NOTE ON A PENNY-SHAPED CRACK UNDER SHEAR

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1. *Introduction.* Sack (3) and Sneddon (4) have considered the problem of a penny-shaped crack in an infinite medium under tension normal to the face of the crack. The same problem has been considered more recently by Green (1) by a different method. In this note we consider the case of the same shaped crack with the medium subjected to a uniform shearing stress parallel to the plane of the crack. It is found that the solution is analogous to that of the corresponding two-dimensional problem considered by Starr (5).

An equivalent problem is that in which the two faces of the crack are subjected to a constant shearing traction, the other two components of stress across it vanishing; the superposition of the appropriate uniform shear stress throughout the medium will then leave both faces free from stress. The plane of the crack divides the medium into two, and it is sufficient to solve the problem for only one of these, provided that the solution is such that the two halves fit together properly, and this is achieved if appropriate continuity conditions at this plane for the displacements and the components of stress across it are satisfied. It is found that the problem reduces to the same boundary problem for a harmonic function as that considered by Green. An expression is also found for the change in strain energy which results when the crack forms in the uniformly sheared material.

2. A sufficiently general solution of the elastic equations of equilibrium can be expressed in the form

$$\left.\begin{aligned} u &= A + z\frac{\partial H}{\partial x}, \\ v &= \quad z\frac{\partial H}{\partial y}, \\ w &= C + z\frac{\partial H}{\partial z}, \end{aligned}\right\} \tag{1}$$

where u, v, w are the Cartesian components of displacement, and A, C, H are harmonic (so that the above expressions are biharmonic), provided, further, that the equations

$$(\lambda+\mu)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\Delta + \mu\nabla^2(u, v, w) = 0 \tag{2}$$

(where λ, μ are the usual elastic constants) are satisfied.

The dilatation Δ is found to be

$$\Delta = \frac{\partial A}{\partial x} + \frac{\partial C}{\partial z} + \frac{\partial H}{\partial z},$$

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which is, correctly, harmonic. The equations (2) are satisfied if we take

$$(\lambda+\mu)\left(\frac{\partial A}{\partial x}+\frac{\partial C}{\partial z}+\frac{\partial H}{\partial z}\right)+2\mu\frac{\partial H}{\partial z}=0. \tag{3}$$

The normal stress across the plane, $z=0$, is found to be

$$\widehat{zz}=\lambda\Delta+2\mu\frac{\partial w}{\partial z}=2\mu\left(\frac{\partial C}{\partial z}+\frac{\mu}{\lambda+\mu}\frac{\partial H}{\partial z}\right),$$

and if we take

$$C=-\frac{\mu H}{\lambda+\mu}$$

this normal stress vanishes.

Equation (3) then becomes

$$(\lambda+\mu)\frac{\partial A}{\partial x}+(\lambda+2\mu)\frac{\partial H}{\partial z}=0,$$

and this is satisfied if we take

$$A=\frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial\phi}{\partial z}\quad\text{and}\quad H=-\frac{\partial\phi}{\partial x},$$

where ϕ is harmonic.

In terms of ϕ the displacements are,

$$u=\frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial\phi}{\partial z}-z\frac{\partial^2\phi}{\partial x^2}, \tag{4, i}$$

$$v=-z\frac{\partial^2\phi}{\partial x\partial y}, \tag{4, ii}$$

$$w=\frac{\mu}{\lambda+\mu}\frac{\partial\phi}{\partial x}-z\frac{\partial^2\phi}{\partial x\partial z}. \tag{4, iii}$$

From these equations, the components of stress can be obtained. In particular, the stresses across the plane $z=0$ are,

$$\frac{\widehat{xz}}{\mu}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\frac{\lambda+2\mu}{\lambda+\mu}\frac{\partial^2\phi}{\partial z^2}-\frac{\lambda}{\lambda+\mu}\frac{\partial^2\phi}{\partial x^2}, \tag{5, i}$$

$$\frac{\widehat{yz}}{\mu}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=-\frac{\lambda}{\lambda+\mu}\frac{\partial^2\phi}{\partial x\partial y}, \tag{5, ii}$$

$$\widehat{zz}=0.$$

3. We consider a state of stress in which there is an applied shear stress, $\widehat{xz}=-S$, acting on the circular area given by $z=0$, $r=a$ (r being the distance from the z-axis); when a uniform shear stress, $\widehat{xz}=+S$, is superposed throughout the medium, both faces of the circle are then free from stress.

On the plane $z=0$ the boundary conditions involving the components of stress that must be satisfied are

(a) within the circle $r=a$ (i) $\widehat{xz}=-S$, (ii) $\widehat{yz}=0$, (iii) $\widehat{zz}=0$, and

(b) outside the circle $r=a$, $\widehat{xz}$, $\widehat{yz}$, $\widehat{zz}$ must be continuous.

Already the conditions involving $\widehat{zz}$ are satisfied since it vanishes everywhere on the plane. If ϕ is an even function of z, the continuity of the remaining stresses outside the circle is achieved. Conditions (a, i) and (a, ii) then remain.

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It is necessary to consider also the displacements at the plane, $z = 0$. This will be done later, but in the meantime we notice that u is an odd function of z and thus if u is to be continuous outside the circle it must vanish there.

Now consider an axially symmetrical function $\phi(r, z)$ even in z for which

$$\frac{\partial^2\phi}{\partial z^2} = -K \quad (r \leqslant a, z = 0),$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad (r > a, z = 0),$$

where K is a constant.

This is the same boundary problem that is considered by Green for a penny-shaped crack under normal tension (1). His solution, for $r \leqslant a$, is

$$\phi(r, 0) = -\tfrac{1}{4}K(2a^2 - r^2) = -\tfrac{1}{4}K(2a^2 - x^2 - y^2),$$

so that on $z = 0$,

$$\frac{\partial\phi}{\partial x} = \frac{Kx}{2}, \quad \frac{\partial^2\phi}{\partial x^2} = \frac{K}{2}, \quad \frac{\partial^2\phi}{\partial x\,\partial y} = 0.$$

Hence it follows that the remaining conditions (a, i) and (a, ii) can also be satisfied. Because of the appearance of the terms

$$-\frac{\lambda}{\lambda+\mu}\frac{\partial^2\phi}{\partial x^2} \quad \text{in (5, i)} \quad \text{and} \quad -\frac{\lambda}{\lambda+\mu}\frac{\partial^2\phi}{\partial x\,\partial y} \quad \text{in (5, ii)},$$

the shear problem is more complicated than the corresponding tension problem, and can be solved in this way only because of the happy results that on $z = 0$ not only $\dfrac{\partial^2\phi}{\partial z^2}$ but also $\dfrac{\partial^2\phi}{\partial x^2}$ reduces to a constant, and that $\dfrac{\partial^2\phi}{\partial x\,\partial y}$ vanishes. All the boundary conditions involving the components of shear can then be satisfied.

4. $\phi(r, z)$, for all values of r and z is conveniently expressed in terms of the 'complex distances' R_1 and R_2, used by Love (2), which are defined as

$$R_1^2 = r^2 + (z - ia)^2, \quad R_2^2 = r^2 + (z + ia)^2,$$

where, for $z \geqslant 0$, the arguments of R_1 and R_2 are so chosen that $R_1 = z - ia$ and $R_2 = z + ia$ when $r = 0$. On $z = 0$, it can be shown that R_1 and R_2 reduce to

$$R_1 = \begin{cases} -i\sqrt{(a^2 - r^2)} & (r \leqslant a), \\ \sqrt{(r^2 - a^2)} & (r \geqslant a), \end{cases} \qquad R_2 = \begin{cases} i\sqrt{(a^2 - r^2)} & (r \leqslant a), \\ \sqrt{(r^2 - a^2)} & (r \geqslant a). \end{cases}$$

It is found that for $z \geqslant 0$,

$$\phi(r, z) = \frac{iK}{4\pi}\left[(2z^2 + 2a^2 - r^2)\log\frac{R_2 + z + ia}{R_1 + z - ia} + 3z(R_1 - R_2) + ia(R_1 + R_2)\right], \quad (6, \text{i})$$

$$\frac{\partial\phi(r, z)}{\partial z} = \frac{iK}{\pi}\left[z\log\frac{R_2 + z + ia}{R_1 + z - ia} + (R_1 - R_2)\right], \quad (6, \text{ii})$$

$$\frac{\partial^2\phi(r, z)}{\partial z^2} = \frac{iK}{\pi}\left[\log\frac{R_2 + z + ia}{R_1 + z - ia} - ia\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]. \quad (6, \text{iii})$$

Now, when $r < a$, $z = 0$,

$$\log\frac{R_2 + z + ia}{R_1 + z - ia} \quad \text{becomes} \quad \log\frac{i\sqrt{(a^2 - r^2)} + ia}{-i\sqrt{(a^2 - r^2)} - ia} = i\pi,$$

while, when $r > a$, $z = 0$,

$$\log\frac{R_2 + z + ia}{R_1 + z - ia} \quad \text{becomes} \quad \log\frac{\sqrt{(r^2 - a^2)} + ia}{\sqrt{(r^2 - a^2)} - ia} = 2i\tan^{-1}\frac{a}{\sqrt{(r^2 - a^2)}} = 2i\sin^{-1}\frac{a}{r}.$$

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Thus at $z = 0$,

$$\phi = \begin{cases} -\frac{K}{4}(2a^2 - r^2) & (r \leqslant a), \\ -\frac{K}{2\pi}\left[(2a^2 - r^2)\sin^{-1}\frac{a}{r} + a\sqrt{(r^2 - a^2)}\right] & (r \geqslant a), \end{cases} \tag{7, i}$$

$$\frac{\partial \phi}{\partial z} = \begin{cases} \frac{2K}{\pi}\sqrt{(a^2 - r^2)} & (r \leqslant a), \\ 0 & (r \geqslant a), \end{cases} \tag{7, ii}$$

$$\frac{\partial^2 \phi}{\partial z^2} = \begin{cases} -K & (r < a), \\ \frac{2K}{\pi}\left[\frac{a}{\sqrt{(r^2 - a^2)}} - \sin^{-1}\frac{a}{r}\right] & (r > a). \end{cases} \tag{7, iii}$$

Hence it is found, from equations (5, i) and (7, i), (7, iii) that in the circle

$$\begin{aligned} \widehat{xz} &= \mu\left[\frac{\lambda + 2\mu}{\lambda + \mu}\frac{\partial^2 \phi}{\partial z^2} - \frac{\lambda}{\lambda + \mu}\frac{\partial^2 \phi}{\partial x^2}\right] \\ &= -\mu\left[\frac{\lambda + 2\mu}{\lambda + \mu} + \frac{\lambda}{2(\lambda + \mu)}\right]K \\ &= -\frac{\mu(3\lambda + 4\mu)K}{2(\lambda + \mu)} = -S, \end{aligned}$$

so that we must take

$$K = \frac{2(\lambda + \mu)S}{\mu(3\lambda + 4\mu)}. \tag{8}$$

From (4, i) and (7, ii) we find that inside the circle

$$\begin{aligned} u &= \frac{\lambda + 2\mu}{\lambda + \mu}\frac{\partial \phi}{\partial z} \\ &= \frac{2(\lambda + 2\mu)K}{\pi(\lambda + \mu)}\sqrt{(a^2 - r^2)} = \frac{4(\lambda + 2\mu)S}{\pi\mu(3\lambda + 4\mu)}\sqrt{(a^2 - r^2)}, \end{aligned} \tag{9}$$

while outside the circle, $u = 0$, of course.

Further, on $z = 0$

$$w = \frac{\mu}{\lambda + \mu}\frac{\partial \phi}{\partial x},$$

so that for $r \leqslant a$,

$$w = \frac{\mu K x}{2(\lambda + \mu)} = \frac{Sx}{(3\lambda + 4\mu)}, \tag{10}$$

while for $r \geqslant a$,

$$\begin{aligned} w &= -\frac{\mu K}{2\pi(\lambda + \mu)}\frac{\partial}{\partial x}\left[(2a^2 - r^2)\sin^{-1}\frac{a}{r} + a\sqrt{(r^2 - a^2)}\right] \\ &= \frac{2S}{\pi(3\lambda + 4\mu)}\left[\sin^{-1}\frac{a}{r} - \frac{a\sqrt{(r^2 - a^2)}}{r^2}\right]x. \end{aligned} \tag{11}$$

v of course vanishes everywhere on the plane.

These results give the displacements at the upper surface of the crack. The corresponding results for the lower surface are

$$u = -\frac{4(\lambda + 2\mu)S}{\pi\mu(3\lambda + 4\mu)}\sqrt{(a^2 - r^2)}, \tag{9'}$$

$$w = \frac{Sx}{3\lambda + 4\mu}. \tag{10'}$$

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The two surfaces of the crack thus appear to cross each other, which is physically impossible. This, however, is corrected by adding the rigid-body displacements

$$w = -\frac{Sx}{3\lambda+4\mu}, \quad u = \frac{Sz}{3\lambda+4\mu},$$

so that the stresses are not altered (though the displacements at infinity are), and the displacements then become

$$w = 0, \quad u = \pm\frac{4(\lambda+2\mu)S}{\pi\mu(3\lambda+4\mu)}\sqrt{(a^2-r^2)}, \tag{12}$$

the + and − signs referring respectively to the upper and lower surfaces of the crack. The displacements so found are continuous across the plane, $z = 0$, outside the circle; inside the circle, v, w vanish, while u is discontinuous, but corresponds to a physically possible relative displacement of the two surfaces of the crack. As in the two-dimensional case (5), the crack remains unwarped.

5. If, when the infinite medium is subjected to a uniform shear throughout, the crack forms, strain energy is released. This loss in strain energy can be calculated by considering two states of strain. First imagine the uniform shear applied when there is no crack; this state of strain is elastically equivalent to that when there is a crack but there is also the appropriate shear tractions applied to both faces; the shear stress can be applied so that there is no displacement in the plane of the crack. Secondly, imagine the crack formed, and its faces free from stress; this state can be obtained by gradually removing the surface tractions on the crack, the displacement thereby arising being given by (12). The loss in strain energy is thus obtained by calculating the work done by the shear forces, $\widehat{xz} = S$, acting through the displacement u. This is obtained by an integration over both faces of the crack. We have, in fact,

$$\int_0^a\int_0^{2\pi} uSr\,dr\,d\theta = \frac{8(\lambda+2\mu)S^2}{\mu(3\lambda+4\mu)}\int_0^a r\sqrt{(a^2-r^2)}\,dr$$
$$= \frac{8(\lambda+2\mu)S^2a^3}{3\mu(3\lambda+4\mu)}. \tag{13, i}$$

In terms of Poisson's ratio σ this becomes

$$\frac{8(1-\sigma)S^2a^3}{3\mu(2-\sigma)}. \tag{13, ii}$$

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