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A characterization of M-Summands

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1. *Definitions.* In the structure theory of Banach spaces as developed in (1), an important role is played by subspaces which are the ranges of projections having norm properties akin to those of the classical Banach spaces. A linear projection e on a Banach space V is called an *M-projection* if

$$\|v\| = \max\{\|ev\|, \|v-ev\|\} \forall v \in V$$

and an *L-projection* if, instead

$$\|v\| = \|ev\| + \|v-ev\| \quad (\forall v \in V).$$

A closed subspace J of V is called an *M-Summand* if it is the range of an M-projection and an *M-Ideal* if J^0 is the range of an L-projection in V'. Every M-Summand is an M-Ideal but the reverse is false.

Alfsen and Effros (1) have shown that a closed subspace J of V is an M-Ideal if and only if one (and so both) of the following two equivalent conditions is satisfied:

(a) if $B_1, \ldots, B_n$ is a finite collection of open balls in V such that $B_i \cap J \neq \varnothing$ $\forall i$ and $\bigcap_1^n B_i \neq \varnothing$ then also $\bigcap_1^n B_i \cap J \neq \varnothing$;

(b) if $D_1, \ldots, D_n$ is a finite collection of closed balls in V such that $D_i \cap J \neq \varnothing$ $\forall i$ and $\operatorname{int} \bigcap_1^n D_i \neq \varnothing$ then also $\bigcap_1^n D_i \cap J \neq \varnothing$.

We show that a strengthening of these conditions characterizes M-Summands. In what follows $D(x,r)$ (resp. $B(x,r)$) represents the closed (resp. open) ball with centre x and radius r.

2. *The characterization.*

THEOREM. *A closed subspace J of a Banach space V is an M-Summand if and only if one (and thus both) of the equivalent conditions (a) and (b) is satisfied:*

(a) *if B_α, $\alpha \in \Lambda$ is a collection of open balls in V such that $B_\alpha \cap J \neq \varnothing$, $\forall \alpha \in \Lambda$ and $\bigcap_{\alpha\in\Lambda} B_\alpha \neq \varnothing$ then also $\bigcap_{\alpha\in\Lambda} B_\alpha \cap J \neq \varnothing$;*

(b) *if D_α, $\alpha \in \Lambda$ is a collection of closed balls in V such that $D_\alpha \cap J \neq \varnothing$, $\forall \alpha \in \Lambda$ and $\bigcap_{\alpha\in\Lambda} D_\alpha \neq \varnothing$ then also $\bigcap_{\alpha\in\Lambda} D_\alpha \cap J \neq \varnothing$.*

Proof. That (a) and (b) are equivalent is clear.

Let J be an M-Summand in V and e *the corresponding M-projection. Let $B_\alpha = B(x_\alpha, r_\alpha)$ $\alpha \in \Lambda$ be a collection of open balls in V as in* (a). *For every α, let j_α be a point in the non-empty set $B_\alpha \cap J$,* and x a point in $\bigcap_{\alpha\in\Lambda} B_\alpha$.

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For each α

$$\|x_\alpha - e(x)\| = \max\{\|e(x_\alpha - x)\|, \|(1-e)x_\alpha\|\}$$
$$= \max\{\|e(x_\alpha - x)\|, \|(1-e)(x_\alpha - j_\alpha)\|\} \leqslant \max\{\|x_\alpha - x\|, \|x_\alpha - j_\alpha\|\},$$

Since $(1-e)j_\alpha = 0$ and both e and $1-e$ have norm 1 or 0. But x and j_α are in B_α and therefore so is $e(x)$. As $e(x)$ also lies in J, we have found a point in $\bigcap_{\alpha\in\Lambda} B_\alpha \cap J$ and so J satisfies condition (a).

Now suppose that J is a closed subspace of V which satisfies (b). For each $v \in V$ define $D(v)$ by

$$D(v) = \{D \mid D \text{ is a closed ball in } V, v \in D, D \cap J \neq \varnothing\}.$$

Then by (b) $\bigcap_{D\in D(v)} D \cap J \neq \varnothing$ for every $v \in V$.

We call this set E_v. Suppose that for some v, E_v contains two distinct points, say x and y. Then for every $\epsilon > 0$, the two balls

$$D_1^\epsilon = D(v - d(v,J)(x-y)/\|x-y\|, \quad d(v,J)+\epsilon),$$
$$D_2^\epsilon = D(v + d(v,J)(x-y)/\|x-y\|, \quad d(v,\ J)+\epsilon)$$

(where $d(v,J) = \inf_{j\in J}\|v-j\|$) lie in $D(v)$. Thus $x \in D_1^\epsilon$ and $y \in D_2^\epsilon$ for every $\epsilon > 0$ and so

$$2d(v,J) + \|x-y\| = \|(2d(v,J)/\|x-y\| + 1)(x-y)\|$$
$$\leqslant \|v - d(v,J)(x-y)/\|x-y\| - x\| + \|v + d(v,J)(x-y)/\|x-y\| - y\|$$
$$\leqslant 2(d(v,J)+\epsilon)$$

again for every $\epsilon > 0$ and therefore $\|x-y\| = 0$.

This shows that, for every v, E_v contains exactly one point which we write $e(v)$. Let e be the map from V into J defined by $v \mapsto e(v)$. The following properties of e are clear from the construction:

(i) $\forall v \in J, \quad e(v) = v$;
(ii) $\forall v \in V, w \in J, \quad e(v+w) = e(v) + w$;
(iii) $\forall \lambda \in \mathbf{R}, v \in V, \quad e(\lambda v) = \lambda e(v)$.

(i) Follows from $D(v,0) \in D(v)$ for $v \in J$;
(ii) from $D(v+w) = \{D + w \mid D \in D(v)\}$ for $w \in J$;
(iii) from $D(\lambda v) = \{\lambda D \mid D \in D(v)\}$ for $\lambda \neq 0$.

The case $\lambda = 0$ is trivial.

It follows from (i) that e is idempotent and that the range of e is J.

We next show that e has the required norm property. It is clear from the construction that $\|v - e(v)\| = d(v,J)$. Also as $D(0, \|v\|) \in D(v)$ for all v, we have that $\|e(v)\| \leqslant \|v\| \ \forall v \in V$

(a) $\|v\| = d(v,J)$.

Then $\|v\| = \max\{\|v-e(v)\|, \|e(v)\|\}$ as required.

(b) $\|v\| > d(v,J)$.

Define $\lambda = \|v\|/(\|v\| - d(v,J))$. Then $(\lambda - 1)\|v\| = \lambda d(v,J)$.

So $D(\lambda v, \lambda d(v,J)) \in D(v)$ and therefore

$$\|\lambda v\| \leqslant \|e(v)\| + \|\lambda v - e(v)\| \leqslant \|e(v)\| + \lambda d(v,J) = \|e(v)\| + (\lambda - 1)\|v\|.$$

Thus $\|v\| = \|e(v)\| = \max\{\|e(v)\|, \|v - e(v)\|\}$ as required.

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It remains to prove that e is linear. As a consequence of (ii) and (iii), it suffices to show that $e(v) = 0$ and $e(w) = 0$ imply that $e(v+w) = 0$. Suppose that $v, w \in V$ with $e(v) = e(w) = 0$ and $e(v+w) = x \neq 0$. Without loss of generality, suppose that $d(v, J) \geqslant d(w, J)$.

Let $D = D(v - d(v, J)\, x/\|x\|,\ d(v, J))$. Now $\|w + d(v, J)\, x/\|x\|\,\| = \max\{d(v, J), \|w\|\}$ by (i), (ii), (iii) and the norm property. And $\|w\| = d(w, J) \leqslant d(v, J)$ as $e(w) = 0$. Therefore $\|w + d(v, J)\, x/\|x\|\,\| = d(v, J)$ which implies that $D \in D(v+w)$ and thus that $x \in D$. But

$$\|v - ([d(v, J)/\|x\|] + 1)\, x\| = \max\{d(v, J) + \|x\|, \|v\|\}$$

again by (i), (ii), (iii) and the norm property. As $\|x\| > 0$, we have $x \notin D$ and thus a contradiction. Hence e is a linear projection whose range is J and which has the required norm property. J is an M-Summand.

Note that for a non-closed subspace J the theorem is trivially satisfied in that J can neither be an M-Summand nor can it satisfy conditions (a) and (b). Strictly speaking, therefore, we could have left out the word 'closed' in the statement of the theorem.

The following corollary gives a positive answer to a problem of Alfsen and Effros ((1), section 7, Problem 2).

COROLLARY. *A w^*-closed M-Ideal in a dual space is an M-Summand.*

Proof. An M-Ideal satisfies condition (b) in section 1. A double application of the compactness of closed balls in the w^*-topology (one to remove 'int' and one to extend to arbitrary collections) shows that it necessarily satisfies condition (b) of our theorem.

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