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# Bounded geodesics on locally symmetric spaces

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*Abstract.* Let $\Gamma$ be a torsion-free subgroup of $\mathrm{SL}_3(\mathbb{R})$ commensurable with $\mathrm{SL}_3(\mathbb{Z})$, and $Y = \mathrm{SO}_3(\mathbb{R})\backslash \mathrm{SL}_3(\mathbb{R})/\Gamma$ be endowed with the natural locally symmetric space structure. We prove that for any point $y \in Y$, the set of directions in which the geodesic ray starting from $y$ is bounded in $Y$, is hyperplane absolute winning.

Key words: Bounded geodesics, hyperplane absolute winning, locally symmetric space, homogeneous dynamics
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## 1. *Introduction*

### 1.1. *Bounded geodesics on complete Riemannian manifolds.*

Let $M$ be a complete Riemannian manifold and let $S(M)$ denote its unit tangent bundle, whose fiber $S_p(M)$ over a point $p \in M$ is the unit sphere in $T_p(M)$ centered at the origin. For any $p \in M$ and any $\xi \in S_p(M)$, let $\gamma(p, \xi)$ denote the geodesic ray through the base point $p$ in the direction $\xi$. Of particular interest is the following subset of $S(M)$:

$$E^+(\infty) := \{(p, \xi) \in S(M) : \gamma(p, \xi) \text{ is bounded in } M\}$$

and its section over a point $p \in M$:

$$E^+(p, \infty) := E^+(\infty) \cap S_p(M) = \{\xi \in S_p(M) : \gamma(p, \xi) \text{ is bounded in } M\}.$$

It is trivial to see that $E^+(\infty) = S(M)$ when $M$ itself is compact. However, for a general non-compact complete Riemannian manifold $M$, it is not even clear whether $E^+(\infty)$ is non-empty. One can find in [37] an affirmative answer for a complete Riemannian manifold with $\dim(M) \geq 3$, curvature bounded by $[-b^2, -1]$ for some $b > 1$ and finite volume.

In fact, one can say more for the special case when $M$ is a locally symmetric space of non-compact type with finite Riemannian volume. Here and hereafter, we say that a subset $E$ in a metric space $S$ is *thick* if $\dim(E \cap U) = \dim(U)$ for any open subset $U$ of $S$,

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where dim stands for the Hausdorff dimension on $S$. Dani [20] proved in the 1980s that $E^+(\infty)$ is thick in $S(M)$ for $M$ as above with a constant negative curvature (in particular of rank 1). A decade later, Aravinda and Leuzinger [7, 8] proved that each $E^+(p, \infty)$ has the winning property for Schmidt's game (see §2) when restricted to a non-constant $C^1$ curve, and hence, is thick in $S_p(M)$ for $M$ as above of rank 1. Similar results were also proved for the Teichmuller geodesic flow on the moduli space of unit-area holomorphic quadratic differentials by Kleinbock and Weiss [26] and by Chaika, Cheung, and Masur [18].

In this paper, we first generalize the previous results as follows.

THEOREM 1.1. *Let $M$ be a locally symmetric space of non-compact type with finite Riemannian volume. Then, for any $p \in M$, the set $E^+(p, \infty)$ is thick in $S_p(M)$.*

We remark here that Theorem 1.1 easily follows from [25, Corollary 5.5] and Marstrand slicing theorem. See §3 for details of the proof.

The main point of this paper is to consider a stronger property of sets than thickness. In the 1960s, Schmidt [36] introduced a certain game played on any metric space, where a winning set for this game is proved to be thick. Later, Schmidt's winning property was upgraded to an even stronger hyperplane absolute winning property (abbreviated as HAW) by Broderick *et al* [15] on any Euclidean space, and by Kleinbock and Weiss [29] on any $C^1$ manifold. It should be stressed here that for any winning property mentioned previously, it has the advantage that any countable intersection of winning sets is still winning. See §2 for all definitions and discussions, and [1–6, 9–14, 16, 27, 28, 33] for other recent results involving winning properties of exceptional sets in Diophantine approximations and dynamical systems.

In this terminology, we raise a stronger conjecture than Theorem 1.1.

*Conjecture 1.2.* Let $M$ be a locally symmetric space of non-compact type with finite Riemannian volume. Then, for any $p \in M$, the set $E^+(p, \infty)$ is HAW on $S_p(M)$.

Now, we briefly review some basic facts concerning locally symmetric spaces. Let $M$ be a locally symmetric space of non-compact type and let $\widetilde{M}$ denote its universal cover. The isometry group $G$ of $\widetilde{M}$ has finitely many connected components, whose identity component is a connected semisimple Lie group without compact factors and with trivial center. Fix a point $p_0 \in M$ and let $\widetilde{p_0} \in \widetilde{M}$ be a preimage of $p_0$. The stabilizer $K := \mathrm{Stab}_G(\widetilde{p_0})$ is a maximal compact subgroup of $G$. We identify the globally symmetric space $\widetilde{M}$ with $K\backslash G$ and view the fundamental group $\Gamma := \pi_1(M)$ as a torsion-free discrete subgroup of $G$ via deck transformations. Then, $M$ can be identified with $K\backslash G/\Gamma$. Conversely, given any semisimple Lie group $G$ without compact factors and with finite center, any maximal compact subgroup $K \subset G$, and any torsion-free discrete subgroup $\Gamma \subset G$, there exists a unique locally symmetric space structure on $K\backslash G/\Gamma$. Moreover, $M$ has finite Riemannnian volume if and only if $\Gamma$ is a lattice in $G$, and the rank of $M$ is the rank of $G$.

In this paper, we verify Conjecture 1.2 for a special kind of locally symmetric space of non-compact type with finite Riemannian volume and of rank 2.

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THEOREM 1.3. *Let $G = \mathrm{SL}_3(\mathbb{R})$, $K = \mathrm{SO}_3(\mathbb{R})$, $\Gamma \leq G$ be a torsion-free subgroup commensurable with $\mathrm{SL}_3(\mathbb{Z})$, and $Y = K\backslash G/\Gamma$ be endowed with the above natural locally symmetric space structure. Then, Conjecture 1.2 holds for Y, that is, for any $y \in Y$, the set $E^+(y, \infty)$ is HAW on $S_y(Y)$.*

1.2. *Relations between homogeneous dynamics and geometry.* Theorem 1.3 can also be reinterpreted from the view of homogeneous dynamics. Let $G$ be a semisimple Lie group without compact factors and with finite center, $\Gamma \subseteq G$ be a non-uniform torsion-free lattice, and $X = G/\Gamma$ be the corresponding homogeneous space. Then, for any maximal compact subgroup $K$ of $G$, there exists a proper projection

$$X = G/\Gamma \to Y = K\backslash G/\Gamma,$$

which realizes a locally symmetric space as the quotient of a homogeneous space with compact fiber.

Now write $\mathfrak{g} = \mathrm{Lie}(G)$. There exists a Cartan involution $\theta$ with respect to the Killing form of $\mathfrak{g}$, which induces a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Here, $\mathfrak{k}$ and $\mathfrak{p}$ are the 1 and $-1$ eigenspaces of $\theta$, respectively. The involution $\theta$ also extends to an involutive automorphism $\widetilde{\theta}$ of $G$. Let $K = G_{\widetilde{\theta}}$ denote the subgroup of all $\widetilde{\theta}$-fixed elements in $G$. Then, $K$ is a maximal compact subgroup of $G$ and there is a diffeomorphism

$$K \times \mathfrak{p} \to G, \quad (k, X) \mapsto k \cdot \exp(X).$$

It follows that the differential at $1_G$ of the projection $G \to K\backslash G$ gives a linear isomorphism

$$\mathfrak{p} \to T_{[K1_G]}(K\backslash G).$$

By homogeneity, the tangent space of each point $[Kg] \in K\backslash G$ is isomorphic to $\mathfrak{p}$. Moreover, the Riemannian metric on it can be seen as the $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{p}$. Then, the unit tangent space equals $\mathfrak{p}^1 := \{\mathbf{v} \in \mathfrak{p} : \|\mathbf{v}\| = 1\}$.

For each base point $[Kg] \in K\backslash G$ and each direction $\xi = (\mathrm{d}R_g)(\mathbf{v}) \in T_{[Kg]}(K\backslash G)$ with $\mathbf{v} \in \mathfrak{p}^1$, the associated geodesic ray on $K\backslash G$ is given by

$$\widetilde{\gamma}([Kg], \xi) = \{K \exp(t\mathbf{v})g : t \geq 0\}.$$

Under the canonical covering map $K\backslash G \to K\backslash G/\Gamma$, the tangent space of each point $y = [Kg\Gamma] \in Y$ can be identified with that of $[Kg] \in K\backslash G$, and each geodesic ray in $Y$ through the base point $y$ in the direction $\xi$ can be written as

$$\gamma(y, \xi) = \{K \exp(t\mathbf{v})g\Gamma : t \geq 0\}.$$

It follows that

$$\begin{aligned} E^+(y, \infty) &= \{\xi \in S_y(Y) : \gamma(y, \xi) \text{ is bounded in } Y\} \\ &\cong \{\mathbf{v} \in \mathfrak{p}^1 : \{K \exp(t\mathbf{v})g\Gamma : t \geq 0\} \text{ is bounded in } Y\} \\ &= \{\mathbf{v} \in \mathfrak{p}^1 : \{\exp(t\mathbf{v})g\Gamma : t \geq 0\} \text{ is bounded in } X\}. \end{aligned}$$

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This reduces Theorem 1.3 into the framework of homogeneous dynamics. Since any countable intersection of HAW sets is still HAW and, hence, thick in the ambient space, we immediately get the following corollary of Theorem 1.3.

COROLLARY 1.4. *Let $G$, $\Gamma$, $K$, $\mathfrak{p}$, $Y$ be as in Theorem 1.3. Then, for any countably many points $\{y_n = [Kg_n\Gamma]\}_{n\in\mathbb{N}}$ in $Y$, the set*

$$\{\mathbf{v} \in \mathfrak{p}^1 : \{K \exp(t\mathbf{v})g_n\Gamma : t \geq 0\} \text{ is bounded in } Y \text{ for all } n \in \mathbb{N}\}$$

*is thick in $\mathfrak{p}^1$.*

One can also consider all bounded diagonal rays through a point on the homogeneous space $X = G/\Gamma$. Let

$$\mathbf{P}^+(\mathfrak{g}) := (\mathfrak{g} \setminus \{0\})/\sim \quad \text{where } \mathbf{v} \sim \mathbf{v}' \quad \text{if and only if } \mathbf{v} \in \mathbb{R}^+\mathbf{v}'$$

denote the positive projective space of $\mathfrak{g}$. For each point $x \in X$, we write

$$E^+(x, \infty) := \{[\mathbf{v}] \in \mathbf{P}^+(\mathfrak{g}) : \{\exp(t\mathbf{v})x : t \geq 0\} \text{ is bounded in } X\}.$$

The counterparts of Theorem 1.1, Conjecture 1.2, Theorem 1.3, and Corollary 1.4 are as follows.

THEOREM 1.5. *Let $G$ be a Lie group, $\Gamma \subseteq G$ be a lattice, and $X = G/\Gamma$ be the corresponding homogeneous space. Then, for any $x \in X$, the set $E^+(x, \infty)$ is thick in $\mathbf{P}^+(\mathfrak{g})$.*

*Conjecture 1.6.* Let $G$ be a Lie group, $\Gamma \subseteq G$ be a lattice, and $X = G/\Gamma$ be the corresponding homogeneous space. Then, for any $x \in X$, the set $E^+(x, \infty)$ is HAW on $\mathbf{P}^+(\mathfrak{g})$.

THEOREM 1.7. *Let $G = \mathrm{SL}_3(\mathbb{R})$, $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, and $X = G/\Gamma$ be the corresponding homogeneous space. Then, Conjecture 1.6 holds for $X$, that is, for any $x \in X$, the set $E^+(x, \infty)$ is HAW on $\mathbf{P}^+(\mathfrak{g})$.*

COROLLARY 1.8. *Let $G$, $\Gamma$, $\mathfrak{g}$, $X$ be as in Theorem 1.7. Then, for any countably many points $\{x_n\}_{n\in\mathbb{N}}$ in $X$, the set*

$$\{[\mathbf{v}] \in \mathbf{P}^+(\mathfrak{g}) : \{\exp(t\mathbf{v})x_n : t \geq 0\} \text{ is bounded in } X \text{ for all } n \in \mathbb{N}\}$$

*is thick in $\mathbf{P}^+(\mathfrak{g})$.*

*Remark 1.9.* Moreover, Conjectures 1.2 and 1.6 also hold for $G = \prod_{i=1}^{s} \mathrm{SL}_2(\mathbb{R})$ with a lattice $\Gamma$ and a maximal compact subgroup $K$ in $G$. This can be done by combining the arguments in this paper and [**4**].

1.3. *Organization of the paper and strategy of the proof.* The paper is organized as follows. In §2, we recall the definitions and properties of Schmidt's game and its variants—the hyperplane absolute game and the hyperplane potential game. In §3, we directly prove Theorems 1.1 and 1.5 by combining [**25**, Corollary 5.5] and Marstrand

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slicing theorem. In §4, we state our key result Theorem 4.1, from which we deduce Theorems 1.3 and 1.7. The proof of Theorem 4.1, which forms the most technical part of this paper, is given in §§5–7.

We highlight here that two of our main theorems, Theorems 1.1 and 1.3, are formulated in a purely geometric way. They can be viewed as applications of the corresponding results in the field of homogeneous dynamics.

Concerning the proof of Theorem 4.1, the target set (4.1) can be regarded as a fiber bundle over an open interval of weight parameters, where each fiber is proved to be HAW on its ambient space (see [**5**]). We manage to show that the winning strategy is 'locally constant' with respect to the parameters, which is reflected in the choices of planes (7.5) in the proof of Lemma 5.5. This requires a detailed analysis of how the strategy used in [**5**] depends on the weight parameter.

## 2. *Preliminaries on Schmidt games*

2.1. *Schmidt's* $(\alpha, \beta)$*-game.* We first recall Schmidt's $(\alpha, \beta)$-game introduced in [**36**]. It involves two parameters $\alpha, \beta \in (0, 1)$, and is played by two players Alice and Bob on a complete metric space $(X, \mathrm{dist})$ with a target set $S \subset X$. Bob starts the game by choosing a closed ball $B_0 = B(x_0, \rho_0)$ in $X$ with center $x_0$ and radius $\rho_0$. After Bob chooses a closed ball $B_i = B(x_i, \rho_i)$, Alice chooses $A_i = B(x_i', \rho_i')$ with

$$\rho_i' = \alpha\rho_i \quad \text{and} \quad \mathrm{dist}(x_i', x_i) \le (1-\alpha)\rho_i \ ,$$

and then Bob chooses $B_{i+1} = B(x_{i+1}, \rho_{i+1})$ with

$$\rho_{i+1} = \beta\rho_i' \quad \text{and} \quad \mathrm{dist}(x_{i+1}, x_i') \le (1-\beta)\rho_i' \ ,$$

etc. This implies that the balls are nested:

$$B_0 \supset A_0 \supset B_1 \supset \cdots \ ;$$

Alice wins the game if the unique point $\bigcap_{i=0}^{\infty} A_i = \bigcap_{i=0}^{\infty} B_i$ belongs to $S$, and Bob wins otherwise. The set $S$ is $(\alpha, \beta)$-*winning* if Alice has a winning strategy, is $\alpha$-*winning* if it is $(\alpha, \beta)$-winning for any $\beta \in (0, 1)$, and is *winning* if it is $\alpha$-winning for some $\alpha$. Schmidt [**36**] proved that:

- winning subsets of Riemannian manifolds are thick;
- if $S$ is $\alpha$-winning and $\varphi : X \to X$ is bi-Lipschitz, then $\varphi(S)$ is $\alpha'$-winning, where $\alpha'$ depends on $\alpha$ and the bi-Lipschitz constant of $\varphi$;
- a countable intersection of $\alpha$-winning sets is again $\alpha$-winning.

As such, Schmidt's game has been a powerful tool for proving thickness of intersections of certain countable families of sets, see e.g. [**1**, **2**, **13–15**, **20–22**]. However, for a fixed $\alpha$, the class of $\alpha$-winning subsets of a Riemannian manifold depends on the choice of the metric, and is not known to be preserved by diffeomorphisms.

2.2. *Hyperplane absolute game on* $\mathbb{R}^d$. Inspired by ideas of McMullen [**32**], the hyperplane absolute game on the Euclidean space $\mathbb{R}^d$ was introduced in [**15**]. It has the advantage that the family of its winning sets is preserved by $C^1$ diffeomorphisms. Let $S \subset \mathbb{R}^d$ be a target set and let $\beta \in (0, \frac{1}{3})$. As before, Bob begins by choosing a

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closed ball $B_0$ of radius $\rho_0$. For an affine hyperplane $L \subset \mathbb{R}^d$ and $\rho > 0$, we denote the $\rho$-neighborhood of $L$ by

$$L^{(\rho)} := \{\mathbf{x} \in \mathbb{R}^d : \operatorname{dist}(\mathbf{x}, L) < \rho\}.$$

Now, after Bob chooses a closed ball $B_i$ of radius $\rho_i$, Alice chooses a hyperplane neighborhood $L_i^{(\rho_i')}$ with $\rho_i' \le \beta\rho_i$, and then Bob chooses a closed ball $B_{i+1} \subset B_i \setminus L_i^{(\rho_i')}$ of radius $\rho_{i+1} \ge \beta\rho_i$. Alice wins the game if and only if

$$\bigcap_{i=0}^{\infty} B_i \cap S \neq \varnothing.$$

The set $S$ is *$\beta$-hyperplane absolute winning* (*$\beta$-HAW* for short) if Alice has a winning strategy, and is *hyperplane absolute winning* (*HAW* for short) if it is $\beta$-HAW for any $\beta \in (0, \frac{1}{3})$.

LEMMA 2.1. [15]

(i) *HAW subsets are winning and hence thick.*
(ii) *A countable intersection of HAW subsets is again HAW.*
(iii) *The image of a HAW set under a $C^1$ diffeomorphism $\mathbb{R}^d \to \mathbb{R}^d$ is HAW.*

2.3. *HAW subsets of a manifold.* The notion of HAW sets has been extended to subsets of $C^1$ manifolds in [29]. This is done in two steps. First, one defines the absolute hyperplane game on an open subset $W \subset \mathbb{R}^d$. It is defined just as the absolute hyperplane game on $\mathbb{R}^d$, except for requiring that Bob's first move $B_0$ be contained in $W$. If Alice has a winning strategy, we say that $S$ is HAW on $W$. Now, let $M$ be a $d$-dimensional $C^1$ manifold and let $\{(U_\alpha, \varphi_\alpha)\}$ be a $C^1$-atlas, that is, $\{U_\alpha\}$ is an open cover of $M$ and each $\varphi_\alpha$ is a $C^1$ diffeomorphism from $U_\alpha$ onto the open subset $\varphi_\alpha(U_\alpha)$ of $\mathbb{R}^d$. A subset $S \subset M$ is said to be *HAW on $M$* if for each $\alpha$, $\varphi_\alpha(S \cap U_\alpha)$ is HAW on $\varphi_\alpha(U_\alpha)$. Note that Lemma 2.1(iii) implies that the definition is independent of the choice of the atlas (see [29] for details).

LEMMA 2.2. [29]

(i) *HAW subsets of a $C^1$ manifold are thick.*
(ii) *A countable intersection of HAW subsets of a $C^1$ manifold is again HAW.*
(iii) *Let $\varphi : M \to N$ be a diffeomorphism between $C^1$ manifolds and let $S \subset M$ be a HAW subset of $M$. Then, $\varphi(S)$ is a HAW subset of $N$.*
(iv) *Let $M$ be a $C^1$ manifold with an open cover $\{U_\alpha\}$. Then, a subset $S \subset M$ is HAW on $M$ if and only if $S \cap U_\alpha$ is HAW on $U_\alpha$ for each $\alpha$.*
(v) *Let $M$, $N$ be $C^1$ manifolds and let $S \subset M$ be a HAW subset of $M$. Then, $S \times N$ is a HAW subset of $M \times N$.*

*Proof.* Items (i)–(iii) appeared as [29, Proposition 3.5] and are clear from Lemma 2.1. Item (iv) is clear from the definition. Item (v) is proved in [29, Proof of Theorem 3.6(a)]. □

We shall also need the following lemma.

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LEMMA 2.3. *Let $M$ be a $C^1$ manifold of dimension $d$, $M' \subset M$ be a finite union of $C^1$ submanifolds of strict lower dimensions. Then, $S \subset M \setminus M'$ is HAW on $M \setminus M'$ if and only if it is HAW on $M$.*

*Proof.* The if direction is a direct consequence of the definition. For the only if direction, by using an induction argument, we may assume that $M'$ itself is a $C^1$ submanifold of strict lower dimension. We choose and fix a $C^1$-atlas $\{(U_\alpha, \phi_\alpha)\}$ of $M$ such that, for any $\alpha$, there exists a hyperplane $L_\alpha \subset \mathbb{R}^d$ satisfying

$$\phi_\alpha(U_\alpha \cap M') \subset L_\alpha \subset \mathbb{R}^d.$$

Since the HAW property is independent of choices of the atlas, to prove $S$ is HAW on $M$, it suffices to prove $\phi_\alpha(S \cap U_\alpha)$ is HAW on $\phi_\alpha(U_\alpha)$ for all $U_\alpha$ in this specific $C^1$-atlas. As $\{(U_\alpha \cap (M \setminus M'), \phi_\alpha)\}$ is a $C^1$-atlas of $M \setminus M'$, $\phi_\alpha(S \cap U_\alpha \cap (M \setminus M'))$ is HAW on $\phi_\alpha(U_\alpha \cap (M \setminus M'))$. In view of these properties, the winning strategy for Alice is clear: for any $\beta$-hyperplane absolute game played on $\phi_\alpha(U_\alpha)$, after Bob chooses his $B_0$, Alice chooses neighborhood $L_\alpha^{\beta\rho(B_0)}$ as her first move. Then, the rest of the game is played on $\phi_\alpha(U_\alpha \cap (M \setminus M'))$, so Alice can play according to her winning strategy on this set. This completes the proof. □

2.4. *Hyperplane potential game.* Finally, we recall the hyperplane potential game introduced in [23]. Being played on $\mathbb{R}^d$, it has the same winning sets as the hyperplane absolute game. This allows one to prove the HAW property of a set $S \subset \mathbb{R}^d$ by showing that it is winning for the hyperplane potential game (see [33]).

Let $S \subset \mathbb{R}^d$ be a target set and let $\beta \in (0, 1)$, $\gamma > 0$. The $(\beta, \gamma)$-*hyperplane potential game* is defined as follows. Bob begins by choosing a closed ball $B_0 \subset \mathbb{R}^d$. After Bob chooses a closed ball $B_i$ of radius $\rho_i$, Alice chooses a countable family of hyperplane neighborhoods $\{L_{i,k}^{(\rho_{i,k})} : k \in \mathbb{N}\}$ such that

$$\sum_{k=1}^{\infty} \rho_{i,k}^{\gamma} \le (\beta\rho_i)^{\gamma},$$

and then Bob chooses a closed ball $B_{i+1} \subset B_i$ of radius $\rho_{i+1} \ge \beta\rho_i$. Alice wins the game if and only if

$$\bigcap_{i=0}^{\infty} B_i \cap \left( S \cup \bigcup_{i=0}^{\infty} \bigcup_{k=1}^{\infty} L_{i,k}^{(\rho_{i,k})} \right) \neq \varnothing.$$

The set $S$ is $(\beta, \gamma)$-*hyperplane potential winning* ($(\beta, \gamma)$-*HPW* for short) if Alice has a winning strategy, and is *hyperplane potential winning* (*HPW* for short) if it is $(\beta, \gamma)$-HPW for any $\beta \in (0, 1)$ and $\gamma > 0$. The following lemma is a special case of [23, Theorem C.8].

LEMMA 2.4. *A subset $S \subset \mathbb{R}^d$ is HPW if and only if it is HAW.*

## 3. *Proof of Theorems 1.1 and 1.5*

In this section, we prove Theorems 1.1 and 1.5 by reducing them to [25, Corollary 5.5]. For the sake of convenience, we state one version of the aforementioned result here.

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THEOREM 3.1. [25, Corollary 5.5] *Let $G$ be a Lie group, $\Gamma$ be a lattice in $G$, $F$ be a one-parameter* Ad*-diagonalizable subgroup of $G$, $H = H(F^+)$ be the unstable horospherical subgroup with respect to $F^+$, and $X = G/\Gamma$. Then, for any $x \in X$, the set*

$$E_H^+(x, \infty) := \{h \in H : F^+hx \text{ is bounded in } X\}$$

*is thick in $H$.*

*Proof of Theorem 1.1.* We first recall some notation in §1. Let $M$ be a locally symmetric space of non-compact type with finite Riemannian volume. Then, $M$ can be identified with $Y = K\backslash G/\Gamma$, where $G$ is a semisimple Lie group without compact factors and with finite center, $K$ is a maximal compact subgroup of $G$, and $\Gamma$ is a lattice in $G$. Let $X = G/\Gamma$, $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{k} = \mathrm{Lie}(K)$, and $\mathfrak{p} \subset \mathfrak{g}$ be the orthogonal complement of $\mathfrak{k}$. Then, it suffices to show that: for any point $y = [Kg\Gamma] \in Y$, the set

$$\{\mathbf{v} \in \mathfrak{p}^1 : \{\exp(t\mathbf{v})g\Gamma : t \geq 0\} \text{ is bounded in } X\}$$

is thick in $\mathfrak{p}^1$.

Let us choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and an open Weyl chamber $\mathfrak{a}_+ \subset \mathfrak{a}$. Let $L = Z_K(\mathfrak{a})$ and $\mathfrak{a}_+^1 = \mathfrak{a}_+ \cap \mathfrak{p}^1$. Then, the map

$$\theta : \mathfrak{a}_+^1 \times L\backslash K \to \mathfrak{p}^1, \quad (\mathbf{v}, Lk) \mapsto (\mathrm{Ad}(k^{-1}))\mathbf{v}$$

is a diffeomorphism onto an open dense subset of $\mathfrak{p}^1$. Then, it suffices to prove that the set

$$\{(\mathbf{v}, Lk) \in \mathfrak{a}_+^1 \times L\backslash K : \{\exp(t\mathbf{v})kg\Gamma : t \geq 0\} \text{ is bounded in } X\} \tag{3.1}$$

is thick in $\mathfrak{a}_+^1 \times L\backslash K$. Moreover, by Marstrand's slicing and projection theorems [30], we only need to prove that: for any $\mathbf{v} \in \mathfrak{a}_+^1$, the set

$$E_K^+([g\Gamma], \infty) := \{k \in K : \{\exp(t\mathbf{v})kg\Gamma : t \geq 0\} \text{ is bounded in } X\} \tag{3.2}$$

is thick in $K$, since the set (3.1) can be decomposed into a union of slices, each of which is a quotient of a set of the form (3.2).

Let $F_{\mathbf{v}}^+ = \{\exp(t\mathbf{v}) : t \geq 0\}$ be the one-parameter subsemigroup of $G$, and $H = H(F_{\mathbf{v}}^+)$ (respectively $H^0 = H^0(F_{\mathbf{v}}^+)$, $H^- = H^-(F_{\mathbf{v}}^+)$) be the unstable (respectively neutral, stable) horospherical subgroup with respect to $F_{\mathbf{v}}^+$. Let $B = H^0H$ and $B^- = H^-H^0$, and consider the action of the subgroup $K$ on the real flag variety $G/B$ by left translation. By Iwasawa decomposition, this action is transitive. It follows that for any $g \in G$, the map

$$\phi_g : K \to G/B, \quad k \mapsto kgB$$

is a projection map with compact fiber. Write $K' = K \cap B^-Bg^{-1} = K \cap B^-Hg^{-1}$. Then, $K \setminus K' = \phi_g^{-1}((G \setminus B^{-1}B)/B)$ is a finite union of manifolds of strict lower dimensions. So, it suffices to show that $E_K^+([g\Gamma], \infty) \cap K'$ is thick in $K'$.

For $k \in K'$, we set $\sigma_1(k) \in B^-$ and $\sigma_2(k) \in H$ to be the unique pair satisfying $kg = \sigma_1(k)\sigma_2(k)$. It is easy to check that the map $\sigma_2$ is a projection map. However,

$$F_{\mathbf{v}}^+kg\Gamma \text{ is bounded} \iff F_{\mathbf{v}}^+\sigma_2(k)\Gamma \text{ is bounded.}$$

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Consequently,

$$E_K^+([g\Gamma], \infty) \cap K' = \sigma_2^{-1}(\{h \in H : F_{\mathbf{v}}^+ h\Gamma \text{ is bounded in } X\}) = \sigma_2^{-1}(E_H^+([1_G\Gamma], \infty)).$$

In view of Theorem 3.1, we see that $E_H^+([1_G\Gamma], \infty)$ is thick in $H$ and, hence, by Marstrand's slicing theorem, $E_K^+([g\Gamma], \infty) \cap K'$ is thick in $K'$. This completes the proof. □

*Proof of Theorem 1.5.* We begin with a simple observation, that is, it suffices to prove Theorem 1.7 for $x = [1_G\Gamma] \in X$. Indeed, for any $x' = [g\Gamma] \in X$ and one-parameter subgroup $F \subset G$, $F^+x'$ is bounded if and only if $g^{-1}F^+gx$ is bounded, which implies

$$E^+(x', \infty) = \mathrm{Ad}(g)(E^+(x, \infty)).$$

Thus, it suffices to prove Theorem 1.5 for $x = [1_G\Gamma] \in X$.

Let $p : \mathfrak{g} \setminus \{0\} \to \mathbf{P}^+(\mathfrak{g})$ denote the projection map. By Marstrand's projection and slicing theorems, to prove Theorem 1.5 is equivalent to show that the set

$$\{\mathbf{v} \in \mathfrak{g} : F_{\mathbf{v}}^+\Gamma \text{ is bounded in } X\}$$

is thick in $\mathfrak{g}$. We claim that we may reduce to the case when $G$ is a connected semisimple Lie group with trivial center and without compact factors for the following reasons.

First, let $G^0$ denote the identity component of $G$. It is clear that $\mathrm{Lie}(G^0) = \mathfrak{g}$ and that $X^0 = G^0/(G^0 \cap \Gamma)$ is a connected component of $X$ containing $[1_G\Gamma]$. In particular,

$$F_{\mathbf{v}}^+\Gamma \text{ is bounded in } X \Longleftrightarrow F_{\mathbf{v}}^+\Gamma \text{ is bounded in } X^0.$$

So, we may assume that $G = G^0$ is connected without loss of generality.

Second, let $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{l}$ be the Levi decomposition, where $\mathfrak{r}$ is the solvable radical of $\mathfrak{g}$ and $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$. Let $\mathfrak{l} = \mathfrak{l}_c \oplus \mathfrak{l}_{nc}$ be the decomposition into simple ideals where $\mathfrak{l}_c$ is the direct sum of all compact simple ones. The projection map $\mathrm{d}q : \mathfrak{g} \to \mathfrak{l}_{nc}$ is a Lie algebra homomorphism, which induces a Lie group homomorphism $q : G \to L_{nc}$ and a projection map $X = G/\Gamma \to X_{nc} := L_{nc}/q(\Gamma)$. Note that $q(\Gamma)$ is a lattice in $L_{nc}$, and the fiber of $X \to X_{nc}$ is a compact homogeneous space $RL_c/(RL_c \cap \Gamma)$ (see [19, Lemma 9.1]). Then, for $\mathbf{v} \in \mathfrak{g}$,

$$F_{\mathbf{v}}^+\Gamma \text{ is bounded in } X \Longleftrightarrow q(F_{\mathbf{v}})^+q(\Gamma) = F_{\mathrm{d}q(\mathbf{v})}^+q(\Gamma) \text{ is bounded in } X_{nc}.$$

It follows that

$$\{\mathbf{v} \in \mathfrak{g} : F_{\mathbf{v}}^+\Gamma \text{ is bounded in } X\} = (\mathrm{d}q)^{-1}\{\mathbf{w} \in \mathfrak{l}_{nc} : F_{\mathbf{w}}^+q(\Gamma) \text{ is bounded in } X_{nc}\}.$$

By Marstrand's slicing theorem, it suffices to show that

$$\{\mathbf{w} \in \mathfrak{l}_{nc} : F_{\mathbf{w}}^+q(\Gamma) \text{ is bounded in } X_{nc}\}$$

is thick in $\mathfrak{l}_{nc}$. So, we may assume that $G$ is a semisimple Lie group without compact factors.

Finally, since $X = G/\Gamma$ is a finite cover of $G/(Z(G) \cdot \Gamma)$, we may replace $G$ by $G/Z(G)$ and $\Gamma$ by $(\Gamma \cdot Z(G))/Z(G)$ without loss of generality. These verify the claim.

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Now, let $G$ be a connected semisimple Lie group with trivial center and without compact factors. Let $G = K \cdot \exp(\mathfrak{p})$ be the global Cartan decomposition, $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, and $L = Z_K(\mathfrak{a})$. We choose an open Weyl chamber $\mathfrak{a}_+ \subset \mathfrak{a}$, and for any $\mathbf{v} \in \mathfrak{a}_+$, let $H = H(F^+_{\mathbf{v}})$ (respectively $H^0 = H^0(F^+_{\mathbf{v}})$, $H^- = H^-(F^+_{\mathbf{v}})$) be the unstable (respectively neutral, stable) horospherical subgroup with respect to $F^+_{\mathbf{v}}$. Note that the subgroups $H$, $H^0$, $H^-$ only depend on the choice of the open Weyl chamber $\mathfrak{a}_+$.

Consider the map

$$\theta_0 : \mathfrak{a}_+ \times Z_G(\mathfrak{a}_+)\backslash G \to \mathfrak{g}, \quad (\mathbf{v}, Z_G(\mathfrak{a}_+)g) \mapsto \mathrm{Ad}(g^{-1})\mathbf{v}.$$

It is easy to check that $\theta$ is a diffeomorphism onto $\mathfrak{g}$. Using Bruhat decomposition, there is a diffeomorphism onto its image:

$$H \times L \times H^- \to Z_G(\mathfrak{a}_+)\backslash G, \quad (h_1, h_2, h_3) \mapsto Z_G(\mathfrak{a}_+)h_3h_2h_1.$$

It follows that the composition map

$$\theta : \mathfrak{a}_+ \times H \times L \times H^- \to \mathfrak{g}, \quad (\mathbf{v}, h_1, h_2, h_3) \mapsto \mathrm{Ad}(h_1^{-1}h_2^{-1}h_3^{-1})\mathbf{v}$$

is also a diffeomorphism onto its image. Moreover, we see that $p$ is a projection map when restricted to $\mathrm{Im}(\theta)$ and that $\mathfrak{g} \setminus \mathrm{Im}\,\theta$ is a finite union of submanifolds of strict lower dimensions. Thus, it suffices to prove that

$$\{(\mathbf{v}, h_1, h_2, h_3) \in \mathfrak{a}_+ \times H \times L \times H^- : F^+_{\mathbf{v}}h_3h_2h_1\Gamma \text{ is bounded in } X\}$$

is thick on $\mathfrak{a}_+ \times H \times L \times H^-$. However, we have for any $\mathbf{v} \in \mathfrak{a}_+$,

$$F^+_{\mathbf{v}}h_3h_2h_1\Gamma \text{ is bounded in } X \Longleftrightarrow F^+_{\mathbf{v}}h_1\Gamma \text{ is bounded in } X.$$

So, the conclusion follows from Marstrand's slicing theorem and Theorem 3.1. This completes the proof. □

## 4. *Proof of Theorems 1.3 and 1.7*

In this section, we prove Theorems 1.3 and 1.7 by assuming the truth of Theorem 4.1 in the following. The proof of Theorem 4.1 will be postponed to the next section.

For any $\lambda \in \Pi := (1/2, 1)$, set

$$\mathbf{v}_\lambda = \mathrm{diag}(\lambda, 1-\lambda, -1) \in \mathfrak{g}, \quad F^+_\lambda = \{\exp(t\mathbf{v}_\lambda) : t \geq 0\} \quad \text{and} \quad F^-_\lambda = \{\exp(-t\mathbf{v}_\lambda) : t \geq 0\}.$$

Let $U \subset G$ (respectively $U^- \subset G$) be the subgroup of unipotent upper triangular (respectively lower triangular) matrices and $A \subset G$ be the subgroup of diagonal matrices with positive entries. Then, $U$ (respectively $U^-$) is the expanding (respectively contracting) subgroup of $F^+_\lambda$ for any $\lambda \in \Pi$. Define

$$E_U(F^+_\lambda) := \{u \in U : F^+_\lambda u\Gamma \text{ is bounded in } X\}.$$

Both Theorems 1.3 and 1.7 will be deduced from the following theorem.

THEOREM 4.1. *The set*

$$S := \{(\lambda, u) \in \Pi \times U : u \in E_U(F^+_\lambda)\} \tag{4.1}$$

*is HAW on* $\Pi \times U$.

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*Remark 4.2.* In [5], it is proved that $E_U(F_\lambda^+)$ is HAW on $U$ for any $\lambda \in \Pi$. However, Theorem 4.1 does not follow from this result. It is generally believed that for a general Borel subset in a product space, winning properties of all its slices do not necessarily imply the winning property of itself.

*Proof of Theorem 1.3 modulo Theorem 4.1.* We first recall basic facts about geodesic flows on $Y$ from [31]. Let $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{p} \subset \mathfrak{g}$ be its orthogonal complement. Then, for any $y = [Kg\Gamma] \in Y$, $S_y(Y)$ can be identified with $\mathfrak{p}^1 = \{\mathbf{v} \in \mathfrak{p} : \|\mathbf{v}\| = 1\}$. Moreover, for any $y = [Kg\Gamma] \in Y$ and $\mathbf{v} \in \mathfrak{p}^1$, the associated geodesic ray is given by

$$\gamma(y, \mathbf{v}) = \{K \exp(t\mathbf{v})g\Gamma : t \geq 0\}.$$

Since $K$ is compact, we have

$$E^+(y, \infty) = \{\mathbf{v} \in \mathfrak{p}^1 : F_{\mathbf{v}}^+ g\Gamma \text{ is bounded in } X\}.$$

The proof of Theorem 1.3 will be given in two steps. To begin, we consider the map

$$\theta : \Pi \times K \to \mathfrak{p}^1, \quad \theta(\lambda, k) = \mathrm{Ad}(k^{-1})\mathbf{v}'_\lambda \quad \text{where } \mathbf{v}'_\lambda = \|\mathbf{v}_\lambda\|^{-1}\mathbf{v}_\lambda \in \mathfrak{p}^1.$$

It is easy to check that $\mathfrak{p}^1 \setminus (\mathrm{Im}\,\theta \cup -\mathrm{Im}\,\theta)$ is a finite union of manifolds of strict lower dimension. Hence, in view of Lemma 2.3, it suffices to prove that $E^+(y, \infty) \cap \pm\mathrm{Im}\,\theta$ is HAW on $\pm\mathrm{Im}\,\theta$. By applying $\overline{\tau} : X \to X$ induced by the outer automorphism $\tau(g) = (g^T)^{-1}$ defined on $G$, we get

$$E^+(y, \infty) \cap -\mathrm{Im}\,\theta = \mathrm{d}\tau(E^+(y', \infty) \cap \mathrm{Im}\,\theta) \quad \text{where } y' = (g^T)^{-1}\Gamma.$$

Thus, it suffices to show $E^+(y, \infty) \cap \mathrm{Im}\,\theta$ is HAW on $\mathrm{Im}\,\theta$ for any $y \in Y$. However, it is also easy to check that $\theta$ is a cover map. Hence, in view of Lemma 2.2(iii), it suffices to show that

$$E' := \theta^{-1}(E^+(y, \infty)) = \{(\lambda, k) \in \Pi \times K : F_\lambda^+ kg\Gamma \text{ is bounded in } X\}$$

is HAW on $\Pi \times K$.

Let $B \subset G$ (respectively $B^- \subset G$) be the subgroup of upper triangular (respectively lower triangular) matrices, and consider the action of the subgroup $K$ on the real flag variety $G/B$ by left translations. By Iwasawa's decomposition, this action is transitive and has finite stabilizer. It follows that for any $g \in G$, the map $\phi_g : K \to G/B$ defined by $\phi_g(k) = kgB$ is a cover map. Write $K' = K \cap B^- Bg^{-1} = K \cap B^- Ug^{-1}$. Then, $K \setminus K' = \phi_g^{-1}((G \setminus B^{-1}B)/B)$ is a finite union of manifolds of strict lower dimensions. Using Lemma 2.3 again, it suffices to show that $E'' := E' \cap (\Pi \times K')$ is HAW on $\Pi \times K'$. For $k \in K'$, we set $\sigma_1(k) \in B^-$ and $\sigma_2(k) \in U$ to be the unique pair satisfying $kg = \sigma_1(k)\sigma_2(k)$. It is easy to check that the map $\sigma_2$ is a cover map. However,

$$F_\lambda^+ kg\Gamma \text{ is bounded in } X \Longleftrightarrow F_\lambda^+ \sigma_2(k)\Gamma \text{ is bounded in } X.$$

Consequently, $E'' = (\mathrm{Id} \times \sigma_2)^{-1}S$. Thus, in view of Lemma 2.2(iii) and Theorem 4.1, we see that $E''$ is HAW on $\Pi \times K'$. This completes the proof. □

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*Proof of Theorem 1.7 modulo Theorem 4.1.* We begin with a simple observation, that is, it suffices to prove Theorem 1.7 for $x = [1_G\Gamma] \in X$. Indeed, for any $x' = [g\Gamma] \in X$ and one-parameter subgroup $F \subset G$, $Fx'$ is bounded if and only if $g^{-1}Fgx$ is bounded, which implies

$$E^+(x', \infty) = \mathrm{Ad}(g)(E^+(x, \infty)).$$

Hence, in view of Lemma 2.2(iii), it suffices to prove Theorem 1.7 for $x = [1_G\Gamma] \in X$.

Consider the map

$$\theta : \Pi \times U \times U^- \to \mathfrak{g}, \quad (\lambda, u_1, u_2) \mapsto \mathrm{Ad}(u_1^{-1}u_2^{-1})\mathbf{v}_\lambda.$$

It is straightforward to check that the map $\theta$ is a diffeomorphism onto its image. Let $p$ be the projection from $\mathfrak{g} \setminus \{0\}$ to $\mathbf{P}^+(\mathfrak{g})$. Then, it can be checked that $p$ is a diffeomorphism onto its image when restricted on $\mathrm{Im}\,\theta \cup -\mathrm{Im}\,\theta$ and $\mathbf{P}^+(\mathfrak{g}) \setminus (p(\mathrm{Im}\,\theta) \cup -p(\mathrm{Im}\,\theta))$ is a finite union of submanifolds of strict lower dimensions. In view of Lemmas 2.2(iv) and 2.3, it suffices to show that $E^+([1_G\Gamma], \infty) \cap \pm p(\mathrm{Im}\,\theta)$ is HAW on $\pm p(\mathrm{Im}\,\theta)$, respectively. Set

$$E^+(\infty) = \{\mathbf{v} \in \mathfrak{g} : F_{\mathbf{v}}^+\Gamma \text{ is bounded in } X\}.$$

Then, $E^+([1_G\Gamma], \infty) = p(E^+(\infty) \setminus \{0\})$. We claim that Theorem 1.7 can be deduced from the following statement:

$$E^+(\infty) \cap \mathrm{Im}\,\theta \text{ is HAW on } \mathrm{Im}\,\theta. \tag{4.2}$$

Indeed, on one hand, it follows from Lemma 2.2(iii) and (4.2) that

$$E^+([1_G\Gamma], \infty) \cap p(\mathrm{Im}\,\theta) = p(E^+(\infty) \cap \mathrm{Im}\,\theta) \text{ is HAW on } p(\mathrm{Im}\,\theta).$$

On the other hand, by applying the map $\overline{\tau} : X \to X$ induced by the outer automorphism $\tau(g) = (g^T)^{-1}$ defined on $G$, we see that

$$E^+([1_G\Gamma], \infty) \cap -p(\mathrm{Im}\,\theta) = (\mathrm{d}\tau \circ p)(E^+(\infty) \cap \mathrm{Im}\,\theta).$$

It follows again from Lemma 2.2(iii) and (4.2) that

$$E^+([1_G\Gamma], \infty) \cap -p(\mathrm{Im}\,\theta) \text{ is HAW on } (\mathrm{d}\tau \circ p)(\mathrm{Im}\,\theta) = -p(\mathrm{Im}\,\theta).$$

These verify our claim.

Finally, to prove (4.2), it suffices to establish the HAW property of $\theta^{-1}(E^+(\infty))$ on $\Pi \times U \times U^-$. However, we have

$$\begin{aligned}(\lambda, u_1, u_2) \in \theta^{-1}(E^+(\infty)) &\Longleftrightarrow F_\lambda^+ u_2 u_1 \Gamma \text{ is bounded} \\ &\Longleftrightarrow (\lambda, u_1) \in S,\end{aligned}$$

which means that $\theta^{-1}(E^+(\infty)) = S \times U^-$. So, the conclusion follows from Lemma 2.2(v) and Theorem 4.1. This completes the proof. □

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## 5. *Proof of Theorem 4.1*

In this section, we prove Theorem 4.1 modulo two key lemmas.

For $(x, y, z) \in \mathbb{R}^3$, set

$$u_{x,y,z} := \begin{pmatrix} 1 & z & x \\ & 1 & y \\ & & 1 \end{pmatrix} \in U.$$

For technical reasons, we will prove Theorem 4.1 by applying Lemma 2.2(iii) to the diffeomorphism $\Psi : \Pi \times \mathbb{R}^3 \to \Pi \times U$ defined by

$$\Psi(\lambda, (x, y, z)) = (\lambda, u_{x,y,z}^{-1}).$$

For simplicity, we write

$$\Sigma := \Pi \times \mathbb{R}^3.$$

It is directly checked that

$$\Psi^{-1}(S) = \{(\lambda, (x, y, z)) \in \Sigma : F_\lambda^+ u_{x,y,z}^{-1} \text{ is bounded}\}.$$

So, it suffices to prove $\Psi^{-1}(S)$ is HAW on $\Sigma$.

For $\lambda \in \Pi$, $\epsilon > 0$, and $\mathbf{v} = (p, r, q) \in \mathbb{Z}^2 \times \mathbb{N}$, we set

$$\Delta_\lambda(\mathbf{v}, \epsilon) := \left\{(x, y, z) \in \mathbb{R}^3 : \left|x - \frac{p}{q} - z\left(y - \frac{r}{q}\right)\right| < \frac{\epsilon}{q^{1+\lambda}}, \left|y - \frac{r}{q}\right| < \frac{\epsilon}{q^{2-\lambda}}\right\}$$

and

$$S_\lambda(\epsilon) := \mathbb{R}^3 \setminus \bigcup_{\mathbf{v} \in \mathbb{Z}^2 \times \mathbb{N}} \Delta_\lambda(\mathbf{v}, \epsilon). \tag{5.1}$$

Note that if $\mathbf{v}' = t\mathbf{v}$ with $t > 1$, then $\Delta_\lambda(\mathbf{v}', \epsilon) \subset \Delta_\lambda(\mathbf{v}, \epsilon)$. Thus,

$$S_\lambda(\epsilon) = \mathbb{R}^3 \setminus \bigcup_{\mathbf{v} \in \Xi} \Delta_\lambda(\mathbf{v}, \epsilon) \quad \text{where } \Xi = \{(p, r, q) \in \mathbb{Z}^2 \times \mathbb{N} : \gcd(p, r, q) = 1\}.$$

Moreover, we set

$$\Delta(\mathbf{v}, \epsilon) := \bigcup_{\lambda \in \Pi} \{\lambda\} \times \Delta_\lambda(\mathbf{v}, \epsilon) \quad \text{and} \quad S(\epsilon) := \Sigma \setminus \bigcup_{\mathbf{v} \in \Xi} \Delta(\mathbf{v}, \epsilon) = \bigcup_{\lambda \in \Pi} \{\lambda\} \times S_\lambda(\epsilon).$$

According to [5, Lemma 3.2], we know that $\{\lambda\} \times S_\lambda(\epsilon) \subset \Psi^{-1}(S)$ for any $\lambda \in \Pi$ and $\epsilon > 0$. Hence, $S(\epsilon) \subset \Psi^{-1}(S)$ for any $\epsilon > 0$. Therefore, Theorem 4.1 can be deduced from the following lemma.

LEMMA 5.1. *The set*

$$S' := \bigcup_{\epsilon > 0} S(\epsilon)$$

*is HAW on* $\Sigma$.

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The rest of this section is devoted to the proof of Lemma 5.1. For convenience, we endow $\Sigma$ with the product metric. For a closed ball $\Omega \subset \Sigma$, we denote its radius as $\rho_\Omega$ and its center as $(\lambda_\Omega, \omega_\Omega)$ with $\lambda_\Omega \in \Pi$, $\omega_\Omega = (x_\Omega, y_\Omega, z_\Omega) \in \mathbb{R}^3$. Hence, we have

$$\Omega = I_\Omega \times B_\Omega \quad \text{where } I_\Omega = I(\lambda_\Omega, \rho_\Omega) \quad \text{and} \quad B_\Omega = B(\omega_\Omega, \rho_\Omega).$$

Here, $I(\lambda_\Omega, \rho_\Omega)$ and $B(\omega_\Omega, \rho_\Omega)$ denote the closed balls in $\Pi$ and $\mathbb{R}^3$, respectively.

For a closed ball $\Omega \subset \Sigma$ and $\mathbf{v} = (p, r, q) \in \Xi$, consider the set of integral vectors

$$\mathcal{W}(\Omega, \mathbf{v}) := \{(a, b, c) \in \mathbb{Z}^3 : (a, b) \neq (0, 0), ap + br + cq = 0, \\ |a| \leq e^{\sqrt{\rho_\Omega}} q^{\lambda_\Omega}, |b + z_\Omega a| \leq e^{\sqrt{\rho_\Omega}} q^{1-\lambda_\Omega}\}.$$

It follows from Minkowski's linear forms theorem that $\mathcal{W}(\Omega, \mathbf{v}) \neq \varnothing$. We choose and fix

$$\mathbf{w}(\Omega, \mathbf{v}) = (a(\Omega, \mathbf{v}), b(\Omega, \mathbf{v}), c(\Omega, \mathbf{v})) \in \mathcal{W}(\Omega, \mathbf{v})$$

such that

$$\max\{|a(\Omega, \mathbf{v})|, |b(\Omega, \mathbf{v}) + z_\Omega a(\Omega, \mathbf{v})|\} \\ = \min\{\max\{|a|, |b + z_\Omega a|\} : (a, b, c) \in \mathcal{W}(\Omega, \mathbf{v})\}, \tag{5.2}$$

and define

$$H_\Omega(\mathbf{v}) := q \max\{|a(\Omega, \mathbf{v})|, |b(\Omega, \mathbf{v}) + z_\Omega a(\Omega, \mathbf{v})|\}.$$

LEMMA 5.2. *For any closed ball $\Omega \subset \Sigma$ and $\mathbf{v} = (p, r, q) \in \Xi$, we have*

$$q \leq H_\Omega(\mathbf{v}) \leq q^{1+\lambda_\Omega}. \tag{5.3}$$

*Proof.* The first inequality is obvious. By Minkowski linear forms theorem, $\mathcal{W}(\Omega, \mathbf{v})$ contains a vector $(a_0, b_0, c_0)$ with $|a_0| \leq q^{\lambda_\Omega}$ and $|b_0 + z_\Omega a_0| \leq q^{1-\lambda_\Omega}$. Thus, it follows from (5.2) that

$$\max\{|a(\Omega, \mathbf{v})|, |b(\Omega, \mathbf{v}) + z_\Omega a(\Omega, \mathbf{v})|\} \\ \leq \max\{|a_0|, |b_0 + z_\Omega a_0|\} \leq \max\{q^{\lambda_\Omega}, q^{1-\lambda_\Omega}\} = q^{\lambda_\Omega}.$$

Hence, the second inequality. □

Let $\Omega_0 \subset \Sigma$ be a closed ball of radius $\rho_0$. Then, $\rho_0 < 1/4$. Let $\kappa > 2$ be such that

$$\max\{|x|, |y|, |z|\} \leq \kappa - 1 \quad \text{for all } (x, y, z) \in B_{\Omega_0}. \tag{5.4}$$

Let $\beta \in (0, 1)$, and $R$ and $\epsilon$ be positive numbers such that

$$R \geq \max\{100\beta^{-1}, 10^4\kappa^4\} \quad \text{and} \quad \sqrt{R^{-1}} \log R^4 \leq 1, \tag{5.5}$$

and

$$\epsilon \leq R^{-24}\rho_0. \tag{5.6}$$

Let $\mathscr{B}_0 = \{\Omega_0\}$. For $n \geq 1$, let $\mathscr{B}_n$ be the family of closed balls defined by

$$\mathscr{B}_n := \{\Omega \subset \Omega_0 : \beta R^{-n}\rho_0 < \rho_\Omega \leq R^{-n}\rho_0\}.$$

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For a closed ball $\Omega$, if

$$\Omega \in \mathscr{B}_n \quad \text{for some } n \geq 0, \tag{5.7}$$

we define

$$\mathcal{V}_\Omega := \{\mathbf{v} \in \Xi : H_n \leq H_\Omega(\mathbf{v}) \leq 3H_{n+1}\},$$

where

$$H_n := 20\epsilon\kappa\rho_0^{-1}R^n, \quad n \geq 0.$$

Note that since $R > \beta^{-1}$, the families $\mathscr{B}_n$ are pairwise disjoint. So, (5.7) is satisfied for at most one integer $n$, and hence, $\mathcal{V}_\Omega$ is well defined. It follows from (5.3) that if $\mathbf{v} = (p, r, q) \in \mathcal{V}_\Omega$, then

$$H_n^{(1/1+\lambda_\Omega)} \leq q \leq 3H_{n+1}. \tag{5.8}$$

Whenever (5.7) is satisfied, we also define

$$\mathcal{V}_{\Omega,1} := \{(p, r, q) \in \mathcal{V}_\Omega : H_n^{(1/1+\lambda_\Omega)} \leq q \leq H_n^{(1/1+\lambda_\Omega)}R^{20}\}$$

and

$$\mathcal{V}_{\Omega,k} := \{(p, r, q) \in \mathcal{V}_\Omega : H_n^{(1/1+\lambda_\Omega)}R^{2k+16} \leq q \leq H_n^{(1/1+\lambda_\Omega)}R^{2k+18}\}, \quad k \geq 2.$$

LEMMA 5.3. *If $\Omega \in \mathscr{B}_n$, then $\mathcal{V}_\Omega = \bigcup_{k=1}^{[n/2]} \mathcal{V}_{\Omega,k}$.*

*Proof.* In view of (5.8), it suffices to show that if $k > [n/2]$, then $\mathcal{V}_{\Omega,k} = \varnothing$. Suppose to the contrary that $\mathcal{V}_{\Omega,k} \neq \varnothing$ for some $k > [n/2]$. Let $(p, r, q) \in \mathcal{V}_{\Omega,k}$. Then,

$$3H_{n+1} \geq q \geq H_n^{(1/1+\lambda_\Omega)}R^{2k+16} > H_n^{(1/1+\lambda_\Omega)}R^{n+15}.$$

However,

$$H_n^{-(1/1+\lambda_\Omega)}R^{-n-15} \cdot 3H_{n+1} = 3(20\epsilon\kappa\rho_0^{-1})^{(\lambda_\Omega/1+\lambda_\Omega)}R^{-(1/1+\lambda_\Omega)n-14} < 1,$$

which is a contradiction. □

Next, we inductively define a subfamily $\mathscr{B}'_n$ of $\mathscr{B}_n$ as follows. Let $\mathscr{B}'_0 = \{\Omega_0\}$. If $n \geq 1$ and $\mathscr{B}'_{n-1}$ has been defined, we let

$$\mathscr{B}'_n := \left\{\Omega \in \mathscr{B}_n : \Omega \subset \Omega' \text{ for some } \Omega' \in \mathscr{B}'_{n-1} \text{ and } \Omega \cap \bigcup_{\mathbf{v}\in\mathcal{V}_\Omega} \Delta(\mathbf{v}, \epsilon) = \varnothing\right\}. \tag{5.9}$$

The following two lemmas concerning $\mathscr{B}'_n$ are key steps in the proof of Theorem 4.1. Their proofs are technical and postponed to the next two sections.

LEMMA 5.4. *Let $n \geq 0$, $\Omega \in \mathscr{B}'_n$, and $\mathbf{v} = (p, r, q) \in \Xi$. If $q^{1+\lambda_\Omega} \leq 2H_{n+1}$, then $\Delta(\mathbf{v}, \epsilon) \cap \Omega = \varnothing$.*

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LEMMA 5.5. *Let $n \geq 0$, $\Omega \in \mathscr{B}'_n$, and $k \geq 1$. There exists an affine hyperplane $L_k(\Omega) \subset \Sigma$ such that for any $\Omega' \in \mathscr{B}_{n+k}$ with $\Omega' \subset \Omega$ and any $\mathbf{v} \in \mathcal{V}_{\Omega',k}$, we have*

$$\Delta(\mathbf{v}, \epsilon) \cap \Omega' \subset L_k(\Omega)^{(R^{-(n+k)}\rho_0)}.$$

We now prove Lemma 5.1 assuming the truth of Lemmas 5.4 and 5.5.

*Proof of Lemma 5.1.* By Lemma 2.4, it suffices to prove that $S'$ is HPW on $\Sigma$. Let $\beta \in (0, 1)$, $\gamma > 0$. We prove that Alice has a strategy to win the $(\beta, \gamma)$-hyperplane potential game on $\Sigma$ with target set $S'$. In the first round of the game, Bob chooses a closed ball $\Omega_0 \subset \Sigma$. By [5, Remark 2.4], we may without loss of generality assume that Bob will play so that $\rho_i := \rho(\Omega_i) \to 0$. Let $\kappa$ and $R$ be positive numbers satisfying (5.4) and (5.5). We also require that

$$(R^\gamma - 1)^{-1} \leq (\beta^2/2)^\gamma. \tag{5.10}$$

Let $\epsilon$, $\mathscr{B}_n$, $\mathcal{V}_\Omega$, $\mathcal{V}_{\Omega,k}$, and $\mathscr{B}'_n$ be as above. Then, Lemmas 5.3–5.5 hold. Let Alice play according to the following strategy. Suppose that $i \geq 0$ and Bob has chosen the closed ball $\Omega_i$. If there is $n \geq 0$ such that

$$\Omega_i \in \mathscr{B}'_n, \text{ and } i \text{ is the smallest non-negative integer with } \Omega_i \in \mathscr{B}_n, \tag{5.11}$$

then Alice chooses the family of neighborhoods $\{L_k(\Omega_i)^{(2R^{-(n+k)}\rho_0)} : k \in \mathbb{N}\}$, where $L_k(\Omega_i)$ are the hyperplanes given in Lemma 5.5. Otherwise, Alice makes an arbitrary move. Since $\Omega_i \in \mathscr{B}_n$, we have that $\rho_i > \beta R^{-n}\rho_0$. This, together with (5.10), implies that

$$\sum_{k=1}^{\infty} (2R^{-(n+k)}\rho_0)^\gamma = (2R^{-n}\rho_0)^\gamma (R^\gamma - 1)^{-1} \leq (\beta\rho_i)^\gamma.$$

So, Alice's move is legal. We prove that this guarantees a win for Alice, that is, the unique point $\mathbf{x}_\infty = (\lambda_\infty, \omega_\infty)$ in $\bigcap_{i=0}^{\infty} \Omega_i$ lies in

$$S' \cup \bigcup_{n \in \mathcal{N}} \bigcup_{k=1}^{n} L_k(\Omega_{i_n})^{(2R^{-(n+k)}\rho_0)},$$

where

$$\mathcal{N} := \{n \geq 0 : \text{ there exists } i = i_n \text{ such that (5.11) holds}\}.$$

There are two cases.

*Case (1).* Assume $\mathcal{N} = \mathbb{N} \cup \{0\}$. For any $\mathbf{v} = (p, r, q) \in \Xi$, there is $n$ such that $q^2 \leq 2H_{n+1}$. Then, it follows that $q^{1+\lambda_n} \leq q^2 \leq 2H_{n+1}$. Since $n \in \mathcal{N}$, we have that $\Omega_{i_n} \in \mathscr{B}'_n$. Then, Lemma 5.4 implies that $\Delta(\mathbf{v}, \epsilon) \cap \Omega_{i_n} = \varnothing$. Thus, $\mathbf{x}_\infty \notin \Delta(\mathbf{v}, \epsilon)$. It then follows from (5.1) and the arbitrariness of $\mathbf{v}$ that $\mathbf{x}_\infty \in S(\epsilon) \subset S'$. Hence, Alice wins.

*Case (2).* Assume $\mathcal{N} \neq \mathbb{N} \cup \{0\}$. Let $n$ be the smallest non-negative integer with $n \notin \mathcal{N}$. Since $B_0 \in \mathscr{B}'_0$, we have $0 \in \mathcal{N}$. Hence, $n \geq 1$. Since $\rho_i \to 0$ and $\rho_{i+1} \geq \beta\rho_i$, there must exist $i \geq 1$ with $\Omega_i \in \mathscr{B}_n$ and $\Omega_{i-1} \notin \mathscr{B}_n$. It follows that $\Omega_i \notin \mathscr{B}'_n$. Since $n - 1 \in \mathcal{N}$, we have $\Omega_{i_{n-1}} \in \mathscr{B}'_{n-1}$. Thus, it follows from (5.9) that $\Omega_i \cap \bigcup_{\mathbf{v} \in \mathcal{V}_{\Omega_i}} \Delta(\mathbf{v}, \epsilon) \neq \varnothing$. By Lemma 5.3, there is $k \in [1, [n/2]]$ and $\mathbf{v} \in \mathcal{V}_{\Omega_i,k}$ such that $\Omega_i \cap \Delta(\mathbf{v}, \epsilon) \neq \varnothing$.

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Applying Lemma 5.5 to $\Omega = \Omega_{i_{n-k}}$ and $\Omega' = \Omega_i$, we obtain $\Omega_i \cap \Delta(\mathbf{v}, \epsilon) \subset L_k(\Omega_{i_{n-k}})^{(R^{-n}\rho_0)}$. So, $\Omega_i \cap L_k(\Omega_{i_{n-k}})^{(R^{-n}\rho_0)} \neq \varnothing$. In view of $\rho_i \leq R^{-n}\rho_0$, it follows that $\mathbf{x}_\infty \in \Omega_i \subset L_k(\Omega_{i_{n-k}})^{(2R^{-n}\rho_0)}$. Hence, Alice also wins.

This completes the proof of Theorem 4.1 modulo Lemmas 5.4 and 5.5. □

## 6. *Proof of Lemma 5.4*

We shall need the following simple but useful lemma.

LEMMA 6.1. *Let $n \geq 0$ and $\rho \in (0, R^{-n}\rho_0]$, then*

$$\rho \log H_{2n+2} \leq \tfrac{1}{2}\sqrt{\rho} \leq 1. \tag{6.1}$$

*Proof.* The second inequality is straightforward. For the first inequality, the case $n = 0$ is trivial since $H_2 < 1$. Let $n \geq 1$. Since the function $x^{-1} \log x$ is monotonically decreasing when $x > e$, it follows that

$$2\sqrt{\rho} \log H_{2n+2} \leq \sqrt{R^{-n}} \log R^{2n+2} \leq \sqrt{R^{-n}} \log R^{4n} \leq \sqrt{R^{-1}} \log R^4 \leq 1.$$

This completes the proof of the first inequality. □

*Proof of Lemma 5.4.* The case $n = 0$ is trivial since $2H_1 < 1$. Let $n \geq 1$. We denote $\Omega_n = \Omega$ and let $\Omega_n \subset \cdots \subset \Omega_1 \subset \Omega_0$ be such that $\Omega_k \in \mathscr{B}'_k$. Write $\lambda_{\Omega_k} = \lambda_k$ and $\rho_{\Omega_k} = \rho_k$ for simplicity. Suppose in contrast that the conclusion of the lemma is not true. Then, there exists $\mathbf{v} = (p, r, q) \in \mathbb{Z}^2 \times \mathbb{N}$ with $q^{1+\lambda_n} \leq 2H_{n+1}$ such that $\Delta(\mathbf{v}, \epsilon) \cap \Omega_k \neq \varnothing$ for every $1 \leq k \leq n$. It follows from the definition of $\mathscr{B}'_k$ that $\mathbf{v} \notin \mathcal{V}_{\Omega_k}$, that is,

$$H_{\Omega_k}(\mathbf{v}) \notin [H_k, 3H_{k+1}], \quad 1 \leq k \leq n. \tag{6.2}$$

Let $1 \leq m \leq n$ be such that

$$2H_m \leq q^{1+\lambda_n} \leq 2H_{m+1}. \tag{6.3}$$

We inductively prove that

$$H_{\Omega_k}(\mathbf{v}) < H_k, \quad \left[\frac{m}{2}\right] \leq k \leq m. \tag{6.4}$$

By letting $k = [m/2]$ in (6.4), we have

$$q \leq H_{\Omega_{[m/2]}}(\mathbf{v}) < H_{[m/2]} < \sqrt{H_m} < q,$$

which is a contradiction. So Lemma 5.4 follows from (6.4). Since

$$H_{\Omega_m}(\mathbf{v}) \leq q^{1+\lambda_m} \leq q^{|\lambda_m - \lambda_n|} q^{1+\lambda_n} \leq H_{m+1}^{\rho_m} \cdot 2H_{m+1} = e^{\rho_m \log H_{m+1}} \cdot 2H_{m+1} \leq 3H_{m+1},$$

it follows from (6.2) that (6.4) holds for $k = m$. Suppose that $[m/2] \leq k \leq m - 1$ and (6.4) holds if $k$ is replaced by $k + 1$. We prove that

$$H_{\Omega_k}(\mathbf{v}) \leq 2H_{\Omega_{k+1}}(\mathbf{v}). \tag{6.5}$$

To prove (6.5), the first step is to establish

$$\mathbf{w}(\Omega_{k+1}, \mathbf{v}) \in \mathcal{W}(\Omega_k, \mathbf{v}). \tag{6.6}$$

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Write $\mathbf{w}(\Omega_k, \mathbf{v}) = (a_k, b_k, c_k)$ and $z_{\Omega_k} = z_k$. Since $\mathbf{w}(\Omega_{k+1}, \mathbf{v}) \in \mathcal{W}(\Omega_{k+1}, \mathbf{v})$, we have

$$|a_{k+1}| \le e^{\sqrt{\rho_{k+1}}} q^{\lambda_{k+1}} \quad \text{and} \quad |b_{k+1} + z_{k+1} a_{k+1}| \le e^{\sqrt{\rho_{k+1}}} q^{1-\lambda_{k+1}}.$$

Note that $\sqrt{\rho_{k+1}} \le \sqrt{\beta^{-1} R^{-1}} \sqrt{\rho_k} \le (1/10)\sqrt{\rho_k}$ and by Lemma 6.1,

$$q^{|\lambda_{k+1} - \lambda_k|} \le e^{\rho_k \log q} \le e^{\rho_k \log H_{m+1}} \le e^{(1/2)\sqrt{\rho_k}}.$$

Now, we are in position to prove (6.6). First, we have

$$\begin{aligned} |a_{k+1}| &\le e^{\sqrt{\rho_{k+1}}} q^{\lambda_{k+1}} \\ &\le (e^{\sqrt{\rho_{k+1}} - \sqrt{\rho_k}} q^{|\lambda_{k+1} - \lambda_k|}) e^{\sqrt{\rho_k}} q^{\lambda_k} \\ &\le (e^{\sqrt{\rho_{k+1}} - (1/2)\sqrt{\rho_k}}) e^{\sqrt{\rho_k}} q^{\lambda_k} \\ &\le e^{\sqrt{\rho_k}} q^{\lambda_k}. \end{aligned}$$

Second, we have

$$\begin{aligned} |b_{k+1} + z_k a_{k+1}| &\le |b_{k+1} + z_{k+1} a_{k+1}| + |a_{k+1}||z_k - z_{k+1}| \\ &\le |b_{k+1} + z_{k+1} a_{k+1}| + |a_{k+1}| \rho_k \\ &\le (1 + |a_{k+1}| \rho_k) e^{\sqrt{\rho_{k+1}}} q^{1-\lambda_{k+1}} \\ &\le (1 + q^{-1} H_{\Omega_{k+1}}(\mathbf{v}) \rho_k)(e^{\sqrt{\rho_{k+1}} - \sqrt{\rho_k}} q^{|\lambda_{k+1} - \lambda_k|}) e^{\sqrt{\rho_k}} q^{1-\lambda_k} \\ &\le (1 + \sqrt{H_m^{-1} H_{k+1} \rho_k})(e^{\sqrt{\rho_{k+1}} - (1/2)\sqrt{\rho_k}}) e^{\sqrt{\rho_k}} q^{1-\lambda_k} \\ &\le (1 + \sqrt{20 \epsilon \kappa \beta^{-1} R^2 \sqrt{\rho_{k+1}}})(e^{\sqrt{\rho_{k+1}} - (1/2)\sqrt{\rho_k}}) e^{\sqrt{\rho_k}} q^{1-\lambda_k} \\ &\le (1 + \sqrt{\rho_{k+1}})(e^{\sqrt{\rho_{k+1}} - (1/2)\sqrt{\rho_k}}) e^{\sqrt{\rho_k}} q^{1-\lambda_k} \\ &\le e^{\sqrt{\rho_k}} q^{1-\lambda_k}. \end{aligned}$$

Hence, (6.6) holds. It then follows from (6.6) and the definition of $\mathbf{w}(\Omega_k, \mathbf{v})$ that

$$\begin{aligned} H_{\Omega_k}(\mathbf{v}) &= q \max\{|a_k|, |b_k + z_k a_k|\} \\ &\le q \max\{|a_{k+1}|, |b_{k+1} + z_k a_{k+1}|\} \\ &\le 2q \max\{|a_{k+1}|, |b_{k+1} + z_{k+1} a_{k+1}|\} \\ &= 2 H_{\Omega_{k+1}}(\mathbf{v}). \end{aligned}$$

This proves (6.5).

It follows from (6.5) and the induction hypothesis that $H_{\Omega_k}(\mathbf{v}) < 2H_{k+1}$. By (6.2), we have $H_{\Omega_k}(\mathbf{v}) < H_k$. Thus, (6.4) is proved. □

## 7. *Proof of Lemma 5.5*

In this section, we prove Lemma 5.5. We first establish the following lemma.

LEMMA 7.1. *Let $n \ge 0$, $\Omega \in \mathscr{B}_n$, and $\mathbf{v} = (p, r, q) \in \Xi$ with $q \le H_{2n+2}$. Then,*

$$(I_\Omega \times \Delta_{\lambda_\Omega}(\mathbf{v}, \epsilon/3)) \cap \Omega \subset \Delta(\mathbf{v}, \epsilon) \cap \Omega \subset (I_\Omega \times \Delta_{\lambda_\Omega}(\mathbf{v}, 3\epsilon)) \cap \Omega. \tag{7.1}$$

*Proof.* Let $(\lambda, x, y, z) \in \Delta(\mathbf{v}, \epsilon) \cap \Omega$. Then,

$$\left| x - \frac{p}{q} - z\left(y - \frac{r}{q}\right) \right| < \frac{\epsilon}{q^{1+\lambda}}, \quad \left| y - \frac{r}{q} \right| < \frac{\epsilon}{q^{2-\lambda}}.$$

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Hence, to show (7.1), it suffices to show that, for any $\lambda \in I_\Omega$, $q^{|\lambda-\lambda_\Omega|} \le 3$. According to Lemma 6.1, we have

$$q^{|\lambda-\lambda_\Omega|} \le e^{\rho_\Omega \log q} \le e^{\rho_\Omega \log H_{2n+2}} \le 3.$$

This completes the proof. □

LEMMA 7.2. *Let $n \ge 0$, $\Omega \in \mathscr{B}'_n$, $k \ge 1$, and $\Omega_j \in \mathscr{B}_{n+k}$ with $\Omega_j \subset \Omega$, $j = 1, 2$. Let $\mathbf{v}_j = (p_j, r_j, q_j) \in \mathcal{V}_{\Omega_j,k}$ be such that $\Delta(\mathbf{v}_j, \epsilon) \cap \Omega \ne \varnothing$ and $\mathbf{w}_j = \mathbf{w}(\Omega_j, \mathbf{v}_j)$. Then,*

$$\max\{|\mathbf{v}_1 \cdot \mathbf{w}_2|,\ |\mathbf{v}_2 \cdot \mathbf{w}_1|\} \le \begin{cases} R^{k+22}\epsilon, & k = 1, \\ R^{k+4}\epsilon, & k \ge 2. \end{cases} \tag{7.2}$$

*Proof.* For simplicity, we write $\rho_\Omega = \rho$ and $\lambda_\Omega = \lambda$. We first give a bound on $q_1$ and $q_2$ in terms of $\lambda$. Indeed, for any $\lambda' \in I_\Omega$, we have

$$H_{n+k}^{|(1/1+\lambda')-(1/1+\lambda)|} \le H_{2n}^{|\lambda'-\lambda|} \le e^{\rho \log H_{2n}} \le 3. \tag{7.3}$$

By applying (7.3) to $\lambda_{\Omega_1}$ and $\lambda_{\Omega_2}$, we get

$$\begin{aligned} &q_1, q_2 \in [\tfrac{1}{3} H_{n+k}^{(1/1+\lambda)},\ 3H_{n+k}^{(1/1+\lambda)} R^{20}], && k = 1, \\ &q_1, q_2 \in [\tfrac{1}{3} H_{n+k}^{(1/1+\lambda)} R^{2k+16},\ 3H_{n+k}^{(1/1+\lambda)} R^{2k+18}], && k \ge 2. \end{aligned} \tag{7.4}$$

Let $(\lambda'_j, x_j, y_j, z_j) \in \Delta(\mathbf{v}_j, \epsilon) \cap \Omega$. Since $\Omega_j \in \mathscr{B}_{n+k}$ and $\mathbf{v}_j \in \mathcal{V}_{\Omega_j}$, we have

$$q_j \le H_{\Omega_j}(\mathbf{v}_j) \le 3H_{n+k+1} \le 3H_{2n+1} \le H_{2n+2}.$$

Thus, it follows from Lemma 7.1 that $(x_j, y_j, z_j) \in \Delta_\lambda(\mathbf{v}_j, 3\epsilon)$. Hence,

$$\left| x_j - \frac{p_j}{q_j} - z_j\left(y_j - \frac{r_j}{q_j}\right)\right| < \frac{3\epsilon}{q_j^{1+\lambda}}, \quad \left| y_j - \frac{r_j}{q_j}\right| < \frac{3\epsilon}{q_j^{2-\lambda}}.$$

The latter inequality implies that

$$\left|\frac{r_j}{q_j}\right| \le |y_j| + \frac{3\epsilon}{q_j^{2-\lambda}} \le \kappa.$$

Thus,

$$\begin{aligned} &\left|\frac{p_1}{q_1} - \frac{p_2}{q_2} - z_{\Omega_2}\left(\frac{r_1}{q_1} - \frac{r_2}{q_2}\right)\right| \\ &\quad = \left| -\left(x_1 - \frac{p_1}{q_1} - z_1\left(y_1 - \frac{r_1}{q_1}\right)\right) + \left(x_2 - \frac{p_2}{q_2} - z_2\left(y_2 - \frac{r_2}{q_2}\right)\right)\right. \\ &\qquad \left. + (x_1 - x_2) + \frac{r_1}{q_1}(z_1 - z_{\Omega_2}) + \frac{r_2}{q_2}(z_{\Omega_2} - z_2) - (y_1 z_1 - y_2 z_2)\right| \end{aligned}$$

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$$\begin{aligned}
&\le \left|x_1 - \frac{p_1}{q_1} - z_1\left(y_1 - \frac{r_1}{q_1}\right)\right| + \left|x_2 - \frac{p_2}{q_2} - z_2\left(y_2 - \frac{r_2}{q_2}\right)\right| \\
&\quad + |x_1 - x_2| + \left|\frac{r_1}{q_1}\right||z_1 - z_{\Omega_2}| + \left|\frac{r_2}{q_2}\right||z_{\Omega_2} - z_2| + |y_1||z_1 - z_2| + |z_2||y_1 - y_2| \\
&\le \frac{3\epsilon}{q_1^{1+\lambda}} + \frac{3\epsilon}{q_2^{1+\lambda}} + 10\kappa\rho
\end{aligned}$$

and

$$\left|\frac{r_1}{q_1} - \frac{r_2}{q_2}\right| = \left| -\left(y_1 - \frac{r_1}{q_1}\right) + \left(y_2 - \frac{r_2}{q_2}\right) + (y_1 - y_2)\right| \le \frac{3\epsilon}{q_1^{2-\lambda}} + \frac{3\epsilon}{q_2^{2-\lambda}} + 2\rho.$$

Write $\mathbf{w}_j = (a_j, b_j, c_j)$. Then, according to Lemma 6.1, we have

$$|a_j| \le e^{\sqrt{\rho_{\Omega_j}}} q_j^{\lambda_{\Omega_j}} \le 9q_j^{\lambda} \quad \text{and} \quad |b_j + z_{\Omega_j} a_j| \le e^{\sqrt{\rho_{\Omega_j}}} q_j^{1-\lambda_{\Omega_j}} \le 9q_j^{1-\lambda}.$$

Set $d_k$ to be a function such that $d_k = 20$ if $k = 1$, and $d_k = 2$ otherwise. It follows that

$$\begin{aligned}
|\mathbf{v}_1 \cdot \mathbf{w}_2| &= q_1|(q_1^{-1}\mathbf{v}_1 - q_2^{-1}\mathbf{v}_2) \cdot \mathbf{w}_2| \\
&= q_1\left|a_2\left(\frac{p_1}{q_1} - \frac{p_2}{q_2}\right) + b_2\left(\frac{r_1}{q_1} - \frac{r_2}{q_2}\right)\right| \\
&= q_1\left|a_2\left(\frac{p_1}{q_1} - \frac{p_2}{q_2} - z_{\Omega_2}\left(\frac{r_1}{q_1} - \frac{r_2}{q_2}\right)\right) + (b_2 + z_{\Omega_2}a_2)\left(\frac{r_1}{q_1} - \frac{r_2}{q_2}\right)\right| \\
&\le q_1|a_2|\left(\frac{3\epsilon}{q_1^{1+\lambda}} + \frac{3\epsilon}{q_2^{1+\lambda}} + 10\kappa\rho\right) + q_1|b_2 + z_{\Omega_2}a_2|\left(\frac{3\epsilon}{q_1^{2-\lambda}} + \frac{3\epsilon}{q_2^{2-\lambda}} + 2\rho\right) \\
&\le 9q_1q_2^{\lambda}\left(\frac{3\epsilon}{q_1^{1+\lambda}} + \frac{3\epsilon}{q_2^{1+\lambda}}\right) + 9q_1q_2^{1-\lambda}\left(\frac{3\epsilon}{q_1^{2-\lambda}} + \frac{3\epsilon}{q_2^{2-\lambda}}\right) \\
&\quad + 12\kappa\rho q_1 \max\{|a_2|, |b_2 + z_{\Omega_2}a_2|\} \\
&\le 27\epsilon\left(\frac{q_2^{\lambda}}{q_1^{\lambda}} + \frac{q_1}{q_2} + \frac{q_2^{1-\lambda}}{q_1^{1-\lambda}} + \frac{q_1}{q_2}\right) + 12\kappa R^{-n}\rho_0 H_{\Omega_2}(\mathbf{v}_2)\frac{q_1}{q_2} \\
&\le 108R^{d_k}\epsilon + 720\kappa^2 R^{d_k+k+1}\epsilon \le R^{d_k+k+2}\epsilon.
\end{aligned}$$

This proves (7.2) for $|\mathbf{v}_1 \cdot \mathbf{w}_2|$. A similar argument shows the same inequality for $|\mathbf{v}_2 \cdot \mathbf{w}_1|$. □

For a closed ball $\Omega \subset \Sigma$ and $\mathbf{v} = (p, r, q) \in \Xi$, we consider the hyperplane

$$L(\Omega, \mathbf{v}) = \{(\lambda, x, y, z) \in \Sigma : a(\Omega, \mathbf{v})x + b(\Omega, \mathbf{v})y + c(\Omega, \mathbf{v}) = 0\}. \tag{7.5}$$

The next lemma states that many pairs $(\Omega, \mathbf{v})$ share the same hyperplane $L(\Omega, \mathbf{v})$.

LEMMA 7.3. *Let $n \ge 0$, $\Omega \in \mathscr{B}_n'$, $k \ge 1$. Then either:*

(i) *there is a hyperplane $L_k(\Omega)$ such that for any $\Omega' \in \mathscr{B}_{n+k}$ with $\Omega' \subset \Omega$, if $\mathbf{v} \in \mathcal{V}_{\Omega',k}$ and $\Delta(\mathbf{v}, \epsilon) \cap \Omega \ne \varnothing$, then $L(\Omega', \mathbf{v}) = L_k(\Omega)$; or*

(ii) *$k = 1$, and there is $\mathbf{v}_0 = (p_0, r_0, q_0) \in \Xi$ with $\frac{1}{3}H_{n+1}^{(1/1+\lambda_\Omega)} \le q_0 \le 3H_{n+2}$ such that for any $\Omega' \in \mathscr{B}_{n+1}$ with $\Omega' \subset \Omega$, if $\mathbf{v} \in \mathcal{V}_{\Omega',1}$ and $\Delta(\mathbf{v}, \epsilon) \cap \Omega \ne \varnothing$, then $\mathbf{v} = \mathbf{v}_0$.*

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*Proof.* We proceed by considering two separate cases.

*Case (1).* Suppose $k = 1$. We consider two subcases.

*Case (1.1).* The linear span of the set

$$\mathcal{V} := \bigcup_{\substack{\Omega' \in \mathscr{B}_{n+1} \\ \Omega' \subset \Omega}} \{\mathbf{v} \in \mathcal{V}_{\Omega',1} : \Delta(\mathbf{v}, \epsilon) \cap \Omega \neq \varnothing\}$$

is of dimension $\geq 2$. We prove that condition (i) holds. It suffices to prove that for $j = 1, 2$, if $\Omega_j \in \mathscr{B}_{n+1}$, $\Omega_j \subset \Omega$, $\mathbf{v}_j \in \mathcal{V}_{\Omega_j,1}$, $\Delta(\mathbf{v}_j, \epsilon) \cap \Omega \neq \varnothing$, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, then $L(\Omega_1, \mathbf{v}_1) = L(\Omega_2, \mathbf{v}_2)$. Write $\mathbf{w}_j = \mathbf{w}(\Omega_j, \mathbf{v}_j)$. It follows from Lemma 7.2 that

$$\max\{|\mathbf{v}_1 \cdot \mathbf{w}_2|, |\mathbf{v}_2 \cdot \mathbf{w}_1|\} \leq R^{23}\epsilon < 1.$$

Since both $\mathbf{v}_2 \cdot \mathbf{w}_1$ and $\mathbf{v}_1 \cdot \mathbf{w}_2$ are integers, they must be 0. Hence, both $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal to $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Thus, $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent. This means that $L(\Omega_1, \mathbf{v}_1) = L(\Omega_2, \mathbf{v}_2)$.

*Case (1.2).* The linear span of $\mathcal{V}$ is of dimension 1. We prove that condition (ii) holds. Let $\mathbf{v}_0 = (p_0, r_0, q_0)$ be any element in $\mathcal{V}$. It follows from $\mathcal{V} \subset \Xi$ that $\mathcal{V} = \{\mathbf{v}_0\}$. Moreover, since $\mathbf{v}_0 \in \mathcal{V}_{\Omega',1}$ for some $\Omega' \in \mathscr{B}_{n+1}$ with $\Omega' \subset \Omega$, we have $\frac{1}{3}H_{n+1}^{(1/1+\lambda_\Omega)} \leq q_0 \leq 3H_{n+2}$. This completes the proof of Case (1).

*Case (2).* Suppose $k \geq 2$. We prove that condition (i) holds. It suffices to prove that if $\Omega_j \in \mathscr{B}_{n+k}$, $\Omega_j \subset \Omega$, $\mathbf{v}_j \in \mathcal{V}_{\Omega_j,k}$, $\Delta(\mathbf{v}_j, \epsilon) \cap \Omega \neq \varnothing$, and $\mathbf{w}_j = \mathbf{w}(\Omega_j, \mathbf{v}_j)$ $(j = 1, 2)$, then $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly dependent. Let $\mathbf{v}_j = (p_j, r_j, q_j)$, $\mathbf{w}_j = (a_j, b_j, c_j)$. Then,

$$\begin{aligned}\max\{|a_j|, |b_j + z_{\Omega_j}a_j|\} = q_j^{-1}H_{\Omega_j}(\mathbf{v}_j) &\leq (\tfrac{1}{3}H_{n+k}^{(1/1+\lambda)}R^{2k+16})^{-1} \cdot 3H_{n+k+1} \\ &= 9H_{n+k}^{(\lambda/1+\lambda)}R^{-2k-15} \leq H_{n+k}^{(\lambda/1+\lambda)}R^{-2k-14}.\end{aligned}$$

Moreover, for any $(x, y, z) \in B_\Omega$, we have

$$\begin{aligned}|b_j + za_j| &\leq |b_j + z_{\Omega_j}a_j| + |a_j||z - z_{\Omega_j}| \leq 3q_j^{1-\lambda_j} + 2\rho|a_j| \\ &\leq 9q_j^{1-\lambda} + H_{n+k}^{(\lambda/1+\lambda)}R^{-2k-14} \cdot 2R^{-n}\rho_0 \leq 9q_j^{1-\lambda} + 2H_{n+k}R^{-n-2k-14}\rho_0 \\ &\leq 9q_j^{1-\lambda} + 1 \leq 10q_j^{1-\lambda} \leq 10(3H_{n+k}^{(1/1+\lambda)}R^{2k+18})^{1-\lambda} \\ &\leq 30H_{n+k}^{(1-\lambda/1+\lambda)}R^{k+9} \leq H_{n+k}^{(1-\lambda/1+\lambda)}R^{k+10}.\end{aligned}$$

Let

$$\mathbf{v}_0 = (p_0, r_0, q_0) = \mathbf{w}_1 \times \mathbf{w}_2.$$

It suffices to prove that $\mathbf{v}_0 = 0$. By the triple cross product expansion, we have

$$\mathbf{v}_1 \times \mathbf{v}_0 = \mathbf{v}_1 \times (\mathbf{w}_1 \times \mathbf{w}_2) = (\mathbf{v}_1 \cdot \mathbf{w}_2)\mathbf{w}_1.$$

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Comparing the first two components of the vectors on both sides, we obtain

$$q_0\frac{r_1}{q_1} - r_0 = q_1^{-1}(\mathbf{v}_1 \cdot \mathbf{w}_2)a_1, \tag{7.6}$$

$$q_0\frac{p_1}{q_1} - p_0 = -q_1^{-1}(\mathbf{v}_1 \cdot \mathbf{w}_2)b_1. \tag{7.7}$$

Suppose in contrast that $\mathbf{v}_0 \neq 0$. There are two cases.

*Case (2.1).* Suppose $q_0 = 0$. Then, $(p_0, r_0) \neq (0, 0)$. It then follows from (7.6) and (7.7) that

$$\begin{aligned}
1 &\leq \max\{|r_0|, |p_0 - z_{\Omega_1} r_0|\} \\
&= q_1^{-1}|\mathbf{v}_1 \cdot \mathbf{w}_2| \max\{|a_1|, |b_1 + z_{\Omega_1} a_1|\} \\
&\leq (\tfrac{1}{3} H_{n+k}^{(1/1+\lambda)} R^{2k+16})^{-1} \cdot R^{k+4}\epsilon \cdot H_{n+k}^{(\lambda/1+\lambda)} R^{-2k-14} \\
&\leq 3H_{n+k}^{(\lambda-1/1+\lambda)} R^{-3k-26}\epsilon < 1,
\end{aligned}$$

which is a contradiction.

*Case (2.2).* Suppose $q_0 \neq 0$. Without loss of generality, we may assume that $q_0 > 0$. Then,

$$\begin{aligned}
q_0 &= |a_1 b_2 - a_2 b_1| = |a_1(b_2 + z a_2) - a_2(b_1 + z a_1)| \\
&\leq 2 \cdot H_{n+k}^{(\lambda/1+\lambda)} R^{-2k-14} \cdot H_{n+k}^{(1-\lambda/1+\lambda)} R^{k+10} = 2H_{n+k}^{(1/1+\lambda)} R^{-k-4}.
\end{aligned} \tag{7.8}$$

It follows that

$$q_0/q_1 \leq 2H_{n+k}^{(1/1+\lambda)} R^{-k-4} \cdot 3H_{n+k}^{-(1/1+\lambda)} R^{-2k-16} = 6R^{-3k-20} \leq 1/6.$$

We prove that $\Delta(\mathbf{v}_1, \epsilon) \cap \Omega \subset \Delta(\mathbf{v}_0, \epsilon)$. In view of (7.8) and Lemma 7.1, it suffices to show that

$$\Delta_\lambda(\mathbf{v}_1, 3\epsilon) \cap B_\Omega \subset \Delta_\lambda(\mathbf{v}_0, \epsilon/3).$$

Let $(x, y, z) \in \Delta_\lambda(\mathbf{v}_1, 3\epsilon) \cap B_\Omega$. It follows from (7.6) and (7.7) that

$$\begin{aligned}
& q_0^{1+\lambda}\left|x - \frac{p_0}{q_0} - z\left(y - \frac{r_0}{q_0}\right)\right| \\
&\leq q_0^{1+\lambda}\left|x - \frac{p_1}{q_1} - z\left(y - \frac{r_1}{q_1}\right)\right| + q_0^{1+\lambda}\left|\frac{p_1}{q_1} - \frac{p_0}{q_0} - z\left(\frac{r_1}{q_1} - \frac{r_0}{q_0}\right)\right| \\
&< q_0^{1+\lambda}\frac{\epsilon}{q_1^{1+\lambda}} + q_0^{\lambda} q_1^{-1}|b_1 + z a_1||\mathbf{v}_1 \cdot \mathbf{w}_2| \\
&\leq \epsilon/6 + 2H_{n+k}^{(\lambda/1+\lambda)} \cdot 3H_{n+k}^{-(1/1+\lambda)} R^{-2k-16} \cdot H_{n+k}^{(1-\lambda/1+\lambda)} R^{k+10} \cdot R^{k+4}\epsilon \\
&\leq (1/6 + 6R^{-2})\epsilon < \epsilon/3
\end{aligned}$$

and

$$\begin{aligned}
q_0^{2-\lambda}\left|y - \frac{r_0}{q_0}\right| &\leq q_0^{2-\lambda}\left|y - \frac{r_1}{q_1}\right| + q_0^{2-\lambda}\left|\frac{r_1}{q_1} - \frac{r_0}{q_0}\right| \\
&< q_0^{2-\lambda}\frac{\epsilon}{q_1^{2-\lambda}} + q_0^{1-\lambda} q_1^{-1}|a_1||\mathbf{v}_1 \cdot \mathbf{w}_2|
\end{aligned}$$

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$$\le \epsilon/6 + 2H_{n+k}^{(1-\lambda/1+\lambda)} \cdot 3H_{n+k}^{-(1/1+\lambda)} R^{-2k-16} \cdot H_{n+k}^{(\lambda/1+\lambda)} R^{-2k-14} \cdot R^{k+4}\epsilon$$
$$\le (1/6 + 6R^{-26})\epsilon < \epsilon/3.$$

Thus, $(x, y, z) \in \Delta_\lambda(\mathbf{v}_0, \epsilon/3)$. This proves $\Delta(\mathbf{v}_1, \epsilon) \cap \Omega \subset \Delta(\mathbf{v}_0, \epsilon)$. It then follows from $\Delta(\mathbf{v}_1, \epsilon) \cap \Omega \neq \varnothing$ that $\Delta(\mathbf{v}_0, \epsilon) \cap \Omega \neq \varnothing$. By Lemma 5.4, we have $q_0^{1+\lambda} > 3H_{n+1}$. However, in view of (7.8), we have

$$q_0^{1+\lambda} \le 2^{1+\lambda} H_{n+k} R^{-(1+\lambda)(k+4)} \le 4R^{-4} H_{n+1} \le 3H_{n+1},$$

which leads to a contradiction. This completes the proof of the lemma. □

We are now prepared to prove Lemma 5.5.

*Proof of Lemma 5.5.* Let $n \ge 0$, $\Omega \in \mathscr{B}'_n$, $k \ge 1$. We need to prove that there is a hyperplane $L_k(\Omega)$ such that for any $\Omega' \in \mathscr{B}_{n+k}$ with $\Omega' \subset \Omega$ and any $\mathbf{v} \in \mathcal{V}_{\Omega',k}$, we have $\Delta(\mathbf{v}, \epsilon) \cap \Omega' \subset L_k(\Omega)^{(R^{-(n+k)}\rho_0)}$. We need only to consider the case that $\Delta(\mathbf{v}, \epsilon) \cap \Omega' \neq \varnothing$. By Lemma 7.3, one of the following statements holds:

(i) there is a plane $L_k(\Omega)$ such that for any $\Omega' \in \mathscr{B}_{n+k}$ with $\Omega' \subset \Omega$, if $\mathbf{v} \in \mathcal{V}_{\Omega',k}$ and $\Delta(\mathbf{v}, \epsilon) \cap \Omega \neq \varnothing$, then $L(\Omega', \mathbf{v}) = L_k(\Omega)$;

(ii) $k = 1$ and there is $\mathbf{v}_0 = (p_0, r_0, q_0) \in \Xi$ with $\frac{1}{3} H_{n+1}^{(1/1+\lambda_\Omega)} \le q_0 \le 3H_{n+2}$ such that for any $\Omega' \in \mathscr{B}_{n+1}$ with $\Omega' \subset \Omega$, if $\mathbf{v} \in \mathcal{V}_{\Omega',1}$ and $\Delta(\mathbf{v}, \epsilon) \cap \Omega \neq \varnothing$, then $\mathbf{v} = \mathbf{v}_0$.

Suppose that statement (i) holds. Let us fix such a $\Omega'$ and denote $\lambda_{\Omega'} = \lambda'$, $\rho_{\Omega'} = \rho'$. By Lemma 7.1, it suffices to show that

$$(I_{\Omega'} \times \Delta_{\lambda'}(\mathbf{v}, 3\epsilon)) \cap \Omega' \subset L(\Omega', \mathbf{v})^{(R^{-(n+k)}\rho_0)} \quad \text{for any } \mathbf{v} = (p, r, q) \in \mathcal{V}_{\Omega'}. \tag{7.9}$$

First, we have

$$\rho q \le R^{-(n+k)}\rho_0 \cdot 3H_{n+k+1} = 60\epsilon\kappa R \le 1/4.$$

Denote $\mathbf{w}(\Omega', \mathbf{v}) = (a, b, c)$. If $(x, y, z) \in \Delta_{\lambda'}(\mathbf{v}, 3\epsilon) \cap B_{\Omega'}$, then

$$\begin{aligned}
|ax + by + c| &= \left| a\left(x - \frac{p}{q} - z\left(y - \frac{r}{q}\right)\right) + (b + za)\left(y - \frac{r}{q}\right)\right| \\
&< |a|\frac{3\epsilon}{q^{1+\lambda'}} + |b + za|\frac{3\epsilon}{q^{2-\lambda'}} \\
&\le 3\epsilon q^{-1}(|a|q^{-\lambda'} + |b + z_{\Omega'}a|q^{-1+\lambda'} + |z - z_{\Omega'}||a|q^{-1+\lambda'}) \\
&\le 3\epsilon q^{-1}(3 + 3 + \rho' q^{-1+2\lambda'}) \\
&\le 3\epsilon q^{-1}(6 + \rho' q) \\
&\le 20\epsilon q^{-1}.
\end{aligned}$$

Note that

$$H_{\Omega'}(\mathbf{v}) = q \max\{|a|, |b + z_{\Omega'}a|\} \le \kappa q \max\{|a|, |b|\}.$$

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Thus,

$$\frac{|ax+by+c|}{\sqrt{a^2+b^2}} < \frac{20\epsilon}{q\max\{|a|,|b|\}} \le \frac{20\epsilon\kappa}{H_{\Omega'}(\mathbf{v})} \le \frac{20\epsilon\kappa}{H_{n+k}} = R^{-(n+k)}\rho_0.$$

This means that $I_{\Omega'} \times \{(x,y,z)\} \subset L(\Omega',\mathbf{v})^{(R^{-(n+k)}\rho_0)}$. This proves (7.9).

Suppose that statement (ii) holds. Let us write $\lambda_\Omega = \lambda,\ \rho_\Omega = \rho$. Consider the hyperplane

$$L_1(\Omega) = \left\{(\lambda,x,y,z) \in \Sigma : x - \frac{p_0}{q_0} - z_\Omega\left(y - \frac{r_0}{q_0}\right) = 0\right\}.$$

We proceed by showing that $\Delta(\mathbf{v}_0,\epsilon)\cap\Omega \subset L_1(\Omega)^{(R^{-n+1}\rho_0)}$. By Lemma 7.1, it suffices to show that

$$I_\Omega \times \Delta_\lambda(\mathbf{v}_0,3\epsilon) \subset L_1(\Omega)^{(R^{-(n+1)\rho_0})}. \tag{7.10}$$

If $(x,y,z) \in \Delta_\lambda(\mathbf{v}_0,3\epsilon)\cap B_\Omega$, then

$$\begin{aligned}
(1+z_\Omega^2)^{-1/2}\left|x - \frac{p_0}{q_0} - z_\Omega\left(y-\frac{r_0}{q_0}\right)\right| &\le \left|x - \frac{p_0}{q_0} - z\left(y-\frac{r_0}{q_0}\right)\right| + |z-z_\Omega|\left|y-\frac{r_0}{q_0}\right| \\
&< \frac{3\epsilon}{q_0^{1+\lambda}} + \rho\frac{3\epsilon}{q_0^{2-\lambda}} \\
&\le 3\epsilon q_0^{-(1+\lambda)}(1+\rho q_0) \\
&\le 27\epsilon H_{n+1}^{-1}(1+R^{-n}\rho_0\cdot 3H_{n+2}) \\
&= 27(20\kappa)^{-1}R^{-(n+1)}\rho_0(1+60\epsilon\kappa R^2) \\
&< R^{-(n+1)}\rho_0.
\end{aligned}$$

This means that $I_\Omega \times \{(x,y,z)\} \subset L_1(\Omega)^{(R^{-(n+1)\rho_0})}$. This proves (7.10). □

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