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Slowly mixing systems and intermittency maps

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Abstract. We consider families of one-dimensional maps on the circle which mix at sub-polynomial rates. Such maps will have an indifferent fixed point, and we show that the rate of mixing of these maps is determined by the precise degeneracy of the fixed point. By constructing suitable induced maps with a countable Markov partition and applying a probabilistic coupling argument we are able to give good estimates on the rate of mixing for these families of maps. The underlying question we pose is the following: given any specific rate of mixing can we construct a map which has that rate of mixing? For a large class of specific rates we explain how to construct the appropriate map.

1. *Introduction*

1.1. *Statement of results.*

Our basic set-up here is a one-dimensional map $\Phi : S^1 \circlearrowleft$ on the circle which is expanding everywhere except at a finite number of points. We work with a class $\mathcal{I}$ of maps with *indifferent* fixed points. Our geometric and dynamic assumptions for a given $\Phi \in \mathcal{I}$ are as follows.

ASSUMPTION 1. *Each* $\Phi : S^1 \circlearrowleft$ *has a unique parabolic fixed point at the origin so that* $\Phi(0) = 0$ *and* $D\Phi(0) = 1$.

ASSUMPTION 2. *On* $S^1 \setminus \{0\}$ *each* Φ *is expanding in a sense that* $D\Phi > 1$, *but* $\lim_{x\to 0} D\Phi(x) = 1$.

ASSUMPTION 3. *Each* $\Phi : S^1 \to S^1$ *is of degree* $d > 1$, *and there is a collection of disjoint intervals* $\{A_j\}_{j\in[1,d]}$ *with* $S^1 = \bigcup_{j=1}^{d} A_j$ *and* $\Phi(A_j) = S^1$.

ASSUMPTION 4. *On a small neighbourhood of* 0 *the map* Φ *takes the form*

$$\Phi(x) = x + x^2\phi(x) \quad \textit{with } \lim_{x\to 0} x\phi(x) = 0, \ \lim_{x\to 0} \phi(x) = \infty.$$

For some $M > 0$ *the function* $\phi(1/x)$ *is chosen to be* C^ω *(real-analytic) on* $[M, \infty)$.

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MAIN RESULT. *We construct maps inside $\mathcal{I}$ which display a rate of mixing that is strictly slower than any polynomial power of time. In particular, we construct maps inside $\mathcal{I}$ with a rate of mixing that is prescribed either to be 'slowly varying' or 'regularly varying'.*

The class of maps we have in mind are those that mix at speeds $1/\log^r n$, for $r \geq 1$ and time $n \to \infty$. Here $\log^r$ denotes the r-fold composition of the logarithm function. However, we also construct maps which mix at various other *slow* speeds, with a more precise meaning of *slow* as given below. These maps arise naturally in parametrized families of maps undergoing a collision of an attractor with a repeller, and are known to display *intermittent* behaviour [**16**]. Another point of interest is that these maps provide examples of dynamical systems admitting sub-exponential decay of correlations, and this is our focus. Recent attention has been made in relation to the special case of Assumption 4, where $\phi(x) = x^{-\alpha}$ for a fixed $\alpha \in (0, 1)$. The ergodic properties of these and related intermittency maps are discussed in [**1, 22, 23**], while mixing properties have been analysed in [**15, 25**], for example. It is shown that the associated maps mix at polynomial speeds. In our situation we show, under an appropriate choice of $\phi(x)$, how even slower rates of mixing can be obtained and, as a consequence of the techniques that are employed, we also construct maps with an intermediate polynomial (regularly varying) rate of mixing: a mixing rate of $1/n(\log n)$, for example. To the best of the author's knowledge, such maps have not been constructed before.

This paper is organized as follows. Below we give a formal description of what we mean by slow mixing, via Definition 1. In terms of the statistical properties of these maps, we analyse their *physical* measures and estimate the rates of decay of correlations. The paper is then devoted to the proof of three theorems. Theorem 1 tells us how to deduce the properties of the physical measure and how to estimate the rate of mixing when given an arbitrary $\Phi \in \mathcal{I}$, whose degeneracy function $\phi(x)$, as specified in Assumption 4, is slowly varying. Theorem 2 tells us the rate of mixing when given a selection of specific cases. We also need a result, namely Theorem 3, which tells us how to estimate the rate of mixing given that the recurrence of the system is known to be weak.

To clarify the meaning of arbitrarily slow mixing in our context, we introduce the concept of regular and slow variation.

Definition 1.

(i) We say that a measurable function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is regularly varying at ∞ with index $\gamma \in \mathbb{R}$ if for all $y > 0$ one has $\lim_{x\to\infty} \rho(xy)/\rho(x) = y^\gamma$. If $\gamma = 0$, then we say that ρ varies slowly at ∞. Similarly $\rho(x)$ varies regularly with index $\gamma \geq 0$ at 0 if $\rho(1/x)$ varies regularly with index $-\gamma$ at ∞. We let R_γ denote the class of functions which vary regularly (at ∞) with index γ.

(ii) We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ is regularly varying at infinity with index $\gamma \in \mathbb{R}$ if there is a function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$, which is regularly varying at infinity with index γ, and is such that $x_n = \rho(n)$ for each $n \in \mathbb{N}$ (and analogously for regular variation at zero).

It can be shown [**3**] that, subject to mild conditions (even as weak as measurability), regularly varying functions must take asymptotic form $x^\gamma \rho(x)$ with $\rho(x)$ slowly varying.

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Common examples of slowly varying functions are

$$\rho(x) = \exp\{(\log x)^{\alpha_1}(\log^2 x)^{\alpha_2}\dots(\log^r x)^{\alpha_r}\}, \quad \rho(x) = \exp\{(\log x)/(\log^r x)\},$$
$$\rho(x) = \exp\{(\log x)^{\alpha_1}\cos(\log x)^{\alpha_2}\}, \quad \alpha_i \in (0,1),\ r \geq 2.$$

In the latter example $\rho(x)$ displays infinite oscillation with $\liminf_{x\to\infty}\rho(x) = 0$ and $\limsup_{x\to\infty}\rho(x) = \infty$. For more examples see [**3**]. We are interested in slowly varying functions which go to zero. For example, we consider functions of the form $\rho(x) = (\log^r x)^{-1}$, where the rate at which $\rho(x) \to 0$ can be made as slow as desired by increasing values of $r \in \mathbb{N}$. We remark, however, that it is still possible to construct sequences $x_n \to 0$ such that for all $r \geq 1$ we have $x_n(\log^r n) \to \infty$ as $n \to \infty$. Our main result concerns the statistical properties of such maps. We examine two dynamical phenomena. The first is the existence of a physical measure and the second is the rate of mixing.

Definition 2. We say that a Φ-invariant measure ν_Φ is physical if for any continuous $\varphi : S^1 \to \mathbb{R}$, the following holds: there is a positive Lebesgue measure subset $U \subset S^1$ such that for all $x \in U$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\varphi\circ\Phi^k(x) = \int\varphi\,d\nu_\Phi. \tag{1}$$

The set of points for which (1) holds are called ν_Φ-generic points and are said to lie in the basin $B(\nu_\Phi)$ of ν_Φ. In particular, we are interested in absolutely continuous physical measures (with respect to Lebesgue measure m). In terms of the mixing properties we analyse the correlation functions

$$\mathcal{C}(\varphi\circ\Phi^n, \psi, \nu_\Phi) \equiv \left|\int(\varphi\circ\Phi^n)\psi\,d\nu_\Phi - \int\varphi\,d\nu_\Phi\int\psi\,d\nu_\Phi\right|, \tag{2}$$

as $n \to \infty$. This correlation function is sometimes referred to as the *covariance*, for example in statistics. We abbreviate the correlation function to $\mathcal{C}_n(\nu_\Phi)$ when the class of observables are understood. It is usual to choose $\varphi, \psi \in \mathcal{H}$ where $\mathcal{H}$ is the space of Hölder continuous functions, i.e. for all $\psi \in \mathcal{H}$, there exist $0 < \gamma < 1$ and $K > 0$ such that $d(\psi(x), \psi(y)) < Kd(x,y)^\gamma$, for all $x, y \in S^1$. In the definition of the correlation function we take ν_Φ to be the Φ-invariant measure, and when (Φ, ν_Φ) is mixing we are interested in the rate at which $\mathcal{C}_n(\nu_\Phi)$ goes to zero. Closely related to the correlation function is a quantity called *the speed of convergence to equilibrium*. To define this let us introduce the Perron–Frobenius operator $P_\Phi : L^1 \to L^1$ associated to Φ which is given by (for all $\psi \in L^\infty, \varphi \in L^1$):

$$\int(\psi\circ\Phi)\varphi\,dm = \int\psi\cdot P_\Phi(\varphi)\,dm, \tag{3}$$

and, hence, for all $\varphi \in L^1$,

$$P_\Phi(\varphi)(x) = \sum_{y:\Phi y = x}\frac{\varphi(y)}{J\Phi(y)}. \tag{4}$$

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Then the quantity

$$\mathfrak{V}_{n,\Phi}(\varphi,\psi) \equiv \|P_\Phi^n(\varphi) - P_\Phi^n(\psi)\|_1 = \int_{S^1} |P_\Phi^n(\varphi)(x) - P_\Phi^n(\psi)(x)|\, dm(x) \tag{5}$$

is called the speed of convergence to equilibrium, where we assume that $\varphi, \psi \in \mathcal{H}$. This quantity measures the speed at which a given probability measure converges to the physical invariant measure under iteration of the Perron–Frobenius operator.

Let us introduce some conventions. For two functions $f(x)$, $g(x)$ we say that $f = \mathcal{O}(g)$ if there exists $K > 0$ such that $|f(x)/g(x)| \leq K$ (as $x \to \infty$), and we say that $f = o(g)$ if $f(x)/g(x) \to 0 (x \to \infty)$. Moreover, we say that $f \sim g$ if $f(x)/g(x) \to 1 (x \to \infty)$, and $f \approx g$ if there exists $K > 0$ such that $K^{-1} \leq f(x)/g(x) \leq K$, $(x \to \infty)$. These conventions also apply if we replace functions by sequences.

de Bruijn conjugates. For a slowly varying function $\rho \in R_0$ which is monotonic on an interval $[M, \infty)$ it can be shown that there exists a $\rho^\sharp \in R_0$ which is unique up to asymptotic equivalence such that

$$\rho(x)\rho^\sharp(x\rho(x)) \to 1, \quad \rho^\sharp(x)\rho(x\rho^\sharp(x)) \to 1 \quad (x \to \infty)$$

and $\rho^{\sharp\sharp} = \rho$. Moreover if $f(x) = x\rho(x)$ then $f^{-1}(x) \sim x\rho^\sharp(x)$ (as $x \to \infty$). We call $(\rho(x), \rho^\sharp(x))$ a *de Bruijn conjugate pair*. Given $\rho(x)$ it is usually of interest to calculate $\rho^\sharp(x)$ and we do this in many particular cases.

THEOREM 1. (Slow convergence to equilibrium) *Let $\Phi \in \mathcal{I}$ and suppose that $\phi(x)$, as defined in Assumption 4, is slowly varying at 0. Let $\bar{\phi} = \phi(1/x)^{-1}$ with $\bar{\phi}^\sharp$ denoting the de Bruijn conjugate of $\bar{\phi}(x)$. Let $n_0 \in \mathbb{N}$ be fixed and suppose that $\sum_{k\geq n_0}(1/k)\bar{\phi}^\sharp(1/k)^{-1} < \infty$. Then (Φ, S^1) has an invariant probability measure ν_Φ which is absolutely continuous and physical. Let $\varphi_1, \varphi_2 \in \mathcal{H}$ and let ρ_ϕ denote the invariant density. Then*

$$\mathcal{C}(\varphi_1 \circ \Phi^n, \varphi_2, \nu_\Phi) = \mathcal{O}\left(\sum_{k\geq n} \frac{1}{k}\bar{\phi}^\sharp\left(\frac{1}{k}\right)^{-1}\right) \quad \text{and} \quad \mathfrak{V}_{n,\Phi}(\varphi_1,\varphi_2) \approx \sum_{k\geq n} \frac{1}{k}\bar{\phi}^\sharp\left(\frac{1}{k}\right)^{-1},$$

and these quantities are slowly varying in n as $n \to \infty$. If $\sum_{k\geq n_0}(1/k)\bar{\phi}^\sharp(1/k)^{-1} = \infty$ then (ϕ, S^1) has a unique physical measure which is the Dirac measure supported on the indifferent fixed point.

To prove Theorem 1 we associate a Markov tower (a tower in the sense of Young [**25**]) to each $\Phi \in \mathcal{I}$ which in turn allows us to deduce the rate of mixing. This requires a local analysis of the behaviour of Φ in the neighbourhood of $x = 0$. The construction of the measure ν_Φ is also needed and we prove that this measure is absolutely continuous when $\sum_{k\geq n_0}(1/k)\bar{\phi}^\sharp(1/k)^{-1} < \infty$. Again, a detailed analysis of the behaviour of Φ near $x = 0$ is required together with a bounded distortion estimate for the derivative along typical orbits in the vicinity of the indifferent fixed point, see §2.2. Due to the general nature of $\phi(x)$ a more detailed analysis is required in contrast to the classical situation in which $\phi(x) = x^{-\alpha}$. A formal treatment is presented, which involves knowledge of the behaviour of the class of slowly varying functions. When $\sum_{k\geq n_0}(1/k)\bar{\phi}^\sharp(1/k)^{-1} = \infty$, we prove

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that the physical measure is the Dirac measure supported on the indifferent fixed point in our non-standard setting, §3. For maps which are C^2 in the vicinity of the indifferent fixed point $x = 0$, for example $\phi(x) = 1$, this result is well known, see [**1, 23, 24**].

After constructing the Markov tower, we are not done however. We need to know how the recurrence time statistics can be used to deduce results on the decay of correlations. In contrast to the Markov tower model analysed in [**25**], we assume very slow (e.g. logarithmic) recurrence assumptions. Theorem 3 in §1.2 gives us the expected result. The method of proof is via probabilistic coupling, as described in [**25**] and we use an enhanced version of this method in order to account for the weak recurrence assumptions. The proof of Theorem 3 is given in §4.

THEOREM 2. (Special cases) *For $\Phi \in \mathcal{I}$, with degeneracy function $\phi(x)$ defined as in Assumption 4, the following special cases hold.*

(i) Logarithmic decay. *Suppose that*

$$\phi\left(\frac{1}{x}\right) = (\log^r(x))^{1+\alpha} \prod_{j=1}^{r-1} \log^j(x).$$

Then, for $\alpha \in (0, \infty)$ and $\varphi_1, \varphi_2 \in \mathcal{H}$,

$$\mathcal{C}(\varphi_1 \circ \Phi^n, \varphi_2, \nu_\Phi) = \mathcal{O}\left(\frac{1}{(\log^r n)^\alpha}\right).$$

(ii) Intermediate logarithmic decay. *Let*

$$\phi\left(\frac{1}{x}\right) = e^{(\log^r x)^\alpha} (\log^r x)^{1-\alpha} \prod_{j=1}^{r-1} \log^j(x)$$

with $\alpha \in (0, 1)$ and $r > 1$. Then, for $\varphi_1, \varphi_2 \in \mathcal{H}$,

$$\mathcal{C}(\varphi_1 \circ \Phi^n, \varphi_2, \nu_\Phi) = \mathcal{O}(e^{-(\log^r n)^\alpha}).$$

(iii) Regularly varying decay. *Let $\phi(x) = x^{\alpha-1}\rho(x)$ where $\alpha \in (0, 1)$ and $\rho(x)$ is slowly varying at 0. Then, for $\varphi_1, \varphi_2 \in \mathcal{H}$,*

$$\mathcal{C}(\varphi_1 \circ \Phi^n, \varphi_2, \nu_\Phi) = \mathcal{O}(n^{1-1/\alpha}((\bar{\rho}^{1/\alpha})^\sharp(n^{1/\alpha}))^{-1}).$$

In each case the invariant measure ν_Φ is absolutely continuous and physical.

Remark. Theorem 2 is a selection of special cases. To prove Theorem 2 we must estimate the asymptotic nature of $\bar{\phi}^\sharp$ when given ϕ. This is not always a straightforward procedure, and the proof of Theorem 2 brings together various interesting techniques in the subject of *asymptotic inversion*.

1.2. *The Markov tower.* Let us now describe the Markov tower. Such a system has proved fruitful in studying rates of mixing for non-uniformly hyperbolic systems, notably in [**25, 26**]. Adopting a similar framework here we describe the formal set-up.

Suppose that a reference set $\mathcal{T}_0$ is fixed (which we consider as a subset of a measure space) and define a dynamical system $(F, \mathcal{T})$ where F is piecewise differentiable (but not

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necessarily invertible) and $\mathcal{T}$ is a *tower* built over $\mathcal{T}_0$. To define $\mathcal{T}$ precisely we suppose that there exists a countable partition $\mathcal{P}^* = \{\omega_{0,i}\}_{i\in\mathbb{N}}$ on $\mathcal{T}_0$ and a return function $R : \mathcal{T}_0 \to \mathbb{N}$ which is constant on each element of $\mathcal{P}^*$. R is defined in such a way that F^R maps each $\omega_{0,i}$ bijectively onto $\mathcal{T}_0$. $\mathcal{T}$ is defined to be the set $\{(x,\ \ell) \in \mathcal{T}_0 \times \mathbb{N} : \ell < R(x)\}$ and the map F acts as follows:

$$F(x,\ell) = \begin{cases} (x, \ell+1) & \text{if } \ell + 1 < R(x), \\ (F^R x, 0) & \text{if } \ell + 1 = R(x). \end{cases} \tag{6}$$

By definition

$$R(x) \equiv \min\{n \geq 1 : F^n x \in \mathcal{T}_0, x \in \mathcal{T}_0\},$$

and analogous to R there is a return function $\hat{R} : \mathcal{T} \to \mathbb{N}$ which is given by

$$\hat{R}(z) \equiv \min\{n \geq 0 : F^n z \in \mathcal{T}_0, z \in \mathcal{T}\}.$$

The levels $\mathcal{T}_\ell$ can also be partitioned into pieces $\{\omega_{\ell,i}\}$ with F^ℓ mapping $\omega_{0,i}$ bijectively onto $\omega_{\ell,i}$ when $\ell + 1 < R(\omega_{0,i})$. The function $\hat{R}$ is then constant on each $\omega_{\ell,i}$. We now extend $\mathcal{P}^*$ on $\mathcal{T}_0$ so that it contains all partition elements $\{\omega_{\ell,i}\}$ in $\mathcal{T}$. Denote the extended partition by $\mathcal{P}$. For each $n \geq 1$ the refined partitions are given by $\mathcal{P}_n = \bigvee_{i=1}^n F^{(-i+1)}\mathcal{P}$ where elements of $\mathcal{P}_n$ get mapped bijectively onto a union of elements of $\mathcal{P}$ at time n. To turn $\mathcal{T}$ into a measure space we introduce a reference measure m and a σ-algebra $\mathcal{B}$ generated by subsets of $\mathcal{T}$. If $\int_{\mathcal{T}_0} R\, dm < \infty$ then m is a finite measure on $\mathcal{T}$. To get some notion of distance between two points $x_1,\ x_2 \in \mathcal{T}_0$ we define their separation time $s(x_1, x_2)$ to be

$$s(x_1, x_2) = \inf\{n > 0 : (F^R)^n(x_1), (F^R)^n(x_2) \text{ lie in distinct } \omega_{0,i}\}.$$

If $s(x_1, x_2)$ is large then x_1 and x_2 are considered to be fairly close. For $x_1, x_2 \in \omega_{\ell,i}$ the separation time of x_1, x_2 is defined to be equal to the separation time of $F^{-\ell}x_1$, $F^{-\ell}x_2 \in \omega_{0,i}$. Note that separation can only occur when points have returned back to $\mathcal{T}_0$.

Dynamical properties of $(F, \mathcal{T})$. We now list three dynamical assumptions on $(F, \mathcal{T})$.

ASSUMPTION 5. (Bounded distortion) *There are constants $\beta \in (0, 1)$ and $K > 0$ such that for all $x_1, x_2 \in \mathcal{T}_0$*

$$\left|\frac{JF^R(x_1)}{JF^R(x_2)} - 1\right| \leq K\beta^{s(F^R(x_1), F^R(x_2))}.$$

Here JF^R denotes the Jacobian of F^R with respect to m (which we assume to be non-singular) and $JF \equiv 1$ on $\mathcal{T} - F^{-1}\mathcal{T}_0$. In addition, it is assumed that $JF > 0$, m-almost everywhere.

ASSUMPTION 6. (Topological mixing) *Let $R_i = R(\omega_{0,i}) \in \mathbb{N}$. Then $gcd\{R_i\} = 1$.*

ASSUMPTION 7. (Finite reference measure) *We have $\int_{\mathcal{T}_0} R\, dm < \infty$.*

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In our situation the asymptotics of $m\{\hat{R} > n\}$ are prescribed to be sub-polynomial or slowly varying. This means that for all $\alpha > 0$, $m\{\hat{R} > n\}$ decays to 0 at a rate slower than $n^{-\alpha}$ as $n \to \infty$. Turning to the statistical properties of $(F, \mathcal{T})$, it will be useful to define function spaces $\mathcal{H}_\beta$ and $\mathcal{H}_\beta^+$, where

$$\mathcal{H}_\beta = \{\varphi : \mathcal{T} \to \mathbb{C} : |\varphi(x) - \varphi(y)| \le \beta^{s(x,y)} \ \forall x, y \in \mathcal{T}\},$$

$$\mathcal{H}_\beta^+ = \Bigg\{\varphi \in \mathcal{H}_\beta : \text{ on each } \omega_{\ell,i} \text{ either } \varphi \equiv 0 \text{ or } \varphi > 0,$$
$$\text{and } \exists K_\varphi > 0 \text{ such that } \left|\frac{\varphi(x)}{\varphi(y)} - 1\right| \le K_\varphi \beta^{s(x,y)} \ \forall x, y \in \mathcal{T}\Bigg\}.$$

Observables on $\mathcal{T}$ and densities of probability measures will be chosen to lie in $\mathcal{H}_\beta$ and $\mathcal{H}_\beta^+$, respectively.

Now for a given measure ν let $|\nu|$ denote the L^1 norm of its density $d\nu/dm$, and for all $A \in \mathcal{B}$, let $F_*\nu(A)$ denote $\nu(F^{-1}A)$. For probability measures ν_1, ν_2 on $\mathcal{T}$ we let $\mathcal{V}_{n,F}(\nu_1, \nu_2)$ denote the quantity $|F_*^n\nu_1 - F_*^n\nu_2|$ which, by definition, is equal to the quantity $\mathfrak{V}_{n,F}(d\nu_1/dm, d\nu_2/dm)$.

It will emerge that most of the asymptotic properties of $\mathcal{V}_{n,F}$ are passed on to the correlation function $\mathcal{C}_n(\nu)$. Let us first recall some useful facts concerning the existence and properties of probability measures for F. We assume that Assumptions 5–7 hold.

FACT 1. [**25**] *$F : \mathcal{T} \mapsto \mathcal{T}$ admits a finite probability measure ν which is absolutely continuous with respect to m. Its density with respect m is strictly positive and lies in $\mathcal{H}_\beta^+$. (F, ν) is exact provided* $\gcd\{R_i\} = 1$.

FACT 2. [**25**] *There are many probability measures $\tilde{\nu}$ on $\mathcal{T}$ with $d\tilde{\nu}/dm \in \mathcal{H}_\beta^+$ such that the speed of convergence to equilibrium is bounded from below, i.e. $\mathcal{V}_{n,F}(\tilde{\nu}, \nu) \ge Km\{\hat{R} > n\}$ for some constant $K = K(\nu) > 0$.*

Our next theorem concerning the rate of decay of correlations for $(F, \mathcal{T})$ is as follows.

THEOREM 3. *Assume that Assumptions 5–7 hold for $(F, \mathcal{T})$. Take an arbitrary measure $\tilde{\nu}$ with density inside $\mathcal{H}_\beta^+$, and consider observables $\varphi_1, \varphi_2 \in \mathcal{H}_\beta$. Suppose that $m\{\hat{R} > n\} = \rho(n)$ with $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ a slowly varying function at ∞ with $\rho \in C^\omega$ and monotonically decreasing to zero on $[x_0, \infty)$ for some $x_0 > 0$. Then*

$$\mathcal{V}_{n,F}(\tilde{\nu}, \nu) = \mathfrak{V}_{n,F}(d\tilde{\nu}/dm, d\nu/dm) \approx \rho(n) \quad \textit{and} \quad \mathcal{C}(\varphi_1 \circ F^n, \varphi_2, \nu) = \mathcal{O}(\rho(n)).$$

Facts 1 and 2 we proved in [**25**] and estimates on the rate of mixing when the tail $m\{\hat{R} > n\}$ behaves as $\mathcal{O}(n^{-\alpha})$ or $\mathcal{O}(\theta^n)$ with $\alpha > 0$ and $\theta \in (0, 1)$ are given. The correlation functions behave as $\mathcal{O}(n^{-\alpha})$ and $\mathcal{O}(\tilde{\theta}^n)$, respectively (for some $\tilde{\theta} \in [\theta, 1)$). If $m\{\hat{R} > n\} = \mathcal{O}(n^{-\alpha-1})$ with $\alpha > 0$ then a central limit theorem property also holds. The proof of Theorem 3 uses a probabilistic coupling argument similar to that in [**25**], but we have to make further modifications in order to deal with the slow return time asymptotics. In particular, we have to rely on given properties of slowly varying functions in order to estimate the rate of mixing.

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As a corollary to Theorem 3, good bounds on the correlation functions can also be obtained in the regularly varying case, i.e. when $m\{\hat{R} > n\} \sim n^{-\alpha}\rho(n)$ with $\alpha > 0$ and $\rho(x)$ a slowly varying function. The following result holds.

COROLLARY 1. *Let $F, \tilde{\nu}, \varphi_1, \varphi_2$ be defined as in Theorem 3. Suppose that $m\{\hat{R} > n\} \sim n^{-\alpha}\rho(n)$ with $\alpha > 0$ and $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ a slowly varying function at ∞ with $\rho \in C^\omega$ on $[x_0, \infty)$ for some $x_0 > 0$. Then*

$$\mathcal{V}_{n,F}(\tilde{\nu}, \nu) \approx n^{-\alpha}\rho(n) \quad \textit{and} \quad \mathcal{C}(\varphi_1 \circ F^n, \varphi_2, \nu) = \mathcal{O}(n^{-\alpha}\rho(n)).$$

1.3. *General background, related results and applications.* Classically, maps with indifferent fixed points were studied by Manneville and Pomeau [**16**] in the 1980s. They worked with a generic unfolding of the saddle-node bifurcation, i.e. maps of the form $f_\epsilon(x) = \epsilon + x + x^{1+\alpha}$ with $x \simeq 0, \alpha > 0$ and small ϵ. Typical orbits under f_ϵ display intermittent behaviour. That is, there are long periods of time, going to infinity as $\epsilon \to 0$, where orbits reside in any given neighbourhood of the origin. For a discussion on intermittency see also [**4, 17, 20**]. If $\Phi \in \mathcal{I}$ is embedded into a parametrized family $\Phi_\epsilon(x) = \epsilon + x + x^2\phi(x)$ then similar phenomena are observed, but the bifurcation, corresponding to the parameter $\epsilon = 0$, is of a more degenerate nature. We summarize the known results below for a specific family of maps $\Phi_\alpha : S^1 \to S^1$ given by

$$\Phi_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } 0 \le x \le \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} < x \le 1, \end{cases} \tag{7}$$

where $\alpha > 0$. Assumptions 1–4 are readily verified.

FACT 3. *Let $\Phi_\alpha : S^1 \to S^1$ be the map given by (7). If $\varphi \in L^\infty(S^1, m)$ and $\psi \in \mathcal{H}$, then:*

(a) *for $\alpha = 0$, Φ_α has an invariant measure $\nu \ll m$ which is mixing and $\mathcal{C}(\varphi \circ \Phi_\alpha^n, \psi, \nu) \le K\theta^n$ for some $\theta < 1, K > 0$;*

(b) *for $\alpha \in (0, 1)$, Φ_α has an invariant measure $\nu \ll m$ which is mixing and $\mathcal{C}(\varphi \circ \Phi_\alpha^n, \psi, \nu) = \mathcal{O}(n^{1-1/\alpha})$;*

(c) *for $\alpha \ge 1$, f has no finite invariant measure $\nu \ll m$—the only physical measure is the Dirac measure supported on the indifferent fixed point.*

There have been many contributions to the proof of Fact 3 and many related results have been established. A proof of Fact 3 that is close to the methodology presented here is given in [**25**] and the non-Markov case is also treated there. Using different techniques, a bound on the correlations of the form $\mathcal{O}(n^{\gamma(1-1/\alpha)})$ (up to a logarithmic correction), where γ is the Hölder exponent of ψ, was obtained in [**15**]. A further proof together with a lower bound was achieved in [**12**]. Higher-dimensional analogues were considered in [**13, 18**].

In the case of a singular physical measure, a σ-finite (but infinite) invariant measure may exist. We mention some references which analyse this situation. For Markov maps with indifferent fixed points see [**1, 22, 23**] and for the non-Markov case see [**27, 28**]. In a higher dimensional context there are some interesting articles by [**11, 13, 14**].

Results of the form of the central limit theorem have been established for f above with $\alpha \in (0, \frac{1}{2})$, see [**25**]. For a subset of parameters $\alpha \in (0, \frac{1}{2})$ an *almost sure invariance*

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principle holds, see [**19**]. For $\alpha \geq 1$, other distributional results can hold; see, for example, [**1, 2**]. With reference to Fact 3(b), in [**21**] (and later [**9**]) complete asymptotics for the correlation functions were established. It was recently announced in [**10**] that optimal bounds also hold in the regular and slowly varying situations.

As an example, a realization of a family of maps covered by Theorem 2 is the *logarithmic* family $\Psi_\beta : S^1 \to S^1$, with $\beta \in (0, 2\log 2 - 1)$. These maps are defined as follows:

$$\Psi_\beta(x) = \begin{cases} x + C_\beta x^2 |\log x|^{1+\beta} & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \tag{8}$$

where C_β is a constant that depends only on β, and is chosen to ensure that $\Psi_\beta(1/2^-) = 1$. Assumptions 1– 4 are readily verified for each map Ψ_β. For both families of maps, namely those given by (7) and (8), a numerical investigation into both of the properties of the invariant density and the correlation function is presented in [**5**]. The potential of these maps to model long series of financial returns on market indexes and currency exchange rates, where it is known that absolute values of financial returns exhibit intermittency, slowly decaying correlations and heavy tailed marginal distributions are also presented in [**5**]. Here, each marginal distribution corresponds to a distribution of the absolute returns $R_T(t) = |X(t + T) - X(t)|$, where $X(t)$ relates to the value of an index at any given (discrete or continuous) time t, and T is a fixed time period. See also [**8**] for a discussion on financial returns.

2. *Slowly mixing maps*

We now work with maps $\Phi \in \mathcal{I}$ which satisfy Assumptions 1–4. Our aim now is to prove Theorems 1 and 2. Adopting the approach of [**25**], we construct a Markov tower for these maps and estimate the corresponding rates of mixing. The moral is to pick a subinterval $J \subset S^1$ far from the indifferent fixed point and find a partition $\mathcal{P}_J = \{\omega_i\}_{i\in\mathbb{N}}$ of J together with a suitable return function $R : \omega_i \to \mathbb{N}$ with $\phi^{R_i}\omega_i = J$ (bijectively and with bounded distortion). Then $m\{x \in J : R(x) > n\}$ is estimated. To deduce the corresponding results on decay of correlations and speeds of convergence to equilibrium we use Theorem 3, which is proved later in §4.

2.1. *Behaviour of Φ near the indifferent fixed point.*

We are concerned with the behaviour of $\Phi \in \mathcal{I}$ on the interval A_1, where for definiteness we write $A_j = [a_{j-1}, a_j]$ with $a_0 = 0$ and $a_d = 1$. Since we work on S^1, we identify the point $x = a_d$ with $x = a_0$. On A_1, let $x_n = \Phi(x_{n+1})$ so that $\{x_n\}$ is a sequence of points converging to zero as $n \to \infty$. The sequence $\{x_n\}$ will play an important role in determining the rate of mixing of Φ. We show that $m\{[0, x_n]\}$ and $\sum_{k>n} m\{[0, x_k]\}$ play the same role as $m\{R > n\}$ and $m\{\hat{R} > n\}$, respectively. The following proposition gives us a way to determine the asymptotics of x_n. The proof of the result follows the ideas of [**1**, chapter 4] and [**6**].

PROPOSITION 1. *Let $f : \mathbb{R}^+ \to (0, 1)$ be regularly varying at 0 with index $\gamma > 0$. Assume also that $x^\gamma f(1/x) \in C^\omega$ on $[C_0, \infty)$ for some $C_0 > 0$. Let $x_0 = 1$ and let*

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$x_n = g(x_{n+1})$ *where* $g(x) = x(1 + f(x))$. *Then*

$$x_n \sim \frac{1}{\gamma^{1/\gamma} h^{-1}(n)} \quad \textit{where } h(x) = \frac{1}{f(1/x)}.$$

Moreover, when $f(x) = x\phi(x)$ *with* $\phi(x)$ *slowly varying, we have that*

$$x_n \sim \frac{1}{n\bar{\phi}^{\sharp}(n)} \quad \textit{where } \bar{\phi}(x) = \frac{1}{\phi(1/x)}$$

and $(\bar{\phi}(x), \bar{\phi}^{\sharp}(x))$ *form a de Bruijn conjugate pair.*

Proof. By the definition of regular variation we may write $f(x) = x^{\gamma}\eta(x)$ with $\eta(x)$ a slowly varying function at 0. We then use the following useful lemma regarding the representation of slowly varying functions.

LEMMA 1. (Representation of slowly varying functions) [**3**] *If* $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ *is slowly varying at* ∞ *then there exists* $C_0 > 0$ *and measurable functions* $\zeta, \varepsilon : [C_0, \infty) \to \mathbb{R}$ *with* $\zeta > 0, \zeta(x) \to c \in \mathbb{R}^+$, $\varepsilon(x) \to 0$ *as* $x \to \infty$ *and*

$$\rho(x) = \zeta(x) \exp\left\{ \int_{C_0}^{x} \frac{\varepsilon(t)}{t}\, dt \right\} \quad \textit{for all } x \geq C_0.$$

Hence, we may write

$$\eta(x) = \zeta(x) \exp\left\{ \int_{C_0}^{1/x} \frac{\varepsilon(t)}{t}\, dt \right\}$$

with $\zeta, \varepsilon : [0, \infty) \to \mathbb{R}^+$ defined in Lemma 1. Using the definitions of h and $\{x_n\}$ in the proposition we have

$$\begin{aligned} h(x_n^{-1}) = \frac{1}{f(x_n)} = \frac{1}{x_n^{\gamma}\eta(x_n)} &= \frac{1}{x_{n+1}^{\gamma}(1 + f(x_{n+1}))^{\gamma}\eta(x_n)} \\ &= \frac{\eta(x_{n+1})}{\eta(x_n)}(1 + f(x_{n+1}))^{-\gamma} h(x_{n+1}^{-1}) \\ &= \frac{\eta(x_{n+1})}{\eta(x_n)}(1 - \gamma f(x_{n+1}) + o(f(x_{n+1}))). \end{aligned}$$

By the properties of $\eta(x)$ we have

$$\log\left|\frac{\eta(x_{n+1})}{\eta(x_n)}\right| = \log\left|\frac{\zeta(x_{n+1})}{\zeta(x_n)}\right| + \int_{x_n^{-1}}^{x_{n+1}^{-1}} \frac{\varepsilon(t)}{t}\, dt = o\left(\log\left|\frac{x_{n+1}}{x_n}\right|\right).$$

To justify the asymptotic behaviour of $\log|\zeta(x_{n+1})/\zeta(x_n)|$ we use a Taylor series argument as follows. Write $\zeta(x) = c + \zeta^*(x)$ where $\lim_{x\to 0}\zeta^*(x) = 0$ and without loss of generality we can take $c = 1$. Then

$$\begin{aligned} &\left|\frac{\log|(1 + \zeta^*(x_{n+1}))/(1 + \zeta^*(x_n))|}{\log|x_{n+1}/x_n|}\right| \\ &\quad = \frac{\sum_{r\geq 1}\left(\zeta^*(x_n) - \zeta^*(x_{n+1})\right)\left\{\sum_{k=0}^{r-1}\zeta^*(x_n)^k\zeta^*(x_{n+1})^{r-1-k}\right\}}{\sum_{r\geq 1}((-1)^{r+1}/r) f(x_{n+1})^r}. \end{aligned} \tag{9}$$

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Since $\zeta^*(x) \in C^\omega$ we can write $\zeta^*(x) = \sum_{\ell \geq 1} a_\ell x^\ell$ where this power series has a non-zero radius of convergence. Using the regular variation properties of $f(x)$ we also deduce that

$$\zeta^*(x_n) - \zeta^*(x_{n+1}) = \sum_{\ell \geq 1} a_\ell x_{n+1}^\ell ((1 + f(x_{n+1}))^\ell - 1) = \mathcal{O}(f(x_{n+1}))\zeta^*(x_{n+1})$$

and, in particular, this implies that the ratio above in (9) is of the order $\zeta^*(x_{n+1})$ which goes to zero as $n \to \infty$.

Continuing with the proof, we then obtain $\eta(x_n) = \big(1 - o(f(x_n))\big)\eta(x_{n+1})$ and, therefore,

$$\begin{aligned} h\left(\frac{1}{x_n}\right) &= h\left(\frac{1}{x_{n+1}}\right)(1 - \gamma f(x_{n+1}) + o(f(x_{n+1})))(1 - o(f(x_{n+1}))) \\ &= h\left(\frac{1}{x_{n+1}}\right) - \gamma + o(1). \end{aligned}$$

Proceeding inductively we obtain $h(x_n^{-1}) = \gamma n$ and, therefore,

$$x_n \sim \frac{1}{h^{-1}(\gamma n)} \sim \frac{1}{\gamma^{1/\gamma} h^{-1}(n)}.$$

When $\gamma = 1$ and $f(x) = x\phi(x)$ it is fairly easy to see that $h^{-1}(x) \sim x\bar{\phi}^\sharp(x)$ (as $x \to \infty$) and hence the conclusion of the proposition follows. □

Remark 1. Proposition 1 above gives the leading asymptotic behaviour of x_n. For the specific choice $\phi(x) = x^\alpha$, it is possible to obtain higher order asymptotics for the sequence x_n, namely we can show that

$$x_n = \frac{1}{(\alpha n)^{1/\alpha}} + \mathcal{O}\left(\frac{\log n}{n^{1+1/\alpha}}\right). \tag{10}$$

2.2. *Distortion estimates and density estimates.* We now work towards the construction of the Φ-invariant measure and the estimation of its density. We obtain distortion bounds for the derivative map, and verify that the return map Φ^R (constructed over a suitable subinterval J) satisfies Adler's distortion condition (see [**1, 22, 23**]).

PROPOSITION 2. *On $[x_{n+1}, x_n]$, the following distortion and expansion conditions are satisfied: there exist $K < \infty$ and $\lambda > 1$ such that*

$$\frac{|D^2\Phi^n(x)|}{(D\Phi^n(x))^2} < K \quad \text{and} \quad D\Phi^n(x) > \lambda, \quad \text{for all } x \in [x_{n+1}, x_n].$$

Similarly, these conditions are satisfied for $x \in S^1 \setminus (x_0', x_0)$ when $n = 1$.

Proof. It will be useful to work with the function $\tilde{\phi}(x) = \phi(1/x)$. The function $\tilde{\phi}(x)$ is slowly varying at ∞ and, again by Lemma 3, $\tilde{\phi}(x) = \zeta(x)\exp\{\int_{C_0}^x \varepsilon(t)/t\,dt\}$ where C_0 is a constant and $\varepsilon(x) = o(1)$, $\zeta(x) \to C_1$ as $x \to \infty$. Since $\tilde{\phi}(x) \in C^\omega$ on $[M, \infty)$, it is possible to tweak $\varepsilon(x)$ to make $\zeta(x)$ constant. The tweaked $\varepsilon(x)$ maintains the desired properties [**3**]. (Alternatively, we could do a local analysis on $\zeta(x)$ as in the proof of Proposition 1).

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SUBLEMMA 1. *As $n \to \infty$ the following estimates hold:*

$$\Phi'(x_n) = 1 + \frac{2}{n} + o\left(\frac{1}{n}\right) \quad \textit{and} \quad \Phi''(x_n) \sim 2\bar{\phi}^\sharp(n).$$

Proof. A simple calculation gives the following estimates:

$$\phi'(x) = -\frac{1}{x^2}\tilde{\phi}'\left(\frac{1}{x}\right), \quad \phi''(x) = \frac{2}{x^3}\tilde{\phi}'\left(\frac{1}{x}\right) + \frac{1}{x^4}\tilde{\phi}''\left(\frac{1}{x}\right),$$

$$\tilde{\phi}'(x) = \frac{\varepsilon(x)}{x}\tilde{\phi}(x), \quad \tilde{\phi}''(x) = \left(\frac{\varepsilon(x)^2}{x^2} + \frac{\varepsilon'(x)}{x} - \frac{\varepsilon(x)}{x^2}\right)\tilde{\phi}(x).$$

Using the fact that $\Phi'(x) = 1 + 2x\phi(x) + x^2\phi'(x)$, we estimate the two terms $x_n\phi(x_n)$ and $x_n^2\phi'(x_n)$ in the following way (as $n \to \infty$):

$$x_n\phi(x_n) \sim \frac{\tilde{\phi}(n\bar{\phi}^\sharp(n))}{n\bar{\phi}^\sharp(n)} \sim (\bar{\phi}^\sharp(n)\bar{\phi}(n\bar{\phi}^\sharp(n))) \sim \frac{1}{n},$$

$$|x_n^2\phi'(x_n)| \sim \frac{\varepsilon(n\bar{\phi}^\sharp(n))}{n} = o(n^{-1}).$$

Since $\Phi''(x) = 2\phi(x) + 2x\phi'(x) + x^2\phi''(x)$ it suffices to estimate $x_n^2\phi''(x_n)$ as $n \to \infty$. We have

$$x_n^2\phi''(x_n) = 2\varepsilon(1/x_n)\phi(x_n) + \phi(x_n)\big(\varepsilon(1/x_n)^2 - \varepsilon(1/x_n) + x_n^{-1}\varepsilon'(1/x_n)\big) = o(\phi(x_n)),$$

where we have used the fact that $\varepsilon'(x) = o(x^{-1})$ as $x \to \infty$. The conclusion of the sublemma follows. □

For $x \in [x_{n+1}, x_n]$, we know from the above sublemma that

$$|D\Phi^k(x)| \sim \left(\frac{n}{n-k}\right)^2 \hat{\phi}\left(\frac{n}{n-k}\right),$$

where $\hat{\phi}$ is slowly varying (at infinity). A simple computation also tells us that

$$\frac{|D^2\Phi^{R(x)}(x)|}{(D\Phi^{R(x)}(x))^2} \le \frac{1}{|D\Phi^n(x)|}\sum_{k=0}^{n-1}\frac{|D^2\Phi(\Phi^k(x))\cdot D\Phi^k(x)|}{D\Phi(\Phi^k(x))}, \tag{11}$$

and so for $x \in [x_{n+1}, x_n]$, we have that

$$\begin{aligned}
\frac{|D^2\Phi^n(x)|}{(D\Phi^n(x))^2} &\le \frac{K}{n^2\hat{\phi}(n)}\sum_{k=0}^{n-1}\left(\frac{n}{n-k}\right)^2\hat{\phi}\left(\frac{n}{n-k}\right)\bar{\phi}^\sharp(n-k) \\
&= \frac{K}{n^2\hat{\phi}(n)}\sum_{k=1}^{n}\frac{n^2}{k^2}\hat{\phi}\left(\frac{n}{k}\right)\bar{\phi}^\sharp(k) \sim \frac{K}{\hat{\phi}(n)}\int_1^n \frac{1}{t^2}\hat{\phi}\left(\frac{n}{t}\right)\bar{\phi}^\sharp(t)\,dt \\
&\le \frac{K}{\hat{\phi}(n)}\int_1^n \frac{1}{t^{2-\delta}}\hat{\phi}\left(\frac{n}{t}\right)dt = \frac{Kn^{\delta-1}}{\hat{\phi}(n)}\int_1^n \frac{1}{t^\delta}\hat{\phi}(t)\,dt \\
&\approx K\int_0^1 \frac{\hat{\phi}(ns)}{\hat{\phi}(n)}\mathcal{X}_{[1/n,1]}s^{-\delta}\,ds \to K(1-\delta)^{-1} \quad \text{as } n \to \infty,
\end{aligned}$$

where in the last line we have used the dominated convergence theorem. Here $\delta > 0$ is fixed small and $K > 0$ is a uniform constant. Hence, the distortion condition stated in the proposition is satisfied. The expansion estimate for $\Phi^n(x)$ on $x \in [x_{n+1}, x_n]$ as stated immediately follows and similarly for $x \in S^1 \setminus (x_0', x_0)$ when $n = 1$.

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Calculation of the invariant density and construction of the Markov tower. Here we explain how the results of Proposition 3 lead to the construction of the Markov tower with the appropriate return time asymptotics. As mentioned at the beginning of this section we can assume that Φ is Markov with respect to the partition $S^1 = \bigcup_{i=1}^d A_i$. The indifferent fixed point $x = 0$ will be a common end point of A_1 and A_d on the circle. x_0 and x_0' denote the other end points of A_1 and A_d, respectively (i.e. the points a_1 and a_{d-1}). We then let $B_n = [x_{n+1}, x_n]$ with $x_n = \Phi(x_{n+1})$ so that $A_1 = \bigcup B_n$. B_n' denotes an analogous interval inside A_d. Finally put $\mathcal{Q} = \{A_2, \ldots, A_{d-1}; B_n, B_n' : n \in \mathbb{N}\}$. To define the tower map and tower space we let $\mathcal{T}_0 = S^1$ and sub-partition $\mathcal{T}_0$ into sets $\omega_{0,i}$ which almost cover $\mathcal{T}_0$ under Φ^R. (For distortion reasons we do not take R to be the time associated with a covering of all of $\mathcal{T}_0$.) For definiteness we put $\mathcal{Q} = \{\omega_{0,i}\}$. To describe the upper levels $\bigcup_{\ell \geq 1} \mathcal{T}_\ell$ of the tower we specify the return function R. On each A_i for $i \in [2, d-1]$ and on B_0, B_0' we put $R = 1$ while $R|B_n = R|B_n' = n + 1$. The intervals $A_2, \ldots, A_{d-1}$ cover $\mathcal{T}_0$ under Φ^R while all of the other partition elements of $\mathcal{Q}$ under Φ^R cover $S^1 \setminus A_1$ or $S^1 \setminus A_d$. For the tower map F on $\mathcal{T} = \{(x, \ell) \in \mathcal{T}_0 \times \mathbb{N} : \ell < R(x)\}$ one has $F(\omega_{\ell,i}) = \omega_{\ell+1,i}$ when $\ell + 1 < R|\omega_{0,i}$ and $F(\omega_{R_i-1,i}) \subset \mathcal{T}_0$ (with this containment possibly strict). $F \mid \omega_{R_i-1,i}$ can be identified with $\Phi^R \omega_{0,i}$. The reference measure m on $\mathcal{T}_0$ can be assumed to be a Lebesgue measure, and taking $JF = 1$ on $\mathcal{T} \setminus \bigcup_i \mathcal{T}_{R_i-1,i}$ gives us a reference measure on the whole tower $\mathcal{T}$ which we also call m. To get the correct notion of separation time we let $s(x, y)$ denote the first time that $(\Phi^R)^n x$ and $(\Phi^R)^n y$ are in different elements of $\mathcal{Q}$. It will be useful to consider the partition $\mathcal{Q}_n = \bigvee_{k=0}^n (\Phi^R)^{-k} \mathcal{Q}$ where elements $\omega \in \mathcal{Q}_n$ contain points which only separate after n returns to $\mathcal{T}_0$. By Proposition 3 we can deduce that JF^R satisfies a bounded distortion condition (in the sense of Assumption 5) with respect to the separation times. Let us now prove an elementary estimate regarding the invariant density of Φ^R, see for example [**7**].

LEMMA 2. *Φ^R admits an absolutely continuous invariant measure ν_{Φ^R} with density σ_{Φ^R} which is Lipschitz.*

Proof. Let $\omega \in \mathcal{Q}_n$ so that for all $x, y \in \omega$ one has $s(x, y) \geq n$. Then, for $x \in \omega$, we claim that

$$\frac{|D^2(\Phi^{R(x)})^n(x)|}{(D(\Phi^{R(x)})^n(x))^2} < M_1$$

for some uniform $M_1 > 0$. To see this we know that Φ^R satisfies Adler's condition for almost all $x \in S^1$. By iteration one obtains

$$\frac{|D^2(\Phi^R)^n(x)|}{(D(\Phi^R)^n(x))^2} \leq \frac{1}{|D(\Phi^R)^n(x)|} \sum_{k=0}^{n-1} \frac{|D^2\Phi^R((\Phi^R)^k(x)) \cdot D(\Phi^R)^k(x)|}{D\Phi^R((\Phi^R)^k(x))}$$
$$\leq K \sum_{k=0}^{n-1} \frac{1}{|D(\Phi^R)^{n-k}(x)|} \leq \frac{K\lambda}{1-\lambda},$$

which is what we want. Let $(\Phi^R)_\omega^{-n}$ denote the inverse of $(\Phi^R)^n$ restricted to $\omega \in \mathcal{Q}_n$. Then the equivalent condition for the (local) inverse map $(\Phi^R)_\omega^{-n}$ is that

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$|D^2(\Phi^R)_\omega^{-n}/D(\Phi^R)_\omega^{-n}| < M_1$. Hence, for all $x, y \in S^1$ we have that

$$\log\left|\frac{D(\Phi^{-R})_\omega^n(x)}{D(\Phi^{-R})_\omega^n(y)}\right| \le \int_y^x \left|\frac{D^2(\Phi^{-R})_\omega^n(t)}{D(\Phi^{-R})_\omega^n(t)}\right| dm(t) \le M_1|x-y|.$$

Let

$$\sigma_n = \frac{1}{n}\sum_{j=0}^{n-1}\frac{d(\Phi^R)_*^j m}{dm} \equiv \frac{1}{n}\sum_{j=0}^{n-1} P_{\Phi^R}^j 1,$$

where P_{Φ^R} is the Perron–Frobenius operator associated with Φ^R. We wish to analyse the limiting behaviour of the sequence $\{\sigma_n\}_{n\ge 0}$ as $n \to \infty$. First of all for each $n \in \mathbb{N}$ and $x, y \in S^1$ we have

$$\begin{aligned}|P_{\Phi^R}^n 1(x) - P_{\Phi^R}^n 1(y)| &\le \sum_{\omega\in\mathcal{Q}_n} |D(\Phi^R)_\omega^{-n}(x) - D(\Phi^R)_\omega^{-n}(y)| \\ &\le \sum_{\omega\in\mathcal{Q}_n} M'|D(\Phi^R)_\omega^{-n}| \cdot |x-y| \le M''|x-y|,\end{aligned}$$

and hence $\{\sigma_n\}_{n\ge 0}$ forms a family of equicontinuous functions. By the Arzela–Ascoli theorem there is a subsequence $\{\sigma_{n_i}\}_{i\ge 0}$ converging to a continuous function σ_∞. This function is the invariant density associated with Φ^R. The absolute continuity properties of the measure ν_{Φ^R} easily follow. □

LEMMA 3. *If* $\sum_{n=1}^\infty 1/(n\bar{\phi}^\sharp(n)) < \infty$ *then* Φ *admits an absolutely continuous invariant measure* ν_Φ *with density:* $\sigma_\Phi(x) \approx 1/x\phi(x)$.

The invariant density for Φ can be calculated using the expression

$$\frac{d\nu_\Phi}{dm}(x) = \sigma_{\Phi^R}(x) + \sum_{k=1}^\infty \sum_{j=1,d} \sigma_{\Phi^R}(\Phi_j^{-k}x)\cdot D\Phi_j^{-k}(x), \tag{12}$$

where σ_{Φ^R} is the invariant density for ν_{Φ^R} and Φ_1^{-k} is the local inverse of Φ^k restricted to the interval A_1 and Φ_d^{-k} is the local inverse of Φ^k restricted to the interval A_d. To understand the form of the density near $x = 0$ it is easier to work with the system $(F, \mathcal{T})$ and a projection mapping $\pi : \mathcal{T} \to S^1$. Recall from fact 1 that F has an invariant measure $\nu \ll m$ and the Φ-invariant measure ν_Φ is given by $\nu_\Phi = \pi_*\nu$. Let us understand the action of π on $\mathcal{T}$. To do this we identify $\mathcal{T}_0$ with $\mathcal{Q}$ and identify the upper levels $\mathcal{T}_\ell \subset \mathcal{T}$ $(\ell > 0)$ as a union of sets of the form $\{\omega_{\ell,k}\}$ where $\omega_{\ell,k} = F^\ell\omega_{0,k}$ and $\omega_{0,k}$ correspond to sets B_k, B_k' with $k > \ell$. Thus sets of the form $\omega_{k+\ell,\ell}$ correspond to the sets B_k, B_k'. This fact may be used to analyse the density $\sigma_\Phi = d\nu_\Phi/dm$ on each B_k'. We obtain $\nu(\pi^{-1}B_n) = \nu\big(\bigcup_{i\ge n} B_n\big)$ and therefore σ_Φ has the property that for $x \in [x_{n+1}, x_n]$

$$\sigma_\Phi(x) \approx \frac{\nu(\pi^{-1}B_n)}{m(B_n)} \sim \frac{x_n}{x_n - x_{n+1}} \sim \frac{1}{x_n\phi(x_n)} \sim n.$$

Hence, by asymptotic inversion

$$\sigma_\Phi(x) \approx \frac{1}{x\phi(x)} \quad \text{since } x_n \sim \frac{1}{n\bar{\phi}^\sharp(n)}.$$

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2.3. *Mixing properties.* Let us turn to the estimation of the correlation function. For lower bounds (in terms of the speed of convergence to equilibrium) we have the following lemma.

LEMMA 4. *There exist measures* $\mu_1 \circ \pi$, $\mu_2 \circ \pi \in \mathcal{H}_\beta^+$ *such that* $\mathcal{V}_{n,\Phi}(\mu_1, \mu_2) \approx m\{\hat{R} > n\}$.

Proof. Consider the measure $\bar{\mu} = (\varphi \mathcal{X}_{\mathcal{T}_0})m$ on $\mathcal{T}$. Here $\varphi \in L^\infty(S^1, m)$ (with $\varphi \geq 0$) and $\mathcal{X}$ is the characteristic function. If $\mu = \pi_* \bar{\mu}$, then

$$\Phi_*^n \mu(B_k) = \bar{\mu}(F^{-n}\pi^{-1}B_k) = \bar{\mu}\left(F^{-n}\left(\bigcup_{\ell \geq 0} \omega_{\ell,k+\ell}\right)\right) \leq K|\varphi|_\infty m\left(\bigcup_{j=k}^{n+k} B_j\right)$$

where the right-hand inequality results from the fact that $F_*^n \bar{\mu}\left(\bigcup_{\ell > n} \mathcal{T}_\ell\right) = 0$. Hence,

$$\begin{aligned}
\mathcal{V}_{n,\Phi}(\mu, \nu_\Phi) &= \int \left| \frac{d(\pi_* F_*^n \bar{\mu})}{dm} - \sigma \right| dm \\
&\approx \sum_{k \geq 0} \left| K|\varphi|_\infty \frac{m\left(\bigcup_{j=k}^{n+k} B_j\right)}{m(B_k)} - \frac{m\left(\bigcup_{j=k}^{\infty} B_j\right)}{m(B_k)} \right| m(B_k) \\
&\approx \sum_{k \geq 0} \left| \frac{|K|\varphi|_\infty}{k\bar{\phi}^\sharp(k)} \left(1 - \frac{k\bar{\phi}^\sharp(k)}{(k+n)\bar{\phi}^\sharp(k+n)}\right) - \frac{1}{k\bar{\phi}^\sharp(k)} \right| \\
&\geq \sum_{k \geq N_0 n} K(N_0) \frac{1}{k\bar{\phi}^\sharp(k)} \\
&\approx \sum_{k \geq N_0 n} m[0, x_k] \approx m\{\hat{R} > N_0 n\} \approx m\{\hat{R} > n\}
\end{aligned}$$

where in the second line we have used the fact that $\bar{\phi}^\sharp(x)$ is slowly varying to deduce

$$|k\bar{\phi}^\sharp(k)((k+n)\bar{\phi}^\sharp(k+n))^{-1} - 1| < \epsilon(N_0) \quad \text{for all } k > N_0 n.$$

Here $\epsilon(N_0) \ll 1$ if N_0 is chosen sufficiently large. The slowly varying property of the tail is also used in the last line. Sharp bounds for $\mathcal{V}_{n,\Phi}(\cdot, \nu_\Phi)$ are now achieved. □

Estimating an upper bound for the speed of mixing. To get bounds on the correlation function and the speed of convergence to equilibrium, we bring to bear the results of §1.2. The reader is also referred to §4. To achieve these bounds we build a Markov tower $\tilde{F} : \tilde{\mathcal{T}} \circlearrowleft$ over the subinterval A_1. Define a return function $\tilde{R}$ on $(\tilde{F}, \tilde{\mathcal{T}})$ as follows: for each $x \in S^1$ let

$$\tilde{R}(x) = \inf_n \{n \geq R(x) : \Phi^n(x) \in A_1\}.$$

If $\{\tilde{T}_i\}_{i \in \mathbb{N}}$ is such that $\tilde{T}_1 = R$ and $\tilde{T}_i = \tilde{T}_{i-1} + R \circ \Phi^{\tilde{T}_{i-1}}$, then $\tilde{R}(x) = \tilde{T}_i(x)$ where $i = \inf_{j \geq 1}\{\Phi^{\tilde{T}_j}(x) \in A_1\}$. Let $\mathcal{F}_i$ be a σ-algebra on S^1 with the following property

$$\hat{\omega} \in \mathcal{F}_i \quad \text{iff } \exists n : \hat{\omega} \in \bigvee_{j=1}^{n} \Phi^{-j+1}\mathcal{Q} \quad \text{with} \quad \tilde{T}_i \mid \hat{\omega} = n.$$

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In particular, by using Lemma 6 (see §4.2) it is possible to show that for all $\hat{\omega} \in \mathcal{F}_i$

$$m\{\tilde{T}_{i+1} - \tilde{T}_i > n \mid \hat{\omega}\} < Km\{R > n\} \quad \text{and} \quad m\{\tilde{R} = \tilde{T}_i \mid \hat{\omega}\} \geq \tilde{\epsilon}_1.$$

where in the latter inequality we may assume that $\tilde{R} \neq \tilde{T}_j$ on $\hat{\omega}$ for all $j \in [1, i-1]$. To obtain the desired asymptotics of $m\{\tilde{R} > n\}$, follow the calculations performed in §4.3. Moreover A_1 can be partitioned into sets $\tilde{\omega}_i$ such that $\Phi^{\tilde{T}_i}$ maps each $\tilde{\omega}_{0,i}$ bijectively onto A_1 with bounded distortion. This gives the required Markov extension. The system $(\tilde{F}, \tilde{\mathcal{T}})$ also has an invariant measure $\tilde{\nu} = \tilde{\pi}_* \nu$ which is exact (where $\tilde{\pi}$ is the projection from $\tilde{\mathcal{T}}$ to S^1). An upper bound for $\mathcal{C}(\varphi_1 \circ \Phi^n, \varphi_2, \nu_\Phi)$ can then be computed by following the proof of Theorem 3. Here $\varphi_1, \varphi_2 \in \mathcal{H}$, but we can replace these by functions $\tilde{\varphi}_j = \varphi_j \circ \tilde{\pi} \in \mathcal{H}_\beta$ on $\tilde{\mathcal{T}}$ (for $j = 1, 2$) and then write $\mathcal{C}_n(\nu_\Phi) = |\int (\tilde{\varphi}_1 \circ \tilde{F}^n)\tilde{\varphi}_2 \, d\tilde{\nu} - \int \tilde{\varphi}_1 \, d\tilde{\nu} \int \tilde{\varphi}_2 \, d\tilde{\nu}|$. In the same way, the upper bound for $\mathcal{V}_{n,\Phi}(\nu_1, \nu_2) = |\Phi^n_* \nu_1 - \Phi^n_* \nu_2|$ can be obtained by lifting to the tower $\tilde{\mathcal{T}}$ and setting $\tilde{\nu}_j = \pi_* \nu_j$.

2.4. *Asymptotic inversion and the proof of Theorem 2.* For a prescribed function $f(x)$, the inverse function $f^{-1}(x)$ cannot always be made explicit. In this section we present techniques that allow us to analyse the behaviour of $f^{-1}(x)$, at least asymptotically, as $x \to 0$ or $x \to \infty$. We begin by defining the notion of a generalized inverse to a given function $f(x)$.

Definition 3. Let $f^\star(x) = \inf\{y \in [X, \infty) : f(y) > x\}$. Then we call $f^\star$ a generalized inverse of f. If g is a function for which $g \circ f(x) \sim f \circ g(x) \sim x$, then g is called an asymptotic inverse to f with a version $g \sim f^\star$ as $x \to \infty$.

The following result holds.

FACT 4. [**3**]

(i) *If $f \in R_\gamma$ for some $\gamma > 0$ then $f^\star \in R_{1/\gamma}$.*

(ii) *If $f \in R_1$ and $f(x) = x\rho(x)$ with $\rho(x)$ slowly varying, then the de Bruijn $\rho^\sharp(x)$ exists and $f^\star(x) \sim x\rho^\sharp(x)$.*

Computation of asymptotic inverses. In anticipation of proving Theorem 2 it will be useful to consider a function f satisfying the following assumptions.

ASSUMPTION 8. *We have $f \in R_\gamma$ for some $\gamma > 0$ and $f(x) \sim x^\gamma \eta(x)$ (as $x \to \infty$) with $\eta(x)$ a slowly varying function at infinity.*

ASSUMPTION 9. *We have $\eta(x) = g \circ \tilde{\eta}(x)$ with $g \in R_{\tilde{\gamma}}$ for some $\tilde{\gamma} > 0$ (as $x \to \infty$) and $\tilde{\eta} \in R_0$. Moreover $\tilde{\eta}$ is monotonically increasing to ∞ and $\tilde{\eta}(xy) \leq \tilde{\eta}(x) + \tilde{\eta}(y)$, for all $x, y \geq C$, for some constant $C > 0$.*

SUBLEMMA 2. *Suppose that $f(x)$ satisfies Assumptions 8 and 9. Then an asymptotic inverse to $f(x)$ is given by the function*

$$f^\star(x) \sim \left(\frac{x}{\eta(x^{1/\gamma})}\right)^{1/\gamma}. \tag{13}$$

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Proof. For $f^\star(x)$ as stated in Sublemma 2 we compute $f^\star \circ f(x)$ and show that it is $\sim x$ for large x. First of all, by direct substitution $f^\star \circ f(x) \sim x\eta(x)^{1/\gamma}\eta(x\eta(x)^{1/\gamma})^{-1/\gamma}$. We claim that for all $x \geq C$ we have that

$$\eta(x) \leq \eta(x\eta(x)^{1/\gamma}) \leq \eta(x) + o(\eta(x)).$$

If the claim is true then for sufficiently large x we obtain

$$x \geq f^\star \circ f(x) \geq \frac{x\eta(x)^{1/\gamma}}{(\eta(x) + o(\eta(x)))^{1/\gamma}} = x\left(1 + \frac{o(\eta(x))}{\eta(x)}\right)^{-1/\gamma}.$$

The right-hand term is then $\sim x$ and we are then done. So let us prove the claim. It is clear by the monotonicity of $\eta(x)$ that $\eta(x) \leq \eta(x\eta(x)^{1/\gamma})$. For the upper bound we use the conditions placed on $\eta(x)$ and estimate as follows

$$\eta(x\eta(x)^{1/\gamma}) \leq g(\tilde{\eta}(x) + \tilde{\eta}(\eta(x)^{1/\gamma})) \sim g(\tilde{\eta}(x) + \tilde{\eta}(\tilde{\eta}(x)^{\tilde{\gamma}/\gamma}L(\tilde{\eta}(x))^{1/\gamma})),$$

where $L(x)$ is slowly varying and $g(x) = x^{\tilde{\gamma}}L(x)$. Proceeding, we have that

$$\eta(x\eta(x)^{1/\gamma}) \leq g(\tilde{\eta}(x) + \tilde{\eta}(\tilde{\eta}(x)^{\tilde{\gamma}/\gamma+\epsilon})) \sim g(\tilde{\eta}(x)(1 + o(1))) \sim (1 + o(1))\eta(x)$$

as $x \to \infty$ (and for all $\epsilon > 0$). This is the required result. □

Another method for calculating asymptotic inverses is to use the Lagrange inversion formula. Take $\rho \in R_0$ and let $\rho(e^x) = \exp f(x)$ with $\rho^\sharp(e^x) = \exp f^\sharp(x)$. Then the condition $\rho^\sharp(x)\rho(x\rho^\sharp(x)) \to 1$ is equivalent to the condition $f(x + f^\sharp(x)) + f^\sharp(x) \to 0$ as $x \to \infty$.

FACT 5. (Lagrange inversion formula, see [**3**]) *Suppose that f is holomorphic in a domain D including the real interval $[R, \infty)$ and $\rho(e^x) = \exp f(x)$ is slowly varying. Suppose that for all large $x \in \mathbb{R}$, we can find a contour γ_x containing x and such that $|f(z)| < |z - x|$ for all $z \in \gamma_x$. Then*

$$f^\sharp(x) = \sum_{n\geq 0} \frac{1}{(n+1)!}\frac{d^n}{dx^n}(-f(x))^{n+1} + o(1) \quad (x \in R, x \to \infty),$$

and $\rho^\sharp(e^x) = \exp f^\sharp(x)$.

To estimate the asymptotic form of a de Bruijn conjugate we usually just need the first few terms of the Lagrange inversion formula. Let

$$f_m^\sharp(x) = \sum_{n=0}^{m} \frac{1}{(n+1)!}\frac{d^n}{dx^n}(-f(x))^{n+1}$$

denote the mth partial sum. The following result holds.

FACT 6. [**3**] *If $f'(x) \to 0$ and $f(x)f'(x) \to 0$ as $x \to \infty$, then $f^\sharp(x) = f_0^\sharp(x) + o(1)$.*

Using Sublemma 2 and/or the technique of Lagrange inversion, the following corollary is obtained.

COROLLARY 2. *Let $f(x) = x\eta(x)$ where $\eta(x) = \prod_{r=1}^{n}(\log^r x)^{\alpha_r}$ and $\alpha_r \geq 0$. Then $\eta^\sharp(x) \sim x/\eta(x)$ as $x \to \infty$. Similarly, if instead $\log \eta(x) = \prod_{r=2}^{n}(\log^r x)^{\alpha_r}$ and $\alpha_r \in (0, 1)$, then the same conclusion holds.*

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We remark that a non-trivial case arises when $\eta(x) = e^{(\log x)^\alpha}$ for $\alpha \in (0,1)$ and $x \to \infty$. The higher-order derivatives in the Lagrange inversion formula have to be analysed. For the function in question we obtain

$$\eta^\sharp(x) = \exp\left\{ \sum_{n=0}^{\lfloor \alpha/(1-\alpha) \rfloor} \frac{1}{n+1} \binom{n}{\alpha n} (\log x)^{\alpha+(\alpha-1)n} + o(1) \right\}.$$

We leave it as a challenge to find $\eta^\sharp(x)$ when $\eta(x) = e^{\log x/\log^r x}$, $(r \geq 2)$. It is now easy to see that parts (i) and (ii) of Theorem 2 hold. For the third case we use the following lemma.

LEMMA 5. *Suppose that Assumptions 1–3 hold, and regarding Assumption 4 we have $\phi(x) = x^{\gamma-1}\rho(x)$, where ρ is slowly varying near $x = 0$ and $\gamma \neq 1$. Then*

$$x_n \sim \gamma^{-1/\gamma} n^{-1/\gamma} ((\bar{\rho}^{1/\gamma})^\sharp (n^{1/\gamma}))^{-1} \quad \textit{and} \quad \mathcal{C}_n(v_\Phi) = \mathcal{O}(n x_n). \tag{14}$$

Proof. Let $f(x) = x\rho(x)$ and $g(x) = x\rho^\sharp(x)$ (as $x \to \infty$). Then, by definition, we have that $f \circ g(x) \sim x$. Moreover, for all $\alpha, \beta > 0$ we have $f(x^\alpha)^\beta \circ g(x^{1/\beta})^{1/\alpha} \sim x$ and therefore an asymptotic inverse of $x^{\alpha\beta}\rho(x^\beta)^\alpha$ is given by $x^{1/\alpha\beta}\rho^\sharp(x^{1/\alpha})^{1/\beta}$. The required asymptotic for x_n now follows from Proposition 1. To get the bound for the correlation function we should first verify Proposition 2 and then verify that the Markov tower construction is still valid (we leave this as an exercise). We then obtain

$$\mathcal{C}_n(v_\Phi) = \mathcal{O}\left(\int_n^\infty x^{-1/\gamma} ((\bar{\rho}^{1/\gamma})^\sharp (x^{1/\gamma}))^{-1}\, dx \right) = \mathcal{O}\left(\frac{n^{1-1/\gamma}}{(\bar{\rho}^{1/\gamma})^\sharp (n^{1/\gamma})} \right),$$

again using the dominated convergence theorem. □

Remark 2. When $\gamma = 1$ the estimate on the correlation function as stated in Lemma 5 is no longer true. In this case the correlation function depends more on the function $\bar{\rho}^\sharp$. However, it can be shown that the correlations decay in a slowly varying fashion (see [**3**]).

Remark 3. Using the same techniques as in the proof of Lemma 3 we also get an estimate on the invariant density near 0 when $\phi(x) = x^{\gamma-1}\rho(x)$ and obtain $\sigma_\Phi(x) \approx 1/(x^\gamma \rho(x))$.

3. *Existence of σ-finite infinite invariant measures*

3.1. *Singular probability measures.* We suppose now that $\sum_{n=1}^\infty 1/(n\bar{\phi}^\sharp(n)) = \infty$. In this case, we show that the only physical measure is the Dirac measure δ_0 supported on the indifferent fixed point. The following result holds.

PROPOSITION 3. *Suppose that $\sum_{n=1}^\infty 1/(n\bar{\phi}^\sharp(n)) = \infty$ and let $\varphi : S^1 \to \mathbb{R}$ be any continuous function. Then, for Lebesgue almost all $x \in S^1$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ (\Phi^j(x)) = \varphi(0) \quad \textit{and} \quad \lim_{n\to\infty} \frac{1}{n} \log |D\Phi^n(x)| = 0.$$

Proof. In the C^2 case the proof is straightforward, see [**1, 24**]. Since our setting is non-standard, we present the details. We fix an arbitrarily small neighbourhood $A_\epsilon^* = (-\epsilon, \epsilon)$

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of the indifferent fixed point and show that for each fixed ϵ, almost every orbit spends most of its time inside A_ϵ^* (on average). First of all we estimate the quantity

$$\widehat{\Theta}_n(k) = \sharp\{0 \le j < n : (\Phi^R)^j(x) \in B_k, B_k'\}/n.$$

Since ν_{Φ^R} is mixing, we have for each fixed k,

$$\lim_{n\to\infty} \widehat{\Theta}_n(k) = \nu_{\Phi^R}(B_k \cup B_k') \approx x_k^2 \phi(x_k) \sim \frac{1}{k^2 \bar{\phi}^\sharp(k)}.$$

We now estimate

$$\Theta_n(\epsilon) = \sharp\{0 \le j < n : \Phi^j(x) \in S^1 \setminus A_\epsilon^*\}/n$$

and show that this tends to zero as $n \to \infty$. Choose the smallest $k > 0$ so that $(x_k', x_k) \subset A_\epsilon^*$. It suffices to estimate $\Theta_n(x_k)$. For each $x \in S^1$ introduce a sequence $\{\sigma(i)\}$ with $(\Phi^R)^i = \Phi^{\sigma(i)}$ and $\sigma(i+1) = \sigma(i) + j$ when $(\Phi^R)^i(x) \in B_j, B_j'$ (with $\sigma(i+1) = \sigma(i) + 1$ otherwise). In any time interval $[\sigma(j), \sigma(j+1)]$ there can only be at most k iterates inside $S^1 \setminus (x_{k+1}', x_{k+1})$ and similarly for $[\sigma(m), n]$. Let m be such that $\sigma(m) \le n < \sigma(m+1)$. Then up to time n the proportion of iterates a typical orbit $\{\Phi^i x\}$ spends inside $S^1 \setminus (x_{k+1}', x_{k+1})$ is given by

$$\begin{aligned}\Theta_n(x_k) &= \frac{m\widehat{\Theta}_m(0) + \sum_{j=1}^k jm\widehat{\Theta}_m(j) + \sum_{j>k} km\widehat{\Theta}_m(j) + k}{(n - \sigma(m)) + m\widehat{\Theta}_m(0) + \sum_{j\ge 1} jm\widehat{\Theta}_m(j)} \\ &\approx \frac{k}{\widehat{\Theta}_m(0) + \sum_{j\ge 1} j\widehat{\Theta}_m(j)}\end{aligned}$$

with the latter term tending to 0 as $n \to \infty$, i.e. as $n \to \infty$ we can show that $m \to \infty$ (almost everywhere) and, therefore, $\widehat{\Theta}_m(j) \to 1/(j^2\bar{\phi}^\sharp(j))$. Hence, the denominator $\widehat{\Theta}_m(0) + \sum_{j\ge 1} j\hat{t}_m(j)$ converges to infinity. The conclusion of the proposition now follows. □

3.2. *Sigma-finite invariant measures.* When $\sum_{n\ge n_0} 1/(n\bar{\phi}^\sharp(n)) = \infty$, then apart from there being the usual singular physical probability measure, there is also a σ-finite infinite invariant measure. The general results of [**22**] regarding the existence of such a measure together with an estimate on the invariant density will apply in our setting. We summarize with the following result.

FACT 7. *Suppose that $\Phi \in \mathcal{I}$ satisfies Assumptions 1–4 and $\sum_{n\ge n_0} 1/(n\bar{\phi}^\sharp(n)) = \infty$. Then there exists a σ-finite infinite invariant measure $\hat{\nu}_\Phi$ that is absolutely continuous with respect to m. Moreover, its density $\hat{\sigma}_\Phi$ behaves as $H(x)/(x\phi(x))$ for all $x \in S^1 \setminus \{0\}$. Here, $H(x)$ is a continuous function satisfying $C^{-1} \le H(x) \le C$ for some constant $C > 0$.*

4. *Estimating the speed of mixing*

In this section we prove Theorem 3. For completeness we include the proof of Fact 2 which concerns the lower bound for the speed of mixing. We then explain how the coupling argument is used to obtain information on the correlation functions given that the recurrence time asymptotics are weak, that is $m\{\hat{R} > n\}$ is assumed to be slowly varying.

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Our approach is a modification of [**25**], but in order to get the required estimates we need a detailed analysis on the behaviour of slowly varying functions. These technicalities are discussed later in this section.

4.1. *A lower bound for the speed of convergence to equilibrium.* Let $P_F : L^\infty(\mathcal{T}, \nu) \circlearrowleft$ denote the Perron–Frobenius operator associated to F and let $\tilde{\nu} = \tilde{\sigma} m$ denote a measure whose density with respect to m is $\tilde{\sigma}$, and let σ be the density associated to the invariant measure ν. Then $P_F(\sigma) = \sigma$ and $P_F^n(\tilde{\sigma}) = d(F_*^n(\tilde{\sigma} m))/dm$. Now fix a constant $c > 0$ and let $\tilde{\nu} = \tilde{\sigma} m$ be a probability measure on $\mathcal{T}$ (whose density lies in $\mathcal{H}_\beta^+$) with $\tilde{\sigma} \geq \sigma + c$ on $\mathcal{T} \setminus \mathcal{T}_0$. Using the fact that $JF \equiv 1$ on $\mathcal{T} \setminus F^{-1}\mathcal{T}_0$ and $F_*\nu = \nu$, then for all $n \geq 0$ one has $P_F^n(\tilde{\sigma}) \geq \sigma + c$ on $\bigcup_{\ell > n} \mathcal{T}_\ell$. Hence,

$$\mathcal{V}_{n,F}(\tilde{\nu}, \nu) = \int_{\mathcal{T}} |P_F^n(\tilde{\sigma}) - \sigma| \, dm \geq c \sum_{\ell > n} m(\mathcal{T}_\ell) = cm\{\hat{R} > n\},$$

which is the required bound as stated in Fact 2.

4.2. *Probabilistic coupling arguments.* Let ν_1, ν_2 be any two probability measures on $\mathcal{T}$ whose densities lie in $\mathcal{H}_\beta^+$. To estimate an upper bound for $\mathcal{V}_{n,F}(\nu_1, \nu_2)$ it will be useful to couple the measures ν_1 and ν_2 by considering the product map $F \times F : \mathcal{T} \times \mathcal{T} \circlearrowleft$ on the product space $\mathcal{T} \times \mathcal{T}$. Here $m \times m$ is the reference measure on $\mathcal{T} \times \mathcal{T}$ and $\nu \times \nu$ defines the $F \times F$ invariant measure. For $j = 1, 2$ we let $\pi_j : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ denote projections onto the jth coordinates so that $\pi_j \circ (F \times F) = F \circ \pi_j$. The quantity $\mathcal{V}_{n,F}(\nu_1, \nu_2)$ may now be written as

$$\mathcal{V}_{n,F}(\nu_1, \nu_2) = |(\pi_1)_*(F \times F)^n(\nu_1 \times \nu_2) - (\pi_2)_*(F \times F)^n(\nu_1 \times \nu_2)|. \tag{15}$$

Stopping times and matching of measure densities. To estimate the expression in (15) we construct two sequences of dynamical partitions $\{\Theta_i\}_{i \geq 0}$, $\{\hat{\Theta}_i\}_{i \geq 0}$ with associated sequences of stopping times $\{t_i\}_{i \geq 0}$, $\{T_i\}_{i \geq 0}$, respectively. Partition elements inside Θ_i will map injectively onto $\mathcal{T}_0$ in one coordinate but should be spread over $\mathcal{T}$ in the other coordinate at time t_i. Elements of $\hat{\Theta}_i$ will map injectively onto $\mathcal{T}_0 \times \mathcal{T}_0$ under $(F \times F)^{T_i}$. We will call the t_i *partial return times* and the T_i *simultaneous return times* to $\mathcal{T}_0 \times \mathcal{T}_0$. A typical element $\Upsilon \in \hat{\Theta}_i$ (with $T_i < n$ defined) will have an itinerary made up of partial and simultaneous return times. That is, for all $\Upsilon \in \hat{\Theta}_i$ there are times $0 = T_0 < T_1 < \cdots < \mathcal{T}_{i(\Upsilon)}$ for which $(F \times F)^{T_j}\Upsilon \subset \mathcal{T}_0 \times \mathcal{T}_0$, and for all $j \in [0, i]$ there are times $T_{j-1} = t_0 < t_1 < \cdots < t_{k(\Upsilon, j)} = T_j$ for which $(F \times F)^{t_l}\Upsilon$ is contained in $\mathcal{T}_0 \times \mathcal{T}$ or $\mathcal{T} \times \mathcal{T}_0$ for $l \in [0, k]$. Supposing for the moment that these stopping times have been defined we explain how the right-hand side of (15) is estimated. Let $\Psi = d(\nu_1 \times \nu_2)/d(m \times m)$, so that $\nu_1 \times \nu_2 = (\Psi)(m \times m)$, and introduce a decreasing sequence of densities $\{\Psi_k\}_{k \geq 0}$, with $\Psi_0 = \Psi$. The sequence $\{\Psi_k\}$ will depend on Υ and its associated sequence of simultaneous return times $\{T_j\}_{j=0}^i$. We set $\Psi_{T_j} = \hat{\Psi}_j$ and $\Psi_{k-1} = \Psi_k$ if $T_j < k < T_{j+1}$, where $\{\hat{\Psi}_j\}_{j \geq 0}$ is a sequence of densities defined such that the following matching condition holds:

$$((\pi_1)_* - (\pi_2)_*)(F \times F)_*^{T_j}((\hat{\Psi}_{j-1} - \hat{\Psi}_j)((m \times m) \mid \Upsilon_j)) = 0, \tag{16}$$

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for all j and for all $\Upsilon_j \in \hat{\Theta}_j$. For all $z \in \mathcal{T} \times \mathcal{T}$, write $\hat{\Psi}_j(z) = \hat{\Psi}_{j-1}(z) - \hat{\mathcal{E}}_{j-1}(z)$. Then the density $\hat{\Psi}_j$ represents the density part of $\hat{\Psi}_{j-1}$ that has not been matched at the simultaneous return time T_j associated with z. To obtain an expression for $\hat{\mathcal{E}}_{j-1}(z)$ we use (16) as a guide: let $\hat{F}^j = (F \times F)^{T_j}$, then for all $A \in \mathcal{B}$, $\Upsilon \in \hat{\Theta}_j$,

$$(\pi_1)_* \hat{F}^j((\hat{\Psi}_{j-1} - \hat{\Psi}_j)((m \times m) \mid \Upsilon))(A) = \int_{(\hat{F}^{-j}\pi_1^{-1}A)\cap\Upsilon} \hat{\mathcal{E}}_{j-1}(z)\, d(m \times m)$$
$$= \int_{(A\times\mathcal{T})\cap\hat{F}^{T_j}\Upsilon} \frac{\hat{\mathcal{E}}_{j-1}(\hat{F}^{-j}z)}{J\hat{F}^j(\hat{F}^{-j}z)}\, d(m \times m),$$

where $J\hat{F}$ is the Jacobian of $\hat{F}$ with respect to $m \times m$. Replacing π_1 by π_2 in the expression above leads to an integral over $(\mathcal{T} \times A) \cap \hat{F}^{T_j}\Upsilon$. So for (16) to hold we need $\hat{\mathcal{E}}_{j-1}(z) = \mathcal{K}_\Upsilon J\hat{F}^j(z)$ where $\mathcal{K}_\Upsilon$ is a constant on Υ.

Our aim is to estimate $\mathcal{V}_{n,F}(\nu_1, \nu_2)$ purely in terms of Ψ_n. This will allow us to estimate $\mathcal{V}_{n,F}(\nu_1, \nu_2)$ in terms of a distribution of simultaneous return times $\{T_i\}$ for points in $\mathcal{T}\times\mathcal{T}$. The asymptotics of $\mathcal{V}_{n,F}(\nu_1, \nu_2)$ will then depend on the asymptotics of $(\nu_1 \times \nu_1)(T_1 > n)$ and on the regularity of $J\hat{F}$. We summarize the details in the following proposition. For a complete proof, see [**25**].

PROPOSITION 4. *For all $n \in \mathbb{N}$, and for all $\nu_1, \nu_2 \in \mathcal{H}_\beta^+$ we have that*

$$\mathcal{V}_{n,F}(\nu_1, \nu_2) \le 2(\nu_1 \times \nu_2)\{T_{j_0} > n\} + 2\sum_{j=j_0}^{\infty} \theta^{j-j_0+1}(\nu_1 \times \nu_2)\{T_j \le n < T_{j+1}\} \quad (17)$$

where $j_0 \in \mathbb{N}$, and $\theta \in (0, 1)$ are fixed constants depending only on F.

The proof of the proposition follows from the following two sublemmas.

SUBLEMMA 3. [**25**] *Let $\{\Psi_n\}$ be the sequence of densities defined as in (16). Then*

$$\mathcal{V}_{n,F}(\nu_1, \nu_2) \le 2\int \Psi_n. \quad (18)$$

SUBLEMMA 4. [**25**] *There exists a $\theta \in (0, 1)$ and a $\tilde{j}_0 \ge 1$ such that for all $j \ge \tilde{j}_0$, the sequence of densities $\{\Psi_n\}$ decrease geometrically, that is $\hat{\Psi}_j \le \theta\hat{\Psi}_{j-1}$.*

The proof of Sublemma 3 uses the matching condition, namely (16), while the proof of Sublemma 4 relies on the fact that $J\hat{F}$ satisfies a regularity condition similar to that of Assumption 5; see [**25**] for details. Proposition 4 follows from these sublemmas by first observing that

$$\int \Psi_n = \int_{\{n<T_{j_0}\}} \Psi_n + \sum_{i=j_0}^{\infty} \int_{\{T_i\le n<T_{i+1}\}} \Psi_n$$
$$= (\nu_1 \times \nu_2)\{n < T_{j_0}\} + \sum_{i=j_0}^{\infty} \int_{\{T_i\le n<T_{i+1}\}} \hat{\Psi}_i$$
$$\le (\nu_1 \times \nu_2)\{n < T_{j_0}\} + \sum_{i=j_0}^{\infty} \int_{\{T_i\le n<T_{i+1}\}} \theta^{i-j_0+1}\Psi,$$

by using Sublemma 4. The result now follows by Sublemma 3. □

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Distribution of stopping times. In order to make use of the inequality given by (17) we need to understand the asymptotic distribution of the sequences of stopping times $\{T_i\}$ and $\{t_i\}$. We now give the precise definition of $\{\Theta_i\}$, $\{\hat{\Theta}_i\}$. Let $(x_1, x_2) \in \mathcal{T} \times \mathcal{T}$ and let n_0 be the mixing constant defined in such a way that $m(\mathcal{T}_0 \cap F^{-n}\mathcal{T}_0) \geq \gamma_0 > 0$ for all $n \geq n_0$. For (x_1, x_2) we define stopping times $\{t_i\}$ as follows. Put $t_0 = 0$ and let $t_1(x_1, x_2) = n_0 + \hat{R}(F^{n_0}x_1)$. Future stopping times for $i \geq 1$ are defined as

$$t_{2i}(x_1, x_2) = t_{2i-1} + n_0 + \hat{R}(F^{n_0+t_{2i-1}}x_2), \quad t_{2i+1}(x_1, x_2) = t_{2i} + n_0 + \hat{R}(F^{n_0+t_{2i}}x_1).$$

Put $\Theta_1(x_1, x_2) = \bigvee_{i=1}^{t_1(x_1)}(F^{-i+1}\mathcal{P}) \times \mathcal{T}$, and for $i \geq 1$ define the partition Θ_i such that for each element of it, the stopping time t_i is constant. If $\Upsilon \in \Theta_{2i-1}$, then $\Upsilon = A_1 \times A_2$ with $F^{t_{2i-1}}A_1 = \mathcal{T}_0$ and $F^{t_{2i-1}}A_2$ spread over $\mathcal{T}$. To obtain $\Upsilon^* \subset \Upsilon$ with $\Upsilon^* \in \Theta_{2i}$, partition Υ into sets for which t_{2i} is constant let Υ^* be such that $F^{t_{2i}}\pi_2\Upsilon^* = \mathcal{T}_0$ (injectively). A similar process is carried out when passing to Θ_{2i+1}, where we instead partition into sets Υ^* for which $F^{t_{2i+1}}\pi_1\Upsilon^* = \mathcal{T}_0$. We now define the stopping time $T \equiv T_1$ associated with some $\Upsilon \in \Theta_i$. Suppose for definiteness that $\Upsilon = A_1 \times A_2 \in \Theta_i$ with $t_1, \ldots, t_i$ constant on Υ and $\Upsilon \cap \{T \leq t_{i-1}\} = \emptyset$. If i is even (and analogously if i is odd), then $F^{t_{i-1}}A_1 = \mathcal{T}_0$ and $F^{t_{i-1}}A_2 \subset \omega_{\ell_1,\ell_2}$. At time t_i, $F^{t_i}A_2 = \mathcal{T}_0$ while $F^{t_i}A_1$ is spread over $\bigcup\{\mathcal{T}_\ell : \ell \leq t_i - t_{i-1}\}$. We set $T = t_i$ on subsets of $\tilde{\Upsilon} \subset \Upsilon$ with $\pi_1 F^{t_i}\tilde{\Upsilon} = \mathcal{T}_0$, and $\tilde{\Upsilon}$ now becomes an element of $\hat{\Theta}_1$. Recalling that $\hat{F} = (F \times F)^T$ the stopping times T_n for $n > 1$ are defined by the relation $T_n = T_{n-1} + T \circ \hat{F}^{n-1}$ and the partition $\hat{\Theta}_n$ is given by $\hat{\Theta}_n = \bigvee_{j=1}^{n} \hat{F}^{-j+1}\hat{\Theta}_1$. We now consider the distribution of the stopping times. In particular, the t_i can be unbounded, but, due to the integrability of R-almost every $(x_1, x_2) \in \mathcal{T} \times \mathcal{T}$, has $t_1 < \infty$ (and similarly for future stopping times). The following lemma summarizes the essential results. The time delay n_0 also plays a role in ensuring that sufficient mixing has taken place before the next partial return time.

LEMMA 6. [**25**]

(1) *There exists* $\hat{\theta} > 0$ *such that for all* $i \geq 2$ *and for all* $\Upsilon \in \Theta_i$ *with* $T \mid \Upsilon > t_{i-1}$

$$(\nu_1 \times \nu_2)\{T = t_i \mid \Upsilon\} \geq \hat{\theta}.$$

(2) *There exists* $K_1 > 0$ *such that for all* $i \geq 0$, *for all* $\Upsilon \in \Theta_i$ *and for all* $n \geq 0$

$$(\nu_1 \times \nu_2)\{t_{i+1} - t_i > n_0 + n \mid \Upsilon\} \leq K_1 m\{\hat{R} > n\}.$$

(3) *There exists* $K_2 > 0$ *such that for all* i *and for all* $\Upsilon \in \hat{\Theta}_i$

$$(\nu_1 \times \nu_2)\{T_{i+1} - T_i > n \mid \Upsilon\} \leq K_2(m \times m)\{T > n\}.$$

4.3. *Estimating* $\mathcal{V}_{n,F}(\nu_1, \nu_2)$ *in terms of* $m\{\hat{R} > n\}$. We now obtain an upper bound for $\mathcal{V}_{n,F}(\nu_1, \nu_2)$ in the situation where $m\{\hat{R} > n\}$ is slowly varying. We begin by estimating $(\nu_1 \times \nu_2)\{T > n\}$. Let $c_0 = c_0(n_0) < 1$ be a small constant to be fixed and write

$$(\nu_1 \times \nu_2)\{T > n\} = (\nu_1 \times \nu_2)\{T > n : t_{c_0 n} \leq n\} + \sum_{i \leq c_0 n} (\nu_1 \times \nu_2)\{T > n : t_{i-1} \leq n < t_i\}.$$

The first term on the right can be estimated as follows:

$$(\nu_1 \times \nu_2)\{T > n : t_{c_0 n} \leq n\} \leq (\nu_1 \times \nu_2)\{T > t_2\} \cdot \prod_{i=3}^{c_0 n} (\nu_1 \times \nu_2)\{T > t_i \mid T > t_{i-1}\} \leq K\hat{\theta}^{c_0 n}$$

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by Lemma 2. For the second term on the right-hand side we write

$$(\nu_1 \times \nu_2)\{T > n : t_{i-1} \le n < t_i\} \le (\nu_1 \times \nu_2)\{T > n : n < t_i\} \le \sum_{j=1}^{i} (\nu_1 \times \nu_2)\left\{T > t_{i-1} : t_j - t_{j-1} > \frac{n}{i}\right\},$$

and by applying Lemma 6 on appropriate conditional probabilities, we obtain an upper bound for $(\nu_1 \times \nu_2)\{T > n : t_{i-1} \le n < t_i\}$ to be $K \cdot i\hat{\theta}^i m\{\hat{R} > n/i - n_0\}$. Hence,

$$(\nu_1 \times \nu_2)\{T > n\} \le K\hat{\theta}^{c_0 n} + \sum_{i \le c_0 n} K' \cdot i\hat{\theta}^i m\left\{\hat{R} > \frac{n}{i} - n_0\right\},$$

where K, K' are constants independent of i and n. Let us now consider the case when $m\{\hat{R} > n\} = \mathcal{O}(\rho(n))$ with $\rho(n)$ as stated in Theorem 3.

We assume throughout that n is large, so that the given tails are asymptotic expressions as $n \to \infty$. We have the following definition.

Definition 4. A function $\rho : \mathbb{N} \to \mathbb{R}$ satisfies 'geometric summability' with respect to constants $\gamma_1, \tilde{\theta} \in (0, 1)$, $\gamma_2 > 0$ if there exists an $N \in \mathbb{N}$, such that for all $n \ge N$,

$$\sum_{k \in [1, \gamma_1 n]} \tilde{\theta}^k \rho\left(\frac{n}{k} - \gamma_2\right) = \mathcal{O}(\rho(n)). \tag{GS}$$

We show that (GS) is satisfied for the class of slowly varying functions stated in Theorem 3, and moreover if the (GS) condition is satisfied, then we show that this is enough for Theorem 3 to hold. Note that functions of the form $\rho(x) = e^{-ax}$ or $\rho(x) = e^{-ax^b}$ (for $a > 0$ and $b \in (0, 1)$) do not satisfy (GS). We begin by showing that the log-like functions considered in Theorem 2 satisfy (GS). The difficulty here is that $\log n/k$ goes to zero as $k \to n$, so we must verify that there is a suitable choice of constants γ_1 and γ_2 for which (GS) holds. We consider functions $\rho(n) = \mathcal{O}(\exp(-\alpha_1(\log^r n)^{\alpha_2}))$ with $\alpha_1 \in (0, 1)$ and $\alpha_2 \in (0, 1]$. The next sublemma tells us how to remove the dependence on γ_2 in (GS).

SUBLEMMA 5. *Suppose that $\rho(x) \in C^\omega$ is slowly varying at ∞ and is monotonically decreasing on $[x_0, \infty)$ for some $x_0 > 0$. Then there exists a constant K_{γ_1} independent of k or n such that for some constant γ_1 depending on both ρ and γ_2 we have that $\rho(n/k - \gamma_2) \le K_{\gamma_1}\rho(n/k)$ for all $k < \gamma_1 n$.*

Proof. On $[x_0, \infty)$ observe that

$$1 \le \frac{\rho(n/k - \gamma_2)}{\rho(n/k)} = \exp\left\{\log \rho\left(\frac{n}{k} - \gamma_2\right) - \log \rho\left(\frac{n}{k}\right)\right\}.$$

By Lemma 1 and the mean value theorem we have for some $z \in [n/k - \gamma_2, n/k]$ and for all $k < \gamma_1 n$

$$\begin{aligned}\frac{\rho(n/k - \gamma_2)}{\rho(n/k)} &= \exp\left\{\frac{\rho'(z)}{\rho(z)}\gamma_2\right\} \\ &= \exp\left\{\left(\frac{\varepsilon(z)}{z} + o\left(\frac{1}{z}\right)\right)\gamma_2\right\} \\ &\le \exp\{(\varepsilon(\min\{x_0, \gamma_1^{-1}\}) + 1)\gamma_2\gamma_1\} = K_{\gamma_1}.\end{aligned}$$

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We have used the fact that $\rho \in C^\omega$ to assert $\zeta'(x)/\zeta(x) = o(x^{-1})$ as $x \to \infty$. The dependence of γ_1 on γ_2 follows from the requirement that $\gamma_1^{-1} - \gamma_2 > x_0$, otherwise we could lie outside the interval of monotonicity. □

Remark 4. When $\rho(n) = \mathcal{O}(\exp(-\alpha_1(\log^r n)^{\alpha_2}))$ then we should choose $\gamma_1 < (\gamma_2 + \exp^r 0)^{-1}$ in the proof as the function $(\log^r x)^{-\alpha_2}$ is not defined for $x \le \exp^r 0$. (Here $\exp^r$ denotes the r-fold composition of the function exp.)

Considering Theorem 2 we first estimate $(\nu_1 \times \nu_2)\{T > n\}$ when $\rho(x)$ is a log-like function.

SUBLEMMA 6. *If* $\rho(n) = \mathcal{O}(\exp(-\alpha_1(\log^r n)^{\alpha_2}))$, *with* $\alpha_1 \in (0, 1)$ *and* $\alpha_2 \in (0, 1]$, *then*

$$(\nu_1 \times \nu_2)\{T > n\} = \mathcal{O}(\rho(n)).$$

Proof. If we show that $\rho(n)$ satisfies (GS) with $\gamma_2 = n_0$ and an appropriate choice of γ_1 then we are done since we obtain $(\nu_1 \times \nu_2)\{T > n\} \le K_1\theta^{\gamma_1 n} + K_2\rho(n)$ for uniform constants K_1, K_2 independent of n. The term on the right of the inequality is then $\mathcal{O}(\rho(n))$ as $n \to \infty$.

To show that $\rho(n)$ satisfies (GS) it is necessary to choose γ_1 small enough in order that the sum $\sum_{k\in[1,\gamma_1 n]} \tilde{\theta}^k \rho(n/k - \gamma_2)$ decreases eventually as a geometric series on the interval $[1, \gamma_1 n]$. If this happens, the sum of the series can be bounded by a constant times the sum of the first few (finite) terms, which is then of the order $\rho(n)$. The only difficulty is that $\rho(n/k)$ blows up to infinity as k increases to $(\exp^{r-1} 0)^{-1}n$ (especially when $\alpha_2 = 1$) and is no longer defined beyond that value. However, we show that all is fine if γ_1 is sufficiently small. By Sublemma 5 we may take $\gamma_2 = 0$ and we begin by estimating $\rho(n/(k+1))/\rho(n/k)$. We obtain, for all $r \ge 2$,

$$1 \le \frac{\rho(n/(k+1))}{\rho(n/k)} = \exp\left\{\alpha_1\left(\log^r\left(\frac{n}{k}\right)\right)^{\alpha_2} - \alpha_1\left(\log^r\left(\frac{n}{k+1}\right)\right)^{\alpha_2}\right\}.$$

To estimate the right-hand term we use the mean value theorem applied to the function $g(x) = (\log^{r-1} x)^{\alpha_2}$ on the interval $I_{n,k} = [\log(n/(k+1)), \log(n/k)]$. Then, for all $x, y \in I_{n,k}$, we have that

$$|g(x) - g(y)| \le \sup_{z\in I_{n,k}} \alpha_2(z \log z \dots \log^{r-2} z(\log^{r-1} z)^{(1-\alpha_2)})^{-1}|x - y|.$$

Letting $x = \log(n/k)$ and $y = \log(n/(k+1))$ and using the fact that $I_{n,k} \subset [\log \gamma_1^{-1}, \log n]$ we have the bound

$$\begin{aligned}\frac{\rho(n/(k+1))}{\rho(n/k)} &\le \exp(\alpha_1\alpha_2(\log c_1^{-1} \dots \log^{r-1} \gamma_1^{-1}(\log^r \gamma_1^{-1})^{(1-\alpha_2)})^{-1} \log(l + k^{-1})) \\ &= \left(1 + \frac{1}{k}\right)^{\lambda^*}\end{aligned}$$

with $\lambda^* = \alpha_1\alpha_2(\log \gamma_1^{-1} \dots \log^{r-1} \gamma_1^{-1}(\log^r \gamma_1^{-1})^{1-\alpha_2})^{-1}$, and $\lambda^* < \infty$ if γ_1 is chosen sufficiently small. By induction we obtain $\rho(n/k) \le k^{\lambda^*}\rho(n)$ and, hence, that

$$\sum_{k\in[1,\gamma_1 n]} \tilde{\theta}^k \rho\left(\frac{n}{k}\right) \le \sum_{k\in[1,\gamma_1 n]} k^{\lambda^*}\tilde{\theta}^k \rho(n) = \mathcal{O}(\rho(n)).$$

The conclusion of Sublemma 6 now follows. □

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SUBLEMMA 7. *Suppose that* $\rho(x) \in C^\omega$ *is slowly varying at* ∞ *and monotonic on* $[x_0, \infty)$*, then*

$$(\nu_1 \times \nu_2)\{T > n\} = \mathcal{O}(\rho(n)).$$

Proof. In light of Sublemma 6 above it suffices to estimate $\rho(n/(k+1))/\rho(n/k)$ for all $n/k > \max\{x_0, \gamma_1^{-1}\}$. By Lemma 3 and the mean value theorem we have that

$$\frac{\rho(n/k)}{\rho(n/(k+1))} \le \exp\left\{\frac{\rho'(z)}{\rho(z)}\right\} = \exp\left\{\left(\frac{\varepsilon(z)}{z} - \frac{\zeta'(z)}{\zeta(z)}\right)\frac{n}{k(k+1)}\right\}$$

for all $z \in [n/(k+1), n/k]$ and $k < \gamma_1 n$. Now choose γ_1 so that $\varepsilon(\gamma_1^{-1})$ is well defined on $[\gamma_1, \infty)$. Using the smoothness properties of ρ (which implies smoothness for $\zeta(x)$ and $\varepsilon(x)$) we obtain

$$\frac{\rho(n/k)}{\rho(n/(k+1))} \le \exp\left\{\frac{\varepsilon(\gamma_1^{-1})}{k} + o\left(\frac{1}{k}\right)\right\} = 1 + \frac{\varepsilon(\gamma_1^{-1})}{k} + o\left(\frac{1}{k}\right).$$

Hence, $\rho(n/k) \le k^{\lambda^*}\rho(n)$ with λ^* bounded if $\varepsilon(\gamma_1^{-1})$ is bounded. The result now follows using the argument of Sublemma 6. □

The estimation of $\mathcal{V}_{n,F}(\nu_1, \nu_2)$ is now a straightforward procedure. By Proposition 4 and arguing as above with $\rho(n)$ slowly varying we have

$$\begin{aligned}
\mathcal{V}_{n,F}(\nu_1, \nu_2) &\le K \sum_{i=1}^{\infty} \theta^i (\nu_1 \times \nu_2)\{T_{i-1} \le n < T_i\} \\
&\le K_1\theta^{c_0 n} + K_2 \sum_{i \le c_0 n} \theta^i \sum_{j=1}^{i} (\nu_1 \times \nu_2)\left\{T_j - T_{j-1} > \frac{n}{i}\right\} \\
&\le K_2\theta^{c_0 n} + K_2 \sum_{i \le c_0 n} i\theta^i (m \times m)\left\{T > \frac{n}{i}\right\} \\
&\le K_1\theta^{c_0 n} + K_2\rho(n) = \mathcal{O}(\rho(n))
\end{aligned}$$

as required (with K_1, K_2 uniform constants). □

Estimating the correlation function $\mathcal{C}(\varphi_1 \circ F^n, \varphi_2, \nu)$. Take $\varphi_1 \in L^\infty$, $\varphi_2 \in \mathcal{H}_\beta$ and let P_F denote the Perron–Frobenius operator associated with F and m, as before. Recall that $\nu = \sigma m$ and $P_F(\sigma) = \sigma$. Let $\tilde{\varphi}_2$ be a rescaling of φ_2 so that $\int \tilde{\varphi}_2 \sigma \, dm = 1$. Let $\tilde{\nu}$ be the probability measure on $\mathcal{T}$ with $\tilde{\nu} = (\tilde{\varphi}_2\sigma)m$. Then

$$\begin{aligned}
\mathcal{C}(\varphi_1 \circ F^n, \tilde{\varphi}_2, \nu) &= \left| \int (\varphi_1 \circ F^n)\tilde{\varphi}_2 \, d\nu - \int \varphi_1 \, d\nu \int \tilde{\varphi}_2 \, d\nu \right| \\
&= \left| \int \varphi_1 (P_F^n(\tilde{\varphi}_2\sigma) - \sigma) \, dm \right| \\
&\le \int |\varphi_1| \cdot |P_F^n(\tilde{\varphi}_2\sigma) - \sigma| \, dm \le |\varphi_1|_\infty \mathcal{V}_{n,F}(\tilde{\nu}, \nu)
\end{aligned}$$

with $\sigma, \tilde{\varphi}_2\sigma \in \mathcal{H}_\beta^+$. This completes the proof of Theorem 3.

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In light of the calculations performed above we can also obtain good bounds on the correlation functions in the regularly varying case. Hence, we can now prove Corollary 1.

Proof of Corollary 1. Suppose that $m\{\hat{R} > n\} \sim n^{-\alpha}\rho(n)$ with $\alpha > 0$ and $\rho(x)$ a slowly varying function and let $\tilde{\rho}(x) = x^{-\alpha}\rho(x)$ where $\rho(x)$ satisfies (GS). Then it can be readily verified that $\tilde{\rho}(x)$ satisfies (GS), and we leave this as an exercise. □

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