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Weierstrass weights of fixed points of an involution

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Abstract

Let X be a curve with an involution T which fixes r points. We show that the Weierstrass weight of a fixed point is at least $(r-2)(r-4)/8$. Our proof is independent of the recent result of Torres.

We consider the case where $X = F_n$, the nth Fermat curve, and T is any of the involutions of F_n. We find that our bound is equal to the actual weight in all known cases ($n \leqslant 7$) and compute then $n = 8$ case to demonstrate that the equality continues to hold.

Introduction

We are interested in the Weierstrass weights of points which are known to be Weierstrass points. Let X be a curve with an automorphism T. In [**7**], Lewittes proved

THEOREM (Lewittes). *If T fixes at least 5 points, then each fixed point is a Weierstrass point.*

We look at the case when T has order 2. In [**1**], Accola studies Weierstrass points with respect to non-special divisors, in a situation similar to ours. Using ideas similar to those of Accola, in the classical special divisor case, we obtain a simple lower bound on the weights of the fixed points of T.

THEOREM. *Suppose T is an involution with r fixed points. Then if P is a fixed point of T,* $\mathrm{wt}(P) \geqslant (r-2)(r-4)/8$.

This formula was also found by Torres in [**10**]. He uses the techniques suggested by the earlier work of Garcia [**3**] and Kato [**6**]. In particular, the semi-group structure

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of the (classical) non-gaps. (By 'classical' we mean the non-gaps defined via poles of functions on the curve, not those which we will use in the following.) Our proof does not use this semi-group structure at all. Furthermore, the following corollary follows immediately from our proof.

COROLLARY. *The parity of* $\mathrm{wt}(P)$ *must agree with* $\mathrm{wt}_e(P)$. *That is*,

$$\mathrm{wt}(P) \equiv \mathrm{wt}_e(P) \pmod 2.$$

Of particular interest is an application to the Fermat curves, $F_n : X^n + Y^n + Z^n = 0$. The points with $XYZ = 0$ are all Weierstrass points and are well-understood. There are $3n$ involutions on F_n which (for $n \geqslant 5$) give a total of $3n^2$ new Weierstrass points. We obtain a lower bound for their weights:

COROLLARY. *If* $P \in F_n$ *is a fixed point of an involution of* F_n, *then*

$$\mathrm{wt}(P) \geqslant \begin{cases} (n-1)(n-3)/8 & n \text{ odd} \\ (n-2)(n-4)/8 & n \text{ even}. \end{cases}$$

This bound is an equality in all known cases ($n \leqslant 7$). We are able to compute the next case, $n = 8$, and we find that we have another equality.

1. *Preliminaries*

Let X be a curve of genus g and let $\mathscr{K}_X$ be a canonical divisor. Consider $\mathscr{L}(\mathscr{K}_X) = \{f \in K(X) : (f) + \mathscr{K}_X \geqslant 0\}$, where $(f) = (f)_X$ is the divisor of the function f on X. This vector space has dimension g and, given any point P on X, we can find a basis for $\mathscr{L}(\mathscr{K}_X), \{h_0, \dots, h_{g-1}\}$, such that

$$0 = \mathrm{ord}_P((h_0) + \mathscr{K}_X) < \cdots < \mathrm{ord}_P((h_{g-1}) + \mathscr{K}_X) \leqslant 2g-1.$$

We will call such a basis a *basis at* P. Let $n_i = \mathrm{ord}_P((h_i)_X + \mathscr{K}_X)$ for $i = 0, \dots, g-1$. Then the n_i are called the non-gaps of P.

A point P is not a Weierstrass point if and only if $n_i = i$ for all i.

Definition. If n_i is a non-gap of P with $n_i \geqslant g$, then we say n_i is an *exceptional non-gap* (*of* P).

Definition. If j is an integer with $0 \leqslant j \leqslant g-1$ and j is not a non-gap, then j is an *exceptional gap* (*of* P).

Notice that there are the same number of exceptional non-gaps as exceptional gaps for any point.

Definition. The (Weierstrass) weight of P is $\mathrm{wt}(P) = \sum_k (n_k - j_k)$, where n_k and j_k are the exceptional non-gaps and gaps, respectively.

So P is a Weierstrass point if and only if $\mathrm{wt}(P) > 0$. It is well known that $\sum_{P \in X} \mathrm{wt}(P) = g^3 - g$. So, in particular, there are at most $g^3 - g$ Weierstrass points on any curve of genus g.

Now, we suppose that X has an involution T (i.e. an automorphism of order 2). Let $C = X/\langle T \rangle$ be the quotient curve. We suppose T fixes r points, $P_1, \dots, P_r$. Thus, if

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$\phi: X \to C$ is the quotient map, then ϕ is ramified at these r points. Also, since ϕ has degree two (T is an involution), the degree of the ramification divisor is r. Notice that if $r \geqslant 5$, then by Lewittes' theorem [7], all r points are Weierstrass points. We will get lower bounds for the weight of these points.

Consider the effect of T on $K(X)$. We can decompose $K(X)$ according to how T acts on functions of $K(X)$.

Let $K^+(X) = \{f \in K(X) : f \circ T = f\}$ and $K^-(X) = \{f \in K(X)\, f \circ T = -f\}$. In particular, $\mathscr{L}(\mathscr{K}_X) = L^+ \oplus L^-$ where $L^+ = \mathscr{L}(\mathscr{K}_X) \cap K^+(X)$ and $L^- = \mathscr{L}(\mathscr{K}_X) \cap K^-(X)$.

2. *Main results*

LEMMA 1. $L^+ \cong \mathscr{L}(\mathscr{K}_C)$.

Proof. The map is given by $f \mapsto f \circ \phi^{-1}$. We see that $f \circ \phi^{-1}$ is a well-defined function on C since $f \circ T = f$. We need to show that $F = f \circ \phi \in \mathscr{L}(\mathscr{K}_C)$. We know that $(f)_X + \mathscr{K}_X \geqslant 0$, but $(f)_X = \phi^*((F)_C)$, the pull-back of $(F)_C$ by ϕ. Thus,

$$(*) \qquad (f)_X + \mathscr{K}_X = \phi^*((F)_C) + \phi^*(\mathscr{K}_C) + \sum_{m=1}^{r} P_m \geqslant 0$$

by the Riemann–Hurwitz equation. Since $\phi^*((F)_C + \mathscr{K}_C) + \sum P_m \geqslant 0$, the only places where $(F)_C + \mathscr{K}_C$ might have order $\leqslant -1$ is at one of the $\phi(P_m)$. But in that case, $\operatorname{ord}_{P_m}(\phi^*((F)_C + \mathscr{K}_C)) \leqslant -2$ which would contradict $(*)$. Thus, $(F)_C + \mathscr{K}_C \geqslant 0$, so $F \in \mathscr{L}(\mathscr{K}_C)$.

It is easy to see that the inverse map is given by $F \mapsto F \circ \phi$. Thus, we have an isomorphism of vector spaces.

COROLLARY 2. *If $f \in K^+$, then $\operatorname{ord}_{P_m}((f) + \mathscr{K}_X) \equiv 1 \pmod 2$. In particular, this is true for $f \in L^+$.*

Proof. Certainly, for any $f \in K^+$ we can look at $F = f \circ \phi \in K(C)$, as in Lemma 1. Then

$$(f)_X + \mathscr{K}_X = \phi^*((F)_C + \mathscr{K}_C) + \sum P_m.$$

Now, $\operatorname{ord}_P \phi^*(D) = e_P(\phi) \operatorname{ord}_{\phi(P)} D$ for any $D \in \operatorname{Div}(C)$ and $P \in C$, where $e_P(\phi)$ is the ramification index of ϕ at P. Since $e_{P_m}(\phi) = 2$, it is clear that $\operatorname{ord}_{P_m}((f)_X + \mathscr{K}_X) \equiv 1 \pmod 2$.

PROPOSITION 3. *There is a function on X, u, with the divisor of u equal to $(u) = \sum P_m + \phi^*(E)$, for some divisor $E \in \operatorname{Div}(C)$.*

Proof. First, we choose $w \in K(C)$ such that $y = \sqrt{w}$ generates $K(X)$ over $K(C)$. We claim that $\operatorname{ord}_{\phi(P)} w$ is odd if and only if P is a ramified point of ϕ (one of the P_m's). For if $\operatorname{ord}_{\phi(P)} w$ is even, then w is locally a square at P and if $\operatorname{ord}_{\phi(P)} w$ is odd, it is not. Thus, if you like, $[\hat{\mathscr{O}}_{X,P} \otimes \mathbb{Q} : \hat{\mathscr{O}}_{C,\phi(P)} \otimes \mathbb{Q}] = e_\phi(P) f_\phi(P) = e_\phi(P)$. Here, $\hat{\mathscr{O}}_{X,P}$ is the completion of the ring of functions on X, regular at P. The field extension is trivial, and therefore the ramification index is 1, if and only if $\operatorname{ord}_{\phi(P)}(w)$ is even. Note that $f_\phi(P)$ is always 1 since the residue class fields are all $\mathbb{C}$, our field of definition. Thus, we have that $(w)_C = \sum \phi(P_m) + 2E$ for some divisor E.

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Now, we look at the divisor $(y)_X$:

$$\begin{aligned}(u)_X = \tfrac{1}{2}(u^2)_X &= \tfrac{1}{2}\phi^*((w)_C)\\ &= \tfrac{1}{2}\phi^*\left(\sum \phi(P_m) + 2E\right)\\ &= \tfrac{1}{2}\sum \phi^*(\phi(P_m)) + \phi^*(E)\\ &= \sum P_m + \phi^*(E),\end{aligned}$$

since $\phi^*(\phi(P_m)) = e_\phi(P_m)\,P_m = 2P_m$.

LEMMA 4. *Given $f \in L^-$, the function f/u is in $K^+(X)$. Thus, $\operatorname{ord}_{P_m}((f)+K_X)$ is even.*

Proof. It is easy to see that $u \in K^-(X)$. For u and $u \circ T$ have the same divisor, and are thus constant multiples of each other. Suppose $u \circ T = au$. Then $u = u \circ T \circ T = a^2 u$, so $a = \pm 1$. Clearly, $a \neq 1$, or else u would be in $K(C)$.

Thus, $(f/u) \circ T = (-f)/(-u) = f/u$. So $(f/u) \in K^+(X)$. So by Corollary 2,

$$(f/u)_X + \mathscr{K}_X = \phi^*((F)_C + \mathscr{K}_C) + \sum P_m$$

for some $F \in K(C)$. Thus,

$$(f)_X + \mathscr{K}_X = \phi^*((F)_C + \mathscr{K}_C) + \sum P_m + (u)_X = \phi^*((F)_C + \mathscr{K}_C + E) + 2\sum P_m.$$

As noted in the proof of Corollary 2, $\operatorname{ord}_{P_m}\phi^* D \equiv 0 \bmod 2$ (here $D = (F)_C + \mathscr{K}_C + E$), so any P_m appears an even number of times.

THEOREM 5. *If X is any curve with an involution which fixes r points, then the Weierstrass weight of each of the fixed points is at least*

$$\operatorname{wt}(P) \geqslant \frac{(r-2)(r-4)}{8}.$$

After Accola, we call this bound the *expected weight of P*, $\operatorname{wt}_e(P) := (r-2)(r-4)/8$.

Proof. The Riemann–Hurwitz formula says that

$$2g-2 = \deg\phi(2q-2) + r = 4q-4+r,$$

where q is the genus of $C = X/\langle T\rangle$. Note that this says that r must be even. Thus, the theorem is void of content if $r \leqslant 5$, so we assume $r > 5$ throughout.

We claim that we can select a basis at P which consists of a basis of L^+ together with a basis of L^-. Consider a function $h \in \mathscr{L}(\mathscr{K}_X)$. We can write $h = h_+ + h_-$ where $h_+ = (h + h \circ T)/2 \in L^+$ and $h_- = (h - h \circ T)/2 \in L^-$. (Note that since $(h)_X = (h \circ T)_X$ and since $\mathscr{L}(\mathscr{K}_X)$ is a vector space, h_+ and h_- are in $\mathscr{L}(\mathscr{K}_X)$.) Now, we know that $\operatorname{ord}_P h_+$ is even, while $\operatorname{ord}_P h_-$ is odd, by Corollary 2 and Lemma 4. Thus,

$$\operatorname{ord}_P h = \min\{\operatorname{ord}_P h_+, \operatorname{ord}_P h_-\}.$$

Suppose we have a basis $\mathscr{B}$ at P. If $h \in \mathscr{B}$, then we can replace h by either h_+ or h_- without changing the order at P. In this way we are assured of finding a basis at P with elements all in either L^+ or L^-.

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Notice that basis elements coming from L^+ correspond to odd non-gaps by Corollary 2, while basis elements from L^- correspond to even non-gaps by Lemma 4.

Since $\dim L^+ = \dim \mathscr{L}(\mathscr{K}_C) = q$, there are q odd non-gaps. Since $\dim L^- = g-q$, there are $g-q$ even non-gaps. Thus, the smallest set of non-gaps could be for P (i.e. the one giving P the least possible weight) is

$$\{1, 3, \dots, 2q-1\} \cup \{0, 2, \dots, 2(g-q-1)\}.$$

Notice that all of the odd integers strictly between $2q-1$ and $\lfloor (g+1)/2 \rfloor$ are exceptional gaps. Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Assuming that we have this 'minimal' set of non-gaps, we calculate the weight of P_m. The exceptional non-gaps are exactly those even non-gaps greater than $g-1$. We have $g-q = (g-1)/2 + r/4$ even non-gaps, and there should only be (if P_m were not a Weierstrass point) $\lfloor (g+1)/2 \rfloor$ of them. Thus there are

$$(g-1)/2 + r/4 - \left[\frac{g+1}{2}\right] = \begin{cases} \dfrac{r-4}{4} & g \text{ odd}, \\ \dfrac{r-2}{4} & g \text{ even} \end{cases}$$

exceptional non-gaps for P_m. Each exceptional non-gap contributes

$$n_k - j_k = \begin{cases} g+1-(2q+1) & g \text{ odd}, \\ g-(2q+1) & g \text{ even} \end{cases} = \begin{cases} \dfrac{r-2}{2} & g \text{ odd}, \\ \dfrac{r-4}{2} & g \text{ even} \end{cases}$$

to the weight of P_m. Thus, we get

$$\operatorname{wt}(P_m) \geqslant \operatorname{wt}_e(P_m) = \frac{(r-2)(r-4)}{8}.$$

where $\operatorname{wt}_e(P_m)$ is the weight of P_m *if* the non-gaps are exactly those that we have listed. They could be larger numbers, which would increase the weight of P_m.

Remark. Essentially, we made two assumptions about P. First, we assumed that the odd non-gaps are as small as possible: $1, 3, \dots, 2q-1$. This is equivalent to assuming that $\phi(P_m)$ was not a Weierstrass point on C. Secondly, we assumed that the even non-gaps were as small as possible: $0, 2, 4, \dots, 2(g-q-1)$. Accola, in a more general setting, uses the fact that this is equivalent to $\phi(P_m)$ not being a D-Weierstrass point (which we have not defined), for a certain fixed divisor D on C. Suffice it to say that an ordinary Weierstrass point on C (the kind we are discussing) is a $\mathscr{K}_C$-Weierstrass point.

Corollary 6. *The parity of* $\operatorname{wt}(P)$ *must agree with* $\operatorname{wt}_e(P)$. *That is,*

$$\operatorname{wt}(P) \equiv \operatorname{wt}_e(P) \pmod 2.$$

Proof. We have shown that there are exactly q even non-gaps and $g-q$ odd non-gaps. If the non-gaps differ from the ones we gave, they must differ by a multiple of 2. This would add a multiple of 2 to the weight.

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Definition. Let BW be the total weight of the branch points of ϕ. That is,

$$BW = \sum_{T(P)=P} \mathrm{wt}\,(P).$$

We include some interesting formulas relating BW to the total weight of the Weierstrass points on X. These are the same lower bounds obtained by Torres in [**10**]. It should be noted that Torres also obtains upper bounds for the weights.

COROLLARY 7. (*a*)

$$BW \geqslant \frac{r}{2}\frac{(r-2)}{2}\frac{(r-4)}{2} = (g-2q)^3-(g-2q).$$

(*b*)

$$\frac{BW}{g^3-g} \geqslant 1-\frac{6q}{g} = \frac{3}{2}\frac{(r-2)}{g}-2.$$

3. *Examples*

In the case that the genus of the quotient curve q is 0, X is hyperelliptic and it is well known that all Weierstrass points are branch points. That is, $BW = g^3-g$. This matches our expected value. When $q = 1$, Kato showed that $BW \leqslant (g-1)(g^2-5g+10)$ and gave examples of curves with a point with this weight, for any g. Garcia first showed that $BW \geqslant (g-1)(g-2)(g-3)$ and gave examples of that case. He also noted that these are the only possible weights. Additionally, Garcia gave examples of all possible weights when $q = 2$.

Our main interest in this problem is in relation to the Fermat curves.

Consider the nth Fermat curve given in projective coordinates by

$$F_n : X^n + Y^n + Z^n = 0.$$

If T is the involution $[X:Y:Z] \mapsto [Y:X:Z]$, then T fixes the n points $P_m = [\zeta_n^m, \zeta_n^m, (-2)^{1/n}]$ for $m = 1, \ldots, n$. In addition, if n is odd, then T fixes one more point: $[1:-1:0] = [-1:1:0]$. Thus, we can apply the theorem.

Alternatively, one could look at the conjugate (to T) automorphisms of F_n given by

$$[X:Y:Z] \mapsto [\zeta_n^j Y : \zeta_n^{-j} X : Z] \quad \text{for } j = 1, \ldots, n-1,$$

or one could look at the automorphisms which switch X with Z or Y with Z or their conjugates. In each cases, there are n fixed points if n is even. If n is odd, we get an additional point with $XYZ = 0$. However, the points with $XYZ = 0$ are well understood. Their exact Weierstrass weights were calculated by [**4**]. From these $3n$ involutions, we get a total of $3n^2$ 'new' Weierstrass points.

Applying the theorem, we get the following bound.

COROLLARY 8. *If $P \in F_n$ is one of the points fixed by one of the involutions T, then*

$$\mathrm{wt}\,(P) \geqslant \mathrm{wt}_e(P) = \begin{cases} \dfrac{(n-1)(n-3)}{8} & n \text{ odd} \\ \dfrac{(n-2)(n-4)}{8} & n \text{ even.} \end{cases}$$

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Notice that this answers the question of whether or not the weights of these points grow with n. Indeed, we see that the weights grow at least like n^2.

For $n \leqslant 7$, it is known that the weights of these new points are exactly wt_e (see [**8**], for example, especially for $n = 7$).

We compute the $n = 8$ case as follows.

PROPOSITION 9. *The point* $P_0 = [1, 1, (-2)^{\frac{1}{8}}]$ *on the curve* F_8 *has Weierstrass weight equal to* 3. *It follows that the weight of any of the* 192 *conjugates* (*including the point itself*) *also have weight* 3.

Proof. It is easy to check that

$$\{x^i/y^j\,dx \quad \text{where} \quad 0 \leqslant i \leqslant j-2 \quad \text{and} \quad j = 1, \dots, 7\},$$

is a basis of holomorphic differentials on F_8. We can consider the Wronskian matrix, W, whose first row is

$$(*) \qquad \{1/y^7, x/y^7, \dots, x^5/y^7, 1/y^6, \dots, 1/y^2\}.$$

It is well-known that, since x is a uniformizing parameter at P_0, this Wronskian will vanish at P_0 to the order $\mathrm{wt}\,(P_0)$. That is, $\mathrm{ord}_{P_0} W = \mathrm{wt}\,(P_0)$. It is too computationally difficult to simply find the determinant of this Wronskian, so we simplify the problem.

First, let $f_1, f_2, \dots, f_g, h$ be any collection of functions (of x). Suppose W_1 is the Wronskian with first row $(f_1, \dots, f_g)$ and W_2 is the Wronskian with first row $(hf_1, \dots, hf_g)$. It is easy to show (using the product rule) that

$$\det W_2 = h^g \det W_1.$$

Thus, if we let $D = d/dx$, differentiation with respect to x,

$$D^3(\det W_2) = D^3(h^g)\det W_1 + 3D^2(h^g)\,D(\det W_1) + 3D(h^g)\,D^2(\det W_1) + h^g D^3(\det W_1).$$

In our case, $h = y^7$, $g = 21$, and $f_1, \dots, f_g$ are given in $(*)$. Evaluating at P_0, we use the fact that P_0 has weight at least 3, so $D^i(\det W_1)(P_0) = 0$ for $i = 0, 1, 2$. We see that

$$D^3(\det W_2)(P_0) = (y^7)^{2\,1} D^3(\det W_1)(P_0).$$

Since $y(P_0) = (-2)^{(-\frac{1}{8})}$, we see that $\mathrm{wt}\,(P_0) = 3$ if and only if $D^3(\det W_2)(P_0) \neq 0$. Note that W_2 is the Wronskian with first row

$$(**) \qquad \{1, x, \dots, x^5, y, xy, \dots, y^5\}.$$

Next, for any set of functions, $\{f_i(x)\}_{i=1}^g$, let $M[n_1, \dots, n_g]$ be the matrix whose ith row is $(D^{n_i}(f_1), \dots, D^{n_i}(f_g))$ where $D = d/dx$. In particular, if $n_i = i-1$ then M is a Wronskian matrix. We note that using the multi-linearity of the determinant (in the rows), we know that

$$D(\det M[0, \dots, g-1]) = \det M[0, \dots, g-2, g],$$

$$D^2(\det M[0, \dots, g-1]) = \det M[0, \dots, g-3, g-1, g] + \det M[0, \dots, g-2, g+1]$$

and

$$D^3(\det M[0, \dots, g-1]) = \det M[0, \dots, g-4, g-2, g-1, g] + 2\det M[0, \dots, g-3, g-1, g+1] + \det M[0, \dots, g-2, g+2].$$

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In our case, $g = 21$, and the f_is are given by (**). We get

$$\begin{aligned} D^3(\det M[0,\dots,20]) &= \det M[0,\dots,17,19,20,21] \\ &\quad + 2\det M[0,\dots,18,20,22] + \det M[0,\dots,19,23] \\ &= K\det M[6,\dots,17,19,20,21] \\ &\quad + 2K\det M[6,\dots,18,20,22] + K\det M[6,\dots,19,23] \end{aligned}$$

where $K = 5!4!3!2! = 34560$, due to the nature of the first six f_is. (The first six columns have all zero entries below the diagonal.)

Now, these determinants are small enough to handle. Using Maple, we computed the various derivatives, evaluated at P_0 (i.e. at $x = (-2)^{-\frac{1}{8}}$), and took the determinants. The results were that

$$\det M[6,\dots,17,19,20,21] = 0,$$

$$\begin{aligned} \det M[6,\dots,18,20,22] &= (2^{\frac{7}{8}})\,2^{183}3^{86}5^{34}7^{39}11^{11}13^{8}17^{5}19^{2}23^{6}31^{3} \\ &\quad \times (47)(109)(881849)(58820429)(257947912500799), \end{aligned}$$

and

$$\det M[6,\dots,19,23] = 0.$$

Thus, $D^3(\det N)(P_0) \neq 0$ and the weight of P_0 is exactly 3.

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