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Constraints on dynamo action in plasmas

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Upper bounds are derived on the amount of magnetic energy that can be generated by dynamo action in collisional and collisionless plasmas with and without external forcing. A hierarchy of mathematical descriptions is considered for the plasma dynamics: ideal magnetohydrodynamics (MHD), visco-resistive MHD, the double-adiabatic theory of Chew, Goldberger and Low (CGL), kinetic MHD and other kinetic models. It is found that dynamo action is greatly constrained in models where the magnetic moment of any particle species is conserved. In the absence of external forcing, the magnetic energy then remains small at all times if it is small in the initial state. In other words, a small 'seed' magnetic field cannot be amplified significantly, regardless of the nature of flow, as long as the collision frequency and gyroradius are small enough to be negligible. A similar conclusion also holds if the system is subject to external forcing as long as this forcing conserves the magnetic moment of at least one plasma species and does not greatly increase the total energy of the plasma (i.e. in practice, is subsonic). Dynamo action therefore always requires collisions or some small-scale kinetic mechanism for breaking the adiabatic invariance of the magnetic moment.

Key words: astrophysical plasmas, magnetised plasmas

1. Introduction

Dynamo action is believed to be ubiquitous throughout the Universe. It is generally agreed that in planetary and stellar interiors as well as in the interstellar and intergalactic medium, turbulent fluid motions are responsible for the presence of magnetic fields (Zweibel & Heiles 1997; Brandenburg & Subramanian 2005; Kulsrud & Zweibel 2008; Durrer & Neronov 2013; Roberts & King 2013), i.e. pre-existing 'seed' fields have been amplified to their present level and are now maintained by dynamo action. The precise mechanism by which this happens is, however, still a matter of debate. It is therefore of interest to identify the conditions under which dynamo action is (im)possible.

The great majority of all research into dynamo theory has been carried out within the framework of magnetohydrodynamics (MHD), although most interplanetary, interstellar and intergalactic plasmas are too weakly collisional to be accurately

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described by this approximation. In the present paper, we investigate how dynamo action is different in mathematical plasma models that go beyond MHD and capture some of the underlying kinetic dynamics of collisionless plasmas. In order to isolate this consideration from the question of how the geometry of the flow or external forces affect the dynamo, we first consider the free evolution (i.e. without external forces) of a conducting fluid from an arbitrary state at time $t = 0$ and ask to what extent the resulting motion can amplify a pre-existing magnetic field. This question is meaningful even for an ideally conducting fluid, where steady-state dynamo action would inevitably give rise to magnetic fields at infinitely small scales (see e.g. Zeldovich *et al.* 1984). In general, an arbitrarily prepared state at $t = 0$ is not in equilibrium, and the ensuing motion will be chaotic, stretching and bending magnetic-field lines, thereby increasing the magnetic-field energy, even in the absence of resistivity and reconnection.* But how much can the magnetic energy increase above its initial value? To what extent can the thermal and kinetic energy of the fluid be converted into magnetic energy?

This problem was first considered in an early paper by Batchelor (1950), who used the visco-resistive MHD equations and considered homogeneous and isotropic turbulence as the underlying fluid flow. He concluded that, if the conductivity is large and the initial magnetic field weak, the magnetic energy will in general grow exponentially with time until 'the large wave-number components contain comparable amounts of kinetic and magnetic energy'. In a contemporaneous article, Biermann & Schlüter (1951) similarly concluded that 'the magnetic energy density will finally reach the energy density of the turbulence'.† While there was then, and is now, some level of disagreement or uncertainty about the detailed state resulting from turbulent dynamo action in MHD, simplified analytical models (Kazantsev 1968; Zeldovich *et al.* 1984; Chertkov *et al.* 1999; Boldyrev & Cattaneo 2004) and numerical simulations (Meneguzzi, Frisch & Pouquet 1981; Haugen & Brandenburg 2004; Schekochihin *et al.* 2004; Beresnyak 2012) as well as recent laboratory experiments (Monchaux *et al.* 2007; Tzeferacos *et al.* 2016) demonstrate beyond reasonable doubt that a sufficiently chaotic three-dimensional flow of a conducting MHD fluid at large enough magnetic Reynolds numbers will generate tangled magnetic fields with energy densities comparable to that of the fluid motions.

The situation can be very different in mathematical models other than MHD, as was first realised by Kulsrud *et al.* (1997) who pointed out that conservation of the first adiabatic invariant (magnetic moment) for each gyrating particle implied that changing the magnetic-field strength by a finite factor would require changing the perpendicular energy of the particles (i.e. the perpendicular pressure) by a similar factor, which is usually not possible due to energy-source constraints.‡ In this work, we extend this argument by exploring a hierarchy of plasma models of increasing complexity,

*In a steady-state situation, the question of taking these effects into account is usually about whether the field can survive in their presence (Zeldovich *et al.* 1984; Chertkov *et al.* 1999; Schekochihin *et al.* 2004). The basic amplification mechanism for a turbulent, small-scale dynamo is ideal and relies on chaotic stretching of field lines.

†Although their view was that it would be the total energy density of the turbulence that the magnetic energy would become comparable to — this indeed appears to be the case (Haugen & Brandenburg 2004; Beresnyak 2012).

‡They further argued that in the presence of pressure anisotropy, which would arise from any local increase of the magnetic-field strength, the magnetic mirror force might modify the velocity field in such a way as to prevent any further field amplification, independently of the absolute magnitude of the field (i.e. even for dynamically weak fields). The microphysical feedback on the macroscopic motions remains poorly understood and a subject of much current interest (Kunz, Schekochihin & Stone 2014; Melville, Schekochihin & Kunz 2016; Riquelme, Quataert & Verscharen 2016; Squire, Quataert & Schekochihin 2016).

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starting from ideal and resistive MHD and proceeding to the double-adiabatic theory of Chew, Goldberger and Low (CGL) (Chew, Goldberger & Low 1956), kinetic MHD (Kulsrud 1983) and more complete kinetic models. In each case, it is possible to derive a rigorous upper bound on the magnetic energy that is valid at all times, and this bound turns out to depend crucially on the conservation properties of the equations, particularly on the equation of state. In all models appropriate for a collisionless plasma with vanishingly small gyroradii, the growth of the magnetic energy is severely limited.

These conclusions follow essentially from the following physical argument. In the absence of external forcing, the total energy

$$W = \left\langle p_\perp + \frac{p_\parallel}{2} \right\rangle + K + M, \tag{1.1}$$

is conserved, where $p_\perp$ and $p_\parallel$ are the pressures perpendicular and parallel to the magnetic field, K the kinetic energy, $M = \langle B^2/2 \rangle$ magnetic energy and angular brackets denote spatial integration. Most of the energy is assumed to be non-magnetic in the initial state, $M \ll W$. Now suppose that the plasma is collisionless and moves in such a way that all quantities vary slowly on the time scale of the gyromotion and weakly on the length scale of the gyroradius of all particle species. The magnetic moment, $\mu = mv_\perp^2/(2B)$, of each particle will then remain constant, and so will therefore the sum

$$\sum_{particles} \mu = \left\langle \frac{p_\perp}{B} \right\rangle. \tag{1.2}$$

It then becomes difficult to amplify the magnetic field, for if the field strength is increased everywhere by a factor of ten, say, the perpendicular pressure must increase by a similar factor. But since the total energy is conserved, and most of it was already in the form of thermal energy at $t = 0$, this is impossible unless the Mach number is so large that there is a significant reservoir of kinetic energy. For this reason, dynamo action becomes virtually impossible in any plasma model that assumes adiabatic invariance of μ.

Having established these results, we consider the effects of finite resistivity, viscosity and external forcing, and find that these do not change the underlying physics significantly, at least as long as we restrict our attention to subsonic flows and dynamical (in contrast to heating) time scales. The constraints found on the growth of magnetic energy are still valid if the energy input from the external forces is not too large. These conclusions are tested by performing and comparing numerical simulations of the CGL equations (cf. Santos-Lima *et al.* 2011) and the conventional MHD equations. Striking differences between these models are seen in the magnetic-field evolution, confirming the conclusion that dynamo action is impossible in a gyrotropic plasma where the magnetic moment of at least one species is conserved. Implications are discussed in the concluding section.

2. MHD and CGL equations

To begin the formal analysis, we first consider the question whether the flow of a conducting fluid can cause amplification of a 'seed' magnetic field when evolving freely from an initial state according to the inviscid, ideal MHD equations (Kulsrud 1983)

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla \cdot \boldsymbol{V} = 0, \tag{2.1}$$

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$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{j} \times \boldsymbol{B} - \boldsymbol{\nabla} \cdot \mathsf{P}, \tag{2.2}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B}), \tag{2.3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{j}, \tag{2.4}$$

where the symbols have their usual meaning, $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{V} \cdot \boldsymbol{\nabla}$ denotes the convective derivative and the pressure tensor is gyrotropic,

$$\mathsf{P} = p_\perp \boldsymbol{I} + (p_\parallel - p_\perp)\boldsymbol{bb}, \tag{2.5}$$

with $\boldsymbol{b} = \boldsymbol{B}/B$. The components of P are determined either by ideal MHD entropy conservation,

$$p_\perp = p_\parallel = p, \tag{2.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho^\gamma}\right) = 0, \tag{2.7}$$

where $\gamma = 5/3$, or by the CGL equations (Chew *et al.* 1956)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_\perp}{\rho B}\right) = 0, \tag{2.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_\parallel B^2}{\rho^3}\right) = 0. \tag{2.9}$$

The adiabatic law (2.7) holds in this latter case, too, if it is understood that p is formally replaced by $(p_\perp^2 p_\parallel)^{1/3}$.

The boundary conditions are either periodic in all three directions or it is assumed that the fluid is surrounded by a rigid, ideally conducting boundary, at which the normal components of $\boldsymbol{V}$ and $\boldsymbol{B}$ vanish. In both cases, and for both the MHD and the CGL models, the total energy, defined by

$$W = \left\langle \frac{\rho V^2}{2} + p_\perp + \frac{p_\parallel}{2} + \frac{B^2}{2} \right\rangle, \tag{2.10}$$

is then conserved, where the angular brackets denote a volume average. The magnetic flux is also conserved (Arnold *et al.* 1982), but we shall not make use of this property.

2.1. *Available energy in ideal MHD*

Eliminating the density from the continuity equation (2.1) and the adiabatic law (2.7) gives

$$\frac{\partial p^{1/\gamma}}{\partial t} + \boldsymbol{\nabla} \cdot \left(p^{1/\gamma} \boldsymbol{V}\right) = 0, \tag{2.11}$$

which implies that the quantity

$$S = \left\langle p^{1/\gamma} \right\rangle \tag{2.12}$$

is conserved, $\mathrm{d}S/\mathrm{d}t = 0$. We thus have two conserved quantities, and may seek the maximum of the magnetic energy

$$M = \left\langle \frac{B^2}{2} \right\rangle \tag{2.13}$$

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under the constraint of constant W and S. Since $M < W$, it is clear that this maximum indeed exists and represents a mathematical upper bound on the magnetic energy at all times $t \geqslant 0$. It may, however, be inaccessible from the initial conditions, because there is an infinity of other constraints associated with the topology of the magnetic-field lines (Taylor & Newton 2015), but we ignore this issue as we only seek an upper bound on M that need not be the best possible one.

Thus, we consider the maximum of the functional

$$T[\rho, \boldsymbol{V}, \boldsymbol{B}, p; \lambda, \mu] = M - \lambda(W - W_0) - \mu(S - S_0), \tag{2.14}$$

where λ and μ are Lagrange multipliers, and initial conditions are denoted by a '0' subscript. Since this functional does not involve any derivatives of the fields $(\rho, \boldsymbol{V}, \boldsymbol{B}, p)$, these must be constant at the maximum in question; this becomes obvious if one writes down the Euler–Lagrange equations for the minimising fields.[§] Denoting these quantities by the subscript '1', we thus have

$$S = \left\langle p_0^{1/\gamma} \right\rangle = p_1^{1/\gamma} \tag{2.15}$$

and $V_1 = 0$, so that

$$M_1 = W - \frac{\left\langle p_0^{1/\gamma} \right\rangle^{\gamma}}{\gamma - 1}. \tag{2.16}$$

The difference between the upper bound M_1 and the initial value of the magnetic energy, M_0, is thus

$$A = M_1 - M_0 = \frac{\langle p_0 \rangle - \left\langle p_0^{1/\gamma} \right\rangle^{\gamma}}{\gamma - 1} + \left\langle \frac{\rho_0 V_0^2}{2} \right\rangle. \tag{2.17}$$

This quantity is always positive because, for $\gamma > 1$,

$$\left\langle p_0^{1/\gamma} \right\rangle \leqslant \langle p_0 \rangle^{1/\gamma} \tag{2.18}$$

by Hölder's inequality.

The expression (2.17) represents an upper bound on the amount of thermal and kinetic energy that is available for conversion into magnetic energy, and will in the following be referred to as the 'available energy' for dynamo action. This upper bound is usually inaccessible: not only does it ignore topological constraints on the evolution of the magnetic field, but it is also in general incompatible with the boundary conditions. As we shall see, it can nevertheless be surprisingly restrictive.

If the initial state is such that either the Mach number is of order unity or the pressure p_0 varies substantially (by order unity) across the domain, then a substantial fraction of the thermal and kinetic energy is available for conversion into magnetic energy, $A/W = O(1)$. If, however, the initial pressure fluctuations are small,

$$p_0(\boldsymbol{r}) = P + \delta p(\boldsymbol{r}), \tag{2.19}$$

[§]The Euler–Lagrange equation to minimise a functional $\int L[y(\boldsymbol{x}), \boldsymbol{x}]\,\mathrm{d}\boldsymbol{x}$ depending on a field $y(\boldsymbol{x})$, its first-order derivatives $\partial y/\partial x_i = y_{x_i}$ and the coordinates $\boldsymbol{x}$, is $\sum_i (\partial/\partial x_i)(\partial L/\partial y_{x_i}) = (\partial L/\partial y)$. If L neither depends on the derivatives $\partial y/\partial x_i$ nor explicitly on the coordinates $\boldsymbol{x}$, it follows that $\partial L/\partial y = 0$. This defines an equation for y that is independent of $\boldsymbol{x}$, and so are therefore its solutions.

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where $P = \langle p_0 \rangle$ and $\delta p \ll P$, then the available energy from these fluctuations is quadratic in their amplitude and is relatively small:

$$A_{\delta p} = \frac{\langle \delta p^2 \rangle}{2\gamma P} \ll W. \tag{2.20}$$

If we consider specifically the limit of small Mach numbers (subsonic motions):

$$Ma \equiv \frac{V_0}{\sqrt{\gamma p_0/\rho_0}} \ll 1, \tag{2.21}$$

the dynamics is pressure balanced and, typically, pressure perturbations are $\delta p/p_0 \sim Ma^2$. This means that, to lowest order in Ma, the available energy is just the kinetic energy in the initial state, $K_0 = \langle \rho_0 V_0^2/2 \rangle$:

$$\frac{A}{K_0} = 1 + O(Ma). \tag{2.22}$$

2.2. *Finite resistivity and viscosity*

Most dynamos considered in the literature involve resistivity, which enables field line diffusion and reconnection to take place. A justified question is, therefore, to what extent the above conclusions remain valid if a finite resistivity is introduced. The induction equation (2.3) then becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}, \tag{2.23}$$

and the entropy conservation law (2.7) is replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho^\gamma}\right) = \frac{(\gamma - 1)\eta j^2}{\rho^\gamma}. \tag{2.24}$$

The energy W is still conserved, but the quantity $S = \langle p^{1/\gamma} \rangle$ now increases with time

$$S(t) > S_0, \tag{2.25}$$

reflecting positive entropy production. However, regardless of how much S increases, we may still seek the maximum magnetic energy M_1 at a given W and $S = S_1$ and thus find an upper bound on the available energy

$$A = M_1 - M_0 = \frac{\langle p_0 \rangle - S_1^\gamma}{\gamma - 1} + \left\langle \frac{\rho_0 V_0^2}{2} \right\rangle. \tag{2.26}$$

Since $S_1 > S_0$, this bound is lower than in the ideal MHD case, and we conclude that less energy is available for conversion to magnetic energy. In other words, the bound (2.17) still holds, but is even tighter than before.

The addition of viscosity to the MHD equations has a similar effect, with viscosity now contributing to entropy production (2.24). Since the total energy (2.10) is conserved and $S(t)$ increases with time, the magnetic energy is again bounded from above by (2.17).

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2.3. *Available energy in the double-adiabatic model*

As we shall now see, the situation is very different in the double-adiabatic model of Chew, Goldberger and Low (Chew *et al.* 1956; Kulsrud 1983). Eliminating the density from the continuity equation (2.1) and from the equations of state (2.8) and (2.9) gives

$$\frac{\partial}{\partial t}\left(\frac{p_\perp}{B}\right) + \boldsymbol{\nabla}\cdot\left(\frac{p_\perp \boldsymbol{V}}{B}\right) = 0, \tag{2.27}$$

$$\frac{\partial}{\partial t}\left(p_\parallel^{1/3} B^{2/3}\right) + \boldsymbol{\nabla}\cdot\left(p_\parallel^{1/3} B^{2/3}\boldsymbol{V}\right) = 0, \tag{2.28}$$

implying that the following two quantities are conserved[¶]

$$I = \left\langle \frac{p_\perp}{B} \right\rangle, \tag{2.29}$$

$$J = \left\langle p_\parallel^{1/3} B^{2/3} \right\rangle. \tag{2.30}$$

We may thus ask for the maximum of the magnetic energy M under the constraint that W, I and J are fixed. Proceeding as before, we find that all the fields $(p_\perp, p_\parallel, B, \ldots)$ are constant and the flow velocity vanishes, so in the state of maximum magnetic energy we have

$$W = IB_1 + \frac{J^3}{2B_1^2} + \frac{B_1^2}{2}. \tag{2.31}$$

This equation is a quartic for B_1 as a function of the invariants I, J and W, and in terms of the normalised magnetic field $b = B_1/\sqrt{W}$, it becomes

$$b^4 + \frac{2Ib^3}{\sqrt{W}} - 2b^2 + \frac{J^3}{W^2} = 0. \tag{2.32}$$

The sum of all four roots is equal to $-2I/\sqrt{W} < 0$, while their product equals $J^3/W^2 > 0$. There are, therefore, two positive and two negative roots. The negative ones can obviously be discarded, and the largest positive root is the upper bound that we are seeking. The other positive root is a lower bound on the magnetic energy, for the constancy of J implies that B cannot be made arbitrarily small at constant W, since small B implies large $p_\parallel$. This is different from the ideal MHD case, where the analogous calculation yields no lower bound on M other than $B = 0$.

In the initial state, the ratio of thermal to magnetic energy is equal to

$$\frac{3\beta_0}{2} = \frac{\langle p_{\perp 0} + p_{\parallel 0}/2\rangle}{M_0} = \frac{W - \rho_0 V_0^2/2}{M_0} - 1, \tag{2.33}$$

and in the state of maximum magnetic energy it is

$$\frac{3\beta_1}{2} = \frac{W}{M_1} - 1 = \frac{2}{b^2} - 1. \tag{2.34}$$

[¶]Additional invariants are derived in appendix A, but we do not consider the additional constraints implied by their existence.

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The general solution of (2.32) is not particularly edifying, but it does yield interesting information in the limit of high β_0. In this limit,

$$\frac{I}{\sqrt{W}} \sim \sqrt{\beta_0} \gg 1, \tag{2.35}$$

$$\frac{J^3}{W^2} \sim \frac{1}{\beta_0} \ll 1, \tag{2.36}$$

and it is helpful to write $x = b\sqrt{\beta_0}$, so that (2.32) becomes

$$\frac{x^4}{\beta_0} + 2ax^3 - 2x^2 + c = 0, \tag{2.37}$$

where $a = I/\sqrt{\beta_0 W}$ and $c = \beta_0 J^3/W^2$ are of order unity. Since β_0 is large, one of the negative roots is obtained by balancing the first two terms, $x = -2a\beta_0$, whereas the three remaining roots are of order unity and are found by neglecting the first term. We thus come to the conclusion that if the initial state is one with little magnetic field, so that β_0 is large, then β will remain large at all times. Indeed, β can only change by order unity in either direction because there is a lower as well as an upper bound on the magnetic energy, both of order $b^2 \sim 1/\beta_0$ since $x = O(1)$. In other words, in the high-β limit, the available energy is a small fraction of the total energy,

$$A \sim \frac{W}{\beta_0}, \tag{2.38}$$

in contrast to conventional MHD, where $A \sim W$.

In the small Mach number limit (2.21), the fact that only a small fraction of the total energy is available to be converted into magnetic field is not by itself very surprising and it is relevant to ask how much of the kinetic energy in the initial state can be converted into magnetic energy. Since

$$\frac{A}{K_0} \sim \frac{1}{Ma^2\beta_0}, \tag{2.39}$$

only a small fraction of K_0 is available for conversion if $\beta_0 \gg Ma^{-2}$. This is in contrast to MHD, where all of K_0 is available: see (2.22). The point is that, in the CGL model, if a certain amount of energy is taken from K_0 to increase M, a much larger amount (roughly $\sqrt{\beta_0}$ as much) must additionally be spent on increasing the perpendicular pressure. For a fluid mechanical picture, one can argue as follows. In an incompressible conducting fluid, the magnetic field is amplified when flux tubes are stretched in the direction along $\boldsymbol{B}$ and compressed in the perpendicular direction. Because of the invariance of magnetic moments, any local increase of B is accompanied by an increased perpendicular pressure, which counteracts the compression. The latter only becomes possible at the expense of a work large enough to overcome the increased pressure.

3. Available energy in kinetic plasma models

The upper bound on the magnetic energy that we have derived arises because the invariant I dictates that the magnetic field cannot be increased without a similar relative increase in $p_\perp$, just as anticipated in Kulsrud *et al.* (1997). Thus, even if the magnetic field is very weak, it 'costs' a significant amount of energy to increase it by a finite factor. This property of the CGL equations only relies on the constancy of the magnetic moment and is, therefore, shared by any plasma model that conserves this quantity.

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3.1. *Kinetic MHD*

An example of such a model is kinetic MHD, which is obtained by expanding the Vlasov–Maxwell system of equations in the small gyroradius limit, ordering the plasma flow velocity to be at most comparable to the ion thermal speed, $V \sim v_{Ti}$ (Kulsrud 1983; Hazeltine & Waelbroeck 1998). The fluid equations then obtained coincide with our equations (2.1)–(2.5), but the components $(p_\perp, p_\parallel)$ of the pressure tensor are determined by the kinetic equation

$$\frac{\partial f_s}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{V}\right) \cdot \boldsymbol{\nabla} f_s + \dot{\epsilon} \frac{\partial f_s}{\partial \epsilon} = 0, \tag{3.1}$$

$$\dot{\epsilon} = e_s v_\parallel E_\parallel - m_s v_\parallel \boldsymbol{b} \cdot \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} - \mu B \boldsymbol{\nabla} \cdot \boldsymbol{V} - (m_s v_\parallel^2 - \mu B) \boldsymbol{bb} : \boldsymbol{\nabla V}, \tag{3.2}$$

rather than by (2.8) and (2.9) (which follow from kinetic MHD only if heat fluxes are neglected). Here, f_s denotes the distribution function of the particles of species s, whose charge is denoted e_s and mass by m_s, the magnetic moment is $\mu = m_s v_\perp^2 / 2B$ and the particle velocity $\boldsymbol{v}$ is measured relative to the mean velocity $\boldsymbol{V}(\boldsymbol{r}, t)$, so that the laboratory-frame velocity of a particle is $\boldsymbol{u} = \boldsymbol{V} + \boldsymbol{v}$. From the solution of the kinetic equation (3.1), the pressures needed in the equation of motion (2.2) are computed by

$$\begin{pmatrix} p_\perp \\ p_\parallel \end{pmatrix} = \sum_s \int \begin{pmatrix} \mu B \\ m_s v_\parallel^2 \end{pmatrix} f_s \, \mathrm{d}^3 v. \tag{3.3}$$

Another difference with conventional MHD is that $E_\parallel$, the component of the electric field that is parallel to $\boldsymbol{B}$, appears in (3.2). As in MHD, $E_\parallel$ is relatively small, as follows from the observation that the first term on the right-hand side of (3.2) is comparable to the others when $E_\parallel / B \sim \rho_i v_{Ti} / L$, where $\rho_i = m_i v_{Ti} / e_i B$ is the ion gyroradius. However, unlike in MHD, $E_\parallel$ affects the motion of the plasma and must be determined by the quasi-neutrality condition

$$\sum_s e_s \int f_s \, \mathrm{d}^3 v = 0, \tag{3.4}$$

which closes the kinetic MHD system of equations.

This system conserves both the energy (2.10) and the total magnetic moment (2.29), the latter for each species individually. This is most easily shown by first writing the kinetic equation (3.1) in conservative form,

$$\frac{\partial}{\partial t}\left(\frac{B f_s}{v_\parallel}\right) + \boldsymbol{\nabla} \cdot \left[\left(\boldsymbol{B} + \frac{B\boldsymbol{V}}{v_\parallel}\right) f_s\right] + \frac{\partial}{\partial \epsilon}\left(\frac{\dot{\epsilon} B f_s}{v_\parallel}\right) = 0. \tag{3.5}$$

Multiplying this equation by

$$\frac{\mu v_\parallel}{B} \mathrm{d}^3 v = \mu \sum_\sigma \frac{2\pi}{m_s^2} \, \mathrm{d}\mu \, \mathrm{d}\epsilon, \tag{3.6}$$

where $\sigma = v_\parallel / |v_\parallel|$, and integrating over velocity and real space gives $\mathrm{d}I_s / \mathrm{d}t = 0$. Obviously, the total magnetic moment

$$I = \sum_s I_s \tag{3.7}$$

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is also conserved. To prove energy conservation, one multiplies equation (3.5) by $(\epsilon v_\parallel/B)\,\mathrm{d}^3v$ integrates similarly, and sums over species, giving

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle p_\perp + \frac{p_\parallel}{2}\right\rangle = \left\langle j_\parallel E_\parallel - p_\perp \boldsymbol{\nabla}\cdot\boldsymbol{V} + (p_\perp - p_\parallel)\boldsymbol{bb}:\boldsymbol{\nabla V}\right\rangle, \tag{3.8}$$

where

$$j_\parallel = \sum_s e_s \int v_\parallel f_s\,\mathrm{d}^3v = 0 \tag{3.9}$$

to the requisite order (Kulsrud 1983). The evolution of the kinetic energy can be computed from the continuity and momentum equations (2.1)–(2.2),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{\rho V^2}{2}\right\rangle = \left\langle \boldsymbol{V}\cdot(\boldsymbol{j}\times\boldsymbol{B}) + p_\perp\boldsymbol{\nabla}\cdot\boldsymbol{V} - (p_\perp - p_\parallel)\boldsymbol{bb}:\boldsymbol{\nabla V}\right\rangle, \tag{3.10}$$

and the magnetic energy evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{B^2}{2}\right\rangle = \langle(\boldsymbol{V}\times\boldsymbol{B})\cdot\boldsymbol{j}\rangle, \tag{3.11}$$

as found from the induction equation (2.3). The sum of these energy relations implies that the total energy (2.10) is conserved.

Knowing that W and I are conserved, we again proceed to seek the state of maximum magnetic energy

$$M = W - L - \langle p_\perp\rangle, \tag{3.12}$$

where we have denoted

$$L = \left\langle \frac{\rho V^2}{2} + \frac{p_\parallel}{2}\right\rangle. \tag{3.13}$$

As before, this state has constant magnetic-field strength, so $M = B_1^2/2$ and

$$B_1^2 + 2IB_1 - 2(W-L) = 0, \tag{3.14}$$

i.e.

$$B_1 = I\left(\sqrt{1 + \frac{2(W-L)}{I^2}} - 1\right). \tag{3.15}$$

At high β, $W \ll I^2$, we conclude that $B_1 \simeq (W-L)/I$, and the maximum magnetic energy is a small fraction of the total energy,

$$\frac{M_1}{W} = \frac{(W-L)^2}{2WI^2} \leqslant \frac{W}{2I^2} \sim \frac{1}{\beta_0} \ll 1. \tag{3.16}$$

As in the case of the CGL model (to which the present argument also applies), if the Mach number of the initial flow is small, the fraction of kinetic energy that can be converted to magnetic energy is

$$\frac{M_1}{K_0} \sim \frac{1}{Ma^2\beta_0}, \tag{3.17}$$

and is small at high enough β_0. This is the same result as (2.39).

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3.2. *More general plasma models*

The analysis leading to the bound (3.16) on the magnetic energy shows that it is the conservation of the magnetic moment that leads to the available energy being so limited at high beta. The limit (3.16) is thus applicable beyond the approximations made in kinetic MHD and will hold in any plasma model where the total magnetic moment and energy are conserved and the latter can be written in the form

$$W = L + \langle p_\perp \rangle + M, \tag{3.18}$$

where L is a positive definite quantity. Mathematically, the argument is exactly the same as that just given for kinetic MHD. Any kinetic description of the plasma in which the gyroradius is small, all frequencies are lower than the ion cyclotron frequency, and collisions are negligible has this property. Drift kinetics, which is similar to kinetic MHD but treats the plasma flow velocity as smaller than the ion thermal speed is an example (Hazeltine & Waelbroeck 1998),[‖] under the proviso that collisions can be ignored, so that the total magnetic moment I is conserved. The magnetic field can therefore only grow appreciably on time scales longer than the collision time, at which point I is no longer conserved.

The perpendicular pressure $p_\perp$ entering in (3.18) and the definition (2.29) of I need not refer to the entire plasma, but could denote just the perpendicular pressure of one of its components. For the bound (3.16) to hold, it is sufficient that the magnetic moments for the particles of one plasma component be conserved (as long as its number density is not so small as to render the high-β approximation invalid). For instance, if we consider dynamo action on time scales that are longer than the collision time for electrons but shorter than that for ions, the collision operator needs to be retained in the electron kinetic equation but can be ignored in the ion dynamics. If, in this situation, the initial ion β is large and $p_\perp$ denotes the perpendicular ion pressure, the bound (3.16) implies that the magnetic energy cannot grow significantly, even though the conservation of the magnetic moment of the electrons is broken by collisions.

4. Case of external forcing

4.1. *MHD model*

Numerical dynamo simulations usually involve external forcing (Meneguzzi *et al.* 1981; Haugen & Brandenburg 2004; Schekochihin *et al.* 2004; Beresnyak 2012; Santos-Lima *et al.* 2011), and we now ask how this may affect our results. If a force is added to the equation of motion,

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{j} \times \boldsymbol{B} - \boldsymbol{\nabla} \cdot \mathsf{P} + \boldsymbol{F}(\boldsymbol{r}, t), \tag{4.1}$$

the total energy W is no longer conserved, but increases (or decreases), as the force adds kinetic energy to the system. In a dissipative plasma, this energy is continually processed: it is transferred to thermal energy by viscosity and resistivity.[**] Since

[‖]This is the kind of ordering that can, for example, be argued to be appropriate for the turbulent plasma in galaxy clusters (Rosin *et al.* 2011).

[**]It can also then be radiated, possibly leading to a statistically constant W, as, for example, is believed to happen in galaxy cluster cores (Zhuravleva *et al.* 2014). Examining further arguments in this section, one might expect this approximate conservation of W to help establish better upper bounds on the attainable magnetic energy. However, in subsonically turbulent systems, this steady state is achieved on the heating/cooling time scales, which, as we will argue in what follows, are much longer than the dynamical time scales on which dynamo action matters.

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entropy is constantly produced by these dissipation processes, S is also no longer conserved. If, over some period of time, the total energy increases from W_0 to W_1 and entropy from S_0 to S_1 and we maximise M_1 at given values of W_1 and S_1, we find (similarly to the calculation in § 2.2)

$$M_1 = W_1 - \frac{S_1^\gamma}{\gamma - 1}, \tag{4.2}$$

and the available energy becomes

$$A = M_1 - M_0 = W_1 - W_0 + \frac{\langle p_0 \rangle - S_1^\gamma}{\gamma - 1} + \left\langle \frac{\rho_0 V_0^2}{2} \right\rangle. \tag{4.3}$$

Compared to the ideal, unforced case (2.17), this makes additional energy available for conversion into magnetic energy (the work done by the external force can go into magnetic energy), attenuated, as in (2.26), by the fact that $S_1 > S_0 = \langle p_0^{1/\gamma} \rangle$ (continuous forcing will produce motions and magnetic fields that will eventually reach dissipative scales and be thermalised, producing entropy). If F is arbitrary, the above calculation does not establish any bound on the increase of the magnetic energy.

Considering again the low Mach number limit (2.21), we can, as we did at the end of § 2.1, neglect the term in (4.3) that contains S_1 and can be bounded from above by (2.20) because $S_1 > S_0$. For simplicity, let us assume an initially motionless state ($V_0 = 0$), which, for a forced system, does not restrict generality in any significant way. If external forcing injects energy into the system at the mean rate ε, then the available energy after time t is

$$A \approx W_1 - W_0 = \varepsilon t. \tag{4.4}$$

In a subsonically forced and, therefore, low Mach number system, the time that it takes the external forcing to inject an amount of energy comparable to the total energy is asymptotically long. If the kinetic energy of the plasma flows at time t is $K_1 \sim Ma^2 W_1$, we may define the 'dynamical' time as the typical time over which external forcing can build up motions with this energy, $\tau_{dyn} = K_1/\varepsilon$. The total energy can only change by an amount of order unity after the 'heating' time $t \sim W_1/\varepsilon \sim \tau_{dyn}/Ma^2 \gg \tau_{dyn}$. The interesting question in the context of dynamo action in such a forced system is how much energy is available for conversion into magnetic fields over times of order τ_{dyn}, which immediately gives us

$$A \sim \varepsilon \tau_{dyn} \sim K_1. \tag{4.5}$$

This is just a (perhaps overcomplicated) way of stating that in subsonic MHD turbulence, over dynamical times, a possible dynamo mechanism has at its disposal energies of order the kinetic energy of the motions, a result analogous to (2.22).

4.2. *CGL and other kinetic plasma models*

Let us assume that the external forcing does not directly break adiabatic invariance, i.e. that it occurs on sufficiently long scales in time and spaces. Nevertheless, the dissipation of the injected energy will in general occur on small scales and via processes (e.g. collisional viscosity and resistivity) that do not conserve I and J. Thus, just like in the case of the MHD model, we are left without exact conservation

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laws that would allow us to constrain dynamo action. Let us, however, again focus on the limit of low Mach numbers and consider the evolution of the system over dynamical (rather than heating) time scales. Over such times, both the total energy of the plasma and the CGL invariants can only change by small amounts: at most,

$$W_1 - W_0 \sim \varepsilon \tau_{dyn} \sim Ma^2 W_1, \tag{4.6}$$

$$I_1 - I_0 \sim \frac{\varepsilon \tau_{dyn}}{B_0} \sim Ma^2 \sqrt{\beta_0 W_0}, \tag{4.7}$$

and similarly for J. This implies that if we seek to maximise magnetic energy in a subsonically forced CGL model subject to some fixed values of W_1, I_1 and J_1, the argument presented in § 2.3 continues to be valid, with the coefficients a and c in (2.37) still of order unity because they only differ by $O(Ma^2)$ from the values they would have had if I and J had been precisely conserved. We conclude that the upper bound (2.39) survives, with K_0 replaced with K_1, the kinetic energy of the forced plasma flows.[††]

Obviously, the same line of argument can be used to extend to the forced case the arguments that we have proposed for kinetic MHD (§ 3.1) and other kinetic models that conserve the magnetic moment of one of the bulk particle species (§ 3.2).

4.3. *Numerical tests*

To illustrate the above considerations of the dependence of dynamo action on the equation of state, we contrast a series of numerical simulations of the conventional MHD equations with isotropic pressure against the anisotropic-pressure CGL equations, both in the subsonically forced regime. Equations (2.1)–(2.4) are solved numerically in three spatial dimensions using a MUSCL-type scheme with a van Leer flux limiter (Kurganov & Tadmor 2000). A divergence-free magnetic field is ensured by applying a flux-constrained approach (also known as a specific variation of staggered mesh or Yee grid method, Balsara & Spicer (1999)). Either the MHD energy equation (2.7) with isotropic pressure or the CGL equations (2.8)–(2.9) are used. No resistive or viscous terms are included explicitly, so dissipation is due only to a small amount of numerical diffusion introduced by the numerical scheme. Periodic boundary conditions are applied in all three spatial directions. The simulation grid is uniform, with resolution of 128^3 points (which is enough for the purposes of capturing dynamo action, (Meneguzzi *et al.* 1981; Haugen & Brandenburg 2004; Schekochihin *et al.* 2004)).

A stochastic flow in the simulation box is established by including a forcing term in the momentum equation (Alvelius 1999) that is uniform in wave–vector space and concentrated to the two smallest wave numbers. The forcing is random and white in time. Solenoidal ($\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0$) forcing is used because this provides the most efficient amplification of the initial magnetic field by the turbulent small-scale dynamo (Federrath *et al.* 2011). Note that, although the forcing is solenoidal, the equations that we solve allow for compressible dynamics and small ($\boldsymbol{\nabla} \cdot \boldsymbol{V} \sim$ Ma^2) compressive fluctuations are present in the system. The injected power is constant in time and relatively small to ensure that the generated velocity fluctuations are subsonic. A small

[††]It is not hard to see that the upper bound (2.38) still holds in the somewhat more general case of W changing by an order-unity (rather than small) amount if one can argue that the dynamics nevertheless preserves I and J. The upper bound (2.38) is only broken if the total energy increases by a large factor.

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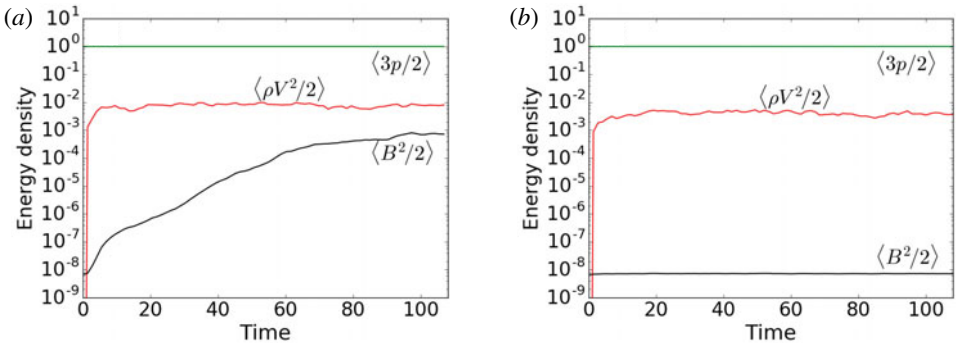


FIGURE 1. Typical time evolution of the total thermal (green), kinetic (red) and magnetic (black) energies in (*a*) forced MHD equations with isotropic pressure and (*b*) forced CGL equations. The energies are in the units of the initial thermal energy $3p_0/2$. The time is in the units of the sound-crossing time $L/\sqrt{\gamma p_0/\rho_0}$, where $L = 1$ is the size of the box. The largest-eddy turnover time, referred to as 'dynamical time' in §§ 4.1 and 4.2, is approximately $\tau_{dyn} \sim 10$ units. Note that the MHD dynamo operates on the time scale of order τ_{dyn}.

uniform mean seed magnetic field is set up in the initial condition,[‡‡] and different strengths are used in the simulations described below, whereby the dependence on initial β_0 is investigated. All fields (velocity, density, magnetic field, perpendicular and parallel pressures) are initially constant across the simulation box, so their fluctuations are driven solely by forcing.

Since we are interested in the consequences of accurate conservation of S in the case of isotropic pressure and I and J in the CGL case, we use either the isotropic-pressure relation (2.7) or the CGL equations (2.8)–(2.9) instead of the total energy conservation equation in the explicit form.[§§]. This implies that the part of the kinetic energy injected into the system that is dissipated numerically rather than being transformed into magnetic energy or into thermal energy via adiabatic mechanisms (compressional heating or, in CGL, pressure-anisotropy heating, i.e. parallel viscosity), is lost and thus constitutes an implicit energy sink in the total energy budget. As we argued in § 4.1, this is a small effect over a finite number of dynamical times. Various other schemes for solving our pressure equation(s) (enforcing total energy conservation or depositing the balance of numerically dissipated energy into $p_\parallel$ and/or $p_\perp$ according to physical assumptions about the nature of sub-grid heating) do not result in any change of the results on dynamo action or lack thereof reported below.

Figure 1 shows the evolution of the kinetic $\langle \rho V^2/2 \rangle$, magnetic $\langle B^2/2 \rangle$ and thermal $\langle 3p/2 \rangle$ energies in a simulation starting from $\beta_0 = 10^8$. In the MHD model, the magnetic energy grows exponentially until a saturated state is achieved, as it always does in dynamo simulations (Meneguzzi *et al.* 1981; Haugen & Brandenburg 2004; Schekochihin *et al.* 2004). In contrast, in the CGL model, the dynamo action is

[‡‡]In a periodic box, such a field cannot, of course, decay, and a certain amount of its amplification will be due just to the turbulent tangling of field lines, rather than to bona fide dynamo action (Schekochihin *et al.* 2007). However, if the system does support dynamo action, the latter will quickly take over and bring the field energy to within a finite factor of the kinetic energy of the motions. Starting with small random magnetic perturbations whose mean is zero did not change any of our results.

[§§]Since our initial state is homogeneous, equations (2.7), (2.8) or (2.9) can be solved simply by enforcing pointwise conservation of p/ρ^γ, $p_\perp/\rho B$ or $p_\parallel B^2/\rho^3$, respectively.

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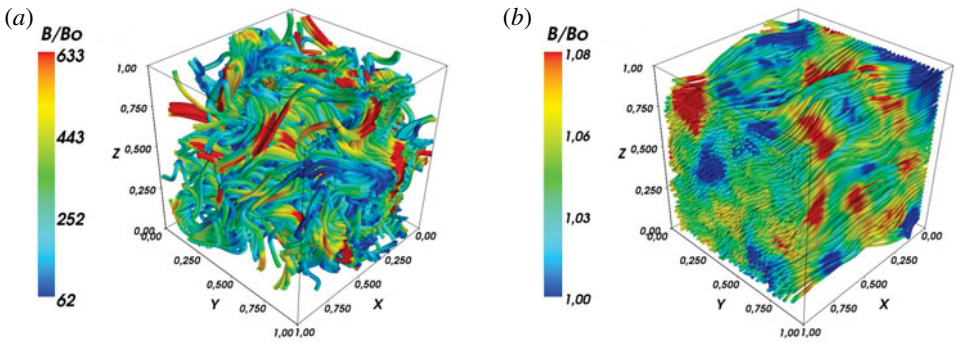


FIGURE 2. Snapshots from the two simulations shown in figure 1, taken at $t = 107$ (in the saturated state): (*a*) MHD, (*b*) CGL. Three-dimensional rendering of field lines is used to exhibit their spatial structure, while the colour shows the local value of the magnetic-field strength B (normalised to the initial value B_0). The initial mean seed field was in the x direction. The field lines that are displayed start at (*a*) $x = 0.5$, (*b*) $x = 0$.

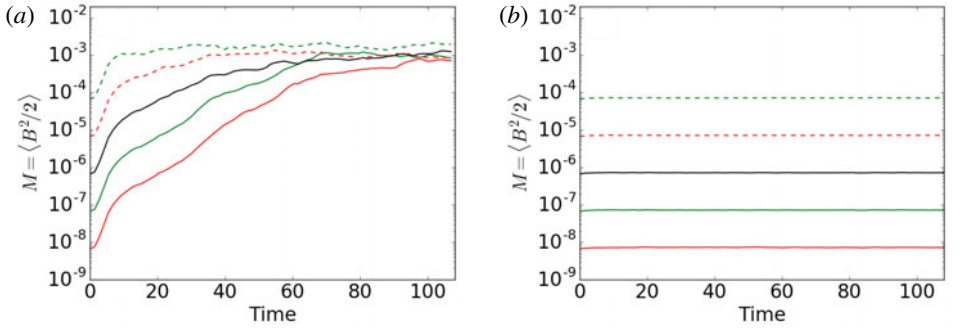


FIGURE 3. Time evolution of the magnetic energy for (*a*) the MHD model with isotropic pressure and (*b*) the CGL model, for five runs starting with different initial magnetic fields. The units are the same as in figure 1.

suppressed, and no significant increase of the magnetic energy density is obtained, confirming the result of Santos-Lima *et al.* (2011). A three-dimensional rendering of the magnetic field in the saturated state of these two simulations is given in figure 2. In the isotropic-pressure MHD plasma, plasma motions were able to amplify the magnetic field via the standard stretch-and-fold mechanism, giving rise to a characteristic pattern of intertwined and folded magnetic-field lines and to intermittent high- and low field regions. The contrast between the highest- and the lowest-B regions is of order of 10, and even the lowest magnitude regions have been amplified by a factor of $\sim$60 from the initial field. In the CGL case, the same initial condition and forcing resulted in very weak bending of the lines and very small amplification of the magnetic-field strength (a few per cent), the spatial alignment of the lines displays no signature of stretching or folding, just gentle wave-like perturbations (these are CGL slow waves, which propagate at the sound speed even at high β, Abraham-Shrauner 1967).

Figure 3 shows the evolution of the magnetic energy for the two models in simulations starting from the magnetic-field strength B_0 corresponding to the range

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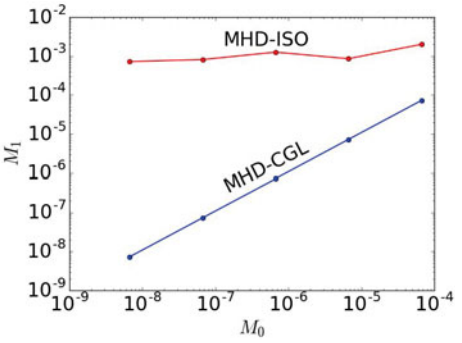


FIGURE 4. The saturated magnetic energy M_1 versus the initial one M_0 for the MHD (red) and CGL (blue) models.

$10^4 < \beta_0 < 10^8$. The amplitude of the forcing and the thermal pressure are the same for all these simulations, implying approximately the same Mach number of the velocity fluctuations in all of them. We see that the saturated magnetic energy is essentially independent of β_0 in the MHD model and entirely determined by it in the CGL model. This point is reinforced by figure 4, which shows the final saturated magnetic energy M_1 vs. the initial one M_0. Thus, the initial magnetic energy cannot be increased in a double-adiabatic, pressure-anisotropic plasma even in the presence of external forcing and the upper bound (2.38) is tight.

5. Conclusions

As we have seen, any mathematical plasma model that conserves the magnetic moment of at least one particle species is subject to an anti-dynamo theorem in the sense that there is a strict limitation on the growth of the total magnetic energy, as indeed anticipated by Kulsrud *et al.* (1997). If the initial magnetic field is small enough that $\beta_0 \gg 1$ (or $\beta_0 \gg Ma^{-2}$ for plasma flows in which the Mach number is small), the magnetic energy cannot grow by more than a factor of order unity as long as the external forcing does not significantly increase the total energy of the system.

These results have immediate implications for dynamo simulations of the collisionless CGL or kinetic MHD equations. If such a simulation should exhibit significant growth of the magnetic energy, it must be due to the accumulation of numerical errors that effectively break the adiabatic invariance of the plasma. Our own simulations of the CGL equations, as well as those of Santos-Lima *et al.* (2011), find that dynamo action is, indeed, practically absent at high β. Moreover, the modest growth of the magnetic energy that does occur is observed to stop at magnetic energy inversely proportional to β_0, as expected from the bound (2.38), suggesting that this scaling is the correct one and that the bound is tight within a factor of order unity. In simulations that instead use the conventional MHD equation of state (2.7), the magnetic energy grows to much larger values, independent of the initial β.

These results do not, however, imply the absence of dynamo action in a real plasma, even if collisions are rare and the gyroradii of all particle species are small. There are at least two reasons why such a conclusion cannot be drawn. First, whether or not external forcing is present, the flow of the plasma will usually be turbulent. The ensuing free-energy cascade will tend to create small-scale structures both in

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real space and in velocity space, with fluctuations arising on Larmor scales and collisions eventually becoming important (Schekochihin *et al.* 2008). The precise adiabatic invariance of the magnetic moment will thus be broken, which may unchain the dynamo.[¶¶] Secondly, the growth of pressure anisotropies caused by magnetic-field changes can lead to the excitation of kinetic mirror and firehose instabilities that have a similar effect (Kunz *et al.* 2014; Melville *et al.* 2016). Without a detailed understanding of these processes, it is not possible to rule out dynamo action. Indeed, it would be unwise to do so, given that the Universe is magnetised, the observed magnetic fields tend to be of dynamical strength (and so likely result from plasma motions), and first numerical evidence of collisionless plasma dynamo action has appeared (Rincon *et al.* 2016). What we can conclude with certainty is that plasma dynamo action must be a multiscale process, with breaking of adiabatic invariance at microscales playing an existentially important role.

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Appendix A. Invariants of the CGL equations

The double-adiabatic equations of state can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\alpha \ln\left(\frac{p_\perp}{\rho B}\right) + (1-\alpha)\ln\left(\frac{p_\parallel^{1/3}B^{2/3}}{\rho}\right)\right] = 0, \tag{A 1}$$

where α is arbitrary. Since $\mathrm{d}\ln\rho/\mathrm{d}t = -\boldsymbol{\nabla}\cdot\boldsymbol{V}$ from the continuity equation, we can thus write

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(p_\perp^{\alpha}p_\parallel^{(1-\alpha)/3}B^{(2-5\alpha)/3}\right) + \boldsymbol{\nabla}\cdot\boldsymbol{V} = 0, \tag{A 2}$$

and conclude that there is an infinity of conserved quantities that can be constructed from the three fields $p_\perp$, $p_\parallel$ and B, namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle p_\perp^{\alpha}p_\parallel^{(1-\alpha)/3}B^{(2-5\alpha)/3}\right\rangle = 0. \tag{A 3}$$

Three of these invariants depend on only two of the fields: $I = \langle p_\perp/B\rangle$ and $J = \langle p_\parallel^{1/3}B^{2/3}\rangle$ correspond to $\alpha = 1$ and $\alpha = 0$, respectively, and $\alpha = 2/5$ yields an invariant that is independent of B,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle p_\perp^{2/5}p_\parallel^{1/5}\right\rangle = 0. \tag{A 4}$$

[¶¶]In fact, small scales in $\boldsymbol{B}$ can also be created by large-scale flows, e.g. because of magnetic-field advection along chaotic streamlines (Zeldovich *et al.* 1984; Schekochihin *et al.* 2004; Rincon *et al.* 2016), again breaking the invariance of μ.

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