For two arbitrary sets of non-zero real numbers, $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, substitute $m_i = a_i^2$, $v_i = \dfrac{b_i}{a_i}$ (so that v_i might be negative) and re-order so that $v_1 > v_2 > \ldots > v_n$. Then (*) rewrites as the Cauchy-Schwarz inequality,

$$\left(\sum a_i b_i\right)^2 \leqslant \left(\sum a_i^2\right)\left(\sum b_i^2\right), \qquad (**)$$

with equality when $v_1 = v_2 = \ldots = v_n$ so that the two sets of numbers are proportional. For two sets of real numbers where some of a_i or b_i might be zero, excluding the pairs (a_i, b_i) with $a_i b_i = 0$ leaves the LHS of (**) unchanged, whereas the RHS cannot decrease when these pairs are put back in so that Cauchy-Schwarz holds in this case as well.

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The constant acceleration formulae imply constant acceleration!

In this note we show that each of the five formulae for motion under constant acceleration, interpreted as differential equations describing the motion, imply that the acceleration is constant.

1. For example, $v = u + at$ rewrites as the general differential equation of motion $\dot{x} = \dot{x}_0 + \ddot{x}t$. Differentiating this gives $\ddot{x} = \ddot{x} + \dddot{x}\,t$ from which $\dddot{x} = 0$ and $\ddot{x}$ is constant.
2. $v^2 = u^2 + 2as$ gives the differential equation $\dot{x}^2 = \dot{x}_0^2 + 2\ddot{x}x$. Differentiating this gives $2\dot{x}\ddot{x} = 2\dot{x}\ddot{x} + 2\dddot{x}\,x$ from which $x\dddot{x} = 0$ so $\dddot{x} = 0$ whenever $x \neq 0$ and $\ddot{x}$ is constant.
3. $s = \frac{1}{2}(u + v)t$ gives the differential equation $x = \frac{1}{2}\left(\dot{x}_0 + \dot{x}\right)t$. Differentiating this twice then gives $\dot{x} = \frac{1}{2}\left(\dot{x}_0 + \dot{x}\right) + \frac{1}{2}\ddot{x}t$ and $\ddot{x} = \frac{1}{2}\ddot{x} + \frac{1}{2}\ddot{x} + \frac{1}{2}\dddot{x}\,t$ from which $\dddot{x} = 0$ and $\ddot{x}$ is constant.
4. $s = vt - \frac{1}{2}at^2$ gives the differential equation $x = \dot{x}t - \frac{1}{2}\ddot{x}t^2$. Differentiating this gives $\dot{x} = \dot{x} + \ddot{x}t - \ddot{x}t - \frac{1}{2}\dddot{x}\,t^2$ from which $\dddot{x} = 0$ and $\ddot{x}$ is constant.
5. Now for the surprise! The fifth equation of motion $s = ut + \frac{1}{2}at^2$ gives the differential equation $x = \dot{x}_0 t + \frac{1}{2}\ddot{x}t^2$. Differentiating this twice gives $\dot{x} = \dot{x}_0 + \ddot{x}t + \frac{1}{2}\dddot{x}\,t^2$ and $\ddot{x} = \ddot{x} + \dddot{x}\,t + \dddot{x}\,t + \frac{1}{2}\ddddot{x}\,t^2$ so that $t^4\ddddot{x} + 4t^3\dddot{x} = 0$. Thus $t^4\dddot{x}$ is constant giving $\ddot{x} = A + \dfrac{B}{t^3}$ for constants A and B. $B = 0$ then avoids the singularity at $t = 0$ and implies constant acceleration.

This way of thinking about the constant acceleration formulae can be used to derive them from one another. For example, start with

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$s = \frac{1}{2}(u + v)t$ in the forms (a) $x = \frac{1}{2}(\dot{x}_0 + \dot{x})t$ and (b) $t = \dfrac{2x}{\dot{x}_0 + \dot{x}}$. Differentiating (a) gives $\dot{x} = \frac{1}{2}(\dot{x}_0 + \dot{x}) + \frac{1}{2}\ddot{x}t$ or $\dot{x} = \dot{x}_0 + \ddot{x}t$ and substituting this back into (a) gives $x = \dot{x}_0 t + \frac{1}{2}\ddot{x}t^2$ and $x = \dot{x}t - \frac{1}{2}\ddot{x}t^2$. And differentiating (b) gives $1 = \dfrac{2\dot{x}(\dot{x}_0 + \dot{x}) - 2x\ddot{x}}{(\dot{x}_0 + \dot{x})^2}$ which simplifies to $\dot{x}^2 = \dot{x}_0^2 + 2\ddot{x}x$.

Finally, and I am very grateful to the referee for pointing this out, the five general differential equations under discussion in this note can all be solved by standard methods. Of course, the results above mean that there are no surprises in the solutions obtained for the initial conditions $\dot{x} = u$ and $\ddot{x} = a$ when $t = 0$, but finding them makes a good exercise. With the numbering above:

1. $t\ddot{x} - \dot{x} = -u$ is a linear differential equation in $\dot{x}$;
2. $2x\ddot{x} - \dot{x}^2 = -u^2$ gives a separable differential equation after the substitutions $p = \dot{x}, \ddot{x} = pp'$;
3. $t\dot{x} - 2x = -ut$ is a linear differential equation in x;
4. $t^2\ddot{x} - 2t\dot{x} + 2x = 0$ is an Euler-type differential equation which can be linearised by the usual substitution $t = e^z$;
5. $t^2\ddot{x} - 2x = -2ut$ is also an Euler-type differential equation which can be solved in the same way as (4).

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A rich problem on digital roots

The digital root function is defined by iterating the 'sums of digits' function until a single digit is reached. For example, the successive digit sums $275 \to 14 \to 5$ show that 275 has digital root 5, written $dr(275) = 5$.

In the course of an introductory lesson on proof a Sixth Form student, Max Chester, asked whether it was true that $dr(12n)$ always divides $12n$. I managed to cobble together a proof and what follows is a more considered analysis of the general problem suggested by this specific case, namely:

Which positive integers k have the property that $dr(kn)$ divides kn for all positive integers n?

The mixture of straightforward and subtle patterns described below makes this a rich problem suitable for investigation by a wide range of age groups. For brevity, we shall call an integer k *good* if it has the property sought and *bad* otherwise. Notice that all factors of bad integers are bad and all multiples of good integers are good. We shall use without proof the following standard properties of the digital root function:

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