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SIMPLICIAL COENDOMORPHISM OPERADS AND COALGEBRAS

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Abstract

In a recent work of Moreno-Fernandez, Wierstra and the present author ['A recognition principle for iterated suspensions as coalgebras over the little cubes operad', Preprint, 2022, arXiv:2210.00839], a coendomorphism operad in the category of pointed topological spaces endowed with the wedge sum was introduced. In this paper, we construct an analogue completely internal to the category of simplicial sets with the goal of defining simplicial coalgebras. As an application, we show that simplicial n-fold suspensions are coalgebras up to coherent homotopy over the Barratt–Eccles E_n-operad provided they have finitely many nondegenerate simplices.

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1. Introduction

The little n-cubes operad C_n was first introduced by May in his 1972 book, *The Geometry of Iterated Loop Spaces* [11], although earlier similar notions appeared in the work of Stasheff [13] and Boardman and Vogt [3, 4]. May noticed that n-fold loop spaces carry a natural monoidal (up to homotopy) structure induced by concatenation of loops, and invented operads to capture this underlying structure without reference to the space itself. This approach proved its utility immediately, when he was able to show that any group-like algebra over C_n is weakly homotopic to an n-fold loop space, a famous result known as *May's recognition principle*.

The recognition principle enables the systematic development of *(co)homology operations* on iterated loop spaces. For example, it was shown that the homology of the little n-cubes operad is the shifted Poisson operad Pois_n in chain complexes [5]. This immediately implies that the homology of n-fold loop spaces possesses not just the Pontryagin product induced by the concatenation of loops, but also a binary product of degree $1-n$ called the Browder bracket, which is a shifted Lie bracket and compatible with the concatenation of loops in the sense that the Gerstanhaber relation

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holds. Integral operations such as the Dyer–Lashof and Kudo–Araki operations arise in this framework as well [7].

Eckmann–Hilton duality suggests that iterated suspensions should possess a parallel theory supporting the development of *homotopy operations*. Moreno-Fernandez and Wierstra, together [12] and with the present author [8], have started the development of such a theory. For each pointed topological space X, we define the *coendomorphism operad* $\mathsf{CoEnd}(X)$. This operad has arity n component

$$\mathsf{CoEnd}(X)(n) := \underline{\mathsf{Top}_*}(X, X^{\vee n}).$$

Given a topological operad $\mathscr{P}$, a $\mathscr{P}$-coalgebra is defined to be a pair (X, ϕ), where X is a space and ϕ is an operadic morphism $\mathscr{P} \to \mathsf{CoEnd}(X)$. An analogue of May's recognition principle holds: the C_n-coalgebras are, up to homotopy, precisely the n-fold suspensions.

So far, this theory has been developed in topological spaces. However, sometimes simplicial methods can be more effective, and one would like a tractable theory of combinatorial coendomorphism operads and coalgebras internal to the category sSet of simplicial sets, which is equipped with symmetric monoidal structure via the wedge sum, to provide a more convenient context for studying homotopy operations. However, the theory does not extend as naively as one might hope, as the wedge sum of Kan complexes is not a Kan complex in general. As a consequence, the obvious choice of coendomorphism operad in simplicial sets defined in arity n by the mapping simplicial set $\underline{\mathsf{sSet}_*}(X, X^{\vee n})$ is not the same as $\underline{\mathsf{Top}_*}(X, X^{\vee n})$ in the common homotopy category of topological spaces and simplicial sets. For example, the simplicial set $\underline{\mathsf{sSet}_*}(S^1, (S^1)^{\vee n})$ contains n disjoint points, one for each copy of S^1 in the wedge sum. In contrast, $\pi_0(\mathsf{Top}_*(S^1, (S^1)^{\vee n}))$ is the amalgamated product $\mathbb{Z}^{*n}$, since it includes the pinch map $S^1 \to S^1 \vee S^1$.

There are two approaches to obtaining a valid coendomorphism operad in the category of simplicial sets.

(1) One can pass from a simplicial set X to its geometric realisation $|X|$, form $\mathsf{CoEnd}_{\mathsf{Top}_*}(|X|)$ and return to simplicial sets via the singular simplicial set functor $\mathsf{Sing}_\bullet$, obtaining the operad $\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}_*}(X))$. This produces a very geometrically transparent operad, but it is very large, typically uncountable even for finite X. Significantly, it is not purely combinatorial or internal to sSet.

(2) By using Kan's Ex^∞ functor as a convenient fibrant replacement, one obtains a coendomorphism operad $\mathsf{CoEnd}_{\mathsf{sSet}_*}(X)$ for every simplicial set X. This operad mirrors several properties of the Ex^∞ functor itself: it is functorial, combinatorially tractable and remains entirely internal to sSet, while still preserving the correct homotopical behaviour.

The main contribution of this paper is the definition of $\mathsf{CoEnd}_{\mathsf{sSet}_*}(X)$ and the proof that it is indeed an operad. For convenience, we state this as a theorem. The precise definition is given later on.

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THEOREM 1.1. *For every finite simplicial set X, the coendomorphism operad, denoted by $\mathsf{CoEnd}_{\mathsf{sSet}_*}(X)$, is an operad in simplicial sets.*

We then show that the two constructions of the coendomorphism operad explained above agree up to homotopy. This justifies our model as the correct one to simplicially model the topological coendomorphism operad on finite simplicial sets. By convention, a finite simplicial set is a simplicial set with a finite number of nondegenerate simplices.

THEOREM 1.2. *For finite simplicial sets X, the coendomorphism operads, denoted by $\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}_*}(X))$ and $\mathsf{CoEnd}_{\mathsf{sSet}_*}(X)$, are weakly equivalent.*

Finally, this framework permits a definition of coalgebras internal to simplicial sets. Using model-categorical arguments, we establish the following result.

THEOREM 1.3. *The n-fold reduced suspension $\Sigma^n X$ of a finite pointed simplicial set X is an E_n-coalgebra.*

1.1. Notation and conventions. All topological spaces are compactly generated and Hausdorff. We refer to the monoidal category of such spaces equipped with the Kelley product as Top and the category of simplicial sets as sSet. The symmetric group on n letters is denoted by $\mathbb{S}_n$. A simplicial set is finite if it has finitely many nondegenerate simplices. We use $\underline{C}(A,B)$ to denote the internal hom-object of closed categories and $C(A,B)$ to denote the underlying set of morphisms. Our references are as follows: for operads [10], for simplicial sets [9], for the definition of simplicial suspensions and wedge sums [6], for the Barratt–Eccles operad [1], and for the Boardman–Vogt resolution (the W-construction) of operads [2].

1.2. The structure of this article. First, we recall some preliminaries on topological coalgebras and Kan's Ex^∞-functor in Section 2. The proofs of Theorems 1.1, 1.2, and 1.3 are then carried out in Section 3.

2. Preliminaries

In this section, we collect some of the prerequisites for understanding this paper. First, we recall the theory of coalgebras in topological spaces. Then, we describe Kan's Ex^∞-functor, a fibrant replacement functor in the Kan–Quillen model structure of simplicial sets.

2.1. Coalgebras in topological spaces. In [8, Definition 2.14], the authors show that one can define a *coalgebra over an operad* in pointed topological spaces endowed with the wedge sum and the one-point space as symmetric monoidal structure.

DEFINITION 2.1. Let X be a pointed topological space. The *topological coendomorphism operad* $\mathsf{CoEnd}_{\mathsf{Top}_*}(X)$ has arity r component

$$\mathsf{CoEnd}_{\mathsf{Top}_*}(X)(r) := \underline{\mathsf{Top}_*}(X, X^{\vee r}).$$

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For $r = 0$, set $\mathsf{CoEnd}_{\mathsf{Top}_*}(X)(0) = \underline{\mathsf{Top}_*}(X, *) = *$. The operadic composition maps are defined by

$$\begin{aligned}\gamma : \mathsf{CoEnd}_{\mathsf{Top}_*}(X)(r) \otimes \mathsf{CoEnd}_{\mathsf{Top}_*}(X)(n_1) \otimes \cdots \otimes \mathsf{CoEnd}_{\mathsf{Top}_*}(X)(n_r) \\ \to \mathsf{CoEnd}_{\mathsf{Top}_*}(X)(n_1 + \cdots + n_r) \\ (f, f_1, \ldots, f_r) \mapsto (f_1 \vee \cdots \vee f_r) \circ f.\end{aligned}$$

The symmetric group action permutes the wedge factors in the output.

The operad $\mathsf{CoEnd}_{\mathsf{Top}_*}(X)$ is naturally pointed. We normally choose to ignore this extra structure and regard $\mathsf{CoEnd}_{\mathsf{Top}_*}(X)$ as unpointed for the rest of this article.

The definition above immediately allows us to define a coalgebra as an algebra over the coendomorphism operad.

DEFINITION 2.2. Let $\mathscr{P}$ be an operad in the category of unpointed topological spaces. A *$\mathscr{P}$-coalgebra* is a pointed space X along with an (unpointed) morphism of operads

$$\Delta : \mathscr{P} \to \mathsf{CoEnd}_{\mathsf{Top}_*}(X).$$

THEOREM 2.3 [8, Theorem A]. *Let $\Sigma^n X$ be the n-fold suspension of a pointed space X. Then, there is a natural map of operads $\Delta : C_n \to \mathsf{CoEnd}_{\mathsf{Top}_*}(\Sigma^n X)$ that encodes the homotopy coassociativity and homotopy cocommutativity of the pinch map. That is, n-fold suspensions are coalgebras over the little n-cubes operad. Furthermore, for any based map $X \to Y$, the induced map $\Sigma^n X \to \Sigma^n Y$ extends to a morphism of C_n-coalgebras.*

2.2. Kan's Ex^∞ functor. Not all objects are fibrant in the classical Kan–Quillen model structure on the category of simplicial sets. Kan introduced the fibrant replacement functor Ex^∞ which computes replacements via the combinatorial process of barycentric subdivision. We recall just the most elementary facts with the goal of establishing our notation. For more details, we refer the reader to [9, Ch. III].

Recall that the nondegenerate simplices of the standard n-simplex $\triangle^n$ are exactly the increasing injections $[m] \to [n]$ with $0 \leq m \leq n$. These are in one-to-one correspondence with the subsets of $\{0, 1, \ldots, n\}$ of cardinality $m + 1$ and, thus, form a poset under inclusion which we denote $P\triangle^n$.

DEFINITION 2.4. The *simplicial subdivision* of the standard n-simplex $\triangle^n$ is

$$\mathrm{sd}\triangle^n := \mathcal{N}(P\triangle^n),$$

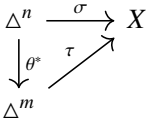
that is, the nerve of the poset $P\triangle^n$ regarded as a small category with morphisms given by inclusions.

LEMMA 2.5 [9, Lemma III.4.1]. *On the level of geometric realisations, there is a homeomorphism $f : |\mathrm{sd}\triangle^n| \xrightarrow{\sim} |\triangle^n|$.*

The notion of subdivision can be extended to any simplicial set, not just the standard simplices. This extension makes use of the notion of a *simplex category*.

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DEFINITION 2.6. The *simplex category* $\triangle \downarrow X$ of a simplicial set X, has for objects all simplicial maps $\sigma : \triangle^n \to X$ and has for morphisms the commutative diagrams of the form



where θ^* is induced by a unique ordinal map $\theta : [m] \to [n]$.

DEFINITION 2.7. Let X be a simplicial set. The *subdivision* $\mathrm{sd}X$ of X is the simplicial set $\mathrm{sd}X = \lim_{\triangle^n \to X} \mathrm{sd}\triangle^n$ with the limit indexed by the simplex category of X.

Our next goal is to define the functor Ex^∞. To do so, we need to introduce the right adjoint of the subdivision functor, for which it is crucial to introduce an auxiliary map. There is a natural simplicial map $\nu_{\triangle^n} : \mathrm{sd}\triangle^n \to \triangle^n$ induced by the map of posets $P\triangle^n \to [n]$ given by $[v_0, v_1, \ldots, v_k] \mapsto v_k$. This map extends to any simplicial set as follows.

DEFINITION 2.8. Let X be a simplicial set. The *last vertex map* $\nu_X : \mathrm{sd}X \to X$ is the simplicial map $\nu_X = \lim_{\triangle^n \to X} \nu_{\triangle^n}$, where the limit is indexed by the simplex category of X.

We define the Ex functor to be the right adjoint of the sd functor. We need the following explicit description of its set of n-simplices.

DEFINITION 2.9. The Ex functor on a simplicial set X has n-simplices

$$\mathrm{Ex}(X)_n := \mathsf{sSet}(\mathrm{sd}\triangle^n, X).$$

There is a morphism $\mu_X : X \to \mathrm{Ex}(X)$ that is adjoint to the last vertex map. Repeatedly applying Ex to this map, we obtain a diagram

$$X \longrightarrow \mathrm{Ex}(X) \longrightarrow \mathrm{Ex}^2(X) \longrightarrow \cdots$$

DEFINITION 2.10. The Ex-functor is the endofunctor on the category of simplicial sets defined as the sequential colimit of the iterated Ex-functor. Explicitly, for any simplicial set X, $\mathrm{Ex}^\infty(X) = \mathrm{colim}(X \to \mathrm{Ex}(X) \to \mathrm{Ex}^2(X) \to \cdots)$.

The key properties of the Ex^∞-functor that we use are summarised in the following result.

THEOREM 2.11 [9, Theorem 4.8]. *Let X be a simplicial set. Then:*

(1) $\mathrm{Ex}^\infty(X)$ *is a Kan complex;*
(2) *the canonical map* $\eta_X : X \to \mathrm{Ex}^\infty(X)$ *is an injective weak homotopy equivalence;*
(3) Ex^∞ *preserves Kan fibrations;*
(4) Ex^∞ *preserves finite limits.*

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3. Coalgebras in simplicial sets

This section contains the main results of this article. We first construct the simplicial coendomorphism operad (Theorem 3.11), then show that it has the correct homotopy type (Theorem 3.21), and, finally, generalise [8, Proposition 2.23], by showing that n-fold suspensions of finite simplicial sets are coalgebras up to coherent homotopy over the Barratt–Eccles operad (Theorem 3.17).

3.1. The simplicial coendomorphism operad. Our goal in this section is to extend the notion of coalgebras to the category of simplicial sets. As in topological spaces [8], we are going to do this by defining the notion of a coendomorphism operad.

As discussed in the introduction, the operad defined in arity n by $\mathsf{sSet}_*(X, X^{\vee n})$ does not have the correct homotopy type due to $X^{\vee n}$ not being a Kan complex even when X is. This hints at the underlying problem. Since not all simplicial sets are Kan complexes, not all maps in the homotopy category can be represented as maps between any pair of representative objects in the original category. To ensure this, we must take a fibrant replacement of $X^{\vee n}$. To stay within the category of simplicial sets, we use Kan's Ex^∞ functor for this task. The underlying symmetric sequence of the desired operad is very easy to describe and we can do this immediately.

DEFINITION 3.1. Let X be a pointed simplicial set. The *simplicial coendomorphism symmetric sequence* of X in arity r is

$$\mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(r) := \underline{\mathsf{sSet}_*}(X, \mathrm{Ex}^\infty(X^{\vee r})).$$

Each $\sigma \in \mathbb{S}_r$ induces a map $\sigma^* : X^{\vee r} \to X^{\vee r}$ by permutation of the factors of the wedge sum. This induces the symmetric group right actions on the symmetric sequence given by

$$- * \sigma : \mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(r) \to \mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(r), \quad f \mapsto \sigma^* \circ f.$$

REMARK 3.2. It is obvious that $- * \sigma$ is a *bona fide* simplicial map because the degeneracy and face maps of the simplicial mapping space act only on the domain of an n-simplex $f : X \times \triangle^m \to \mathrm{Ex}^\infty(X^{\vee r})$ and not on the codomain.

REMARK 3.3. A very natural question to ask is whether it is possible to dually define the coendomorphism operad via subdivision directly, that is, with maps $\mathrm{sd}^k X \to X^{\vee r}$. Unfortunately, this would require arbitrary subdivision of X and the Ex^∞-functor does not have a left adjoint sd^∞, although each finite stage Ex^n does.

Next, we define the operadic composition maps. We start with some notation.

REMARK 3.4. Since $\mathrm{Ex}^\infty(X)$ is the colimit of the chain of injective weak homotopy equivalences, $X \xrightarrow{\sim} \mathrm{Ex}(X) \xrightarrow{\sim} \mathrm{Ex}^2(X) \xrightarrow{\sim} \cdots$, it follows that for all $x \in \mathrm{Ex}^\infty(X)$, there exists an $N > 0$ such that $x \in \mathrm{Ex}^n(X)$ for all $n > N$. Of course, we are implicitly identifying each $\mathrm{Ex}^n(X)$ with its image in $\mathrm{Ex}^\infty(X)$, where they form an exhaustive filtration.

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Recall that we consistently say that a simplicial set is finite whenever it has finitely many nondegenerate simplices.

DEFINITION 3.5. Let X be a finite, pointed simplicial set and let $f : X \times \triangle^m \to \mathrm{Ex}^\infty(X^{\vee r})$ be an n-simplex of $\mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(r)$. According to Remark 3.4, we can associate a unique integer N_σ to every nondegenerate simplex $\sigma \in X \times \triangle^m$, this being the smallest N such that $f(\sigma) \in \mathrm{Ex}^N(X^{\vee r})$. We define the integer

$$N_f := \max\{N_\sigma \mid \sigma \text{ is a nondegenerate simplex of } X \times \triangle^m\}.$$

REMARK 3.6. The integer N_f is well defined because $X \times \triangle^m$, the domain of f, has finitely many nondegenerate simplices.

REMARK 3.7. It is easy to check the following three properties of the integer N_f:

(1) the map f factors through $\mathrm{Ex}^{N_f}(X^{\vee r})$;
(2) the integer N_f is the smallest one with the property in item (1);
(3) for all $N \geq N_f$, the map f factors through $\mathrm{Ex}^N(X^{\vee r})$.

Our definition of the coendomorphism operad makes heavy use of the adjunction between Ex and sd. For ease of reading, we introduce two pieces of helpful notation.

NOTATION 3.8. Recall that sd^n is left adjoint to Ex^n for all n. In particular, for any simplicial set X and $N > 0$,

$$\mathsf{sSet}(\mathrm{sd}^N(X \times \triangle^m), X^{\vee r}) \cong \mathsf{sSet}(X \times \triangle^m, \mathrm{Ex}^N(X^{\vee r})).$$

The adjoint of a map $f : \mathrm{sd}^N(X \times \triangle^m) \to X^{\vee r}$ is denoted f^c. Via the colimit map $\ell_N : \mathrm{Ex}^N(X^{\vee r}) \to \mathrm{Ex}^\infty(X^{\vee r})$, this adjoint $f^c \colon X \times \triangle^m \to \mathrm{Ex}^N(X^{\vee r})$ uniquely extends to a map $\ell_n \circ f^c = \overline{f} \colon X \times \triangle^m \to \mathrm{Ex}^\infty(X^{\vee r})$ of $\mathsf{sSet}(X \times \triangle^m)$. This is precisely an m-simplex in $\underline{\mathsf{sSet}}(X, \mathrm{Ex}^\infty(X^{\vee r}))$.

NOTATION 3.9. Let $f : X \times \Delta^m \to \mathrm{Ex}^\infty(X^{\vee r})$ be an m-simplex in $\underline{\mathsf{sSet}}(X, \mathrm{Ex}^\infty(X^{\vee r}))$. Then, it follows from Remark 3.6 that for all $N \geq N_f$, there is a unique map $(f, N) \colon \mathrm{sd}^N(X \times \triangle^m) \to X^{\vee r}$ such that $\overline{(f, N)} = f$.

We are now in a position to define the composition maps of the simplicial coendomorphism operad. Since the subdivision functor is a left adjoint, it preserves colimits. In particular, it commutes with wedge sums. We make use of this fact.

DEFINITION 3.10. Let X be a finite, pointed simplicial set. Let $f \in \mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(r)_m$ and $f_i \in \mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(n_i)_m$ for $1 \leq i \leq r$. The composition map

$$\gamma \colon \mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(r) \times \mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(n_1) \times \cdots \times \mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(n_r) \to \mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(n_1 + \cdots + n_r)$$

is defined as $\gamma(f; f_1, \ldots, f_r) = \overline{F}$, where F is the map

$$F : \mathrm{sd}^{N+N_f}(X \times \triangle^m) \xrightarrow{\delta_{\mathrm{sd}^{N_f}(X \times \triangle^m)}} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m))$$

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$$\xrightarrow{\mathrm{sd}^N(\mathrm{id}\times\mathrm{sd}^{N_f}(\pi_2))} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X\times\triangle^m)\times\mathrm{sd}^{N_f}(\triangle^m)) \xrightarrow{a} \mathrm{sd}^N(\mathrm{sd}^{N_f}(X\times\triangle^m)\times\triangle^m)$$
$$\xrightarrow{(f,N_f)} \mathrm{sd}^N(X^{\vee r}\times\triangle^m) \xrightarrow{b} \mathrm{sd}^N(X\times\triangle^m)^{\vee r} \xrightarrow{(f_1,N)\vee\cdots\vee(f_r,N)} X^{\vee n_1+\cdots+n_r}.$$

Here:

(1) N is the integer $\max(N_{f_1},\dots,N_{f_r})$;
(2) $\delta_{\mathrm{sd}^{N_f}(X\times\triangle^m)} : \mathrm{sd}^{N_f}(X\times\triangle^m)\to\mathrm{sd}^{N_f}(X\times\triangle^m)\times\mathrm{sd}^{N_f}(X\times\triangle^m)$ is the diagonal map;
(3) $\pi_2 : X\times\triangle^m\to\triangle^m$ is the projection;
(4) $a : \mathrm{sd}^{N_f}(\triangle^m)\to\mathrm{sd}^N(\mathrm{sd}^{N_f}(X\times\triangle^m)\times\triangle^m)$ is the map $\mathrm{sd}^N(\mathrm{id}\times\nu^{(N_f)}_{\triangle^m})$ where $\nu^{(N_f)}_{\triangle^m} := \nu_{\triangle^m}\circ\cdots\circ\nu_{\mathrm{sd}^{N_f-1}\triangle^m}$ and $\nu_Z : \mathrm{sd}Z\to Z$ is the last vertex map;
(5) b is an isomorphism, as $\times$ is distributive over the wedge sum and the wedge sum commutes with subdivision.

We need to check that the definition above gives rise to a well-defined operad. We phrase this result as a theorem.

THEOREM 3.11. *Let X be a finite, pointed simplicial set. Then, the composition maps of Definition 3.10 induce an operad structure on the symmetric sequence* $\mathsf{CoEnd}_{\mathsf{sSet}_*}(X)$.

Before proving this theorem, we wish to make two useful remarks and introduce a final piece of notation.

REMARK 3.12. Our first remark concerns the relationship between (f,N) and (f,M) for $M>N\geq N_f$. From the definition of Ex, for all simplicial sets Z and Z', the simplicial morphism $\mathsf{sSet}(\nu_Z,Z')$ is adjoint to $\mathsf{sSet}(Z,\mu_{Z'})$, where both $\mu_Z : Z\to\mathrm{Ex}(Z)$ and $\nu_Z : \mathrm{sd}Z\to Z$ are the maps induced by the last vertex map. Thus, we have the relation

$$(f,N)\circ\nu_{\mathrm{sd}^N(X\times\triangle^m)} = (f,N+1)$$

for all $N\geq N_f$ and its obvious extension by induction. A second useful well-known result about ν_Z is that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{sd}Z & \xrightarrow{\nu_Z} & Z \\ \downarrow{\scriptstyle \mathrm{sd}f} & & \downarrow{\scriptstyle f} \\ \mathrm{sd}Z' & \xrightarrow{\nu_{Z'}} & Z'. \end{array} \tag{3.1}$$

NOTATION 3.13. We define $\nu^{(k)}_Z := \nu_Z\circ\cdots\circ\nu_{\mathrm{sd}^{k-1}Z}$.

REMARK 3.14. Another useful thing to note is that we can replace N_f in the definition of F with any integer $K\geq N_f$ and F will not change. To see why, call this new map $F(K)$ and then observe, with the help of Diagram (3.1), that $F(K) = F\circ\nu^{(K-N_f)}_{\mathrm{sd}^{N_f}(X\times\triangle^m)}$. By our previous remark,

$$\overline{F\circ\nu^{(K-N_f)}_{\mathrm{sd}^{N_f}(X\times\triangle^m)}} = \overline{F}.$$

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Similarly, if we replace N in the definition with a larger integer K', the function F in Definition 3.10 will become another function which we will call $F(K')$. It once again follows from Remark 3.12 and Diagram (3.1) that this function will be related to F by the identity $f(K') = F \circ \nu_Z^{(K'-N)}$, and so we can also replace N with any larger integer in Definition 3.10 without changing the operad structure.

PROOF OF THEOREM 3.11. We need to verify that this defines an operad, starting with the associativity axiom. Thus, we wish to show that

$$\gamma(\gamma(f, f_1, \dots, f_r), f_{1,1}, \dots, f_{r,n_r}) = \gamma(f, \gamma(f_1, f_{1,1}, \dots f_{1,n_1}), \dots, \gamma(f_r, f_{r,1}, \dots f_{r,n_r}))$$

for all $f \in \mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(r)_m$, $f_i \in \mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(n_i)_m$, and $f_{i,j} \in \mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(n_{i,j})_m$. Expanding the left-hand side, we obtain

$$\left(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij}, M)\right) \circ \left(\bigvee_{k=1}^{r} \mathrm{sd}^M((f_k, M') \times \nu_{\Delta^m}^{(M')} \circ \mathrm{sd}^{M'}(\pi_2))\right) \circ \mathrm{sd}^{M+M'}((f, N_f) \times (\nu_{\Delta^m}^{(N_f)} \circ \mathrm{sd}^{N_f}(\pi_2))), \tag{3.2}$$

where $M = \max\{N_{f_{ij}}\}_{1\le i\le r, 1\le j\le r_i}$ and $M' = \max\{N_{f_i}\}_{1\le i\le r}$. Now, let $M_i = \max\{N_{f_{i,j}}\}_{0\le j\le n_i}$ and recall that $(f, M) = (f, M_i) \circ \nu_{\mathrm{sd}^M(X\times\Delta^m)}^{(M-M_i)}$. From this, (3.2) can be written as

$$\left(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij}, M_i) \circ \nu_{\mathrm{sd}^{M_i}(X\times\Delta^m)}^{(M-M_i)}\right) \circ \left(\bigvee_{k=1}^{r} \mathrm{sd}^M((f_k, M') \times \nu_{\Delta^m}^{(M')} \circ \mathrm{sd}^{M'}(\pi_2))\right) \circ \mathrm{sd}^{M+M'}((f, N_f) \times (\nu_{\Delta^m}^{(N_f)} \circ \mathrm{sd}^{N_f}(\pi_2))).$$

This can be written as

$$\left(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij}, M_i)\right) \circ \left(\bigvee_{k=1}^{r} \nu_{\mathrm{sd}^M(X^{\vee r_k}\times\Delta^m)}^{(M-M_k)} \circ \mathrm{sd}^M((f_k, M') \times \nu_{\Delta^m}^{(M')} \circ \mathrm{sd}^{M'}(\pi_2))\right) \circ \mathrm{sd}^{M+M'}((f, N_f) \times (\nu_{\Delta^m}^{(N_f)} \circ \mathrm{sd}^{N_f}(\pi_2))).$$

Using the commutativity of Diagram (3.1), we see that this is equal to

$$\left(\bigvee_{i=1}^{r}\bigvee_{j=1}^{r_i}(f_{ij}, M_i)\right) \circ \left(\bigvee_{k=1}^{r} \mathrm{sd}^{M_k}((f_k, M') \times \nu_{\Delta^m}^{(M')} \circ \mathrm{sd}^{M'}(\pi_2)) \circ \nu_{\mathrm{sd}^{M'+M_k}(X\times\Delta^m)}^{M-M_k}\right) \circ \mathrm{sd}^{M+M'}((f, N_f) \times (\nu_{\Delta^m}^{(N_f)} \circ \mathrm{sd}^{N_f}(\pi_2))).$$

Once again using Diagram (3.1), we can rewrite this as

$$\left(\bigvee_{i=1}^{r}\left(\bigvee_{j=1}^{r_i}(f_{ij}, M_i)\right) \circ \mathrm{sd}^{M_i}((f_i, M_{f_i}) \times (\nu_{\Delta^m}^{(M_{f_i})} \circ \mathrm{sd}^{M_{f_i}}(\pi_2))) \circ \nu_{\mathrm{sd}^{M'+M_i}(X\times\Delta^m)}^{M+M'-M_i-M_{f_i}}\right) \circ \mathrm{sd}^{M+M'}((f, N_f) \times (\nu_{\Delta^m}^{(N_f)} \circ \mathrm{sd}^{N_f}(\pi_2))).$$

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The above expression is equal to

$$\bigvee_{i=1}^{r} (\overline{\gamma(f_i, f_{i1}, \ldots, f_{ir_i})}, M + M') \circ \mathrm{sd}^{M+M'}((f, N_f) \times (\nu^{(N_f)}_{\mathrm{sd}^{M+M'}(\triangle^m)} \circ \mathrm{sd}^{N_f}(\pi_2))).$$

By our previous argument, this is equal to

$$\gamma(f, \gamma(f_1, f_{1,1}, \ldots f_{1,n_1}) \ldots, \gamma(f_1, f_{r,1}, \ldots f_{r,n_r})),$$

as desired.

The identity element of the operad is the canonical map $\mu_X : X \to \mathrm{Ex}^\infty(X)$. Verifying the equivariance axioms is straightforward; it is almost exactly the same as verifying them for the topological coendomorphism operad. Therefore, we have defined an operad. □

It remains only to define simplicial coalgebras, which proceeds exactly as one would expect.

DEFINITION 3.15. Let $\mathscr{P}$ be an operad in simplicial sets. A *$\mathscr{P}$-coalgebra* is a finite, pointed simplicial set X together with an operadic morphism $\mathscr{P} \to \mathsf{CoEnd}_{\mathsf{sSet}_*}(X)$.

Lastly, we define E_n-algebras in sSet. The W-construction of an operad in simplicial sets is defined in [2]. It provides a functorial cofibrant replacement for operads.

DEFINITION 3.16. An *E_n-coalgebra* in simplicial sets is a coalgebra over the W-construction of the Barratt–Eccles E_n-operad.

3.2. Simplicial suspensions are E_n-coalgebras. In this section, we prove that simplicial suspensions of finite simplicial sets are E_n-coalgebras (Theorem 1.3). This is in direct analogy with the corresponding result for topological spaces [8, Theorem A]. The strategy is as follows. First, we use the simplicial chains functor $\mathsf{Sing}_\bullet$ to convert the topological little n-cubes operad C_n, the topological coendomorphism operad and any operad map Φ from the former to the latter into the simplicial setting. Then, we apply the homotopy transfer principle to lift this morphism from a cofibrant replacement of C_n to the simplicial coendomorphism operad.

The precise statement of the simplicial version of Theorem 2.3 is as follows.

THEOREM 3.17. *The n-fold reduced suspension $\Sigma^n X$ of a finite pointed simplicial set X is an E_n-coalgebra.*

Our proof of this theorem requires that the product of simplicial sets commutes with the geometric realisation functor. This is not true in full generality. Therefore, we shall need to restrict from the category of all topological spaces to the category of compactly generated Hausdorff spaces and we take our product therein to be the Kelley product.

We need to make topological operads into simplicial operads. To do so, we need the following definition.

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DEFINITION 3.18. Let $\mathscr{P}$ be an operad in Top_*. We define the operad $\mathsf{Sing}_\bullet\mathscr{P}$ in sSet with arity n component $(\mathsf{Sing}_\bullet\mathcal{P})(n) := \mathsf{Sing}_\bullet(\mathcal{P}(n))$, where $\mathsf{Sing}_\bullet$ is the singular chains functor. The action of $\sigma \in \mathbb{S}_n$ on $(\mathsf{Sing}_\bullet\mathcal{P})(n)$ is given by $(\mathsf{Sing}_\bullet\mathcal{P})(n) * \sigma := \mathsf{Sing}_\bullet(\mathcal{P}(n) * \sigma)$. The operadic composition map is $\gamma_{\mathsf{Sing}_\bullet\mathcal{P}} := \mathsf{Sing}_\bullet(\gamma_{\mathcal{P}})$ and the unit is the simplex $[\triangle^0 \to 1_{\mathsf{Top}_*}] \in \mathsf{Sing}_\bullet\mathcal{P}(1)$.

REMARK 3.19. The operad composition map in this definition is well defined because $\mathsf{Sing}_\bullet$ is right adjoint to the geometric realisation. This implies that it preserves limits and, in particular, products.

As mentioned in the introduction, we can define $\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}_*}(|X|))$ to be an alternative coendomorphism operad. The following theorem gives us a precise description of it.

LEMMA 3.20. *Let X be a finite pointed simplicial set. The operad $\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}_*}(|X|))$ is isomorphic to the simplicial operad $Q(X)$ with arity r component equal to*

$$Q(X)(r) := \underline{\mathsf{sSet}_*}(X, \mathsf{Sing}_\bullet|X^{\vee r}|).$$

The operadic composition map

$$\gamma : Q(X)(r) \times Q(X)(n_1) \times \cdots \times Q(X)(n_r) \to Q(X)(n_1 + \cdots + n_r)$$

is explicitly given as follows. For $f \in Q(X)(r)_m$ and $f_i \in Q(X)(n_i)_m$ for $1 \le i \le r$, the composition $\gamma(f; f_1, \ldots, f_r)$ is the adjoint under the Top–sSet adjunction of the map $F : |X \times \triangle^m| \to |X^{\vee n_1 + \cdots + n_r}|$. Here, F is defined by

$$|X \times \triangle^m| \xrightarrow{|\mathrm{id} \times \delta_{\triangle^m}|} |X \times \triangle^m \times \triangle^m| \xrightarrow{a} |X \times \triangle^m| \times |\triangle^m| \xrightarrow{|f| \times \mathrm{id}} |\mathsf{Sing}_\bullet|X^{\vee r}|| \times |\triangle^m| \xrightarrow{\epsilon_{X^{\vee r}} \times \mathrm{id}}$$
$$|X^{\vee r}| \times |\triangle^m| \xrightarrow{b} |X \times \triangle^m|^{\vee r} \xrightarrow{\bigvee_{i=1}^r |f_i|} \bigvee_{i=1}^r |\mathsf{Sing}_\bullet|X^{\vee n_i}|| \xrightarrow{\bigvee_{i=1}^r \epsilon_{X^{\vee n_i}}} \bigvee_{i=1}^r |X^{\vee n_i}| \xrightarrow{c} |X^{\vee n_1 + \cdots + n_r}|,$$

where:

(1) *$\delta_{\triangle^m} : \triangle^m \to \triangle^m \times \triangle^m$ is the diagonal map;*
(2) *for a topological space Y, the map $\epsilon_Y : |\mathsf{Sing}_\bullet(Y)| \to Y$ is the counit of the adjunction given by geometric realisation and singular simplicial set;*
(3) *$a : |X \times \triangle^m \times \triangle^m| \to |X \times \triangle^m| \times |\triangle^m|$ is an isomorphism, as $\times$ commutes with geometric realisation;*
(4) *$b : |X^{\vee r}| \times |\triangle^m| \to |X \times \triangle^m|^{\vee r}$ is an isomorphism, as both $\times$ and the wedge sum commute with geometric realisation;*
(5) *$c : \bigvee_{i=1}^r |X^{\vee n_i}| \to |X^{\vee n_1 + \cdots + n_r}|$ is an isomorphism, as the wedge sum commutes with geometric realisation.*

For each $\sigma \in \mathbb{S}_r$, there is a map $\sigma^* : X^{\vee r} \to X^{\vee r}$ given by permuting the terms of the wedge sum by σ. The symmetric structure on $Q(X)(r)$ is defined by post-composition with the morphism $\mathsf{Sing}_\bullet|\sigma^*|$.

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PROOF. First, we show that

$$\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}_*}(|X|))(r) = \mathsf{Sing}_\bullet \underline{\mathsf{Top}_*}(|X|, |X^{\vee r}|) \cong \underline{\mathsf{sSet}_*}(X, \mathsf{Sing}_\bullet |X^{\vee k}|). \tag{3.3}$$

This is proven by an argument on m-simplices. For all $K \in \mathsf{sSet}$ and $Y \in \mathsf{Top}_*$,

$$\mathsf{Top}_*(|\triangle^m|, \underline{\mathsf{Top}_*}(|K|, Y)) \cong \mathsf{Top}_*(|\triangle^m| \times |K|, Y)$$

by the tensor–hom adjunction. By the identity $|X| \times |Y| \cong |X \times Y|$,

$$\mathsf{Top}_*(|\triangle^m| \times |K|, Y) \cong \mathsf{Top}_*(|\triangle^m \times K|, Y).$$

Finally, by adjunction,

$$\mathsf{Top}_*(|\triangle^m \times K|, Y) \cong \mathsf{sSet}(\triangle^m \times K, \mathsf{Sing}_\bullet Y).$$

Since these isomorphisms are functorial, they commute with the face and degeneracy maps, so with the choice of $Y = X^{\vee k}$, we have established (3.3).

Second, it remains to check that operad morphisms are as described in the statement of the lemma. For $f \in \mathsf{Top}_*(|\triangle^m \times X|, |X^{\vee r}|)$ and $f_i \in \mathsf{Top}_*(|\triangle^m \times X|, |X^{\vee n_i}|)$, the composite $\gamma(f, f_1, \dots f_n)$ is the function

$$F : |X \times \triangle^m| \xrightarrow{|\mathrm{id} \times \delta_{\triangle^m}|} |X \times \triangle^m \times \triangle^m| \xrightarrow{a} |X \times \triangle^m| \times |\triangle^m| \xrightarrow{f \times \mathrm{id}}$$
$$\times |X^{\vee r}| \times |\triangle^m| \xrightarrow{b} |X \times \triangle^m|^{\vee r} \xrightarrow{\vee_{i=1}^r f_i} \bigvee_{i=1}^r |X^{\vee n_i}| \xrightarrow{c} |X^{\vee n_1 + \cdots + n_r}|.$$

The isomorphism $\mathsf{sSet}_*(\triangle^m \times X, \mathsf{Sing}_\bullet |X^{\vee r}|) \xrightarrow{\sim} \mathsf{Top}_*(|\triangle^m \times X|, |X^{\vee r}|)$ can be written as $f \mapsto \epsilon_{X^{\vee r}} \circ |f|$. Therefore, the composition map is exactly as described. □

The simplicial coendomorphism operad and the operad $\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}_*}(|X|))$ are equivalent in the sense of the following result.

THEOREM 3.21. *Let X be a finite, pointed simplicial set. Then, the simplicial coendomorphism operad and the operad $\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}_*}(|X|))$ are weakly equivalent.*

This result implies that in the common homotopy category of topological spaces and simplicial sets, the simplicial coendomorphism operad is isomorphic to the topological coendomorphism operad. We prove this result by constructing a zig-zag involving a third operad, which we define first.

DEFINITION 3.22. Let X be a finite, pointed simplicial set. Then, the *mixed coendomorphism operad $R(X)$* has arity r component

$$R(X)(r) = \underline{\mathsf{sSet}_*}(X, \mathrm{Ex}^\infty(\mathsf{Sing}_\bullet |X^{\vee r}|)).$$

For each $\sigma \in \mathbb{S}_r$, there is a map $\sigma^* : X^\vee \to X^\vee$ given by permuting the terms of the wedge sum by σ. The symmetric structure on $R(X)(r)$ is defined by post-composition with the morphism $\mathrm{Ex}^\infty(\mathsf{Sing}_\bullet |\sigma^*|)$. We define the operadic composition map using both the sd–Ex and the simplicial chains–geometric realisation adjunctions consecutively. Let $f \in Q(X)(r)_m$ and $f_i \in Q(X)(n_i)_m$ for $1 \le i \le r$. Then, the operadic

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composition map

$$\gamma : R(X)(r) \times R(X)(n_1) \times \cdots \times R(X)(n_r) \to R(X)(n_1 + \cdots + n_r)$$

is defined to be $\overline{F}$ which is the adjoint, under the sd–Ex adjunction, of the morphism, $F : \mathrm{sd}^{N_f}(X \times \triangle^m) \to \mathsf{Sing}_\bullet |X^{\vee n_1 + \cdots + n_r}|$. Here, F is itself an adjoint, this time under the geometric realisation–simplicial chains adjunction, of a morphism

$$G : |\mathrm{sd}^{N+N_f}(X \times \triangle^m)| \to |X^{\vee n_1 + \cdots + n_r}|,$$

which we define to be the composite

$$\begin{aligned}
|\mathrm{sd}^{N+N_f}(X \times \triangle^m)| &\xrightarrow{|\mathrm{sd}^N(\delta_{\triangle^m})|} |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m))| \\
&\xrightarrow{|\mathrm{sd}^N(\mathrm{id} \times \mathrm{sd}^{N_f}(\pi_2))|} |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(\triangle^m))| \xrightarrow{a} |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)| \\
&\xrightarrow{\mathrm{sd}^N((f, N_f) \times \mathrm{id})} |\mathrm{sd}^N(\mathsf{Sing}_\bullet |X^{\vee r}| \times \triangle^m)| \xrightarrow{b} |\mathsf{Sing}_\bullet |X^{\vee r}| \times \triangle^m| \xrightarrow{c} |X^{\vee r} \times \triangle^m| \\
&\xrightarrow{d} |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| \xrightarrow{e} |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)|^{\vee r} \xrightarrow{\vee_{i=1}^r |f_i|} \bigvee_{i=1}^r |\mathsf{Sing}_\bullet |X^{\vee n_i}|| \xrightarrow{\vee_{i=1}^r \epsilon_{X^{\vee n_i}}} \bigvee_{i=1}^r |X^{\vee n_i}|,
\end{aligned}$$

where:

(1) N is the integer $\max(N_{f_1}, \ldots, N_{f_n})$;
(2) for Y a topological space, the map $\epsilon_Y : |\mathsf{Sing}_\bullet(X^{\vee r})| \to Y$ is the counit of the adjunction between topological spaces and simplicial sets;
(3) $\delta_{\mathrm{sd}^{N_f}(X \times \triangle^m)} : \mathrm{sd}^{N_f}(X \times \triangle^m) \to \mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(X \times \triangle^m)$ is the diagonal map;
(4) $\pi_2 : X \times \triangle^m \to \triangle^m$ is the projection;
(5) $a : |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \mathrm{sd}^{N_f}(\triangle^m))| \to |\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)|$ is the map $|\mathrm{sd}^N(\mathrm{id} \times \nu_{\triangle^m} \circ \cdots \circ \nu_{\mathrm{sd}^{N_f - 1}\triangle^m})|$;
(6) $b : |\mathrm{sd}^N(\mathsf{Sing}_\bullet |X^{\vee r}| \times \triangle^m)| \to |\mathsf{Sing}_\bullet |X^{\vee r}| \times \triangle^m|$ is a homeomorphism from Lemma 2.5, which states that there is a homeomorphism $h_Z : |\mathrm{sd}(Z)| \to |Z|$ for every simplicial set Z (although this homeomorphism is not necessarily natural for simplicial morphisms $Z \to Z'$);
(7) $c : |\mathsf{Sing}_\bullet |X^{\vee r}| \times \triangle^m| \xrightarrow{c} |X^{\vee r} \times \triangle^m|$ is the composite

$$|\mathsf{Sing}_\bullet |X^{\vee r}| \times \triangle^m| \xrightarrow{p} |\mathsf{Sing}_\bullet |X^{\vee r}|| \times |\triangle^m| \xrightarrow{|\epsilon_{X^{\vee r}}| \times \mathrm{id}} |X^{\vee r}| \times |\triangle^m| \xrightarrow{q} |X^{\vee r} \times \triangle^m|,$$

where p and q are isomorphisms as the Kelley product commutes with geometric realisation;
(8) $d : |X^{\vee r} \times \triangle^m| \to |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)|$ is the homeomorphism that exists by Lemma 2.5;
(9) $e : |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| \to |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)|^{\vee r}$ is a homeomorphism because the wedge sum commutes with geometric realisation;
(10) $f : \vee_{i=1}^r |X^{\vee n_i}| \to |X^{\vee n_1 + \cdots + n_r}|$ is a homeomorphism, as the wedge sum commutes with geometric realisation.

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PROOF OF THEOREM 3.21. Since, by Lemma 3.20, the operad $\mathsf{Sing}_\bullet(\mathsf{CoEnd}_{\mathsf{Top}_*}(|X|))$ is isomorphic to $Q(X)(r)$, it suffices to construct a zig-zag of weak equivalences

$$\mathsf{CoEnd}(X) \xrightarrow{p} R(X) \xleftarrow{q} Q(X).$$

We define $p(r)$ to be the morphism

$$\underline{\mathsf{sSet}_*}(X, \mathrm{Ex}^\infty(\upsilon_{X^{\vee r}})) : \underline{\mathsf{sSet}_*}(X, \mathrm{Ex}^\infty(X^{\vee r})) \to \underline{\mathsf{sSet}_*}(X, \mathrm{Ex}^\infty(\mathsf{Sing}_\bullet|X^{\vee r}|)),$$

where $\upsilon_{X^{\vee r}} : X^{\vee r} \to \mathsf{Sing}_\bullet|X^{\vee r}|$ is the unit of the singular chains–geometric realisation adjunction. Observe that $\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}}) : \mathrm{Ex}^\infty(X^{\vee r}) \to \mathrm{Ex}^\infty(\mathsf{Sing}_\bullet|X^{\vee r}|)$ is a weak equivalence between fibrant simplicial sets. Hence, it is a homotopy equivalence and the functor $\mathsf{sSet}_*(X, -)$ preserves homotopy equivalences. Hence, p is a weak equivalence.

It remains to check that it induces a morphism of operads. We check this directly. Note first that $\underline{\mathsf{sSet}_*}(X, \mathrm{Ex}^\infty(\upsilon_{X^{\vee r}}))(f) = \mathrm{Ex}^\infty(\upsilon_{X^{\vee r}}) \circ f$. Then, observe that $N_{\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}})\circ f} = N_f$ and that we have $\max(N_{\upsilon_{\mathrm{Ex}^\infty(X^{\vee r})}\circ f_1}, \ldots, N_{\upsilon_{\mathrm{Ex}^\infty(X^{\vee r})}\circ f_n}) = \max(N_{f_1}, \ldots, N_{f_n})$. Then, observe that the morphism

$$|\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)| \xrightarrow{\mathrm{sd}^N((\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}})\circ f, N_f)\times \mathrm{id})} |\mathrm{sd}^N(\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m)|$$

factors as

$$|\mathrm{sd}^N(\mathrm{sd}^{N_f}(X \times \triangle^m) \times \triangle^m)| \xrightarrow{\mathrm{sd}^N((f, N_f)\times \mathrm{id})} |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| \xrightarrow{\mathrm{sd}^N(\upsilon_{X^{\vee r}}\times \mathrm{id})} |\mathrm{sd}^N(\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m)|.$$

Moreover, having first observed that the following diagram is commutative:

$$\begin{array}{ccc} |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| & \xrightarrow{|\mathrm{sd}^N(\upsilon_{X^{\vee r}}\times \mathrm{id})|} & |\mathrm{sd}^N(\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m)| \\ \downarrow{\scriptstyle h_{(X^{\vee r}\times \triangle^m)}} & & \downarrow{\scriptstyle h_{\mathsf{Sing}_\bullet|X^{\vee r}|\times \triangle^m}} \\ |(X^{\vee r} \times \triangle^m)| & \xrightarrow{|(\upsilon_{X^{\vee r}}\times \mathrm{id})|} & |(\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m)|, \end{array}$$

where $h_Z : |\mathrm{sd}Z| \to |Z|$ is the map that exists by Lemma 2.5, we see that the composite

$$\begin{aligned} |\mathrm{sd}^N(X^{\vee r} \times \triangle^m)| &\xrightarrow{|\mathrm{sd}^N(\upsilon_{X^{\vee r}}\times \mathrm{id})|} |\mathrm{sd}^N(\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m)| \xrightarrow{b} |\mathsf{Sing}_\bullet|X^{\vee r}| \times \triangle^m| \\ &\xrightarrow{c} |\mathsf{Sing}_\bullet|X^{\vee r}|| \times |\triangle^m| \xrightarrow{|\epsilon_{X^{\vee r}}|\times \mathrm{id}} |X^{\vee r}| \times |\triangle^m| \xrightarrow{d} |X^{\vee r} \times \triangle^m| \xrightarrow{e} |\mathrm{sd}^N(X \times \triangle^m)|^{\vee r} \end{aligned}$$

is an isomorphism by the *triangle identities* for the $\mathsf{Sing}_\bullet|-|-|$ adjunction. Explicitly, the (left) triangle identity for an adjunction $L \dashv R$ with unit $\eta : \mathrm{id}_X \to R \circ L$ and counit $\epsilon : L \circ R \to \mathrm{id}_Y$ states that the natural transformation of functors defined as the composite $L \xrightarrow{L\eta} LRL \xrightarrow{\epsilon L} L$ is the identity transformation. Upon further observing that, for the same reason, the composite

$$|\mathrm{sd}^N(X \times \triangle^m)|^{\vee r} \xrightarrow{\bigvee_{i=1}^r |\mathrm{Ex}^\infty(\upsilon_{X^{\vee r}})\circ f_i|} \bigvee_{i=1}^r |\mathsf{Sing}_\bullet|X^{\vee n_i}|| \xrightarrow{\bigvee_{i=1}^r \epsilon_{X^{\vee n_i}}} \bigvee_{i=1}^r |X^{\vee n_i}|$$

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is exactly the map

$$|\mathrm{sd}^N(X \times \triangle^m)|^{\vee r} \xrightarrow{\bigvee_{i=1}^r |f_i|} \bigvee_{i=1}^{r} |X^{\vee n_i}|,$$

it becomes obvious that γ commutes with p and so p is a weak equivalence of operads.

Similarly, we define $q(r)$ to be the morphism

$$\underline{\mathsf{sSet}_*}(X, \mu_{\mathsf{Sing}_\bullet |X^{\vee r}|}) : \underline{\mathsf{sSet}_*}(X, \mathsf{Sing}_\bullet |X^{\vee r}|) \to \underline{\mathsf{sSet}_*}(X, \mathrm{Ex}^\infty(\mathsf{Sing}_\bullet |X^{\vee r}|)).$$

This is a weak equivalence of simplicial sets for exactly the same reasons that $p(r)$ is. Observe that $N_{q(r)(f)} = 0$ for all $f \in Q(X)(r)$. It follows from the form of the operad maps that the morphism q identifies $Q(X)$ with a suboperad of $R(X)(r)$. In particular, q is a morphism of operads and so a weak equivalence of operads. □

Finally, we can prove the main result of this section.

PROOF OF THEOREM 3.17. Let $\Sigma^n X$ be the n-fold suspension of a simplicial set X. As $|\Sigma X|$ is a CW-complex, it is in Top_*. Suspensions are a particular kind of finite colimit and the geometric realisation functor commutes with all colimits as it is a right adjoint, so suspensions commute with geometric realisation and, thus, $|\Sigma^n X|$ is a coalgebra over the little n-cubes operad in Top. This coalgebra structure is an operadic morphism $\Phi : C_n \to \mathrm{CoEnd}_{\mathsf{Top}_*}(|\Sigma^n X|)$. As discussed previously, we can use $\mathsf{Sing}_\bullet$ to transfer these operads and this algebra structure to the category of simplicial sets, producing the morphism of operads

$$\mathsf{Sing}_\bullet(\Phi) : \mathsf{Sing}_\bullet(C_n) \to \mathsf{Sing}_\bullet(\mathrm{CoEnd}_{\mathsf{Top}_*}(|\Sigma^n X|)).$$

There is a weak equivalence between $\mathsf{CoEnd}_{\mathsf{sSet}_*(X)}$ and $\mathsf{Sing}_\bullet(\mathrm{CoEnd}_{\mathsf{Top}_*}(|\Sigma^n X|))$ by Theorem 3.21. In each arity, $\mathrm{CoEnd}_{\mathsf{sSet}_*}(X)(n)$ is a mapping space where the target is a Kan complex; hence, Kan itself and a fibrant operad in the operadic model structure. By its construction, in each arity $\mathsf{Sing}_\bullet \mathrm{CoEnd}_{\mathsf{Top}_*}(|\Sigma^n X|)$ is a singular complex and thus as an operad, it is also fibrant. Since we have a weak equivalence between fibrant operads, over the cofibrant replacement $(\mathsf{Sing}_\bullet C_n)_\infty$ of $\mathsf{Sing}_\bullet C_n$, we have an induced bijection between the homotopy classes of morphisms of operads

$$[(\mathsf{Sing}_\bullet C_n)_\infty, \mathrm{CoEnd}_{\mathsf{sSet}_*}(\Sigma^n X)] \cong [(\mathsf{Sing}_\bullet C_n)_\infty, \mathsf{Sing}_\bullet \mathrm{CoEnd}_{\mathsf{Top}_*}(|\Sigma^n X|)].$$

Thus, we can choose a morphism $\phi : (\mathsf{Sing}_\bullet C_n)_\infty \to \mathrm{CoEnd}_{\mathsf{sSet}_*}(\Sigma^n X)$ such that ϕ is homotopy equivalent to $\mathsf{Sing}_\bullet \Phi$.

Finally, to prove that n-fold suspensions are E_n-algebras, it suffices to note that all topological operads are fibrant and so the weak equivalence between the little n-cubes operad and the geometric realisation of the Barratt–Eccles E_n-operad remains one when taking the $\mathsf{Sing}_\bullet$ functor. The Barratt–Eccles E_n-operad $\Gamma^{(n)}$ is weakly equivalent to $\mathsf{Sing}_\bullet |\Gamma^{(n)}|$ and, in particular, $(\mathsf{Sing}_\bullet C_n)_\infty$ can be taken to be the Boardman–Vogt resolution of $\Gamma^{(n)}$, that is, the operad $W(\triangle^1, \Gamma^{(n)})$. □

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