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Polyhedral immersions

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1. We work in the category of *compact* polyhedral spaces and polyhedral maps ((5)). Given spaces M and Q and a map $f\colon M \to Q$, we define $U_r(f)$ to be the set of points $x \in M$ such that $f^{-1}f(x)$ contains at least r points. $S_r(f)$ is the closure of $U_r(f)$ in M. f is an *embedding* if it is a homeomorphism into, and an *immersion* if it is locally an embedding. We shall call f a *simple immersion* if $S_3(f) = \varnothing$ and the connected components of $S_2(f)$ are individually embedded by f. Obviously a simple immersion is an immersion. If M and Q are manifolds (as they will be for the rest of the paper) $f\colon M \to Q$ is *proper* if it takes ∂M, the boundary of M, into ∂Q. In (2) the following result was proved:

THEOREM 1. *Let M^m be $(2m-q)$-connected and Q^q be $(2m-q+1)$-connected, $m \leqslant q-3$. Then any proper map $f\colon M \to Q$ which embeds ∂M is homotopic rel ∂M to an embedding.*

Here the superscript denotes the dimension of the space. It is possible to weaken the connectivity conditions if we only require f to be homotopic to an immersion. We prove

THEOREM 2. *Let M^m and Q^q be without boundary and $(2m-q)$-connected, $3m \leqslant 2q-2$. Then any map $f\colon M \to Q$ is homotopic to a simple immersion.*

THEOREM 3. *Let M^m and Q^q be without boundary, let M be $(2m-q-1)$-connected and Q be $(2m-q)$-connected. Then, if $3m \leqslant 2q-2$ and the stable homotopy group $\pi_{2m-q-1} = 0$, any map $f\colon M \to Q$ is homotopic to a simple immersion.*

N.B. The assumption that Q is compact without boundary is only for convenience in the proofs. Modifications to the theorems when M has a boundary will be discussed in section 5. Counter Example 3 of (2) gives an f which satisfies the hypotheses of Theorem 2 but not Theorem 1 and is homotopic to a simple immersion but not an embedding.

To prove both theorems we construct balls A^m in M and B^q in Q such that f maps $M-A$, ∂A, $\operatorname{int} A$ into $Q-B$, ∂B, $\operatorname{int} B$, respectively, and such that $f|M-\operatorname{int} A$ is a simple immersion. We then extend the immersion $f|\partial A$ to a proper simple immersion of A in B. In Theorem 3, the unlinking of certain pairs of spheres in ∂B ensures that this is possible. In Theorem 2, we perform an operation which may be described as reflecting the self intersections of $f(M)$ outside B into the interior of B. We also make use of Theorem 1.

2. We recall that a space K *collapses* to a subspace L if we can go from K to L by a finite sequence of *elementary collapses*. An elementary collapse of K to K_1 occurs

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if $K = K_1 \cup B$, $K_1 \cap B = C$ where B is a ball, of dimension r say, and C is an $(r-1)$-ball in ∂B. K is *collapsible* if L is a point. If K is a subspace of a manifold M, K collapses to $K \cap \partial M$, and $K \cap \partial M$ is collapsible, then K is said to be *admissibly collapsible* in M. We shall make use of Whitehead's theorem ((4)) on second derived neighbourhoods of collapsible subspaces of manifolds. Let K be a subspace of M^m. Suppose M is triangulated so that K is a subcomplex. Then

WHITEHEAD'S THEOREM. *If K is collapsible, $N(K, M'')$ is an m-ball.*

COROLLARY. *If K is admissibly collapsible in M, then*

$$N(K, M'') \cap \partial M = N(K \cap \partial M, (\partial M)'')$$

is an $(m-1)$-ball.

Here, as throughout the paper, we use the same symbol for a space (map) in the polyhedral category and for a chosen representative complex (piecewise linear map) in the combinatorial category ((5)). M'' denotes a second derived subdivision of the complex M. $N(K, M'')$, the closed simplicial neighbourhood of K in M'', consists of all closed simplexes in M'' which meet K.

Zeeman's engulfing theorem ((5), p. 68) tells us that any subspace K^k in the interior of a k-connected manifold M^m, $k \leqslant m-3$, is contained in a collapsible subspace J^{k+1} in $\operatorname{int} M$. In (2) it was proved that if M is k-connected and ∂M is $(k-1)$-connected, $k \leqslant m-3$, then any subspace K^k such that $\dim (K \cap \partial M) = k-1$ is contained in an admissibly collapsible subspace J^{k+1} in M. We can also arrange that $\dim (J \cap \partial M) = k$ in this case.

We quote a general position theorem of Zeeman ((5), p. 59). A map f of a space H^h into a manifold M^m is in *general position* if $\dim S_r(f) \leqslant rh - (r-1)m$, all r.

GENERAL POSITION THEOREM. *Any continuous map $g\colon H \to M$ is arbitrary close to and homotopic to a (polyhedral) map $f\colon H \to M$ in general position. Furthermore, if F is a subspace of H and $g|F$ is polyhedral and in general position we may require $f|F = g|F$.*

It is also true that if G is another subspace of H and $g|G$ is polyhedral and an embedding we may require $f|G$ to be an embedding. Suppose now that we have subspaces K^k, L^l in the interior of a k-connected manifold M^m, $k \leqslant m-3$. It follows from the engulfing theorem, the general position theorem and the remark following that we can find a collapsible subspace J^{k+1} containing K in $\operatorname{int} M$ such that

$$\dim (J-K) \cap L \leqslant (k+1) + l - m.$$

We call J a collapsible subspace *on K in general position with respect to L.*

Given two disjoint subspaces K_1^k and K_2^k in $\operatorname{int} M^m$, where M is k-connected and $2(k+1) \leqslant m$ we can put disjoint collapsible subspaces J_1^{k+1} on K_1 and J_2^{k+1} on K_2 in $\operatorname{int} M$. (Notice $2(k+1) \leqslant m$ implies $k \leqslant m-3$ except when $k = 0$, $m = 2$, when the engulfing theorem is still true, as is well known.) For we may first put J_1 on K_1 in general position with respect to K_2, so that $J_1 \cap K_2 = \varnothing$ ($(k+1)+k < m$), then remove a second derived neighbourhood (an m-ball) of J_1, and finally put J_2 on K_2 in the manifold left, which is k-connected. A similar argument shows that if M is k-connected, ∂M

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is $(k-1)$-connected, $2(k+1) \leqslant m$, then we may put disjoint admissibly collapsible subspaces J_1^{k+1}, J_2^{k+2} on disjoint subspaces K_1^k, K_2^k such that

$$\dim (K_1 \cap \partial M) = \dim (K_2 \cap \partial M) = k-1.$$

Let A^m and B^q be balls, for any m and q. We can choose a proper extension $f: A \to B$, which we shall call a *linear extension*, of any given map $f: \partial A \to \partial B$ as follows: Choose triangulations $s: C \to A$ and $t: D \to B$ of A and B by simplexes C and D. Choose points $x \in \operatorname{int} C$ and $y \in \operatorname{int} D$. We have a map $h = t^{-1}fs: \partial C \to \partial D$. Define $h(x) = y$, and extend h linearly over C, which is the join of x with ∂C. Finally, define $f: A \to B$ by $f = ths^{-1}$. If $f|\partial A$ is an embedding, so is $f: A \to B$. Similarly, we may choose a linear extension $f: A_1 \cup A_2 \to B$ of $f: \partial A_1 \cup \partial A_2 \to \partial B$, where $A_1 \cup A_2$ is the disjoint union of two balls. If $f|\partial A_1$ and $f|\partial A_2$ are embeddings, then $f: A_1 \cup A_2 \to B$ is a simple immersion; the connected components of $S_2(f)$ are cones on $S_2(f)|\partial A_1$ and $S_2(f)|\partial A_2$.

Now suppose that A^m and B^m are balls such that $A \cap B = \partial A \cap \partial B = C^{m-1}$, a ball. Both $\partial A - \operatorname{int} C$ and $\partial B - \operatorname{int} C$ are $(m-1)$-balls ((1)). We may extend the identity on ∂C linearly to a homeomorphism of $\partial A - \operatorname{int} C$ onto $\partial B - \operatorname{int} C$. This, together with the identity on C, gives us a homeomorphism of ∂A onto ∂B. We may extend this linearly to a homeomorphism of A onto B. We shall call a homeomorphism of this type a *reflexion of A onto B through C*.

3. In this section we state and prove two lemmas which go most of the way towards proving Theorems 2 and 3. Throughout the section M^m and Q^q are manifolds without boundary, A^m and B^q are balls in M and Q, respectively, and $f: M \to Q$ is a map which takes $M - A$, ∂A, $\operatorname{int} A$ into $Q - B$, ∂B, $\operatorname{int} B$, respectively. Further, we suppose $M_0 = Cl(M - \operatorname{int} A)$ and $Q_0 = Cl(Q - \operatorname{int} B)$, so $\partial M_0 = \partial A$ and $\partial Q_0 = \partial B$. The index i takes the values 1 and 2.

LEMMA 1. *Suppose $f|M_0$ is a simple immersion and $S_2(f|M_0)$ has as connected components two subspaces C_i^{2m-q} such that* $\dim C_i \cap \partial M_0 = 2m-q-1$. *Then, if M_0 is $(2m-q)$-connected and $3m \leqslant 2q-2$, f is homotopic to a simple immersion.*

Proof. If $2m \leqslant q$, then moving f into general position gives us a simple immersion of M in Q. So suppose $2m-q \geqslant 1$. Put disjoint admissibly collapsible subspaces D_i^{2m-q+1} on C_i in M_0 $(2(2m-q+1) \leqslant m)$ and a collapsible subspace D^{2m-q+1} on $f((D_1 \cup D_2) \cap \partial A)$ in ∂B in general position with respect to $f(\partial A)$, so that

$$D \cap f(\partial A) = f((D_1 \cup D_2) \cap \partial A) \quad ((2m-q+1)+(m-1) < q-1).$$

Since D_1 collapses to $D_1 \cap \partial A$ and $f|D_1$ is an embedding, $f(D_1)$ collapses to $f(D_1 \cap \partial A)$, i.e. $f(D_1) \cap \partial B$, so $E = f(D_1) \cup D$ is admissibly collapsible in Q_0. Triangulate M and Q so that A, B, D_i and E are subcomplexes and $f: M \to Q$ is simplicial. With these triangulations let

$$F_i^m = N(D_i, M_0''), \quad G_i^m = N(D_i \cap \partial A, A''), \quad H_i^{m-1} = N(D_i \cap \partial A, (\partial A)''),$$
$$F^q = N(E, Q_0''), \quad G^q = N(D, B''), \quad H^{q-1} = N(D, (\partial B)''),$$

where K'' denotes the second barycentric derived complex of K. All these subspaces are balls by Whitehead's Theorem. $F_i \cap G_i = \partial F_i \cap \partial G_i = H_i$ and $F \cap G = \partial F \cap \partial G = H$. We may, then, choose reflexion homeomorphisms r_i of F_i onto G_i through H_i and r of F onto G through H.

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The balls $F_1 \cup G_1$ and $F_2 \cup G_2$ are disjoint, and f embeds F_1 properly in F because $f^{-1}(E) = D_1 \cup (C_2 \cup (D_2 \cap \partial A))$. It also embeds the submanifold

$$J^m = N(C_2 \cup (D_2 \cap \partial A), M_0'')$$

properly in F. Further F contains no other points of $f(M)$.

We now produce a simple immersion g of M in Q. We define g on M_0 by $g = f$, on G_1 by $g = rfr_1^{-1}$ and on $r_2(J)$ by $g = rfr_2^{-1}$. At this stage we have a well-defined proper simple immersion of $M_0 \cup G_1 \cup r_2(J)$ in $Q_0 \cup G$ taking interior points into interior points. The connected components of $S_2(g)$ are $C_i \cup r_i(C_i)$, which are situated in the interior of $M_0 \cup G_1 \cup r_2(J)$. Let M_1 and Q_1 be the closed complements of $M_0 \cup G_1 \cup r_2(J)$ and $Q_0 \cup G$ in M and Q, respectively. We assert that we can extend the embedding of ∂M_1 in ∂Q_1 to an embedding of M_1 in Q_1. First, ((1)) $Q_1 = Cl(B - G)$ is a q-ball, because $\partial G \cap \partial B = H$ is a $(q-1)$-ball. Similarly, $Cl(A - G_1)$ is an m-ball, and it is easy to see that $M_1 = Cl(A - G_1 \cup r_2(J))$ is $(m-(2m-q)-2)$-connected, i.e. $(q-m-2)$-connected (M_1 is effectively A minus a second derived neighbourhood of $r_2(C_2)$ and a $(q-m-1)$-complex in general position in A does not meet $r_2(C_2)$). We may extend $g|\partial M_1$ to a continuous map g_1 of M_1 into Q_1, as Q_1 is a ball. By the general position theorem we may deform g_1 rel ∂M_1 to a (polyhedral) map g_2 of M_1 into Q_1. Then, by Theorem 1, g_2 is homotopic rel ∂M_1 to an embedding $g: M_1 \to Q_1$ (M_1 is certainly $(2m-q)$-connected, as $2m - q \leqslant q - m - 2$). Obviously, as we have only altered f on the m-ball A inside the q-ball B, g is homotopic to f.

LEMMA 2. *If* $3m \leqslant 2q-2$ *and* $\pi_{2m-q-1} = 0$, *any simple immersion* g: $\partial A \to \partial B$ *such that* $S_2(g)$ *has just two connected components* C_i^{2m-q-1} *may be extended to a simple immersion of* A *in* B.

COROLLARY. *Suppose* $f|M_0$ *is a simple immersion and* $S_2(f|M_0)$ *has as connected components two subspaces* C_i^{2m-q} *such that* $\dim (C_i^{2m-q} \cap \partial M_0) = 2m-q-1$. *Then if*

$$3m \leqslant 2q-2 \quad \textit{and} \quad \pi_{2m-q-1} = 0,$$

f is homotopic to a simple immersion.

The corollary follows immediately from the lemma. Define $g|M_0 = f|M_0$ and extend $g|\partial A$ to $g: A \to B$ by the lemma. Then the simple immersion g is homotopic to f. Of course it may happen that $S_2(g|\partial A)$ has more than two connected components, but it is easy to see that we can deal separately with each pair which is identified under g.

Proof of Lemma 2. If $m < 4$, $3m \leqslant 2q-2$ implies C_i^{2m-q-1} are empty, and we may extend the embedding of ∂A in ∂B linearly to an embedding of A in B. Suppose $m \geqslant 4$. Put disjoint collapsible subspaces D_i^{2m-q} on C_i in ∂A $(2(2m-q) < m-1)$ and a collapsible subspace D^{2m-q+1} on $g(D_1 \cup D_2)$ in ∂B in general position with respect to $g(\partial A)$, so that $D \cap g(\partial A) = g(D_1 \cup D_2)$ $((2m-q+1)+(m-1) < q-1)$. Triangulate A and B so that D_i and D are subcomplexes and $g: \partial A \to \partial B$ is simplicial. With these triangulations let $E_i^m = N(D_i, A'')$, $F_i^{m-1} = N(D_i, (\partial A)'')$, $E^q = N(D, B'')$ and $F^{q-1} = N(D, (\partial B)'')$. These subspaces are all balls by Whitehead's Theorem. Clearly $E_i \cap \partial A = F_i$ and $E \cap \partial B = F$. The map g embeds F_i properly in F, and takes no

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other points of ∂A into F. Also $C_i \subset \operatorname{int} F_i$. By the Corollary on p. 191 of (6), the conditions $$m-2 \geqslant 2, \quad 3m \leqslant 2q-2 \quad \text{and} \quad \pi_{m-2}(S^{q-m-1}) = 0$$ imply that the $(m-2)$-spheres $g(\partial F_i)$ are unlinked in the $(q-2)$-sphere ∂F. Hence, they bound disjoint $(m-1)$-balls, G_i say, properly embedded in $\partial E - \operatorname{int} F$ which is a $(q-1)$-ball ((1)). Also $\partial E_i - \operatorname{int} F_i$ are $(m-1)$-balls ((1)).

We extend the homeomorphisms $g|\partial F_i : \partial F_i \to \partial G_i$ linearly to homeomorphisms of $\partial E_i - \operatorname{int} F_i$ onto G_i. These together with $g|F_i$ give a simple immersion of $\partial E_1 \cup \partial E_2$ in ∂E. We may extend this linearly to a proper simple immersion of $E_1 \cup E_2$ in E. It remains now to define g on $M_1 = Cl(M - E_1 \cup E_2)$. But M_1 is an m-ball ((1)), and already its boundary is embedded in that of the q-ball ((1)) $Q_1 = Cl(Q-E)$. Hence another linear extension completes the proof of Lemma 2.

4. *Proofs of Theorems* 2 *and* 3. We may assume that f is in general position, so that $\dim S_2(f) \leqslant 2m-q$ and $S_3(f) = \varnothing$. Triangulate M and Q so that f is simplicial. Then $S_2(f)$ is a subcomplex of the complex M, and the branch locus $S_2(f) - U_2(f)$ is contained in the $(2m-q-1)$-skeleton, X say, of $S_2(f)$. Put a collapsible subspace Y^{2m-q} on X in M in general position with respect to $S_2(f)$, so that $Y \cap S_2(f) = X \quad (2(2m-q) < m)$, and a collapsible subspace Z^{2m-q+1} on $f(Y)$ in Q in general position with respect to $f(M)$, so that $Z \cap f(M) = f(Y) \quad ((2m-q+1)+m < q)$. Subdivide the given triangulations of M and Q so that Y and Z are subcomplexes and f is simplicial. In the subdivided complexes M and Q let $A^m = N(Y, M'')$ and $B^q = N(Z, Q'')$. Then A and B are balls and f maps $M-A, \partial A, \operatorname{int} A$ into $Q-B, \partial B, \operatorname{int} B$, respectively. The branch locus of f is contained in $\operatorname{int} A$. In fact $f|M - \operatorname{int} A$ is a simple immersion. The connected components of $S_2(f|M - \operatorname{int} A)$ are certain $(2m-q)$-balls properly embedded in $M - \operatorname{int} A$; these are exactly the $(2m-q)$-simplexes of $S_2(f)$ (in the original triangulation of M), subdivided and with a second derived neighbourhood of their boundaries removed. It is clear that we may deal separately with each pair of balls which is identified under f. All operations performed in the proofs of Lemmas 1 and 2 may be carried out well clear of any other pair. We now appeal to these lemmas and the corollary to complete the proofs of the theorems. (N.B. If $\dim S_2(f) < 2m-q$, then $S_2(f) = X \subset \operatorname{int} A$, f embeds ∂A in ∂B and may be extended linearly to an embedding of A in B. This gives an embedding of M in Q.)

The content of this section may be put in the form.

THEOREM 4. *If for given m and q with $3m \leqslant 2q-2$ any simple immersion of ∂A^m in ∂B^q may be extended to an immersion of A in B, where A and B are balls, then any map $f: M^m \to Q^q$ is homotopic to an immersion, where M is $(2m-q-1)$-connected and Q is $(2m-q)$-connected. If further any immersion of ∂A in ∂B can be extended to an immersion of A in B we can remove the connectivity condition on Q.*

To see the validity of the last remark, notice that if we had put a collapsible subspace Y on the branch locus itself instead of on X and if Y were in general position with respect to $S_2(f)$, so that $Y \cap S_2(f) =$ the branch locus, then f would embed Y and $f(Y)$ would be collapsible. We could then take second derived neighbourhoods A of Y and B of $f(Y)$. $f|\partial A$ would immerse ∂A in ∂B, but would not in general be a simple immersion.

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5. If the boundary of M is non-empty and $f: M \to Q$ is a proper map, then Theorems 2 and 3 still apply if $f|\partial M$ is an embedding. The simple immersion of M in Q is now homotopic to f rel ∂M. Theorem 3 also holds if $f|\partial M$ is a simple immersion, provided that the connected components of $S_2(f|\partial M)$ have dimension $k \leqslant \frac{1}{2}(m-3)$ and ∂M is k-connected. If f is not proper but $S_2(f) \subset \operatorname{int} M$ then both theorems hold without any connectivity condition on ∂M. Theorem 2 can even be improved to an embedding theorem, as we may pipe the self-intersections off $f(M)$ over $f(\partial M) \cap \operatorname{int} Q$ by the technique of Lemma 2.7 of (3).

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