertible Debt with $\nu^2\left(k'(0.5I)\right)^2 + 4\sigma^2 k(0.5I)k''(0.5I) < 0$

Structuring factors at t	$\theta_{1,t} = \begin{cases} 0, & G_t \le g(t) \\ 2\sigma k(0.5I) + \dfrac{\nu^2\left(k'(0.5I)\right)^2}{2\sigma k''(0.5I)}, & G_t > g(t) \end{cases}$ and $\theta_{2,t} = \nu k(0.5I)$
Max and Min volatilities	$\sigma_{\min} = -\dfrac{\nu^2\left(k'(0.5I)\right)^2}{2\sigma k''(0.5I)}$ and $\sigma_{\max} = 2\sigma k(0.5I)$
Optimal strategy at t	$I^*_{1,t} = I^*_{2,t} = 0.5I$
Owner's value at t	$\xi_t e^{[2(\mu-\sigma\lambda)k(0.5I)-r](T-t)} \times D(G_t, Le^{-b\sigma_{\max}}, Le^{b\sigma_{\min}}, L, T-t, 2(\mu-\sigma\lambda)k(0.5I), \sigma_{\max}, \sigma_{\min})$
Financing restriction on b	$I - W = e^{[2(\mu-\sigma\lambda)k(0.5I)-r]T} \times \left[G_0 - D(G_0, Le^{-b\sigma_{\max}}, Le^{b\sigma_{\min}}, L, T, 2(\mu-\sigma\lambda)k(0.5I), \sigma_{\max}, \sigma_{\min})\right]$

Panel B. Structured Convertible Debt with $\nu^2\left(k'(0.5I)\right)^2 + 4\sigma^2 k(0.5I)k''(0.5I) > 0$

Structuring factors at t	$\theta_{1,t} = \begin{cases} 2\sigma k(0.5I) + \dfrac{\nu^2\left(k'(0.5I)\right)^2}{2\sigma k''(0.5I)}, & G_t \le g(t) \\ 0, & G_t > g(t) \end{cases}$ and $\theta_{2,t} = \nu k(0.5I)$
Max and Min volatilities	$\sigma_{\min} = 2\sigma k(0.5I)$ and $\sigma_{\max} = -\dfrac{\nu^2\left(k'(0.5I)\right)^2}{2\sigma k''(0.5I)}$
Optimal strategy at t	$I^*_{1,t} = I^*_{2,t} = 0.5I$
Owner's value at t	$\xi_t e^{[2(\mu-\sigma\lambda)k(0.5I)-r](T-t)} \times D(G_t, Le^{-b\sigma_{\max}}, Le^{b\sigma_{\min}}, L, T-t, 2(\mu-\sigma\lambda)k(0.5I), \sigma_{\max}, \sigma_{\min})$
Financing restriction on b	$I - W = e^{[2(\mu-\sigma\lambda)k(0.5I)-r]T} \times \left[G_0 - D(G_0, Le^{-b\sigma_{\max}}, Le^{b\sigma_{\min}}, L, T, 2(\mu-\sigma\lambda)k(0.5I), \sigma_{\max}, \sigma_{\min})\right]$

Panel B of Table 5 shows the case when expression (33) is positive and $\theta_{1,t}$ is given by equation (35). The positive value of expression (33) implies that

$$\sigma_{\min} = 2\sigma k(0.5I) \quad \text{and} \quad \sigma_{\max} = -\frac{\nu^2\left(k'(0.5I)\right)^2}{2\sigma k''(0.5I)}.$$

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Letting $K_1 = Le^{-b\sigma_{\max}}$ and $K_2 = Le^{b\sigma_{\min}}$, the owner's claim is again expressed in terms of D. However, this differs from Panel A, since $\sigma_{\min}$ and $\sigma_{\max}$ are reversed. At time 0, the concave and convex regions are again balanced against one another by choosing L such that $g(0) = G_0$, and the financing restriction on b guarantees that the owner captures the first best NPV.

D. Discussion

From equation (30), specifying $\theta_{2,t} = \nu k(0.5I)$ is not only sufficient but also necessary for achieving the first best. This is different than the result in Section II.F, since now the structuring factor must depend on the project-specific parameter ν and the scale function $k(\cdot)$ if the owner wants to solve the asset substitution problem. It is therefore worth exploring how the agency cost of asset substitution is impacted when debt structures that rely on ν and $k(\cdot)$ are precluded.

If $\theta_{2,t} \neq \nu k(0.5I)$, equation (30) shows that the first best is nearly achieved when $f_{GG} \approx 0$. On the other hand, the owner's investment strategy may deviate significantly from the first best when the magnitude of f_{GG} is large. The worst case scenario is characterized by taking the limit as $f_{GG} \to \pm\infty$. In this case, equation (30) reduces to

$$\frac{k'(I_{2,t})}{k'(I_{1,t})} = \frac{\sigma\left(\sigma[k(I_{1,t}) + k(I_{2,t})] - \theta_{1,t}\right)}{\sigma\left(\sigma[k(I_{1,t}) + k(I_{2,t})] - \theta_{1,t}\right) + \nu\left(\nu k(I_{2,t}) - \theta_{2,t}\right)}. \tag{37}$$

Note that the right-hand side of equation (37) tends to 1 as the magnitude of $\theta_{1,t}$ becomes large. Thus, even if the owner is precluded from using ν and $k(\cdot)$ to structure the debt, the agency cost of asset substitution can be mitigated by choosing a large enough value for $\theta_{1,t}$.

This is illustrated in Figure 1, which uses the scale function $k(z) = z^{\gamma}$. Graph A illustrates the agency cost of asset substitution for three different values of γ.[17] For example, if $\gamma = 0.5$ and $I_{1,t} = 0$ for all t, the agency cost is around 10%, which is economically significant. Graph B shows that the cost can be mitigated by structuring the debt with a large value for $\theta_{1,t}$.[18] For example, if the magnitude of $\theta_{1,t}$ is close to 100, $I_{1,t} \approx 50$. According to Graph A, the agency cost is small if $I_{1,t} \approx 50$; this is very close to the first best, since $I = 100$. The ability of structured debt to mitigate the agency cost of asset substitution is similar to that of dynamic debt rebalancing (see Childs et al. (2005), Ju and Ou-Yang (2005)).

IV. Characteristics of Structured Debt

This section contrasts the characteristics of structured debt with those of non-structured debt. The focus is on debt payoffs, default probabilities, yields, default costs, conversion probabilities, and conversion values.

[17]The agency cost in Graph A of Figure 1 is defined as the first best firm value divided by the second best firm value, minus 1. This coincides with the definition in Childs et al. (2005).

[18]The sign of $\theta_{1,t}$ (positive or negative) is chosen to satisfy the second-order condition.

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FIGURE 1

Agency Cost and Investment

Graph A of Figure 1 shows the agency cost of asset substitution when the owner chooses a constant investment strategy that deviates from the first best. The agency cost is defined as the first best firm value divided by the second best firm value, minus 1. The first best firm value is

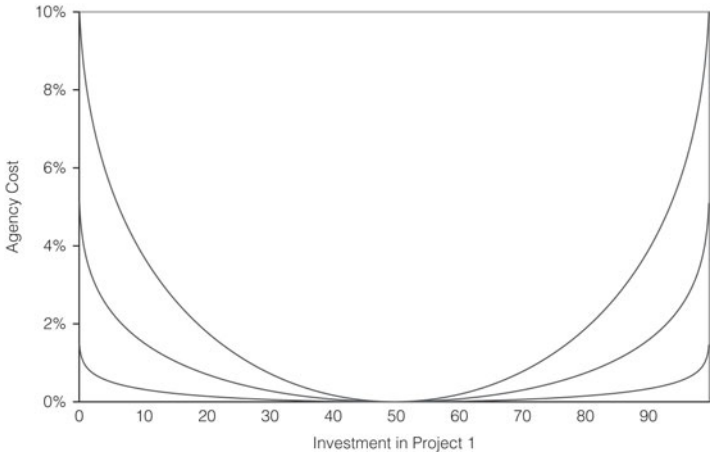
$$X_0 e^{2(\mu-\sigma\lambda)k(0.5I)T-rT},$$

and the second best firm value is

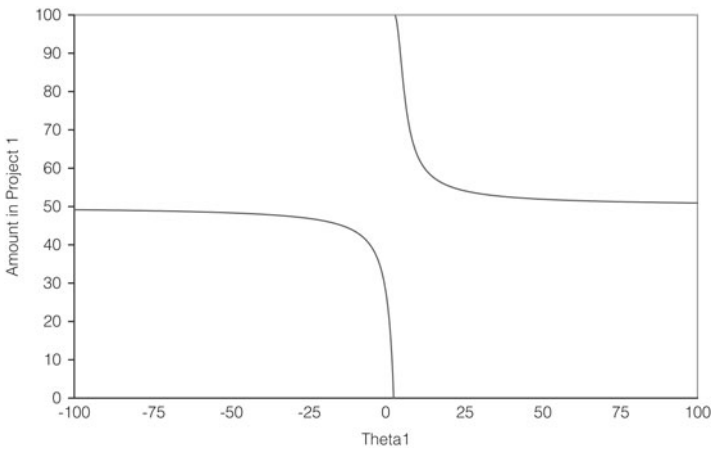
$$X_0 e^{(\mu-\sigma\lambda)[k(I_{1,t})+k(I-I_{1,t})]T-rT}.$$

The scale function is $k(z)=z^{\gamma}$, where $\gamma=0.1$ in the bottom curve, 0.3 in the middle curve, and 0.5 in the top curve. The other parameters are $\mu=0.2$, $\sigma=0.25$, $\lambda=0.3$, $I=100$, and $T=1$. Graph B uses equation (37) to illustrate how $I_{1,t}$ varies with $\theta_{1,t}$. The other parameters for this graph are $\gamma=0.5$, $\sigma=\nu=0.25$, and $\theta_{2,t}=0$.

Graph A. Agency Cost



Graph B. Investment in Project 1



A. Structured versus Nonstructured Payoffs

The payoffs for structured and nonstructured debt are illustrated in Figure 2. Graphs A–D illustrate nonstructured debt (as in Panel A of Table 2), nonstructured convertible debt (Panel A of Table 4), structured debt (Panel A of Table 3), and

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structured convertible debt (Panel B of Table 4), respectively. All of the graphs use $V_0 = 100$ and $W = 50$. Thus, the owner raises 50 via debt financing.

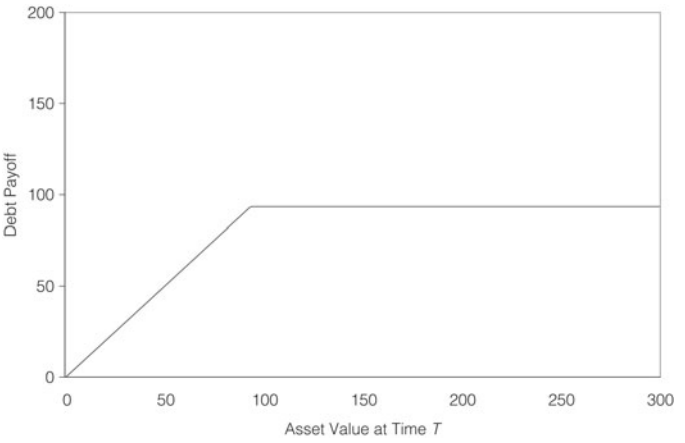
The nonstructured debt in Graph A of Figure 2 is standard. For high asset values, the debtholders receive the promised face value K. For low asset values, the debtholders receive less than K but take ownership of the firm. This standard payoff can be contrasted with the structured debt payoff min $[V_T, \xi_T K]$ that is shown in Graph C. Since the elasticity of ξ_T with respect to V_T is greater than 1,

FIGURE 2

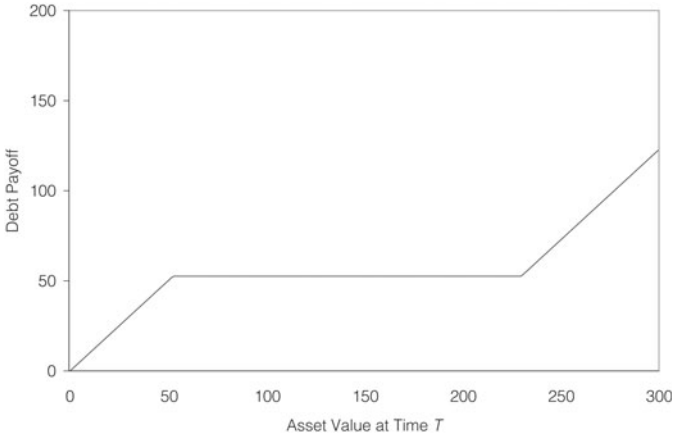
Debt Payoffs

The nonstructured payoffs are min $[V_T, K]$ for Graph A of Figure 2 and min $[V_T, K_1]$ + max $[0, V_T - K_2]$ for Graph B. The structured payoffs are min $[V_T, \xi_T K]$ for Graph C and min $[V_T, \xi_T K_1]$ + max $[0, V_T - \xi_T K_2]$ for Graph D. All graphs use $V_0 = 100$ and $W = 50$; thus the owner raises 50 via external debt financing. To raise 50, $K = 93.36$ in Graph A, $K = 52.60$ in Graph C, and $K_1 = 52.56$ and $K_2 = 230.20$ for both Graphs B and D. The structured securities use $\theta_t = \sigma_1 + \sigma_2$ for all t. The first best pair is $\{\mu_2, \sigma_2\}$. Graph C uses $\sigma_1 = 0.30$ and $\sigma_2 = 0.25$, which is consistent with Table 3. Graphs A, B, and D use $\sigma_1 = 0.25$ and $\sigma_2 = 0.30$, which is consistent with Tables 2 and 4. Other parameters are $\mu_2 - \sigma_2\lambda = 0.125$, $r = 0.05$, and $T = 1$.

Graph A. Nonstructured Debt



Graph B. Nonstructured Convertible Debt

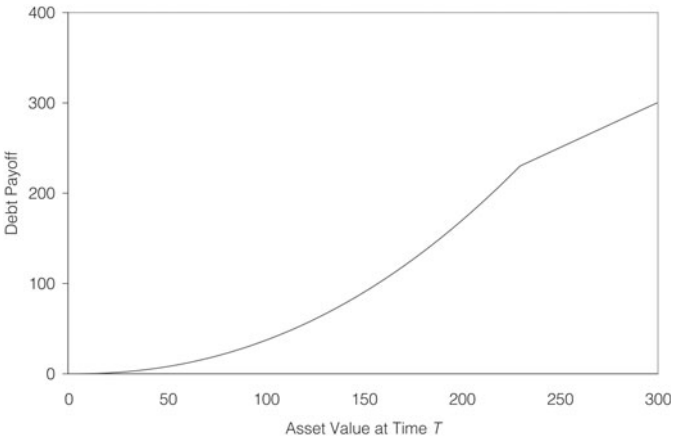


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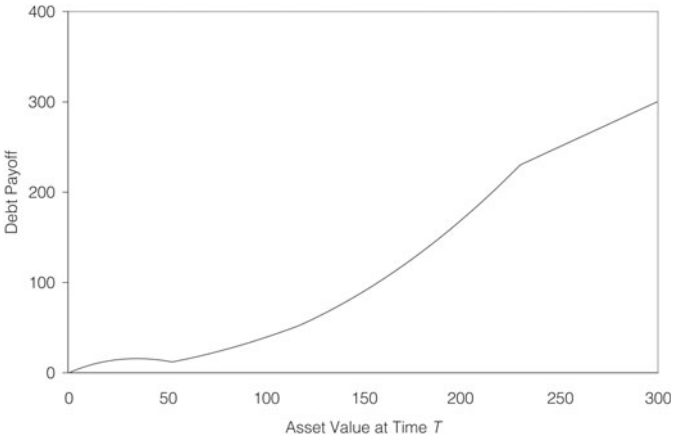
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FIGURE 2 (continued)
Debt Payoffs

Graph C. Structured Debt



Graph D. Structured Convertible Debt



default occurs at high asset values rather than low asset values. For low asset values, the firm is not in financial distress. For this region, the structured debt payoff is increasing and convex. Default first occurs when V_T is about 230, which corresponds to the kink in the graph. For higher asset values, the graph is linear, since the debtholders take ownership of the firm. The graph shows that structured debt is remarkably equity-like. Debtholders have upside participation and take control of the firm in good states. This allows the owner to eliminate the agency cost of asset substitution. However, the owner now has downside exposure and maintains control of the firm in bad states.

Graph B of Figure 2 shows nonstructured convertible debt. For low asset values, the firm defaults, and the debtholders receive less than the debt's face value; for high asset values, the debtholders convert their debt into equity and take control of the firm. This standard payoff can be contrasted with the structured convertible debt payoff in Graph D. Like structured debt, the elasticity of

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ξ_T with respect to V_T is greater than 1. Thus, default occurs at high asset values, while conversion occurs at low asset values. The payoff due to conversion corresponds to the hump in the graph, which occurs when V_T is less than about 50. By structuring the convertible debt to eliminate the agency cost of asset substitution, the debt's payoff characteristics (i.e., default and conversion) are reversed relative to those of nonstructured debt.

B. Default and Conversion Probabilities

The contrast between structured and nonstructured financing is also seen by examining the sensitivity of the default and conversion probabilities to movements in the firm's total assets. First, consider the nonstructured debt contract in Panel A of Table 2. The default probability is

$$\text{(38)} \qquad \mathrm{E}_t\left[1_{\{V_T<K\}}\right] = N\left(\frac{\log\left(\frac{K}{V_t}\right)-\left(\mu_2-\frac{1}{2}\sigma_2^2\right)(T-t)}{\sigma_2\sqrt{T-t}}\right),$$

where $1_{\{\cdot\}}$ is the indicator function for the event in curly brackets and $N(\cdot)$ is the standard normal cumulative distribution function.[19] Since the derivative of equation (38) with respect to V_t is negative, the default probability decreases as the firm's assets increase.

The result for structured financing is remarkably different, since ξ_t is correlated with V_t. To determine the sensitivity of the default probability, both the direct effect due to V_t and the indirect effect due to ξ_t must be analyzed. Consider the structured debt contract in Panel A of Table 3. Since $\theta_t = \sigma_1 + \sigma_2$ for all t, ξ_t is path-independent and can be written in terms of V_t as

$$\text{(39)} \qquad \xi_t = \left(\frac{V_t}{V_0}\right)^{1+(\sigma_1/\sigma_2)} e^{-\frac{1}{2}(\sigma_1+\sigma_2)^2 t+(\sigma_1+\sigma_2)\lambda t-\left(\mu_2-\frac{1}{2}\sigma_2^2\right)(1+(\sigma_1/\sigma_2))t}.$$

The default probability is $\mathrm{E}_t\left[1_{\{V_T<\xi_T K\}}\right]$, which is rewritten as

$$\text{(40)} \qquad \mathrm{E}_t\left[1_{\{G_T<K\}}\right] = N\left(\frac{\log\left(\frac{K}{G_t}\right)-\left(\mu_2-\frac{1}{2}\sigma_1^2+(\sigma_1-\lambda)(\sigma_1+\sigma_2)\right)(T-t)}{\sigma_1\sqrt{T-t}}\right).$$

Using $G_t = V_t\xi_t^{-1}$, the derivative of equation (40) with respect to V_t, including both the direct and indirect effects, is *positive*. This occurs because the elasticity is $(\partial\xi_t/\partial V_t)(V_t/\xi_t) = 1 + (\sigma_1/\sigma_2) > 1$, which can be seen from equation (39). When V_t increases by 1%, the face value of the debt increases by more than 1%. Contrary to the standard intuition, a higher asset level does not produce a lower default probability.

[19] The default probabilities in this section are calculated under the true (i.e., P) probability measure. Calculating these probabilities under Q instead of P does not alter the conclusions.

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For the case of structured convertible debt, the analysis is more complex due to the presence of the boundary $g(t)$. In this case, ξ_t may be path-dependent, since it may depend on the amount of time that G_t spends in the vicinity of $g(t)$, as measured by the local time.[20] Furthermore, the elasticity of ξ_t with respect to V_t may change as G_t crosses $g(t)$. For example, consider Panel B of Table 4. In this case, regardless of whether G_t is above or below $g(t)$, the elasticity is always greater than 1. Thus, the prior conclusion that the default probability is increasing in V_t also applies to structured convertible debt. In addition, the conversion probability is decreasing in V_t. However, from Panels C and D of Table 4, structuring may produce a mix of results, since the elasticity may be greater than 1 on one side of $g(t)$ and may be equal to 0 on the other side.

For the generalized version of Green's (1984) model in Section III, the implications of structured financing for the default probability are equally interesting. For example, suppose expression (33) is negative and the owner finances the firm by issuing a structured debt contract with payoff $\min[X_T, \xi_T K]$, where $\theta_{1,t} = 0$ and $\theta_{2,t} = \nu k(0.5I)$ for all t. Although this capital structure is not shown in the tables, it is verified from expressions (30)–(31) that the first best is achieved. The default probability is

$$\begin{aligned} (41) \qquad & \mathrm{E}_t\left[1_{\{X_T<\xi_T K\}}\right] \\ &= \mathrm{E}_t\left[1_{\{G_T<K\}}\right] \\ &= N\left(\frac{\log\left(\frac{K}{G_t}\right) - 2k(0.5I)\left[\mu - \sigma^2 k(0.5I)\right](T-t)}{2\sigma k(0.5I)\sqrt{T-t}}\right). \end{aligned}$$

Since $\theta_{1,t}$ and $\theta_{2,t}$ are constant, ξ_t is path-independent and can be written as

$$(42) \qquad \xi_t = \left[\frac{V_{2,t}}{V_{2,0}}\frac{V_{1,0}}{V_{1,t}}\right]^{k(0.5I)} e^{-\frac{1}{2}\nu^2 k^2(0.5I)t + \frac{1}{2}\nu^2 k(0.5I)t}.$$

Furthermore, note that X_t can be written as

$$(43) \quad X_t = X_0\left[\frac{V_{2,t}}{V_{2,0}}\frac{V_{1,t}}{V_{1,0}}\right]^{k(0.5I)} e^{-\frac{1}{2}\nu^2 k^2(0.5I)t + \frac{1}{2}\nu^2 k(0.5I)t + \sigma^2 k(0.5I)t - 2\sigma^2 k^2(0.5I)t}.$$

Recalling that $G_t = X_t \xi_t^{-1}$, equations (42)–(43) imply that

$$(44) \qquad G_t = X_0\left[\frac{V_{1,t}}{V_{1,0}}\right]^{2k(0.5I)} e^{\sigma^2 k(0.5I)t - 2\sigma^2 k^2(0.5I)t}.$$

Thus, G_t does not depend on the idiosyncratic risk $z_{2,t}$. This has important implications for the default probability in equation (41). For example, consider a scenario in which $z_{2,t}$ increases but $z_{1,t}$ is unchanged. In this case, $V_{2,t}$ and X_t both

[20] The local time of a continuous stochastic process is defined in Karatzas and Shreve ((1991), p. 218). For the structured convertible debt contracts in Figures 2 and 3, it is assumed that the local time term that affects ξ_t is 0, thus allowing these figures to be constructed without referring to a particular path.

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increase, but $V_{1,t}$ and G_t are unchanged. Thus a change in the firm's assets X_t does not necessarily alter the firm's default probability in equation (41). Alternatively, consider a scenario in which a change in $z_{1,t}$ is offset by a change in $z_{2,t}$ such that X_t is unchanged. Now the firm's assets are unchanged, but the change in $z_{1,t}$ produces a change in both $V_{1,t}$ and equation (41). Thus the default probability may change even if the assets do not change.

A similar phenomenon arises for the structured convertible bonds in Table 5. However, the calculations are considerably more involved due to the switching boundary $g(t)$. But the central result is unaltered: In the presence of structured financing, the default (conversion) probability may no longer be a decreasing (increasing) function of the firm's assets, which runs counter to the standard intuition.

C. Yields

For the reasons discussed above, the yields of the structured and nonstructured securities have remarkably different asset sensitivities. For the nonstructured debt contract in Panel A of Table 2, the time t yield is defined as

$$\frac{K}{e^{(\mu_2-\sigma_2\lambda-r)(T-t)}\left[V_t - C_{\mathrm{BS}}(V_t, K, T-t, \mu_2-\sigma_2\lambda, \sigma_2)\right]},$$

which equals the face value K divided by the time t debt value. It is easy to show that this yield is a decreasing function of V_t. In contrast, for the structured debt contract in Panel A of Table 3, the time t yield is

$$\frac{\mathrm{E}_t\left[\xi_T K\right]}{e^{(\mu_2-\sigma_2\lambda-r)(T-t)}\left[V_t - \xi_t C_{\mathrm{BS}}(G_t, K, T-t, \mu_2-\sigma_2\lambda, \sigma_1)\right]},$$

where the numerator is the expected face value and the denominator is the structured debt's time t value. Since $\theta_t = \sigma_1 + \sigma_2$ for all t, the expected face value is $\mathrm{E}_t\left[\xi_T K\right] = \xi_t K e^{\lambda(\sigma_1+\sigma_2)(T-t)}$. Thus, the yield can be rewritten as

$$\frac{K e^{\lambda(\sigma_1+\sigma_2)(T-t)}}{e^{(\mu_2-\sigma_2\lambda-r)(T-t)}\left[G_t - C_{\mathrm{BS}}(G_t, K, T-t, \mu_2-\sigma_2\lambda, \sigma_1)\right]}.$$

Since the elasticity of ξ_t with respect to V_t is greater than 1, the yield is an *increasing* function of V_t. The yields on structured and nonstructured securities move in opposite directions, which implies that structured securities should be viewed as hedge assets. This potentially explains the popularity of structured securities with investors and third-party issuers.

D. Default Cost

The prior analysis used a zero cost of default. This assumption is now relaxed to investigate the trade-off between the cost of default and the agency cost of asset substitution. For instance, consider the case in which the cost of default is proportional to firm value. Since default for the structured securities in Figure 2 occurs at high asset levels, the default cost may be much higher than for nonstructured

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securities. Thus, by structuring the debt to solve the asset substitution problem, the owner may be trading off a lower agency cost for a higher default cost.

Note that a nonzero default cost does not affect the ability of structured financing to solve the asset substitution problem. The functional form of the owner's payoff (see Tables 2–5) remains valid. Thus, the owner is still able to specify $\{\theta_t : t \in [0, T]\}$ to induce the first best strategy. However, a nonzero default cost does impact the debt's payoff. To raise $V_0 - W$ at time 0, the debt's face value typically will be higher than in the case without costly default. Thus, the financing restrictions on K (for structured debt) and b (for structured convertible debt) must be altered to accommodate the time 0 value of the default cost.

Letting δ denote the cost of default as a proportion of firm value, the default cost for structured debt is $\delta V_T 1_{\{V_T < \xi_T K\}}$, where $1_{\{\cdot\}}$ is the indicator function. Similarly, the default cost for nonstructured debt is $\delta V_T 1_{\{V_T < K\}}$. Replacing K in these expressions with K_1 gives the default cost for structured and nonstructured convertible debt, respectively. The time t value of the default cost is calculated by taking an expectation with respect to the risk-adjusted measure Q and discounting at the rate r. Closed-form expressions are given in the Appendix.

Graphs A–D of Figure 3 illustrate the trade-off between the cost of default and the agency cost of asset substitution. The first best pair is $\{\mu_2, \sigma_2\}$ and $\sigma_1 > \sigma_2$. The solid curves represent structured debt, which achieves the first best. The dashed curves represent nonstructured debt in which the owner engages in asset substitution by shifting to the high-volatility pair $\{\mu_1, \sigma_1\}$ after the debt is in place. Graphs A–D show the agency cost of substitution (which is 0 for structured debt), the default probability, the default cost, and the total cost (agency cost plus default cost), respectively. Since the total cost curves intersect, structured debt may not always turn out to be the lower cost alternative. However, structured debt does allow the owner to eliminate the total cost at time T on the set of states for which the owner retains control of the firm. This is not true for nonstructured debt, since the owner still bears the cost of choosing a second best operating strategy, even if the firm does not default.

Graphs E and F of Figure 3 illustrate the default costs and the conversion values for structured convertible debt (solid curves) and nonstructured convertible debt (dashed curves), respectively. These two graphs use $\mu_1 - \sigma_1\lambda = \mu_2 - \sigma_2\lambda$ and $\sigma_2 > \sigma_1$, consistent with Panels A and B of Table 4. Since the agency cost is 0 for both types of debt, the default cost is also the total cost. Note that the total cost for structured debt may be considerably higher than that for nonstructured debt. However, the set of states for which the total cost is incurred is different for the two types of securities. A similar statement can be made about the set of states for which the conversion option is valuable. This supports the earlier statement that structured securities are hedge assets (i.e., an investor can diversify by holding both types in the same portfolio).

E. Other Characteristics

To design a structured security that solves the asset substitution problem, the owner either must know the project parameters (see Tables 2–4) or must have knowledge of the average project volatility or the range of volatilities (see

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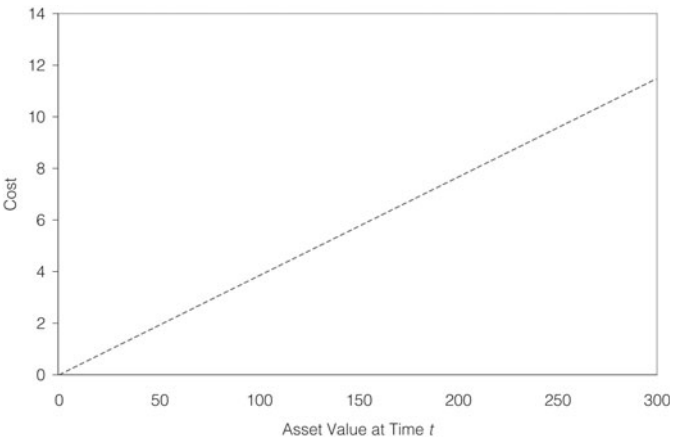
Section II.F). In these cases, the security's design depends either directly or indirectly on parameters other than the first best. For example, suppose $\{\mu_2, \sigma_2\}$ are the first best parameters and $\{\mu_1, \sigma_1\}$ are the second best parameters. The first best firm value is $V_t e^{(\mu_2 - \sigma_2\lambda - r)(T-t)}$, which depends only on the first best parameters. However, the structured debt in Panel A of Table 3 depends on σ_1, the second best

FIGURE 3

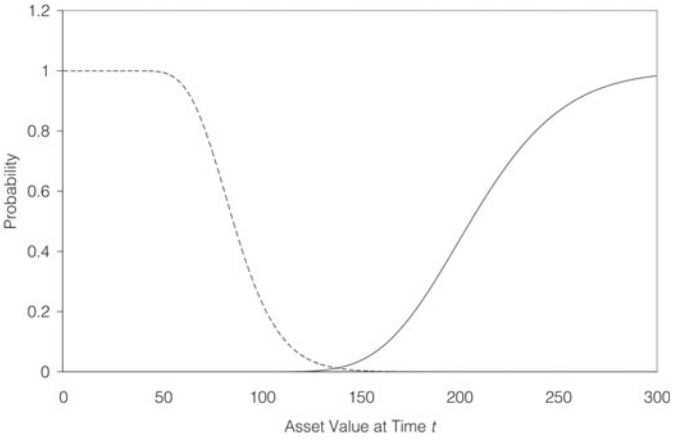
Agency Cost, Default Cost, and Conversion Value

Graphs A–D of Figure 3 compare structured debt (solid curves) to nonstructured debt (dashed). The structured debt achieves the first best; for the nonstructured debt, the owner is assumed to engage in asset substitution by choosing the high-volatility pair $\{\mu_1, \sigma_1\}$ for all t. Graphs A–D show the agency cost of asset substitution (which is 0 for the first best), the default probability, the default cost, and the total cost (agency plus default). Graphs E and F compare structured (solid) and nonstructured (dashed) convertible debt. Both achieve the first best. Graph E (F) shows the default cost (conversion option value). All graphs use $V_0 = 100$ and $W = 50$. To raise $V_0 - W = 50$, the structured debt uses $K = 52.64$, the nonstructured debt uses $K = 92.78$, and both convertible securities use $K_1 = 52.60$ and $K_2 = 230.08$. All of the structured securities use $\theta_t = \sigma_1 + \sigma_2$ for all t. Other parameters are $r = 0.05$, $T - t = 0.5$, and $\delta = 0.1$; Graphs A–D use $\mu_2 - \sigma_2\lambda = 0.125$, $\mu_1 - \sigma_1\lambda = 0.05$, $\sigma_1 = 0.30$, and $\sigma_2 = 0.25$; Graphs E and F use $\mu_2 - \sigma_2\lambda = \mu_1 - \sigma_1\lambda = 0.125$, $\sigma_2 = 0.30$, and $\sigma_1 = 0.25$.

Graph A. Agency Cost of Asset Substitution



Graph B. Default Probability: Structured versus Nonstructured Debt



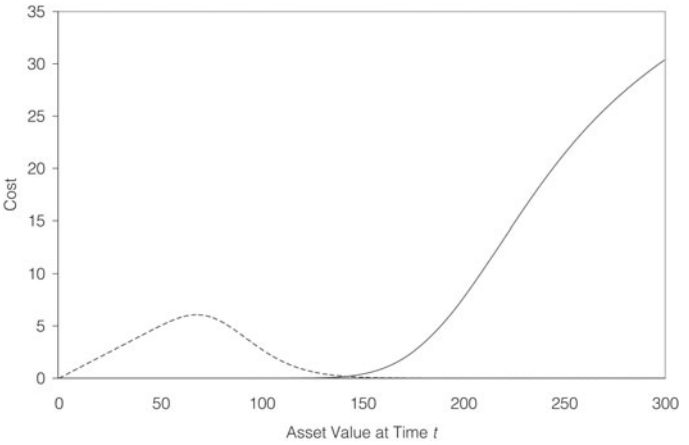
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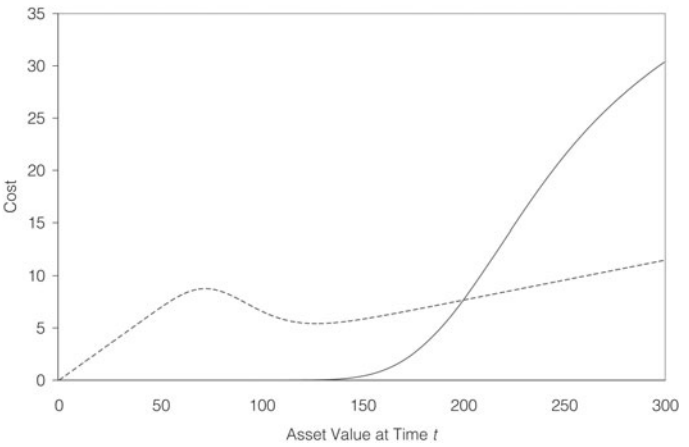
FIGURE 3 (continued)

Agency Cost, Default Cost, and Conversion Value

Graph C. Default Cost: Structured versus Nonstructured Debt



Graph D. Total Cost: Structured versus Nonstructured Debt



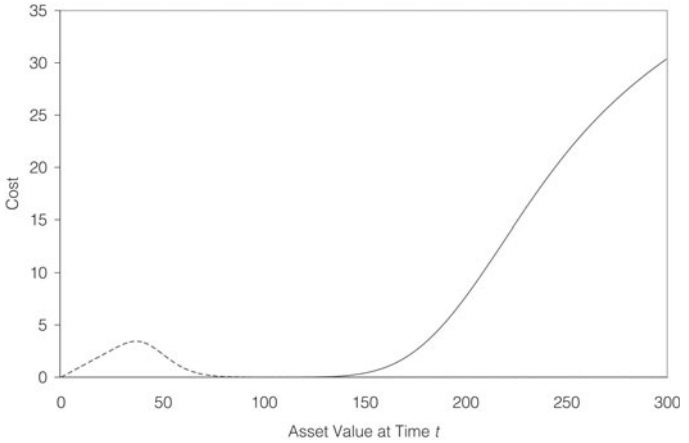
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volatility. As σ_1 increases, K must increase due to the financing restriction. All else being equal, the face value of the issued debt is higher as the second best project becomes more volatile, even though the owner never deviates from the first best. This shows how seemingly innocuous parameters of the investment opportunity set, in this case σ_1, become relevant when the firm issues a structured security. Compared to an all-equity firm, a firm that finances itself with structured debt faces additional exposure to shifts in the investment opportunity set. This suggests that structured financing is less suitable for companies that face unpredictable investment opportunities. On the other hand, firms in relatively stable industries are probably the best candidates for issuing structured debt. This latter statement appears to be consistent with the issuers in Table 1.

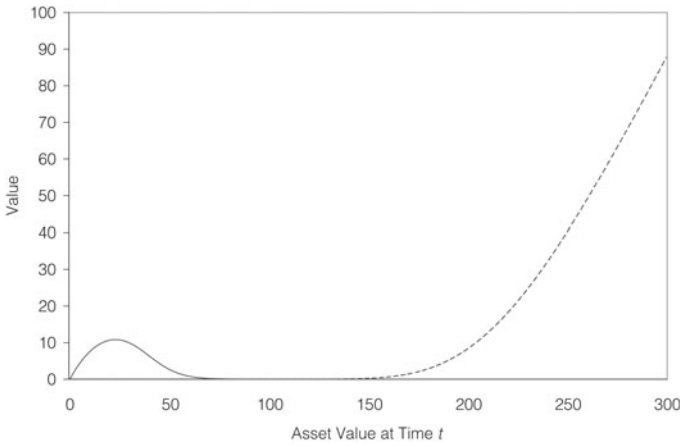
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FIGURE 3 (continued)
Agency Cost, Default Cost, and Conversion Value

Graph E. Default Cost: Structured versus Nonstructured Convertible Debt



Graph F. Value of Conversion Option: Structured versus Nonstructured Convertible Debt



F. Structuring to Avoid Default

For structured debt with a time T face value of $\xi_T K$, default may occur when the total assets are at a relatively high level. This leads to potentially high default costs, as shown in Figure 3. Since the elasticity of ξ_t with respect to V_t depends on θ_t, the question arises as to whether or not a debt contract can be structured to avoid default altogether. This section shows that such a structure is possible. The debt is structured to behave like straight debt when the firm is not in financial distress, but at the first sign of financial distress the debt acquires equity-like characteristics, and it continues to behave in this manner until its final maturity date.

Using the framework from Section II, suppose the firm issues structured debt whose time T payoff is $\min[V_T, \xi_T K]$, where ξ_t is given by equation (6) and

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$V_0 > K$. Suppose the owner chooses the first best pair $\{\mu_2, \sigma_2\}$, and define the stopping time $\tau = \inf\{t \geq 0 : V_t \leq K\}$. In this case, τ is the first time the firm's assets fall to the level K. Let $\theta_t = 0$ for all $t \leq \tau$, and let $\theta_t = \sigma_2$ for all $t > \tau$.

This type of structured security has the following properties: Prior to time τ, $\xi_t = 1$ since $\theta_t = 0$. In this case, the time t face value is K, and the firm is not in financial distress, since $V_t > K$. If V_t never hits K prior to T, the debtholders receive K at time T. On the other hand, if τ occurs before T, the firm is in a state of potential financial distress, since $V_\tau = K$. But now, ξ_t begins to vary, since $\theta_t = \sigma_2$. Because θ_t matches the volatility of V_t, the elasticity of ξ_t with respect to V_t is now equal to 1. The assets V_t continue to grow at the rate $\mu_2 - \sigma_2\lambda > 0$, but random fluctuations in V_t are matched one-for-one by fluctuations in ξ_t. Thus, $V_T > \xi_T K$ at time T, and the firm never defaults on its debt.[21]

Since the firm never defaults, the value of this structured debt contract is

$$\mathrm{E}_t^Q\left[e^{-r(T-t)}\min\left[V_T, \xi_T K\right]\right] \quad = \quad \mathrm{E}_t^R\left[\frac{\xi_t}{\xi_T}e^{-r(T-t)}\xi_T K\right] \quad = \quad \xi_t e^{-r(T-t)}K.$$

At time 0, the value is $e^{-rT}K$. Even though the final payoff is $\xi_T K$, which is risky, the structured debt is valued at time 0 as if it were a riskless security. This is because the potential upside (resulting from a random increase in V_t after time τ) is balanced against the potential downside (resulting from a random decrease in V_t after time τ) such that the time 0 value coincides with that of riskless debt.[22] The owner's value at time t is $\xi_t\left[e^{(\mu_2-\sigma_2\lambda-r)(T-t)}G_t - e^{-r(T-t)}K\right]$, which shows that $\partial^2 f/\partial G^2 = 0$. Thus, equation (12) implies that the owner's optimal choice is the first best pair $\{\mu_2, \sigma_2\}$, which was assumed earlier. This example shows how structuring can simultaneously solve the asset substitution problem without incurring potentially high default costs. This is also a case in which the owner and the debtholders agree on the operating policy of the firm (see Section II.E).

V. Concluding Remarks and Future Research

This paper shows how the asset substitution problem can be solved via capital structure design in a dynamic setting. While this paper enhances our understanding of dynamic asset substitution, several open questions remain. First, it would be interesting to examine whether short-term debt rebalancing, as discussed in Childs, Mauer, and Ott (2005) and Ju and Ou-Yang (2005), is a true substitute for structured financing. For example, do firms with similar characteristics use debt rebalancing and structured financing interchangeably? Leary and Roberts (2005) provide empirical evidence on dynamic rebalancing, but they do not control for structuring. While collateral pledges and covenants are sometimes

[21] This can also be seen by noting that the volatility of G_t under the R measure vanishes when $\theta_t = \sigma_2$ (see equation (11)), since $G_\tau = K$ and $\mu_2 - \sigma_2\lambda > 0$, $G_T > K$ at time T, which implies that $V_T > \xi_T K$.

[22] This debt structure is equivalent to having straight riskless debt with face value K and a forward contract whose value is 0 at time 0 and whose payoff is $\xi_T K - K$ at time T. Decomposing the debt structure in this manner shows why structured debt can be interpreted as a bundled hedge (Smithson and Chew (1992), Chidambaran et al. (2001)).

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used to mitigate asset substitution due to investment flexibility, the evidence in MacKay (2003) suggests that risk shifting due to production flexibility is harder to contract away. It would be interesting to see if firms with high levels of production flexibility are more frequent users of structured debt.

Second, as Myers (1977) has argued, leveraged firms may underinvest in positive NPV projects, since the benefits might go to the debtholders. While the empirical evidence supporting Myers' (1977) argument is mixed (see Barclay and Smith (1995), Stohs and Mauer (1977), and Titman and Wessels (1988)), this article suggests a new insight into the problem. Specifically, a firm should use structured debt whose face value varies with the moneyness of the firm's growth options, thus mitigating the incentive of equityholders to pass up good projects. Along similar lines, it would be interesting to examine how the relationship between debt capacity and growth options, as discussed in Barclay, Smith, and Morellec (2006), is altered in the presence of structured financing.

Third, structuring appears to be a win-win outcome from the perspective of both the issuer and the buyer. The issuer is able to solve the asset substitution problem, which eliminates the agency cost that would otherwise be borne by the firm's equityholders. On the other hand, the yield on structured debt, unlike that of nonstructured debt, may be an increasing function of the firm's assets. In an economy with multiple firms whose assets are cross-sectionally correlated, a bond investor can diversify his portfolio by adding structured securities. Structured securities are hedge assets and should command a high price in practice. It would be interesting to investigate empirically the diversification benefits from holding structured securities.

Lastly, the analysis reveals multiple capital structures that solve the asset substitution problem for each set of parameter restrictions (see Tables 2–5). Although certain practical features such as taxes have been omitted, the multiplicity of capital structures suggests that it should be relatively easy for firms to adopt a capital structure that eliminates the agency cost of dynamic asset substitution. This point of view is consistent with the survey evidence in Graham and Harvey ((2001), table 10). Furthermore, since there are multiple structures that solve the asset substitution problem, it seems unlikely that all of these would be rendered infeasible once the omitted features such as taxes are accommodated. It would be interesting to see if these statements are supported by financial managers in practice.

Appendix

The Function $D(G_t, t)$

The function $D(G_t, t)$ is discussed in Section II.C. For the region $G_t \leq g(t)$, the function $D(G_t, t)$ is

$$\text{(A-1)} \qquad \begin{aligned} D(G_t, t) \;=\; & G_t N(d_1) + e^{-c(T-t)} \left[K_2 N(d_2) - K_1 N(d_3)\right] \\ & + \left(\frac{\sigma_{\max}}{\sigma_{\max} + \sigma_{\min}}\right) G_t \left(\frac{G_t}{g(t)}\right)^{\frac{\sigma_{\min} - \sigma_{\max}}{\sigma_{\max}}} \left[N(d_4) - N(d_5)\right] \end{aligned}$$

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$$- \left(\frac{\sigma_{\max}}{\sigma_{\max} + \sigma_{\min}} \right) G_t \left(\frac{L}{K_1} \right)^{\frac{\sigma_{\min} - \sigma_{\max}}{\sigma_{\max}}} N(d_6)$$

$$- \left(\frac{\sigma_{\min}}{\sigma_{\max} + \sigma_{\min}} \right) G_t \left(\frac{L}{K_2} \right)^{\frac{\sigma_{\max} - \sigma_{\min}}{\sigma_{\min}}} N(d_7),$$

while for the region $G_t > g(t)$, the function $D(G_t, t)$ is

$$\begin{aligned} \text{(A-2)} \qquad D(G_t, t) \quad = \quad & G_t N(-d_8) + e^{-c(T-t)} \left[K_2 N(d_9) - K_1 N(d_{10}) \right] \\ & + \left(\frac{\sigma_{\min}}{\sigma_{\max} + \sigma_{\min}} \right) G_t \left(\frac{G_t}{g(t)} \right)^{\frac{\sigma_{\max} - \sigma_{\min}}{\sigma_{\min}}} \left[N(d_{11}) - N(d_{12}) \right] \\ & - \left(\frac{\sigma_{\max}}{\sigma_{\max} + \sigma_{\min}} \right) G_t \left(\frac{L}{K_1} \right)^{\frac{\sigma_{\min} - \sigma_{\max}}{\sigma_{\max}}} N(d_{13}) \\ & - \left(\frac{\sigma_{\min}}{\sigma_{\max} + \sigma_{\min}} \right) G_t \left(\frac{L}{K_2} \right)^{\frac{\sigma_{\max} - \sigma_{\min}}{\sigma_{\min}}} N(d_{14}), \end{aligned}$$

where $N(\cdot)$ is the standard normal cumulative distribution function. The expressions for d_1–d_{14} are

$$d_1 = \frac{\log\left(\frac{G_t}{K_1}\right) + c(T-t) + \frac{1}{2}\sigma_{\max}^2 (T-t)}{\sigma_{\max}\sqrt{T-t}},$$

$$d_2 = \frac{\log\left(\frac{G_t}{L}\right) - \frac{\sigma_{\max}}{\sigma_{\min}} \log\left(\frac{K_2}{L}\right) + c(T-t) - \frac{1}{2}\sigma_{\max}^2 (T-t)}{\sigma_{\max}\sqrt{T-t}},$$

$$d_3 = \frac{\log\left(\frac{G_t}{K_1}\right) + c(T-t) - \frac{1}{2}\sigma_{\max}^2 (T-t)}{\sigma_{\max}\sqrt{T-t}},$$

$$d_4 = \frac{\log\left(\frac{G_t}{L}\right) + \log\left(\frac{K_1}{L}\right) + c(T-t) - \frac{1}{2}\sigma_{\max}^2 (T-t) + \sigma_{\min}\sigma_{\max} (T-t)}{\sigma_{\max}\sqrt{T-t}},$$

$$d_5 = \frac{\log\left(\frac{G_t}{L}\right) - \frac{\sigma_{\max}}{\sigma_{\min}} \log\left(\frac{K_2}{L}\right) + c(T-t) - \frac{1}{2}\sigma_{\max}^2 (T-t) + \sigma_{\min}\sigma_{\max} (T-t)}{\sigma_{\max}\sqrt{T-t}},$$

$$d_6 = \frac{\log\left(\frac{G_t}{L}\right) + \log\left(\frac{K_1}{L}\right) + c(T-t) + \frac{1}{2}\sigma_{\max}^2 (T-t)}{\sigma_{\max}\sqrt{T-t}},$$

$$d_7 = \frac{\log\left(\frac{G_t}{L}\right) - \frac{\sigma_{\max}}{\sigma_{\min}} \log\left(\frac{K_2}{L}\right) + c(T-t) + \frac{1}{2}\sigma_{\max}^2 (T-t)}{\sigma_{\max}\sqrt{T-t}},$$

$$d_8 = \frac{\log\left(\frac{G_t}{K_2}\right) + c(T-t) + \frac{1}{2}\sigma_{\min}^2 (T-t)}{\sigma_{\min}\sqrt{T-t}},$$

$$d_9 = \frac{\log\left(\frac{G_t}{K_2}\right) + c(T-t) - \frac{1}{2}\sigma_{\min}^2 (T-t)}{\sigma_{\min}\sqrt{T-t}},$$

$$d_{10} = \frac{\log\left(\frac{G_t}{L}\right) - \frac{\sigma_{\min}}{\sigma_{\max}} \log\left(\frac{K_1}{L}\right) + c(T-t) - \frac{1}{2}\sigma_{\min}^2 (T-t)}{\sigma_{\min}\sqrt{T-t}},$$

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$$d_{11} = \frac{\log\left(\frac{L}{G_t}\right) - \log\left(\frac{K_2}{L}\right) - c(T-t) + \frac{1}{2}\sigma_{\min}^2(T-t) - \sigma_{\min}\sigma_{\max}(T-t)}{\sigma_{\min}\sqrt{T-t}},$$

$$d_{12} = \frac{\log\left(\frac{L}{G_t}\right) + \frac{\sigma_{\min}}{\sigma_{\max}}\log\left(\frac{K_1}{L}\right) - c(T-t) + \frac{1}{2}\sigma_{\min}^2(T-t) - \sigma_{\min}\sigma_{\max}(T-t)}{\sigma_{\min}\sqrt{T-t}},$$

$$d_{13} = \frac{\log\left(\frac{L}{G_t}\right) + \frac{\sigma_{\min}}{\sigma_{\max}}\log\left(\frac{K_1}{L}\right) - c(T-t) - \frac{1}{2}\sigma_{\min}^2(T-t)}{\sigma_{\min}\sqrt{T-t}},$$

$$d_{14} = \frac{\log\left(\frac{L}{G_t}\right) - \log\left(\frac{K_2}{L}\right) - c(T-t) - \frac{1}{2}\sigma_{\min}^2(T-t)}{\sigma_{\min}\sqrt{T-t}},$$

where $\log(\cdot)$ denotes the natural logarithm. Equation (14) can be verified by direct substitution of equations (A-1)–(A-2) and its partial derivatives. It can also be verified that $\partial D/\partial G$ and D are continuous at $g(t)$. Letting $K_1 = Le^{-b\sigma_{\max}}$ and $K_2 = Le^{b\sigma_{\min}}$ for some $b > 0$, it is straightforward to verify that $d_4 = d_5$, $d_6 = d_7$, $d_{11} = d_{12}$, and $d_{13} = d_{14}$, which simplifies equations (A-1) and (A-2). Furthermore, with $K_1 = Le^{-b\sigma_{\max}}$ and $K_2 = Le^{b\sigma_{\min}}$, the first and second derivatives of $D(G_t, t)$ can be written as

$$\frac{\partial D}{\partial G} = \begin{cases} N(d_1) - e^{-b(\sigma_{\max}-\sigma_{\min})}N(d_6), & G_t \le g(t) \\ N(-d_8) - e^{-b(\sigma_{\max}-\sigma_{\min})}N(d_{13}), & G_t > g(t) \end{cases}$$

and

(A-3)
$$\frac{\partial^2 D}{\partial G^2} = \begin{cases} \left(\frac{1}{G_t\sigma_{\max}\sqrt{2\pi(T-t)}}\right)e^{-\frac{1}{2}(d_1)^2}\left[1 - \left(\frac{G_t}{g(t)}\right)^{\frac{2b}{\sigma_{\max}(T-t)}}\right], & G_t \le g(t) \\ \left(\frac{1}{G_t\sigma_{\min}\sqrt{2\pi(T-t)}}\right)e^{-\frac{1}{2}(d_8)^2}\left[\left(\frac{g(t)}{G_t}\right)^{\frac{2b}{\sigma_{\min}(T-t)}} - 1\right], & G_t > g(t) \end{cases}.$$

The sign of $\partial D/\partial G$ is always positive, while the sign of $\partial^2 D/\partial G^2$ in equation (A-3) is determined by the term in square brackets. It is obvious that $\partial^2 D/\partial G^2 > 0$ if $G_t < g(t)$, $\partial^2 D/\partial G^2 < 0$ if $G_t > g(t)$, and $\partial^2 D/\partial G^2 = 0$ if $G_t = g(t)$. □

Default Cost

Analytical expressions for the proportional default cost, which is discussed in Section IV.D, are presented here. Let $\mu_2 - \sigma_2\lambda > \mu_1 - \sigma_1\lambda$, so that $\{\mu_2, \sigma_2\}$ is the first best pair. If $(\sigma_t - \theta_t)^2$ is constant, then Black-Scholes (1973) pricing applies. Letting β^2 denote the constant value of $(\sigma_t - \theta_t)^2$, the time t value of the default cost is

$$\mathrm{E}_t^Q\left[e^{-r(T-t)}\delta V_T 1_{\{V_T < \xi_T K\}}\right] = \delta V_t e^{(\mu_2 - \sigma_2\lambda - r)(T-t)} N\left(\frac{\log\left(\frac{\xi_t K}{V_t}\right) - \left(\mu_2 - \sigma_2\lambda + \frac{1}{2}\beta^2\right)(T-t)}{\beta\sqrt{T-t}}\right),$$

where δ is the default cost as a proportion of time T firm value. For the structured debt (solid curve) in Graph C of Figure 3, $\theta_t = \sigma_1 + \sigma_2$ for all t, and thus $\beta = \sigma_1$. For the

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nonstructured debt (dashed curve) in Graph C, let $\xi_t = 1$, $\beta = \sigma_1$, and replace $\mu_2 - \sigma_2\lambda$ with $\mu_1 - \sigma_1\lambda$.

If $(\sigma_t - \theta_t)^2$ switches values, then Black-Scholes (1973) pricing no longer applies. Following Section II.C, let $\sigma_{\max} = \max[\sigma_1, \sigma_2]$, let $\sigma_{\min} = \min[\sigma_1, \sigma_2]$, and define $g(t)$ as in equation (13). If the owner finances the firm with convertible debt, the time t value of the default cost is $\mathrm{E}_t^Q\left[e^{-r(T-t)}\delta V_T 1_{\{V_T < \xi_T K_1\}}\right]$. For the region $G_t \leq g(t)$, this value is equal to

$$\delta\xi_t G_t e^{(c-r)(T-t)}\left\{N(-d_1) + \left(\frac{\sigma_{\min}}{\sigma_{\max}+\sigma_{\min}}\right)\left(\frac{L}{K_1}\right)^{\frac{\sigma_{\min}-\sigma_{\max}}{\sigma_{\max}}} N(d_6) - \left(\frac{\sigma_{\max}}{\sigma_{\max}+\sigma_{\min}}\right)\left(\frac{G_t}{g(t)}\right)^{\frac{\sigma_{\min}-\sigma_{\max}}{\sigma_{\max}}} N(d_4)\right\},$$

while for the region $G_t > g(t)$, this value is equal to

$$\delta\xi_t G_t e^{(c-r)(T-t)}\left(\frac{\sigma_{\min}}{\sigma_{\max}+\sigma_{\min}}\right) \times \left\{\left(\frac{G_t}{g(t)}\right)^{\frac{\sigma_{\max}-\sigma_{\min}}{\sigma_{\min}}} N(d_{12}) + \left(\frac{L}{K_1}\right)^{\frac{\sigma_{\min}-\sigma_{\max}}{\sigma_{\max}}} N(d_{13})\right\},$$

where d_4, d_6, d_{12}, and d_{13} are given above. For Graph E of Figure 3, $c = \mu_2 - \sigma_2\lambda = \mu_1 - \sigma_1\lambda$, $\sigma_{\max} = \sigma_2$, and $\sigma_{\min} = \sigma_1$. For the structured debt (solid curve) in this graph, $\theta_t = \sigma_1 + \sigma_2$ for all t. For the nonstructured debt (dashed curve) in this graph, $\theta_t = 0$ for all t, and thus $\xi_t = 1$ and $G_t = V_t$. □

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